On the combinatorial principle $P(c)$

by

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Abstract. We show that the combinatorial principle $P(c)$ is equivalent to Martin's Axiom restricted to sigma-centered posets. This answers affirmatively a conjecture of F. D. Tall. We also show that $P(c)$ implies the existence of a first countable Dowker space.

0. Introduction. This paper is organized as follows: In Section 1 we introduce necessary definitions and prove that $P(c)$ implies Martin's Axiom for σ-centered posets. This establishes the equivalence of these two statements. In Section 2 we exploit this result to show that $P(c)$ implies there is a first countable Dowker space.

Our set-theoretic notation is standard. ω is the first infinite ordinal and c is the least ordinal of the same cardinality as $\{x\colon x\subseteq\omega\}$ (we assume the axiom of choice). If γ is an ordinal, then $\mathrm{cf}\gamma$ is the least ordinal α such that there exists $f\colon \alpha\to\gamma$ with $\gamma = \bigcup \mathrm{Range}$ of f. A subset C of c is closed and unbounded in c if C is closed and unbounded in the ordinal topology of c. A subset S of c is stationary if S meets each closed and unbounded subset of c.

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1. Martin's Axiom for σ-centered posets is eqivalent to $P(c)$.

1.0. Preliminaries. Let $(P, \leqslant)$ be a partially ordered set, i.e., a poset. $p, q\in P$ are *compatible* if there exists an $r\in P$ such that $r\leqslant p$ and $r\leqslant q$. A subset S of P is *open* if $p\leqslant q$ and $q\in S$ implies that $p\in S$. A subset S of P is *centered* if for each finite $F\subseteq S$, there is a $p\in P$ such that $p\leqslant q$ for all $q\in F$. A subset S of P is *dense* if for each $p\in P$ there is an $s\in S$ with $s\leqslant p$. A *c.c.c. poset* is one in which every uncountable subset contains two compatible elements. A *σ-centered poset* is one that is the union of countably many centered subsets. Let $\mathscr{D}$ be a collection of dense subsets of P. $G\subseteq P$ is *Y-generic* if

(1) if $p\leqslant q$ and $p\in G$, then $q\in G$,

(2) for every $\{p, q\}\subseteq G$ there is an $r\in G$ with $r\leqslant p$ and $r\leqslant q$,

(3) for each $D\in\mathscr{D}$, $G\cap D\neq\emptyset$.

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Martin's Axiom (see Martin and Solovay [9]) states that for every c.c.c. poset P and for every collection $\mathscr{D}$ of fewer than c dense subsets of P, there is a $\mathscr{D}$-generic $G\subseteq P$. An equivalent topological statement is that no compact T_2 space of countable cellularity is the union of fewer than c nowhere dense sets. Martin's Axiom for σ-centered posets is Martin's Axiom with c.c.c. replaced by σ-centered. An equivalent topological statement is that no compact T_2 separable space is the union of fewer than c nowhere dense sets (see Kunen and Tall [6]).

A collection of sets S is called *centered* if $\bigcap F$ is infinite for every finite $F\subseteq S$. Consider the following three combinatorial statements:

$P(c)$: For each centered collection $\mathscr{A}$ of fewer than c subsets of ω, there is an infinite $B\subseteq\omega$ such that $B-A$ is finite for all $A\in\mathscr{A}$.

$\mathrm{BF}(c)$: For each collection $\mathscr{H}$ of fewer than c mappings from ω to ω, there is an $f\colon \omega\to\omega$ such that for each $h\in\mathscr{H}$ there is an $N_h<\omega$ such that $h(n)\leqslant f(n)$ for each $n\geqslant N_h$.

$\mathrm{BB}(c)$: For each collection $\mathscr{H}$ of fewer than c mappings from $\bigcup_{n<\omega} {}^n\omega$ to ω, there is an $f\colon \omega\to\omega$ such that for each $h\in\mathscr{H}$ there is an $N_h<\omega$ such that $h(f\restriction n)\leqslant f(n)$ for each $n\geqslant N_h$.

Martin's Axiom implies Martin's Axiom for σ-centered posets implies $P(c)$ implies $\mathrm{BF}(c)$. For elaborations on these implications see Booth [2], Burke and van Douwen [3], Kunen and Tall [6] and Rothberger [10]. It is also known that $\mathrm{BF}(c)$ does not imply $P(c)$ and that Martin's Axiom for σ-centered posets does not imply Martin's Axiom (cf. Kunen and Tall [6]). For more consequences of $P(c)$ see Rudin [14] and Tall [15]. We will show that $P(c)$ implies Martin's Axiom for σ-centered posets (thus answering affirmatively a conjecture of F. D. Tall). In the process we show that $P(c)$ implies $\mathrm{BB}(c)$.

