On the complete classification of unitary $N = 2$ minimal superconformal field theories

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Abstract

Aiming at a complete classification of unitary $N = 2$ minimal models (where the assumption of space-time supersymmetry has been dropped), it is shown that each candidate for a modular invariant partition function of such a theory is indeed the partition function of a minimal model. A family of models constructed via orbifoldings of either the diagonal model or of the space-time supersymmetric exceptional models demonstrates that there exists a unitary $N = 2$ minimal model for every one of the allowed partition functions in the list obtained from Gannon's work [26].

Kreuzer and Schellekens' conjecture that all simple current invariants can be obtained as orbifolds of the diagonal model, even when the extra assumption of higher-genus modular invariance is dropped, is confirmed in the case of the unitary $N = 2$ minimal models by simple counting arguments.

We find a nice characterisation of the projection from the Hilbert space of a minimal model with k odd to its modular invariant subspace, and we present a new simple proof of the superconformal version of the Verlinde formula for the minimal models using simple currents.

Finally we demonstrate a curious relation between the generating function of simple current invariants and the Riemann zeta function.

Acknowledgements

Dedication

I dedicate this thesis to my late father, who always encouraged my studies, and who is sorely missed.

Contents

Chapter 1

Introduction

Conformal field theories (CFTs) [3, 33, 8, 22, 21] have been a well-studied area of research since they first became a hot topic following the publication of the seminal paper of Belavin, Polyakov and Zamolodchikov in 1984 [3]. In their paper, they laid down the formalism of conformal field theories by combining the representation theory of the Virasoro algebra with the concept of local operators, and discovered the minimal models. The term *minimal* indicates that the Hilbert space of the CFT decomposes into only finitely many irreducible representations of (two commuting copies of) the Virasoro algebra. The existence of null-vectors in the Hilbert spaces of minimal models permit ODEs to be derived, which in turn allow the minimal models to be completely solved.

Miraculously, the minimal models turned out to describe phenomena in statistical mechanics [6]; most notable is their description of 2nd or higher order phase transitions, e.g. the Ising model [51, 3] and the tri-critical Ising model [20]. Once the inequivalent irreducible unitary representations of the Virasoro algebra with central charge $0 \leq \mathbf{c} < 1$ were known, the next problem was to piece them together in a modular invariant way (see section 2.2.4). All modular invariant combinations were found to fall into the well-known $\mathcal{A}$-$\mathcal{D}$-$\mathcal{E}$ meta pattern (see e.g. [71]).

The classification of other classes of conformal field theories has been the aim of much work, and is an ongoing project. Most promising is the study of *rational* theories, whose Hilbert spaces may contain infinitely many irreducible representations of the Virasoro algebras, but which can be organised into a finite sum of representations of some larger so-called W-*algebra*. An important source of rational theories are the WZW models [66, 67]: families of theories, which can be constructed for any semi-simple finite-dimensional Lie algebra $\mathfrak{g}$. Many of the families of WZW models have been at least partially classified [25, 24], the most famous being the complete classification of the $\mathfrak{g} = \mathfrak{su}(2)$ case [5], which again falls into an $\mathcal{A}$-$\mathcal{D}$-$\mathcal{E}$ classification.

Another source of rational CFT is inspired by string theory [2, 36, 37, 53], the most promising candidate for a description of the fundamental forces of the universe. String theorists have developed the notion of supersymmetry, the

idea that there is a symmetry between bosonic and fermionic matter in our universe. In mathematical terms, the Virasoro algebra is enlarged by adding N supersymmetry operators (and their super partners). One can then consider superconformal field theories (SCFTs), theories that fall into representations of this enlarged algebra. The minimal unitary $N = 2$ superconformal field theories, for example, provide building blocks for Gepner models [38].

Contrary to popular belief, to date the unitary $N = 2$ minimal models [4, 10, 9, 68, 47, 43, 57, 55, 54] have not been completely classified. It is commonly stated that they also fall into the $\mathcal{A}$-$\mathcal{D}$-$\mathcal{E}$ meta-pattern, due to the work of [46, 62, 7], in which those unitary $N = 2$ minimal models that enjoy space-time supersymmetry are demonstrated to be in one-to-one correspondence with the $\mathcal{A}$-$\mathcal{D}$-$\mathcal{E}$ simple singularities. But when one quite reasonably drops the condition of space-time supersymmetry, one finds a much larger possible set of solutions.

The condition of space-time supersymmetry means that there should be a fundamental symmetry between space-time bosons and fermions; in a SCFT, the symmetry implies that all information about the space-time anti-periodic fields (the R sector) is encoded by the space-time periodic fields (the NS sector) and vice versa. This relation is encoded by the *spectral flow* (see e.g. [38] and section 3.4), which provides an explicit map from one sector to the other in supersymmetric theories.

Gannon [26] classified the possible partition functions of the unitary $N = 2$ minimal models, showing that in fact there is a much larger playground than previously suspected: there are finitely many partition functions at each level k, but the number is unbounded as k increases, in contrast with the $N = 0$ case. There are also many more "exceptional" cases; $10, 18$ and 8 corresponding to what are somewhat misleadingly termed $\mathcal{E}_6, \mathcal{E}_7$ and $\mathcal{E}_8$ models, respectively.

Two natural questions then arise: do all of these partition functions belong to genuine SCFTs, or are some just mathematical curiosities? And could there be more than one minimal model associated to each partition function? In this thesis we answer the first of these questions. Perhaps surprisingly, it can be resolved using only orbifold-related arguments. It turns out that orbifoldings [13, 14] from every possible partition function to the partition function of one of a small list of well-known and fully understood models can be explicitly calculated, showing that each partition function is indeed that of a fully-fledged SCFT. This is an important step towards the full classification of the unitary $N = 2$ minimal models.

We note that Kreuzer and Schellekens [45] have proved a related result. They construct simple current modular invariant partition functions via orbifoldings of the diagonal model and use the further assumption of higher-genus modular invariance to show that all simple current modular invariant partition functions can be obtained this way. They hypothesise that this extra assumption is unnecessary, which we are able to confirm for the case of unitary $N = 2$ minimal models by simple counting arguments.

The layout of the thesis is as follows. Chapter 2 is a review of unitary $N = 2$ superconformal field theories, and of simple currents. We use simple current arguments to prove the $N = 2$ version of Verlinde's formula for the minimal

models. We then investigate some symmetries of the partition functions of the minimal models.

In chapter 3 we describe Gannon's program of classifying the possible partition functions of the unitary $N = 2$ minimal SCFTs, and present the statement of the result (which did not appear explicitly in [26]) with a few minor errors corrected. We illustrate the classification of the partition functions with examples at the first two levels, and use the list to confirm the classification of the $N = 2$ models which enjoy space-time supersymmetry. A natural characterisation of the projection to the modular invariant subspace of the theories with k odd is given, followed by a review of the parafermion construction of the minimal models due to Zamolodchikov and Fateev [17, 69] and to Qiu [55]. Lastly, we perform some non-trivial checks on the candidate theories from Gannon's list and present as examples the field content of the theories at $k = 1, 2$.

Chapter 4 contains a brief review of orbifold techniques, followed by the statement and proof of the main theorem: every candidate partition function, listed in section 3.2, belongs to a fully-fledged SCFT. The proof is an explicit construction of orbifoldings from any given partition function to one of a few fixed and fully understood SCFTs.