For the rest of Section 1 we denote ${}^n\omega$ by $\mathscr{F}_n$ and $\bigcup_{n<\omega} {}^n\omega$ by $\mathscr{F}$. The following lemma is D. K. Burke and E. K. van Douwen's proof that $P(c)$ implies $\mathrm{BF}(c)$ with minor modifications.

1.1. LEMMA. *$P(c)$ implies* $\mathrm{BB}(c)$.

Proof. Let $\{f_\alpha\colon \alpha<\varkappa\}$ be $\varkappa<c$ mappings from $\mathscr{F}$ to ω. Let $\leqslant^*$ well order $\mathscr{F}$ in order type ω. Without loss of generality, we assume that for each $\alpha<\varkappa$ and for each $s\leqslant^* t$, $f_\alpha(s)\leqslant f_\alpha(t)$, i.e., f_α is increasing with respect to $\leqslant^*$.

Define $\mathscr{E} = \{\{(s,n)\colon f_\alpha(s)\leqslant n\}\colon \alpha<\varkappa\}\cup\{\{t\colon s\leqslant^* t\}\times\omega\colon s\in\mathscr{F}\}$. By $P(c)$, there is an infinite $B\subseteq\mathscr{F}\times\omega$ such that for each $E\in\mathscr{E}$, $B-E$ is finite. For each $s\in\mathscr{F}$ define $A_s = \{s\}\times\omega$. Define $h\colon \mathscr{F}\to\mathscr{F}$ by $h(s)=$ the first t (under $\leqslant^*$) where $s\leqslant^* t$ and $B\cap A_t\neq\emptyset$. Define $f\colon \omega\to\omega$ by $f(n)=\min\{k\in\omega\colon (h(f\restriction n),k)\in B\}$.

This f does the job. Let $\alpha<\varkappa$. There exists s such that for each $s\leqslant^* t$, if $(t,k)\in B$, then $f_\alpha(t)\leqslant k$. Choose N_α such that for each $n\geqslant N_\alpha$, $s\leqslant^* f\restriction n$, and therefore $s\leqslant^* h(f\restriction n)$. Then for each $n\geqslant N_\alpha$, $f_\alpha(f\restriction n)\leqslant f_\alpha(h(f\restriction n))\leqslant f(n)$. ■

Prior to our proof of the theorem, we make one observation. Martin's Axiom

for σ-centered posets is equivalent to the statement that for each σ-centered poset $(P,\leqslant)$ and for each collection $\mathscr{D}$ of fewer than c dense subsets of P, there is a pairwise compatible collection C of elements of P that contains an element from each $D\in\mathscr{D}$. To prove the non-trivial direction, let P be a σ-centered poset and let $\mathscr{D}$ be a collection of fewer than c dense subsets of P. By a Lowenheim–Skolem argument, we may assume that $|P|<c$. For each pair $a,b\in P$ define $D_{ab}=\{q\in P\colon q$ is not compatible with either a or $b\}\cup\{q\in P\colon q\leqslant a$ and $q\leqslant b\}$. Then $\mathscr{D}'=\mathscr{D}\cup\{D_{ab}\colon a,b\in P\}$ is a collection of fewer than c dense subsets of P. Hence, there exists a pairwise compatible collection C of elements of P that contains an element from each $D\in\mathscr{D}'$. Then, $\{p\in P\colon$ there exists a $q\in C$ with $q\leqslant p\}$ is a $\mathscr{D}$-generic subset of P.

1.2. THEOREM. *$P(c)$ implies Martin's Axiom for σ-centered posets.*

Proof. Let $P=\bigcup_{1\leqslant m<\omega} P_m$ where each P_m is centered. We denote the partial order on P by $\leqslant$. Let $\{D_\alpha\colon \alpha<\varkappa\}$ be $\varkappa<c$ dense subsets of P. Without loss of generality we may assume that each D_α is open. We must find a pairwise compatible collection C of elements of P that contains an element from each D_α.

For each $\alpha<\varkappa$ define $B_\alpha=\{m\colon D_\alpha\cap P_m\neq\emptyset\}$. For each $\alpha<\varkappa$ and for each $p\in P$ define $B_\alpha(p)=\{m\colon$ there is a $q\in D_\alpha\cap P_m$ with $q\leqslant p\}$. Without loss of generality, for each m, $\{B_\alpha(p)\colon \alpha<\varkappa$ and $p\in P_m\}$ is centered. Otherwise, there exists $\alpha_1<\alpha_2\ldots<\alpha_n<\varkappa$ and $p\in P$ such that $\bigcap_{i\leqslant n} D_{\alpha_i}\cap\{q\colon q\leqslant p\}\subseteq\bigcup_{i\leqslant n_0} P_i$ for some n_0; for each $i\leqslant n_0$ we may assume that there exists β_i such that $D_{\beta_i}\cap P_i=\emptyset$ (otherwise, we could use this P_i as our compatible collection). Thus, $\bigcap_{i\leqslant n} D_{\alpha_i}\cap\{q\colon q\leqslant p\}\cap\cap\bigcap_{i\leqslant n_0} D_{\beta_i}=\emptyset$. This contradicts open denseness of the D_α's. By $P(c)$, for each m, there is an infinite A_m such that for each $p\in P_m$, $A_m-B_\alpha(p)$ is finite for all $\alpha<\varkappa$. Note that for each m and for each α, A_m-B_α is finite. $\mathrm{BF}(c)$ implies that there is an infinite A_0 such that for each m and for each α, A_m-A_0 is finite and A_0-B_α is finite.

For each $m\geqslant 0$, enumerate A_m as $\{a_m(j)\colon 0\leqslant j<\omega\}$. Define $g\colon \mathscr{F}\to\omega$ by induction as follows:

$g(\varphi)=0$,

if $n>0$ and $s\in\mathscr{F}_n$, then $g(s)=a_{g(s\restriction n-1)}(s(n-1))$.

NOTATION. If $s\in\mathscr{F}_n$ and $r\in\omega$, then $sr=s\cup\{(\operatorname{dom} s,r)\}$. For each $r\in\omega$, $\langle r\rangle$ denotes $s\in\mathscr{F}_1$ where $s(0)=r$.

Fix $\alpha<\varkappa$. We now define $f_\alpha\colon \mathscr{F}\to\omega$. This we do by induction on $n<\omega$. We simultaneously define a set $\{d_\alpha(s,r)\colon s\in\mathscr{F}_n$ and $r\in\omega\}\subseteq P$.

Stage $n=0$: Define $f_\alpha(\varphi)$ so that for each $r\geqslant f_\alpha(\varphi)$ there is a $d\in D_\alpha\cap P_{g(\langle r\rangle)}$. This is possible because $g(\langle r\rangle)=a_0(r)$ and A_0-B_α is finite. For each $r\in\omega$ choose $d_\alpha(\varphi,r)\in P_{g(\langle r\rangle)}$ so that for each $r\geqslant f_\alpha(\varphi)$, $d_\alpha(\varphi,r)\in D_\alpha$.

Assume that for each $i<n$ and for each $s\in\mathscr{F}_i$ we have defined $f_\alpha(s)$ and $\{d_\alpha(s,r)\colon r\in\omega\}$ such that 1° for each $r\in\omega$, $d_\alpha(s,r)\in P_{g(sr)}$ and 2° for each $r\geqslant f_\alpha(s)$, $d_\alpha(s,r)\in D_\alpha$ and $d_\alpha(s,r)\leqslant d_\alpha(s\restriction i-1, s(i-1))$.

Stage $n>0$: For each $s\in\mathscr{F}_n$ define $f_\alpha(s)$ so that for each $r\geqslant f_\alpha(s)$ there is a $d \in D_\alpha \cap P_{h(sr)}$ such that $d \leqslant d_\alpha(s \upharpoonright n-1, s(n-1))$. This is possible since $d_\alpha(s \upharpoonright n-1, s(n-1)) \in P_{g(s)}$, $g(sr) = a_{g(s)}(r)$ and $A_{g(s)} - B_\alpha(d_\alpha(s \upharpoonright n-1, s(n-1)))$ is finite. For each $r \in \omega$, choose $d_\alpha(s, r) \in P_{g(sr)}$ so that for each $r \geqslant f_\alpha(s)$, $d_\alpha(s, r) \in D_\alpha$ and $d_\alpha(s, r) \leqslant d_\alpha(s \upharpoonright n-1, s(n-1))$. This completes the inductive step.

By BB(c), there is an $f\colon \omega \to \omega$ such that for each $\alpha < \varkappa$ there is an N_α such that for each $n \geqslant N_\alpha$, $f_\alpha(f \upharpoonright n) \leqslant f(n)$.

Claim. $\{d_\alpha(f \upharpoonright N_\alpha, f(N_\alpha))\colon \alpha < \varkappa\}$ *is the collection that we are striving for.*

Proof. For each α, $d_\alpha(f \upharpoonright N_\alpha, f(N_\alpha)) \in D_\alpha$. This follows from 2^0 and the fact that $f_\alpha(f \upharpoonright N_\alpha) \leqslant f(N_\alpha)$.