In chapter 5 we calculate the number of simple current physical invariants and confirm a hypothesis of Kreuzer and Schellekens for the special case of the unitary $N = 2$ minimal models; namely, that every simple current invariant is obtained via an orbifold of the diagonal model. Finally we present a mathematical curiosity relating the number of simple current invariants at each level to the Riemann zeta function.

Chapter 6 contains our conclusions.

Chapter 2

$N = 2$ Superconformal Field Theories

2.1 The $N = 2$ super Virasoro algebra

$N = 2$ superconformal field theories (SCFTs) were first constructed by Ademollo et al [1] by introducing $U(1)$ Kac-Moody algebra symmetries along with two supersymmetry generators, in addition to the $N = 0$ Virasoro field. They are quantum field theories that enjoy $N = 2$ supersymmetry, or, in more mathematical terms, theories whose pre-Hilbert spaces form representations of the $N = 2$ super Virasoro algebra (SVA). The approach of Belavin, Polyakov and Zamolodchikov in their seminal paper [3] can be applied to $N = 2$ extended SCFTs, as was done, for example, by [11, 52].

   The $N = 2$ SVA is most succinctly described in the $N = 2$ superspace formalism. In this setup the space for an $N = 2$ supersymmetric quantum field theory in two dimensions has coordinate $Z = (z, \theta^+, \theta^-)$ where z is a complex coordinate and $\theta^\pm$ are Grassmann variables satisfying $\{\theta^+, \theta^-\} = 0$. We define the supercurrent

$$\mathbf{J}(Z) = -2J(z) + \frac{1}{\sqrt{2}}\theta^+ G^-(z) + \frac{1}{\sqrt{2}}\theta^- G^+(z) + \theta^- \theta^+ T(z)$$

where $T(z)$ is the Virasoro field, $G^\pm(z)$ are the supersymmetry generators and $J(z)$ is the Kac-Moody $U(1)$ current. The superfield $\mathbf{J}(Z)$ has the following operator product expansion (OPE) with itself:

$$\mathbf{J}(Z_1)\mathbf{J}(Z_2) = \frac{(\theta_1^- - \theta_2^-)(\theta_1^+ - \theta_2^+)}{Z_1 - Z_2}\partial\mathbf{J}(Z_2) + \frac{\theta_1^- - \theta_2^-}{Z_1 - Z_2}D^-\mathbf{J}(Z_2)$$

$$- \frac{\theta_1^+ - \theta_2^+}{Z_1 - Z_2}D^+\mathbf{J}(Z_2) + \frac{(\theta_1^- - \theta_2^-)(\theta_1^+ - \theta_2^+)}{(Z_1 - Z_2)^2}\mathbf{J}(Z_2)$$

$$+ \frac{\mathbf{c}/3}{(Z_1 - Z_2)^2} + \ldots.$$

We define $Z_1 - Z_2 = z_1 - z_2 - \frac{1}{2}(\theta_1^+ \theta_2^- + \theta_1^- \theta_2^+)$, and understand that where $\theta_i^\pm$ for $i = 1, 2$ appears in the denominator, the expression defined by its (truncated) Taylor series[1]. $D^\pm$ are covariant derivatives defined by

$$D^+ = \frac{\partial}{\partial \theta^+} + \frac{1}{2}\theta^- \frac{\partial}{\partial z}, \qquad\qquad D^- = \frac{\partial}{\partial \theta^-} + \frac{1}{2}\theta^+ \frac{\partial}{\partial z}$$

and $\mathbf{c} \in \mathbb{C}$ is called the central charge. This super-OPE encodes all the OPEs of the constituent fields $T(z), G^\pm(z), J(z)$, which are given by:

$$
\begin{aligned}
T(z_1)T(z_2) &= \frac{\mathbf{c}/2}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{(z_1 - z_2)} + \dots \\
&= T(z_2)T(z_1), \\
T(z_1)J(z_2) &= \frac{J(z_2)}{(z_1 - z_2)^2} + \frac{\partial J(z_2)}{(z_1 - z_2)} + \dots \\
&= J(z_2)T(z_1), \\
J(z_1)J(z_2) &= \frac{\mathbf{c}/12}{(z_1 - z_2)^2} + \dots \\
&= J(z_2)J(z_1), \\
T(z_1)G^\pm(z_2) &= \frac{3}{2}\frac{G^\pm(z_2)}{(z_1 - z_2)^2} + \frac{\partial G^\pm(z_2)}{(z_1 - z_2)} + \dots \\
&= G^\pm(z_2)T(z_1), \\
J(z_1)G^\pm(z_2) &= \pm\frac{1}{2}\frac{G^\pm(z_2)}{(z_1 - z_2)} + \dots \\
&= G^\pm(z_2)J(z_1), \\
G^+(z_1)G^-(z_2) &= \frac{2\mathbf{c}/3}{(z_1 - z_2)^3} + \frac{4J(z_2)}{(z_1 - z_2)^2} + \frac{2T(z_2) + 2\partial J(z_2)}{(z_1 - z_2)} + \dots \\
&= -G^-(z_2)G^+(z_1), \\
G^\pm(z_1)G^\pm(z_2) &= 0 + \dots \\
&= -G^\pm(z_2)G^\pm(z_1),
\end{aligned}
\tag{2.1}
$$

where we have omitted regular terms[2]. We see that $T(z)$ is a Virasoro field with central charge $\mathbf{c}$ and conformal weight 2, $J(z)$ is a $U(1)$ current with conformal weight 1, and $G^\pm(z)$ are (odd) supersymmetry generators with conformal weight $\frac{3}{2}$.

These OPEs encode a Lie superalgebra of the modes of the fields $T(z), G^\pm(z)$ and $J(z)$. To determine the mode expansion of the various fields we must consider the different types of possible boundary conditions. Since the fields are local objects defined on the punctured complex plane, we can consider fields to

---

[1] Since $Z_1 - Z_2$ is quadratic in the totally antisymmetric $\theta_i^\pm$ it commutes with all other terms, so our notation is not ambiguous.

[2] Some authors replace $J(z)$ with $\frac{1}{2}J(z)$.

live on an n-sheeted covering of the punctured complex plane. Then when z circles around the origin the field $A(z)$ can pick up a factor $\xi \in \mathbb{C}$:

$$\lim_{t \to 1} A(e^{2\pi i t} z) = \xi A(z)$$

where $\xi^n = 1$. This yields three possibilities consistent with the OPEs above: the Neveu-Schwarz (NS) sector [50], in which all the fields are periodic

$$\lim_{t \to 1} A(e^{2\pi i t} z) = A(z), \quad \text{for } A = T, J, G^{\pm};$$

the Ramond (R) sector [56], in which $G^{\pm}(z)$ are anti-periodic and the others periodic; and the twisted (T) sector. To investigate the T sector it is convenient to use the basis

$$G^1(z) = \frac{1}{\sqrt{2}}(G^+(z) + G^-(z)), \qquad G^2(z) = \frac{-i}{\sqrt{2}}(G^+(z) - G^-(z)),$$

which yields the following OPEs:

$$
\begin{aligned}
T(z_1)G^j(z_2) &= \frac{3}{2}\frac{G^j(z_2)}{(z_1 - z_2)^2} + \frac{\partial G^j(z_2)}{(z_1 - z_2)} + \dots \\
&= G^j(z_2)T(z_1), \\
J(z_1)G^j(z_2) &= \sum_{k=1,2} \frac{i\epsilon_{jk}}{2}\frac{G^k(z_2)}{(z_1 - z_2)} + \dots \\
&= G^j(z_2)J(z_1), \\
G^j(z_1)G^k(z_2) &= \frac{2\delta_{jk}\mathbf{c}/3}{(z_1 - z_2)^3} + \frac{4i\epsilon_{jk}J(z_2)}{(z_1 - z_2)^2} + \frac{2i\epsilon_{jk}\partial J(z_2) + 2\delta_{jk}T(z_2)}{(z_1 - z_2)} + \dots
\end{aligned}
$$

$$(2.2)$$

where $j, k \in \{1, 2\}$, and ϵ is the totally antisymmetric pseudo-tensor with $\epsilon_{12} = 1$. We then take $J(z)$ and $G^2(z)$ to be anti-periodic and the other fields to be periodic.[3]