To show compatibility, we take $\alpha \neq \beta$. Assume $N_\alpha \leqslant N_\beta$. Therefore, $f_\alpha(f \upharpoonright N_\beta) \leqslant f(N_\beta)$ and $d_\alpha(f \upharpoonright N_\beta, f(N_\beta)) \in P_{g(f \upharpoonright N_\beta + 1)}$ by 1^0. By repeated application of 2^0 we get that $d_\alpha(f \upharpoonright N_\beta, f(N_\beta)) \leqslant d_\alpha(f \upharpoonright N_\alpha, f(N_\alpha))$. Since $P_{g(f \upharpoonright N_\beta + 1)}$ is centered we get a q such that $q \leqslant d_\alpha(f \upharpoonright N_\beta, f(N_\beta))$ and $q \leqslant d_\beta(f \upharpoonright N_\beta, f(N_\beta))$ and we are done. ■

2. A first countable Dowker space from $P(c)$.

2.0. Preliminaries. A *Dowker space* is a normal T_2 space X such that X is not countably metacompact. *Countable metacompactness* is equivalent to the statement that for every decreasing sequence of closed sets $\{C_n\colon n<\omega\}$ with $\bigcap_{n<\omega} C_n = \emptyset$, there is a decreasing sequence of open sets $\{O_n\colon n<\omega\}$ with $C_n \subseteq O_n$ for each $n<\omega$ and such that $\bigcap_{n<\omega} O_n = \emptyset$. *Normality* is equivalent to the statement that for every finite sequence of closed sets $\{C_i\colon i<n\}$, with $\bigcap_{i<n} C_i = \emptyset$ there is a finite sequence of open sets $\{O_i\colon i<n\}$ with $C_i \subseteq O_i$ for each $i<n$ and such that $\bigcap_{i<n} O_i = \emptyset$.

M. E. Rudin [13] has shown that there exists a Dowker space. In [13], she asks if there exist Dowker spaces with small cardinal functions; in particular, is there a first countable Dowker space? This question is still open. However, several investigators, P. de Caux [4], Juhász, Kunen and Rudin [7], Rudin [12] and Weiss [16] have constructed first countable Dowker spaces using additional set-theoretic assumptions known to be consistent with the axioms of Zermelo and Fraenkel for set theory. Moreover, these spaces are small in other respects as well, e.g., density character and cardinality. Our space is of interest due to the relative weakness of the assumption $P(c)$ in this context.

Several acknowledgments are in order; the idea of refining the euclidean topology on a Luzin set is drawn from Juhász, Kunen and Rudin [7], the idea of using stationary and closed unbounded subsets is drawn from Weiss [16], the partial order used in Fact 1 of our theorem is drawn from Kunen [5] and the partial order used in Fact 2 is drawn from Juhász and Weiss [8].

Several definitions are in order; a *generalized Luzin subset* of the Cantor cube 2^ω is a subspace of 2^ω of size c in which each nowhere dense subset has size $<c$ and a *regular refinement* of a topological space (X, t_1) is a topological space (X, t_2) where $t_1 \subseteq t_2$ and for each $x \in X$ and $O_1 \in t_1$ with $x \in O_1$, there is an $O_2 \in t_2$ such that $x \in O_2$ and $\mathrm{Cl}_{t_1} O_2 \subseteq O_1$. $P(c)$ implies that there is a generalized Luzin subset of 2^ω (a straightforward induction following from the fact that 2^ω is not the union of $<c$ nowhere dense subsets; for a proof of this latter fact, see Rothberger [11]), that every regular refinement of a separable metric space of size $<c$ is normal (see Alster and Przymusinski [1]) and that c is a regular cardinal (this follows from the fact that $2^\varkappa = 2^\omega$ for $\omega \leqslant \varkappa < c$ which is essentially proved in Rothberger [10]).

2.1. A Dowker space. Let (Y, e) denote the Cantor cube 2^ω with the euclidean topology. $S \subseteq Y$ is e-nowhere dense if $\mathrm{Int}_e(\mathrm{Cl}_e S) = \emptyset$. Let $\mathscr{B}$ denote a countable basis of clopen sets for Y which is closed under finite unions. Let L be a generalized Luzin subset of Y such that for each non-empty e-open U, $L \cap U$ is generalized Luzin. Enumerate all countable subsets of L as $\{C_\gamma\colon \gamma<c\}$. Define

$$T_c = \{\gamma<c\colon \mathrm{cf}\gamma \leqslant \omega\}.$$

Let $\{S(B, r)\colon B \in \mathscr{B}$ and $r<\omega\}$ be a disjoint partition of T_c where each $S(B, r)$ is a stationary subset of c. For each $r<\omega$, define $S_r = \bigcup\{S(B, r)\colon B \in \mathscr{B}\}$.

Theorem. *$P(c)$ implies that there is a first countable Dowker space.*

Proof. By induction on $\gamma \in T_c$, we define a subset $X = \{x_\gamma\colon \gamma \in T_c\} \subseteq L$, sets $\{U_\gamma(n)\colon \gamma \in T_c$ and $1 \leqslant n < \omega\}$ and topologies t_γ on X_γ; where $X_\gamma = \bigcup_{r<\omega} X_\gamma(r)$ and $X_\gamma(r) = \{x_\alpha\colon \alpha \in T_c,\ \alpha \leqslant \gamma$ and $\alpha \in \bigcup_{i\leqslant r} S_i\}$: such that for every $\alpha<\gamma$ in T_c:

1_γ. $t_\alpha = t_\gamma \cap \{M\colon M \subseteq X_\alpha\}$, t_γ refines the e-topology on X_γ and X_α is t_γ-clopen.

2_γ. $\{U_\gamma(n)\colon 1 \leqslant n < \omega\}$ is a t_γ-basic neighbourhood system at x_γ consisting of t_γ-clopen and e-nowhere dense sets. Furthermore,

a) for each n, $U_\gamma(n+1) \subseteq U_\gamma(n)$,

b) for each n, $U_\gamma(n) = X_\gamma \cap \mathrm{Cl}_e U_\gamma(n)$,

c) $x_\gamma \notin \bigcup\{\mathrm{Cl}_e U_\gamma(1)\colon \alpha<\gamma\}$,

d) $\gamma \in S(B, r)$ implies $U_\gamma(1) \subseteq X_\gamma(r)$.

3_γ. If $\gamma \in S(B, r)$ and $\mathrm{cf}\gamma = \omega$, then there exists an increasing sequence of ordinals less than γ, $\{\gamma_n\colon n<\omega\}$ which converges to γ, such that for every $\{\alpha_n\colon n<\omega\}$ with

(i) for each $n<\omega$, $\alpha_n<\gamma$ and $B \subseteq \mathrm{Cl}_e C_{\alpha_n}$ and

(ii) for each $n<\omega$, $C_{\alpha_n} \subseteq B \cap [X_\gamma(r) - X_{\gamma_n}(r)]$,

we have that $x_\gamma \in \mathrm{Cl}_{t_\gamma}(\bigcup\{C_{\alpha_n}\colon n<\omega\})$.

Assume for the moment, that we can complete the inductive step. Let t be generated by $\bigcup\{t_\gamma\colon \gamma \in T_c\}$. (X, t) is a refinement of (X, e) such that each point has a countable neighbourhood base consisting of (X, t) clopen, (X, e) closed and e-nowhere dense sets.

A. *(X, t) is not countably metacompact.* Define $D_r = \{x_\alpha\colon \alpha \in \bigcup_{r<i} S_i\}$. For each $r<\omega$, D_r is t-closed. $\bigcup_{r<\omega} D_r = \emptyset$. For each $r<\omega$, let $D_r \subseteq U_r$ where U_r is t-open. We claim that for each $r<\omega$, $|X - U_r| < c$. To see this, fix $r<\omega$. We show that $\mathrm{Cl}_e(\bigcup\{B \in \mathscr{B}\colon |B \cap (X - U_r)| < c\}) = Y$. If not, there is a $B \in \mathscr{B}$ such that for every $B' \subseteq B$, $B' \in \mathscr{B}$ we have $|B' \cap (X - U_r)| = c$. Define $C = \{\gamma<c\colon$ for each $\delta<\gamma$

there is $\alpha<\gamma$ with $C_\alpha \subseteq B \cap (X-U_r) \cap \{x_\beta\colon \delta<\beta<\gamma\}$ and $B \subseteq \mathrm{Cl}_e C_\alpha\}$. C is a closed and unbounded subset of c. Since $S(B, r+1)$ is stationary, choose $\gamma \in C \cap S(B, r+1)$. Note that $\mathrm{cf}\gamma = \omega$. By inductive hypothesis 3_γ it follows that $x_\gamma \in \mathrm{Cl}_{t_\gamma}[(X-U_r) \cap X_\gamma] = \mathrm{Cl}_t[(X-U_r) \cap X_\gamma] \subseteq X-U_r$. This implies that $\gamma \in \bigcup_{i\leqslant r} S_i$ which contradicts $\gamma \in S(B, r+1)$. Thus, $\mathrm{Cl}_e(\bigcup \{B \in \mathscr{B}\colon |B \cap (X-U_r)|<c\}) = Y$. Since L is generalized Luzin and $X \subseteq L$, we have that $|X-U_r|<c$. End of proof of claim. Hence $\bigcap_{r<\omega} U_r \neq \emptyset$ because $\mathrm{cf}\, c>\omega$.