We can now expand the fields into their Fourier modes. In every sector we have $T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n$, and in both the NS and R sectors we have $J(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J_n$ and

$$G^{\pm}(z) = \sum_{n \in \tilde{\mathbb{Z}}} z^{-n-\frac{3}{2}} G_n^{\pm},$$

where $\tilde{\mathbb{Z}} = \mathbb{Z} + \frac{1}{2}$ in the NS sector and $\tilde{\mathbb{Z}} = \mathbb{Z}$ in the R sector. Note that the zero mode L_0 exists in every sector, and for the NS and R sectors J_0 also exists.

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[3]We could just as well have chosen $G^1(z)$ to be anti-periodic and $G^2(z)$ periodic: the two resulting Lie algebras are isomorphic.

6

We can now write out the SVA encoded by eqs. (2.1) for the NS and R sectors:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\mathbf{c}}{12}m(m^2-1)\delta_{m+n,0},$$

$$[L_m, J_n] = -nJ_{n+m},$$

$$[J_m, J_n] = \frac{\mathbf{c}}{12}m\delta_{m+n,0},$$

$$[L_n, G_s^\pm] = \left(\frac{n}{2} - s\right)G_{n+s}^\pm, \qquad (2.3)$$

$$[J_n, G_s^\pm] = \pm\frac{1}{2}G_{n+s},$$

$$\{G_r^+, G_s^-\} = 2L_{r+s} + 2(r-s)J_{r+s} + \frac{\mathbf{c}}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0},$$

$$\{G_r^\pm, G_s^\pm\} = 0,$$

where $n, m \in \mathbb{Z}$, and $r, s \in \mathbb{Z} + \frac{1}{2}$ in the NS sector and $r, s \in \mathbb{Z}$ in the R sector. Note that the SVA contains a copy of the $N = 0$ Virasoro algebra at central charge $\mathbf{c}$ and a copy of the $U(1)$ Kac-Moody algebra.

In the twisted sector we write

$$J(z) = \sum_{n\in\mathbb{Z}} z^{-n-\frac{1}{2}} J_{n-\frac{1}{2}}$$

$$G^1(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} G_{n-\frac{1}{2}}^1, \qquad G^2(z) = \sum_{n\in\mathbb{Z}} z^{-n-\frac{3}{2}} G_n^2.$$

Using these Fourier mode expansions, eqs. (2.2) leads to the SVA in the twisted sector:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\mathbf{c}}{12}m(m^2-1)\delta_{m+n,0},$$

$$[L_n, J_s] = -sJ_{n+s},$$

$$[J_r, J_s] = \frac{\mathbf{c}}{12}r\delta_{r+s,0},$$

$$[L_n, G_p^j] = \left(\frac{n}{2} - p\right)G_{n+p}^j,$$

$$[J_r, G_p^j] = \sum_{k=1,2} \frac{i\epsilon_{jk}}{2}G_{r+p}^k,$$

$$\{G_p^j, G_q^k\} = 2\delta_{jk}L_{p+q} + 2i\epsilon_{jk}(p-q)J_{p+q} + \mathbf{c}\left(p^2 - \frac{1}{4}\right)\delta_{jk}\delta_{p+q,0},$$

where $j, k \in \{1, 2\}$, $n, m \in \mathbb{Z}$, $r, s \in \mathbb{Z} + \frac{1}{2}$ and the lower indices of G_p^1 and G_p^2 run over $\mathbb{Z} + \frac{1}{2}$ and $\mathbb{Z}$ respectively.

From now on we will only be interested in the NS and R sectors.

2.1.1 Lowest weight representations of the SVA

In this section we recall some basic results regarding representations of the SVA in the NS and R sectors. The super Virasoro algebra in either the Neveu-

Schwarz or Ramond sector at central charge $\mathbf{c}$ is the Lie superalgebra spanned by $\{\mathbf{1}, L_n, J_n, G_r^\pm \mid n \in \mathbb{Z}, r \in \widetilde{\mathbb{Z}}\}$, where

$$\widetilde{\mathbb{Z}} = \begin{cases} \mathbb{Z} + \frac{1}{2} & \text{NS sector} \\ \mathbb{Z} & \text{R sector} \end{cases}$$

with commutation relations given in eqs (2.3). A *lowest weight representation* (LWR) of the SVA is a representation of the SVA which contains a vector $v \neq 0$ such that

$$L_n v = J_n v = G_n^\pm v = 0 \quad \forall n > 0,$$
$$L_0 v = h v,$$
$$J_0 v = Q v,$$

for some $h, Q \in \mathbb{C}$. h is called the *conformal weight*, Q is called the $U(1)$ *charge* and v is the *lowest weight vector* (LWV).

We will write $\text{SVir}_{\mathbf{c}}$ for the universal enveloping algebra of the SVA at central charge $\mathbf{c}$. Then for any $h, Q \in \mathbb{C}$ the *Verma module* $V(h, Q)$ over $\text{SVir}_{\mathbf{c}}$ at weight h and charge Q is defined to be the vector space quotient of $\text{SVir}_{\mathbf{c}}$ by the left ideal generated by $\{L_n, J_n, G_r^\pm, \mathbf{1} - 1, L_0 - h\mathbf{1}, J_0 - Q\mathbf{1} | n, r > 0\}$.[4] Then $V(h, Q)$ is spanned by the *descendent states* of the vector $|h, Q\rangle$:

$$L_{-n_1}...L_{-n_p} J_{-m_1}...J_{-m_q} G_{-a_1}^+...G_{-a_r}^+ G_{-b_1}^-...G_{-b_s}^- |h, Q\rangle, \qquad (2.4)$$

where $n_i, m_i > 0$, $a_i, b_i \geq 0$. The state (2.4) has L_0 and J_0 eigenvalues

$$h + \sum_{i=1}^{p} n_i + \sum_{j=1}^{q} m_j + \sum_{k=1}^{r} a_k + \sum_{l=1}^{s} b_l, \qquad (2.5)$$

$$t + \sum_{k=1}^{r} \frac{1}{2} - \sum_{l=1}^{s} \frac{1}{2} = t + \frac{r - s}{2},$$

respectively. We note that $V(h, Q)$ is naturally a $\text{SVir}_{\mathbf{c}}$-module, and in fact that any $\text{SVir}_{\mathbf{c}}$-module is a quotient of $V(h, Q)$.

A similar situation occurs in the T sector, except that the primary fields are labelled only by their conformal weight, since in this sector J_0 does not exist.

2.2 The unitary minimal models

2.2.1 Unitary representations of the SVA

A representation $\mathbb{H}$ of $\text{SVir}_{\mathbf{c}}$ is called *unitary* if $\mathbb{H}$ carries an inner product $\langle \cdot | \cdot \rangle$ with respect to which

$$L_n^\dagger = L_{-n} \qquad\qquad J_n^\dagger = J_{-n} \qquad\qquad G_n^{\pm\,\dagger} = G_{-n}^\mp,$$

---

[4] Here $\mathbf{1}$ is an element of the SVA and $1 \in \mathbb{C}$.

where † is the adjoint. This is a rather strong constraint, as we shall now see.