B. (X, t) *is normal*. We first show: If H and K are disjoint t-closed subsets of X and $|H|<c$, then H and K can be separated. Since $P(c)$ implies that c is regular, there is $\gamma \in T_c$ such that $H \subseteq X_\gamma$. Note that X_γ is t-clopen. Inductive hypothesis 2_γ a), b) and c) implies that (X_γ, t_γ) is a regular refinement of (X_γ, e) so $P(c)$ implies that (X_γ, t_γ) is normal. There exists t_γ-open U and V in X_γ, therefore t-open in X, such that $H \subseteq U$, $K \cap X_\gamma \subseteq V$ and $U \cap V = \emptyset$. Then $H \subseteq U$ and $K \subseteq (X-X_\gamma) \cup V$ and we are done.

Now, let H and K be arbitrary disjoint t-closed subsets of X. For each $r<\omega$, define $H_r = \{x_\alpha \in H\colon \alpha \in \bigcup_{i\leqslant r} S_i\}$. Define

$$U(H, r) = Y-\mathrm{Cl}_e(\bigcup \{B \in \mathscr{B}\colon |B \cap H_r|<c\})\,.$$

Then, for every non-empty e-open $V \subseteq U(H, r)$, $|V \cap H_r| = c$. Since L in generalized Luzin, for each $r<\omega$, $|H_r - U(H, r)|<c$ and therefore $|H-\bigcup_{r<\omega} U(H, r)|<c$. Similarly define K_r and $U(K, r)$ for K. $\bigcup_{r<\omega} U(H, r) \cap \bigcup_{r<\omega} U(K, r) = \emptyset$. If not, there is $B \in \mathscr{B}$ with $B \subseteq U(H, r) \cap U(K, s)$ such that for every non-empty e-open $V \subseteq B$, $|H_r \cap V| = |K_s \cap V| = c$. Define $C_H = \{\gamma<c\colon$ for each $\delta<\gamma$ there is $\alpha<\gamma$ with $C_\alpha \subseteq H_r \cap B \cap \{x_\beta\colon \delta<\beta<\gamma\}$ and $B \subseteq \mathrm{Cl}_e C_\alpha\}$ and $C_K = \{\gamma<c\colon$ for each $\delta<\gamma$ there is $\alpha<\gamma$ with $C_\alpha \subseteq K_s \cap B \cap \{x_\beta\colon \delta<\beta<\gamma\}$ and $B \subseteq \mathrm{Cl}_e C_\alpha\}$. Both C_H and C_K are closed and unbounded subsets of c. Since $S(B, r+s+1)$ is stationary, choose $\gamma \in C_H \cap C_K \cap S(B, r+s+1)$. Note that $\mathrm{cf}\gamma = \omega$. By 3_γ it follows that $x_\gamma \in \mathrm{Cl}_{t_\gamma}(H_r \cap B \cap \{x_\gamma\colon \beta<\gamma\}) \cap \mathrm{Cl}_{t_\gamma}(K_s \cap B \cap \{x_\beta\colon \beta<\gamma\})$. Therefore, $x_\gamma \in H \cap K$. Contradiction.

Now, by the first paragraph, there are t-open V_H and V_K such that $H-\bigcup_{r<\omega} U(H, r) \subseteq V_H \subseteq \mathrm{Cl}_t V_H \subseteq X-K$ and $K-\bigcup_{r<\omega} U(K, r) \subseteq V_K \subseteq \mathrm{Cl}_t V_K \subseteq X-H$. Hence H and K are contained in the disjoint t-open sets $(V_H \cup \bigcup_{r<\omega} U(H, r))-\mathrm{Cl}_t V_K$ and $(V_K \cup \bigcup_{r<\omega} U(K, r))-\mathrm{Cl}_t V_H$ respectively.

We return now to complete our induction.

At stage 0, choose $x_0 \in L$ and let $t_0 = \{\varphi, \{x_0\}\}$. For each $n \geqslant 1$, define $U_0(n) = \{x_0\}$.

At stage γ where $\mathrm{cf}\gamma<\omega$ we just add points. Choose $x_\gamma \in L-\bigcup \{\mathrm{Cl}_e U_\alpha(1)\colon \alpha<\gamma\}$ which is non-empty since L is generalized Luzin and c is regular and let t_γ be generated by $\bigcup \{t_\alpha\colon \alpha<\gamma\} \cup \{\{x_\gamma\}\}$. For each $n \geqslant 1$, define $U_\gamma(n) = \{x_\gamma\}$.