Let v be a vector of length 1 in a unitary representation $\mathbb{H}$ of $\mathrm{SVir}_\mathbf{c}$ in the R sector with conformal weight h. Then using unitarity,

$$
\begin{aligned}
0 \leq \|G_0^+ v\|^2 + \|G_0^- v\|^2 &= \langle v | \{G_0^+, G_0^-\} | v \rangle \\
&= \langle v | \left(2L_0 - \frac{\mathbf{c}}{12} \right) | v \rangle = 2h - \frac{\mathbf{c}}{12}.
\end{aligned}
$$

Clearly then $h \geq \frac{\mathbf{c}}{24}$. If $h = \frac{\mathbf{c}}{24}$ then v is a primary state, since otherwise applying L_n, J_n or $G_n^\pm$ with $n > 0$ would decrease the conformal weight below $\frac{\mathbf{c}}{24}$. Further, we see that $G_0^\pm v = 0$. Such a state is called a *Ramond ground state*.

If v is a primary state with $h > \frac{\mathbf{c}}{24}$ then we have a pair of highest weight states $|h, Q_+\rangle, |h, Q_-\rangle$, degenerate with respect to L_0, satisfying

$$
\begin{aligned}
G_0^+ |h, Q_+\rangle = 0, && G_0^- |h, Q_+\rangle \propto |h, Q_-\rangle, \\
G_0^+ |h, Q_-\rangle \propto |h, Q_+\rangle, && G_0^- |h, Q_-\rangle = 0
\end{aligned}
$$

where $Q_+ - Q_- = \frac{1}{2}$. So, generically, an irreducible representation $\mathbb{H}^\mathrm{R}$ of the Ramond sector of $\mathrm{SVir}_\mathbf{c}$ contains two highest weight states, $|h, Q_+\rangle$ and $|h, Q_-\rangle$, either of which generates $\mathbb{H}$ as a $\mathrm{SVir}_\mathbf{c}$-module. In what follows, we will pick a preferred highest weight state and label the representation $\mathbb{H}^{\mathrm{R}, \mp}$ accordingly.

2.2.2 Classification of irreducible unitary representations of SVA

In reference [4], it was shown that unitary $N = 2$ superconformal models with central charge $\mathbf{c} < 3$ can only take a discrete set of central charges

$$
\mathbf{c} = \frac{3k}{\overline{k}}, \quad k \in \mathbb{Z}, \tag{2.6}
$$

where the integer k is called the *level* and we have written $\overline{k} := k + 2$. Furthermore, it was shown that for a fixed central charge $\mathbf{c}$, there are precisely $\frac{1}{2}(k+1)(k+2)$ inequivalent unitary irreducible lowest weight representations in both the NS and R sectors.

In the NS sector we have pre-Hilbert spaces $\mathbb{H}^{NS}_{j_1, j_2}$ whose lowest weight state has lowest weight and $U(1)$ charge[5]

$$
h_{j_1, j_2} = \frac{4 j_1 j_2 - 1}{4 \overline{k}}
$$

$$
Q_{j_1, j_2} = \frac{j_1 - j_2}{2 \overline{k}}
$$

for $(j_1, j_2) \in \{(j_1, j_2) \in \mathbb{Z} + \frac{1}{2} \mid 0 < j_1, j_2, j_1 + j_2 \leq k + 1\}$.

---

[5]Note that a factor of 2 arises in the denominator of the charge Q since, in our notation, relative charge takes half integer values, as opposed to the convention of integer relative charge assumed in e.g. [4] and much of the modern literature.

In the R sector we have pre-Hilbert spaces $\mathbb{H}^{R,\pm}_{j_1,j_2}$, where we label by $\mathbb{H}^{R,\pm}$ the space with lowest weight vector $|h, Q_\mp\rangle$. The lowest weight state has lowest weight and $U(1)$ charge

$$h_{j_1,j_2} = \frac{k + 8j_1 j_2}{8\overline{k}}$$

$$Q_{j_1,j_2} = \pm \left(\frac{j_1 - j_2}{2\overline{k}} - \frac{1}{4} \right)$$

respectively, for $(j_1, j_2) \in \{(j_1, j_2) \in \mathbb{Z} \mid 0 \le j_1 - 1, j_2, j_1 + j_2 \le k + 1\}$.[6]

2.2.3 Characters of representations

Define the character $\mathrm{ch}^{(X)}_{j_1,j_2}$ of the pre-Hilbert space $\mathbb{H}^X_{j_1,j_2}$ by $\mathrm{ch}^{(X)}_{j_1,j_2}(\tau, z) :=$ $\mathrm{Tr}_{\mathbb{H}^X_{j_1,j_2}} \left(q^{L_0 - \frac{c}{24}} y^{J_0} \right)$, where $X \in \{NS, R\pm\}$, $q = \exp(2\pi i \tau)$, $y = \exp(2\pi i z)$. Here $(\tau, z) \in \mathcal{H} \times \mathbb{C}$ where $\mathcal{H}$ is the upper-half complex plane. One can think of the characters either as formal power series in $q, y^{\frac{1}{2}}$ and $y^{-\frac{1}{2}}$ or as functions of $(\tau, z) \in \mathcal{H} \times \mathbb{C}$. The first interpretation is useful since the coefficient $d_{n,m}$ in the sum $\mathrm{ch}(\tau, z) = q^{h - \frac{c}{24}} y^Q \sum_{n,m} d_{n,m} q^n y^m$ gives the dimension of the space of simultaneous L_0, J_0 eigenstates with conformal weight $n + h$ and charge $m + Q$. The second interpretation gives functions which behave well under modular transformations as we will see later. The characters were calculated explicitly by [47, 15, 44].[7]

A useful change of variables is given by $l = j_1 + j_2 - 1$, $m = j_1 - j_2$ in the NS sector and by $l = j_1 + j_2 - 1$, $m = \pm(j_1 - j_2 - 1)$ in the R sector. Thus, labelling the NS sector by $\lambda = 0$ and the R$\pm$ sector by $\lambda = \mp\frac{1}{2}$, we have pre-Hilbert spaces of states $\mathbb{H}^{(\lambda)}_{l,m}$ indexed by

$$P_k := \{(l, m) \mid l = 0, ..., k; \ m = -l, ..., l; \ l + m \equiv 0 \pmod 2\}$$

with

$$h^{(\lambda)}_{l,m} = \frac{l(l+2)}{4\overline{k}} + \frac{\lambda^2}{2} - \frac{(m - 2\lambda)^2}{4\overline{k}} \tag{2.7}$$

$$Q^{(\lambda)}_{l,m} \equiv \frac{m + k\lambda}{2\overline{k}} \pmod 1 \tag{2.8}$$

as expected from the coset construction [34] based on $\left(\widehat{\mathfrak{su}(2)}_k \oplus \widehat{\mathfrak{u}(1)}_2 \right) / \widehat{\mathfrak{u}(1)}_{\overline{k}}$ given in [10].

---

[6]The $-\frac{1}{4}$ term arising in our expression for Q above is absent in the paper of [4], since they define the charge to be the average of the charges $\frac{Q^+ + Q^-}{2}$.

[7]The embedding diagrams conjectured in [47, 15, 44] were false. Dörrzapf [16] derived the correct embedding pattern and showed that the characters derived in [47, 15, 44] were nevertheless correct.

In this more convenient notation, Ravanini and Yang [57] found an elegant expression for the characters, based on the parafermion construction of the $N = 2$ minimal theories ([69, 55] and see also section 3.6.3):

$$\mathrm{ch}_{l,m}^{(k,\lambda)}(\tau, z) = \sum_{m'=-k+1}^{k} c_{l,m'}^{(k)}(\tau) \Theta_{m'\overline{k}-mk+2\lambda k, k\overline{k}}\left(\frac{\tau}{2}, \frac{kz}{2}, 0\right),$$

where $\overline{k} = k + 2$, $c_{l,m}^{(k)}$ are the $\widehat{\mathfrak{su}(2)}_k$ string functions [41] and $\Theta_{a,b}$ is the theta function given by[8]

$$\Theta_{a,b}(\tau, z, u) = \exp(-2\pi i u) \sum_{n \in \mathbb{Z}+a/2b} q^{bn^2} y^n.$$

The following symmetry properties are well known:

$$c_{l,m}^{(k)} = c_{l,m+2k}^{(k)} = c_{l,-m}^{(k)} = c_{k-l,k-m}^{(k)},$$
$$c_{l,m}^{(k)} = 0 \quad \text{unless } l + m \equiv 0 \pmod{2},$$
$$\Theta_{a,b} = \Theta_{a+2nb,b} \quad \forall n \in \mathbb{Z}.$$

They imply the following symmetry properties of the characters:

$$\mathrm{ch}_{l,m+2n\overline{k}}^{(k,\lambda)} = \mathrm{ch}_{l,m}^{(k,\lambda)} \quad \forall n \in \mathbb{Z},$$
$$\mathrm{ch}_{k-l,m+\overline{k}}^{(k,\lambda)} = \mathrm{ch}_{l,m}^{(k,\lambda)},$$
$$\mathrm{ch}_{l,-m}^{(k,\lambda)}(\tau, z) = \mathrm{ch}_{l,m}^{(k,-\lambda)}(\tau, -z).$$

Gepner [30] introduces what Gannon calls 'half-characters' $\chi_{a,c}^{(b)}$ for each triple $(a, b, c) \in \{0, ..., k\} \times \mathbb{Z}_4 \times \mathbb{Z}_{2\overline{k}}$ with $a + b + c \equiv 0 \pmod{2}$. They are given by

$$\chi_{a,c}^{(b)}(\tau, z) = \sum_{j \in \mathbb{Z}_k} c_{a,c+4j-b}^{(k)}(\tau) \Theta_{2c+(4j-b)\overline{k}, 2k\overline{k}}(\tau, kz, 0). \tag{2.9}$$

$\chi_{a,c}^{(b)}$ is invariant under the transformation $(a, b, c) \to (k - a, b + 2, c + \overline{k})$. We can mod out by this symmetry and, after choosing a realisation of the R sector, we can assume that the characters are indexed by $(a, c) \in Q_k := \{0, ..., k\} \times \mathbb{Z}_{2\overline{k}}$, by setting $b = [a + c]$. Here we have defined $[x]$ to be 0 when x is even, and ± 1 when x is odd, where the sign is fixed by the choice of R$\pm$. Then $[a + c] = 0$ labels the NS sector and $[a + c] = \pm 1$ labels the R$\pm$ sector. We will often leave away the b index for notational simplicity.

We have the following relationship between the $\mathrm{ch}_{l,m}^{(k,\lambda)}$ and the $\chi_{a,c}$:

$$\chi_{a,c}(\tau, z) + \chi_{k-a,c+\overline{k}}(\tau, z) = \mathrm{ch}_{l,m}^{(k,\lambda)}(\tau, z) \tag{2.10}$$

---

[8] Once again a factor of two has appeared to bring the relative charge to half-integer values; and our definition of the theta function, following Gepner [30], differs from that of [57] by a factor of b in the y exponent.

$$= \mathrm{Tr}_{\mathbb{H}_{l,m}^{(\lambda)}} \left(q^{L_0 - \frac{\mathbf{c}}{24}} y^{J_0} \right),$$

$$\chi_{a,c}(\tau, z) - \chi_{k-a,c+\overline{k}}(\tau, z) = \widetilde{\mathrm{ch}}_{l,m}^{(k,\lambda)}(\tau, z) \tag{2.11}$$

$$:= \exp \left(\mp 2\pi i Q_{l,m}^{(\lambda)} \right) \mathrm{ch}_{l,m}^{(k,\lambda)}(\tau, z \pm 1)$$

$$= \mathrm{Tr}_{\mathbb{H}_{l,m}^{(\lambda)}} \left((-1)^{2J_0 - 2Q_{l,m}^{(\lambda)}} q^{L_0 - \frac{\mathbf{c}}{24}} y^{J_0} \right).$$

where $a = l$, $b = [a + c] = -2\lambda$, $c = m - 2\lambda$.[9] These relations hold true for $(l, m) \in P_k$ and $\lambda \in \{0, \pm\frac{1}{2}\}$; or equivalently for $(a, c) \in P'_k$ where

$$P'_k := \{(a, c) \mid a = 0, \ldots, k, \quad |c - [a + c]| \le a\}.$$

It is convenient to relabel the pre-Hilbert space and the conformal weight and charge using $a = l$, $b = [a + c] = -2\lambda$, $c = m - 2\lambda$. Then we see from (2.7) and (2.8) that the LWR $\mathbb{H}_{ac}$ has conformal weight and charge given by

$$h_{ac} = \frac{a(a+2) - c^2}{4\overline{k}} + \frac{[a+c]^2}{8}, \tag{2.12}$$

$$Q_{ac} = \frac{c}{2\overline{k}} - \frac{[a+c]}{4} \tag{2.13}$$

for $(a, c) \in P'_k$. Thus we can read off

$$\chi_{a,c}(\tau, z) = \mathrm{Tr}_{\mathbb{H}_{a,c}} \left(\frac{1}{2} \left(1 + (-1)^{2J_0 - 2Q_{a,c}} \right) q^{L_0 - \frac{\mathbf{c}}{24}} y^{J_0} \right), \tag{2.14}$$

$$\chi_{k-a,c+\overline{k}}(\tau, z) = \mathrm{Tr}_{\mathbb{H}_{a,c}} \left(\frac{1}{2} \left(1 - (-1)^{2J_0 - 2Q_{a,c}} \right) q^{L_0 - \frac{\mathbf{c}}{24}} y^{J_0} \right), \tag{2.15}$$

for $(a, c) \in P'_k$. Writing $\mathbf{j}$ for the current $\mathbf{j} \cdot (a, c) = (k - a, c + \overline{k})$ (see section 2.4), we have $Q_k = P'_k \cup \mathbf{j} \cdot P'_k$. Thus every χ_{ac} is characterised by equations (2.14) and (2.15) as the trace of a certain projection operator over a representation of the SVA. We recognise $(-1)^{2(J_0 - Q_{ac})}$ as the chiral world-sheet fermion operator. It is well-defined since J_0 has charge Q_{ac} on the lowest weight state $|a, c\rangle$ of $\mathbb{H}_{ac}$, and since the charge of a descendent state differs from Q_{ac} by a half-integer or an integer. The chiral world-sheet fermion operator commutes with the modes L_n, J_n and anti-commutes with the modes $G_r^\pm$, so $\frac{1}{2}(1 + (-1)^{2(J_0 - Q_{ac})})$ projects to those states created from the lowest weight state $|h_{ac}, Q_{ac}\rangle$ by the application of an even number of fermionic modes $G_r^\pm$, i.e. states of the form

$$L_{-n_1} \ldots L_{-n_\alpha} J_{-m_1} \ldots J_{-m_\beta} G_{-l_1}^+ \ldots G_{-l_\gamma}^+ G_{-k_1}^- \ldots G_{-k_\delta}^- |h, Q\rangle$$

for which $\gamma + \delta$ is even. Similarly $\frac{1}{2}(1 - (-1)^{2(J_0 - Q_{ac})})$ projects to those states with $\gamma + \delta$ odd.