At stage γ where $\mathrm{cf}\gamma = \omega$ choose an increasing sequence $\{\gamma_n\colon n<\omega\}$ of ordinals from T_c that converges to γ from below. There exists unique $B \in \mathscr{B}$ and $r<\omega$ such that $\gamma \in S(B, r)$. Choose $x_\gamma \in (B \cap L)-\bigcup \{\mathrm{Cl}_e U_\alpha(1)\colon \alpha<\gamma\}$. For each $n<\omega$, define $\mathscr{S}_\gamma(n) = \{C_\alpha\colon \alpha<\gamma,\ B \subseteq \mathrm{Cl}_e C_\alpha$ and $C_\alpha \subseteq B \cap [X_\gamma(r)-X_{\gamma_n}(r)]\}$.

Case i: For some $n<\omega$, $\mathscr{S}_\gamma(n) = \emptyset$. Let t_γ be generated by $\bigcup \{t_\alpha\colon \alpha<\gamma\} \cup \{\{x_\gamma\}\}$. For each $n \geqslant 1$, define $U_\gamma(n) = \{x_\gamma\}$.

Case ii: For every $n<\omega$, $\mathscr{S}_\gamma(n) \neq \emptyset$. Now we do work. Choose

$$\{B_\gamma(n)\colon 1 \leqslant n<\omega\} \subseteq \mathscr{B}$$

such that:

a) $n \neq m$ implies $B_\gamma(n) \cap B_\gamma(m) = \emptyset$,

b) $\mathrm{diameter}_e B_\gamma(n)<1/n$ and $B_\gamma(n) \subseteq B$,

c) for every e-open N that contains x_γ there is an m such that for each $n \geqslant m$, $B_\gamma(n) \subseteq N$.

Define $K = \{x_\alpha\colon \alpha<\gamma$ and $\alpha \in \bigcup_{r<i} S_i\}$. K is $[\bigcup \{t_\alpha\colon \alpha<\gamma\}]$-closed for the reasons $\{2_\gamma \mathrm{d})\colon \alpha<\gamma\}$. Note that for each $n<\omega$, $K \cap \bigcup S_\gamma(n) = \emptyset$. $U_\gamma(1)$ must avoid K.

FACT 1. *For each $n \geqslant 1$, there is an e-nowhere dense $N_n \subseteq B_\gamma(n) \cap X_\gamma(r)$ such that for each $C_\alpha \in \mathscr{S}_\gamma(n)$, $N_n \cap C_\alpha \neq \emptyset$. Furthermore, $\mathrm{Cl}_e N_n \cap K = \emptyset$.*

Proof. Fix $n \geqslant 1$. Define $P = \{(F, G)\colon F$ is a finite subset of $B_\gamma(n)-K$, $G \in \mathscr{B}$ and $F \subseteq G\}$. Define $\leqslant$ by $(F_1, G_1) \leqslant (F_2, G_2)$ iff $F_2 \subseteq F_1$ and $G_1 \subseteq G_2$. $(P, \leqslant)$ is a σ-centered poset since $\mathscr{B}$ is countable. For every α such that $C_\alpha \in \mathscr{S}_\gamma(n)$ define $D_\alpha = \{(F, G)\colon F \cap C_\alpha \neq \emptyset\}$. For every $H \in \mathscr{B}$ define $D_H = \{(F, G)\colon H \not\subseteq G\}$. For every $k \in K$ define $D_k = \{(F, G)\colon k \notin G\}$. $\{D_\alpha\colon C_\alpha \in \mathscr{S}_\gamma(n)\} \cup \{D_H\colon H \in \mathscr{B}\} \cup \cup \{D_k\colon k \in K\}$ are $<c$ dense subsets of $(P, \leqslant)$. Let $\mathscr{G}$ be a generic subset of P such that $\mathscr{G}$ intersects each of these dense sets. Define $N_n = B_\gamma(n) \cap X_\gamma(r) \cap \bigcap \{G \in \mathscr{B}\colon$ there is an F with $(F, G) \in \mathscr{G}\}$. D_α's insure that N_n meets each $C_\alpha \in \mathscr{S}_\gamma(n)$, D_H's insure N_n is e-nowhere dense and D_k's insure that $\mathrm{Cl}_e N_n \cap K = \emptyset$.