The notation $\chi_{k-a,c+\overline{k}}$ is natural, since the state(s) with the lowest weight after projection have weight $h_{k-a,c+\overline{k}}$ mod 1 and charge $Q_{k-a,c+\overline{k}}$ mod 1 where we have extended the definition of h and Q in equations (2.12) and (2.13) to the indexing set Q_k.

---

[9]The central charge $\mathbf{c}$ should not be confused with the label c.

2.2.4 Modular invariance

We now investigate the modular properties of the characters. The modular group $\mathrm{SL}(2, \mathbb{Z})$ acts faithfully on $\mathcal{H} \times \mathbb{C} \times \mathbb{C}$ where, as before, $\mathcal{H}$ is the upper-half complex plane. Its action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z, u) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, u + \frac{xcz^2}{c\tau + d} \right)$$

where x is some fixed complex number. Note that the action descends to the usual $\mathrm{PSL}(2, \mathbb{Z})$ action on $\mathcal{H}$. $\mathrm{SL}(2, \mathbb{Z})$ now has a natural action on the theta functions: we find that

$$T \cdot \theta_{a,b}(\tau, z, u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \theta_{a,b}(\tau, z, u)$$
$$= \theta_{a,b}(\tau + 1, z, u)$$
$$= \exp\left(\frac{2\pi i a}{4b} \right) \theta_{a,b}(\tau, z, u),$$

$$S \cdot \theta_{a,b}(\tau, z, u) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \theta_{a,b}(\tau, z, u)$$
$$= \theta_{a,b}\left(-\frac{1}{\tau}, \frac{z}{\tau}, u + \frac{z^2}{4b\tau} \right)$$
$$= \left(\frac{-i\tau}{2b} \right)^{\frac{1}{2}} \sum_{a' \in \mathbb{Z}_{2b}} \exp\left(-\frac{2\pi i a a'}{2b} \right) \theta_{a',b}(\tau, z, u)$$

where the action of S is calculated using Poisson resummation.[10] Similarly $\mathrm{SL}(2, \mathbb{Z})$ acts on the string functions by [41]

$$T \cdot c_{l,m}^{(k)}(\tau) = c_{l,m}^{(k)}(\tau + 1)$$
$$= \exp\left(2\pi i \left(\frac{l(l+2)}{4\bar{k}} - \frac{m^2}{4k} - \frac{\mathbf{c}}{24} \right) \right) c_{l,m}^{(k)}(\tau),$$

$$S \cdot c_{l,m}^{(k)}(\tau) = c_{l,m}^{(k)}\left(-\frac{1}{\tau} \right)$$
$$= (-i\tau)^{-\frac{1}{2}} \sum_{l'=0}^{k} \sum_{m' \in \mathbb{Z}_{2k}} S_{l,l'}^{\widehat{\mathfrak{su}(2)}_k} \left(S_{m,m'}^{\widehat{\mathfrak{u}(1)}_k} \right)^{*} c_{l',m'}^{(k)}(\tau)$$

where the S matrices are given by

$$S_{l,l'}^{\widehat{\mathfrak{su}(2)}_k} = \sqrt{\frac{2}{k}} \sin\left(\frac{\pi(l+1)(l'+1)}{\bar{k}} \right), \tag{2.16}$$

$$S_{m,m'}^{\widehat{\mathfrak{u}(1)}_k} = \sqrt{\frac{1}{2k}} \exp\left(-\frac{\pi i m m'}{k} \right). \tag{2.17}$$

---

[10]We have fixed x here to be $\frac{1}{4b}$

We can read off the action of T on the character $\chi_{a,c}$ directly from (2.14) and (2.15):

$$
\begin{aligned}
T \cdot \chi_{a,c}(\tau, z) &= \chi_{a,c}(\tau + 1, z) \\
&= \exp\left(2\pi i \left(h_{a,c} - \frac{\mathbf{c}}{24}\right)\right) \chi_{a,c}(\tau, z) \\
&= \sum_{(a',c') \in Q_k} T^{\widehat{\mathfrak{su}(2)}_k}_{a,a'} T^{\widehat{\mathfrak{u}(1)}_2}_{[a+c],[a'+c']} \left(T^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}_{c,c'}\right)^* \chi_{a',c'}(\tau, z).
\end{aligned}
\tag{2.18}
$$

As before, the $h_{a,c}$ are defined for all $(a,c) \in Q_k$ by equation (2.12). The T-matrices are those of the $\widehat{\mathfrak{su}(2)}$ and $\widehat{\mathfrak{u}(1)}$ characters arising in the Sugawara [60] construction. They are given by

$$
\begin{aligned}
T^{\widehat{\mathfrak{su}(2)}_k}_{a,a'} &= \delta_{a,a'} \exp\left(2\pi i \left(\frac{(a+1)^2}{4\overline{k}} - \frac{1}{8}\right)\right) \\
T^{\widehat{\mathfrak{u}(1)}_l}_{c,c'} &= \delta_{c,c'} \exp\left(2\pi i \left(\frac{c^2}{4l} - \frac{1}{24}\right)\right).
\end{aligned}
\tag{2.19}
$$

Kac and Wakimoto calculated the S-matrix in [42]. It reads

$$
\begin{aligned}
S \cdot \chi_{a,c}(\tau, z) &:= e^{-\frac{i\pi \mathbf{c} z^2}{3\tau}} \chi_{a,c}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) \\
&= \sum_{(a',c') \in Q_k} 2 S^{\widehat{\mathfrak{su}(2)}_k}_{a,a'} S^{\widehat{\mathfrak{u}(1)}_2}_{[a+c],[a'+c']} \left(S^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}_{c,c'}\right)^* \chi_{a',c'}(\tau, z).
\end{aligned}
\tag{2.20}
$$

2.3   Structure of the physical invariants

2.3.1   Notation

We consider $\mathbb{Z}_{2p}$ to have underlying set $\{-p+1, \ldots, p\}$. Recall that we defined

$$
\begin{aligned}
P_k &:= \{(l, m) \in \{0, \ldots, k\} \times \mathbb{Z}_{2\overline{k}} \mid |m| \leq l, \ l + m \equiv 0 \mod 2\}, \\
P'_k &:= \{(a, c) \in \{0, \ldots, k\} \times \mathbb{Z}_{2\overline{k}} \mid |c - [a+c]| \leq a, \}, \\
Q_k &:= \{0, \ldots, k\} \times \mathbb{Z}_{2\overline{k}},
\end{aligned}
$$

where, throughout the paper, $\overline{k} = k + 2$ and we are using the convention that $[x] = 0$ when x is even and $[x] = \pm 1$ when x is odd, where the sign depends on whether we realise the Ramond sector as R+ or R−.

We now choose for once and for all to realise the Ramond sector via R+. In the notation of the last section, this is equivalent to $b = [a+c] = -2\lambda = 1$. Then each $(l, m) \in P_k$ labels an NS character ch^0_{lm} and a Ramond(+) character $\mathrm{ch}^{-\frac{1}{2}}_{lm}$. The $(a, c) \in Q_k$ label the half-characters χ_{ac}, and $[a+c] = 0$ or 1 label the NS and R+ sectors, respectively.

The conformal weight and charge are given in (2.12) and (2.13). These equations actually define h, Q for all $(a, c) \in Q_k$.

2.3.2 Partition functions and physical invariants

A $N = 2$ SCFT at central charge $\mathbf{c} = \frac{3k}{k}$ is, among other things, a representation of the direct sum of two commuting copies of the $N = 2$ SVA. We are interested in minimal theories, in which case this representation decomposes into a finite number of irreducible representations $\mathbb{H}_{lm}^\lambda \otimes \overline{\mathbb{H}}_{l'm'}^{\lambda'}$, $(lm), (l'm') \in P_k$, $\lambda, \lambda' \in \{0, -\frac{1}{2}\}$, which we saw in section 2.2.2 were classified by [4]. So the full pre-Hilbert space is

$$\mathbb{H} = \bigoplus_{\substack{(l,m) \in P_k \\ (l',m') \in P_k \\ \lambda, \lambda' \in \{0, -\frac{1}{2}\}}} \mathbf{M}_{l,m;\, l',m'}^{\lambda, \lambda'} \mathbb{H}_{lm}^\lambda \otimes \overline{\mathbb{H}}_{l'm'}^{\lambda'} \tag{2.21}$$

where the non-negative integer matrix $\mathbf{M}_{l,m;\, l',m'}^{\lambda, \lambda'}$ counts the multiplicity of the irreducible representation $\mathbb{H}_{lm}^\lambda \otimes \overline{\mathbb{H}}_{l'm'}^{\lambda'}$. One of the axioms of an SCFT is that the vacuum is unique; that is, the vacuum representation appears precisely once:

$$\mathbf{M}_{0,0;\, 0,0}^{0,0} = 1. \tag{2.22}$$

We will define the *full partition function* $Z(\tau, z)$ of a SCFT $\mathbb{H}$ to be the trace over $\mathbb{H}$ of $q^{L_0 - \frac{\mathbf{c}}{24}} y^{J_0} \overline{q}^{\overline{L}_0 - \frac{\mathbf{c}}{24}} \overline{y}^{\overline{J}_0}$ where, as before, $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$, $\mathbf{c}$ is the central charge and L_0, J_0 and $\overline{L}_0, \overline{J}_0$ span the Cartan subalgebra of the left- and right-handed copies of the SVA, respectively (see equations (2.3)). Thus we have

$$Z(\tau, z) = \sum_{\substack{(l,m) \in P_k \\ (l',m') \in P_k \\ \lambda, \lambda' \in \{0, -\frac{1}{2}\}}} \mathbf{M}_{l,m;\, l',m'}^{\lambda, \lambda'} \mathrm{ch}_{lm}^\lambda(\tau, z) \mathrm{ch}_{l'm'}^{\lambda'}(\tau, z)^*$$

where, as in section 2.2.3, $\mathrm{ch}_{lm}^\lambda(\tau, z)$ are the characters of the representation $\mathbb{H}_{lm}^\lambda$.

In $N = 0$ CFTs, the characters of the irreducible representations are required to transform into one another linearly under the action of $SL_2(\mathbb{Z})$. But in the superconformal case the situation is slightly more subtle; here, in order to build an $SL_2(\mathbb{Z})$ module from the characters ch_{lm}^λ we are forced to consider also the 'twisted' characters $\widetilde{\mathrm{ch}}_{lm}^\lambda$ defined in equation (2.11).[11]

As we can see from equations (2.10) and (2.11), the basis $\{\mathrm{ch}_{lm}^\lambda, \widetilde{\mathrm{ch}}_{lm}^\lambda \mid (lm) \in P_k, \lambda \in \{0, -\frac{1}{2}\}\}$ is equivalent to the basis of half characters $\{\chi_{ac} \mid (ac) \in Q_k\}$. We shall work in this latter basis from now on, for several reasons. Firstly this basis diagonalises the T-matrix, as seen in equation (2.18); secondly, if we consider the $\chi_{ac}(\tau, z)$ as a formal power series in q and y multiplied by a factor $q^{h - \frac{\mathbf{c}}{24}} y^Q$, the term $q^n y^m$ has a non-negative integer coefficient, which we can

---

[11]The S and T transformations of the characters and twisted characters can easily be deduced from the those of the half-characters in equations (2.20) and (2.18).

interpret as the dimension of the space of states at level n and relative charge m. Thirdly, the action of the simple currents on the half-characters is more transparent.

We can now state the condition of modular invariance: a SCFT $\mathbb{H}$ is required to have a *bosonic partition function*

$$Z(\tau, z) = \sum_{\substack{(ac) \in Q_k \\ (a'c') \in Q_k}} M_{a,c; a',c'} \chi_{ac}(\tau, z) \chi_{a'c'}(\tau, z)^* \tag{2.23}$$

which is *modular invariant*; that is, Z is invariant under the natural action of $SL_2(\mathbb{Z})$, defined by the action of S and T on χ_{ac}, which was described in section 2.2.4. Using the fact that the χ_{ac} are linearly independent, this is equivalent to the condition that

$$SMS^\dagger = M, \tag{2.24}$$

$$TMT^\dagger = M; \tag{2.25}$$

or, using the unitarity of S and T, equivalent to $SM = MS$ and $TM = MT$. A non-negative integer matrix M with $M_{0,0;\,0,0} = 1$ and satisfying equations (2.24) and (2.25) above is called a *physical invariant*.

We have not yet spelled out exactly how one should associate a modular partition function (2.23) to a given representation (2.21). While it is clear from equations (2.14) and (2.15) that we should project out half the states in each family, it is not clear *which* half, (except in the vacuum representation when we know that the vacuum state must be present). In practise one starts by finding a modular invariant (bosonic) partition function and reconstructing the full Hilbert space. One then reads off the correct projection *a posteriori*.[12]

We will find a nice characterisation of the projection to the bosonic states in section 3.5 for some class of minimal models in terms even sublattices of the charge lattice, but for now we will illustrate what happens for the class of modular invariant partition functions which count states only in the NS-NS and R-R sectors.

We note here one immediate consequence of modular invariance of a physical invariant: using equation (2.18), we see that T-invariance is equivalent to

$$M_{a,c;\,a',c'} \neq 0 \quad \Rightarrow \quad h_{ac} - h_{a'c'} \in \mathbb{Z}. \tag{2.26}$$

---

[12]One might have expected that χ_{ac} should count bosons and $\chi_{k-a,c+\overline{k}}$ should count fermions for $(ac) \in P_k$, and that the projection should be given by keeping only boson×boson and fermion×fermion. In fact this is true only for the $\mathcal{A}$-model. One reason for this is that the choice of χ_{ac} over $\chi_{k-a,c+\overline{k}}$ in the R sector is completely arbitrary, as it depends upon choosing your favourite of the two possible highest weight states. Another reason is that, as we shall see in section 4, we shall perform orbifolds by the fermion number operator $(-1)^F$, which mixes up our notion of bosons and fermions.