FACT 2. *Let $\{N_n\colon n \geqslant 1\}$ be as in Fact 1. For each $n \geqslant 1$ there is a $[\bigcup \{t_\alpha\colon \alpha<\gamma\}]$-clopen e-nowhere dense V_n such that $N_n \subseteq V_n \subseteq B_\gamma(n) \cap X_\gamma(r)$, $V_n \cap K = \emptyset$ and $V_n = \mathrm{Cl}_e V_n \cap X_\gamma$.*

Proof. Fix $n \geqslant 1$. Define $\mathscr{U} = \{$all finite unions of $U_\alpha(i)$'s: $\alpha<\gamma$, $\alpha \in \bigcup_{j\leqslant r} S_j$, $1 \leqslant i<\omega$ and $U_\alpha(i) \subseteq B_\gamma(n)\}$. Note that $|\mathscr{U}|<c$. Define $P = \{(U, G)\colon U \in \mathscr{U},\ G \in \mathscr{B}$ and $U \cap G = U \cap K = G \cap \mathrm{Cl}_e N_n = \emptyset\}$. Define $\leqslant$ by $(U_1, G_1) \leqslant (U_2, G_2)$ iff $U_2 \subseteq U_1$ and $G_2 \subseteq G_1$. Note that (U_1, G_1) and (U_2, G_2) are compatible iff $(U_1 \cup U_2) \cap \cap (G_1 \cup G_2) = \emptyset$. $(P, \leqslant)$ is a σ-centered poset. To see this, define

$$P_G = \{(U, G)\colon U \in \mathscr{U} \text{ and } U \cap G = U \cap K = G \cap \mathrm{Cl}_e N_n = \emptyset\}\,.$$

Then, each P_G is centered and $P = \bigcup \{P_G\colon G \cap \mathrm{Cl}_e N_n = \emptyset$ and $G \in \mathscr{B}\}$. For each $x \in X_\gamma$, define $D_x = \{(U, G)\colon x \in U \cup G\}$. For each $y \in N_n$, define

$$D_y = \{(U, G)\colon y \in U\}\,.$$

For each $H \in \mathscr{B}$, define $D_H = \{(U, G): H \cap G \neq \emptyset\}$. $\{D_x: x \in X_\gamma\} \cup \{D_y: y \in N_n\} \cup \cup \{D_H: H \in \mathscr{B}\}$ are $< c$ dense subsets of $(P, \leqslant)$. Let $\mathscr{G}$ be a generic subset of P such that $\mathscr{G}$ intersects each of these dense sets. Define $V_n = \bigcup \{U$: there is a G with $(U, G) \in \mathscr{G}\}$.

That V_n is $[\bigcup \{t_\alpha: \alpha<\gamma\}]$-open, $V_n \subseteq B_\gamma(n) \cap X_\gamma(r)$ and $V_n \cap K = \emptyset$ follows from the definition of P and $\mathscr{U}$. D_H's insure that V_n is e-nowhere dense. D_y's insure that $N_n \subseteq V_n$. D_x's insure that $V_n = \mathrm{Cl}_e V_n \cap X_\gamma$. Since t_α refines the e-topology on X_α, we have that V_n is $[\bigcup \{t_\alpha: \alpha<\gamma\}]$-closed as well.

Finally, define $U_\gamma(n) = \{x_\gamma\} \cup \bigcup_{n \leqslant i} V_i$. Let t_γ be generated by

$$\bigcup \{t_\alpha: \alpha<\gamma\} \cup \{U_\gamma(n): 1 \leqslant n<\omega\}.$$

That 1_γ and 2_γ hold follows from our choice of x_γ, our choice of the $B_\gamma(n)$'s, Fact 1, Fact 2 and our definition of $U_\gamma(n)$ and t_γ. Let us check 3_γ where $\gamma \in S(B, r)$ and $\mathrm{cf}\gamma = \omega$. We had an increasing sequence of ordinals less than γ, $\{\gamma_n: n<\omega\}$, converging to γ. Assume there is $\{\alpha_n: n<\omega\}$ such that

(i) for each $n<\omega$, $\alpha_n<\gamma$ and $B \subseteq \mathrm{Cl}_e C_{\alpha_n}$ and

(ii) for each $n<\omega$, $C_{\alpha_n} \subseteq B \cap [X_\gamma(r) - X_{\gamma_n}(r)]$.

Choose a t_γ-neighbourhood of x_γ, $U_\gamma(m) = \{x_\gamma\} \cup \bigcup_{m \leqslant i} V_i$. $C_{\alpha_m} \in \mathscr{S}_\gamma(m)$. By Fact 1, $N_m \cap C_{\alpha_m} \neq \emptyset$. By Fact 2, $N_m \subseteq V_m$ and therefore $V_m \cap C_{\alpha_m} \neq \emptyset$, i.e., $U_\gamma(m) \cap C_{\alpha_m} \neq \emptyset$. Hence $x_\gamma \in \mathrm{Cl}_{t_\gamma}(\bigcup \{C_{\alpha_n}: n<\omega\})$. ∎

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