2.3.3   A simplification

We consider the case in which there are no states in the NS-R or R-NS sectors; that is,

$$M_{a,c;\,a',c'} \neq 0 \Rightarrow a + c + a' + c' \equiv 0 \mod 2. \qquad (2.27)$$

We begin by proving a lemma regarding the restrictions placed on a physical invariant by the exclusion of NS-R and R-NS sectors. Denote by $\mathbf{j} : Q_k \to Q_k$ the bijection $(a,c) \mapsto (k-a, c+\bar{k})$, where, as always, $\bar{k} = k + 2$. One checks directly using the S-matrix, or by skipping ahead to equation (2.39), that $S_{\mathbf{j}(ac),a'c'} = (-1)^{a'+c'} S_{ac;\,a'c'}$. This nice interaction of $\mathbf{j}$ with the S-matrix is typical of *simple currents*, of which $\mathbf{j}$ is an example. We shall see many more of these in section 2.4.3.

Lemma 2.3.1. *Let $Z = \sum_{a,c,a',c'} M_{ac;\,a'c'} \chi_{ac} \chi^*_{a'c'}$ be a physical invariant and suppose that there are no NS-R or R-NS states present, i.e. that (2.27) holds. Then $M_{\mathbf{j}(a,c);\,\mathbf{j}(a',c')} = M_{ac;\,a'c'}$ for all $(ac), (a'c') \in Q_k$.*

Proof. Using equation (2.24) we have

$$M_{\mathbf{j}(a,c);\,\mathbf{j}(a',c')} = (SMS^\dagger)_{\mathbf{j}(a,c);\,\mathbf{j}(a',c')}$$

$$= \sum_{\substack{(st)\in Q_k \\ (uv)\in Q_k}} S_{\mathbf{j}(a,c);\,st} M_{st;\,uv} S^\dagger_{uv;\,\mathbf{j}(a',c')}$$

which, by equation (2.39),

$$= \sum_{\substack{(st)\in Q_k \\ (uv)\in Q_k}} (-1)^{s+t+u+v} S_{ac;\,st} M_{st;\,uv} S^\dagger_{uv;\,a'c'}$$

$$= \sum_{\substack{(st)\in Q_k \\ (uv)\in Q_k}} S_{ac;\,st} M_{st;\,uv} S^\dagger_{uv;\,a'c'}$$

$$= (SMS^\dagger)_{ac;\,a'c'}$$

$$= M_{ac;\,a'c'},$$

where we used equations (2.27) and (2.24) again. $\square$

Consider the most general bosonic partition function $Z(\tau, z)$ obeying equation (2.27). We will use lemma 2.3.1 and the identities (2.10) and (2.11) to express it in terms of the characters $\mathrm{ch}_{lm}(\tau, z)$. First recall from section 2.3.1 that the NS sector is labelled by indices $\mathrm{NS}_k := P_k \cup \mathbf{j} \cdot P_k = \{(ac) \in Q_k \mid a + c \equiv 0\}$. Thus Z restricted to the NS-NS sector reads

$$Z^{\mathrm{NS}} = \sum_{\substack{(ac),(a'c')\in Q_k \\ a+c\equiv a'+c'\equiv 0}} M_{a,c;\,a',c'} \chi_{ac} \chi^*_{a'c'}$$

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which, by lemma 2.3.1,

$$
\begin{aligned}
&= \sum_{\substack{(ac)\in P_k \\ (a'c')\in P_k}} M_{a,c;\,a',c'} \left(\chi_{ac}\chi^*_{a'c'} + \chi_{\mathbf{j}(ac)}\chi^*_{\mathbf{j}(a'c')} \right) \\
&\quad + \sum_{\substack{(ac)\in P_k \\ (a'c')\in P_k}} M_{\mathbf{j}(a,c);\,a',c'} \left(\chi_{\mathbf{j}(ac)}\chi^*_{a'c'} + \chi_{ac}\chi^*_{\mathbf{j}(a'c')} \right) \\
&= \sum_{\substack{(lm)\in P_k \\ (l'm')\in P_k}} M_{l,m;\,l',m'} \frac{1}{2} \left(\mathrm{ch}^0_{lm}\mathrm{ch}^{0\,*}_{l'm'} + \widetilde{\mathrm{ch}}^0_{lm}\widetilde{\mathrm{ch}}^{0\,*}_{l'm'} \right) \\
&\quad + \sum_{\substack{(lm)\in P_k \\ (l'm')\in P_k}} M_{\mathbf{j}(l,m);\,l',m'} \frac{1}{2} \left(\mathrm{ch}^0_{lm}\mathrm{ch}^{0\,*}_{l'm'} - \widetilde{\mathrm{ch}}^0_{lm}\widetilde{\mathrm{ch}}^{0\,*}_{l'm'} \right)
\end{aligned}
\tag{2.28}
$$

where in the last line we used equations (2.10) and (2.11). Now equations (2.12) and (2.26) show that either $M_{l,m;\,l',m'}$ or $M_{\mathbf{j}(l,m);\,l',m'}$ is zero for (l,m) and (l',m') in the NS sector, since $h_{\mathbf{j}(l,m)} \equiv h_{l,m} + \frac{1}{2} \mod 1$. Combining this with the interpretation of ch^0_{lm} and $\widetilde{\mathrm{ch}}^0_{lm}$ as traces over the pre-Hilbert space $\mathbb{H}^0_{lm}$ given in equations (2.10) and (2.11), we can read off the full Hilbert space of the NS-NS sector:

$$
\mathbb{H}^{\mathrm{NS}} = \bigoplus_{\substack{(lm)\in P_k \\ (l'm')\in P_k}} \left(M_{l,m;\,l',m'} + M_{\mathbf{j}(l,m);\,l',m'} \right) \mathbb{H}^0_{l,m} \otimes \overline{\mathbb{H}}^0_{l',m'}.
\tag{2.29}
$$

We claim the projection to the subspace with modular invariant partition function is

$$
\mathcal{P}_b = \frac{1}{2} \left(1 + (-1)^{2(L_0 - \overline{L}_0)} \right).
\tag{2.30}
$$

We have already seen this is well-defined on the primary states that can appear in the full Hilbert space of the NS-NS sector. It remains to check that $\mathcal{P}_b$ is well-defined on descendent states. As we saw in equation (2.5), a descendent state of the form (2.4) is an eigenvector of the operator $L_0 - h$ with eigenvalue in $\frac{1}{2}\mathbb{Z}$, where h is the conformal weight of the primary state. The analogue statement is true for right-hand descendent states. Since $h - \overline{h} \in \frac{1}{2}\mathbb{Z}$ for any primary state in $\mathbb{H}^{\mathrm{NS}}$, the operator $(-1)^{2(L_0 - \overline{L}_0)}$ is well-defined on $\mathbb{H}^{\mathrm{NS}}$. In fact $\mathcal{P}_b$ is none other than the projection to the world-sheet bosonic states: it kills all states with half-integer spin and keeps those with integer spin.

To summarise: the NS-NS part of the modular invariant bosonic partition function is related to the bosonic sector of the NS-NS part of the full partition function by:

$$
Z^{\mathrm{NS}} = \mathcal{P}_b \left[\sum_{\substack{(lm)\in P_k \\ (l'm')\in P_k}} N^{\mathrm{NS}}_{l,m;\,l',m'} \mathrm{ch}^0_{lm}\mathrm{ch}^{0\,*}_{l'm'} \right]
$$

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where the projection is understood to be inserted into the trace in the definition of the character ch, and where

$$N^{\mathrm{NS}}_{l,m;l',m'} = M_{l,m;\,l',m'} + M_{\mathbf{j}(l,m);\,l',m'}.$$

The Ramond-Ramond part of the modular invariant partition function is

$$Z^{\mathrm{R}} = \sum_{\substack{(lm)\in P_k \\ (l'm')\in P_k}} M_{l,m+1;\,l',m'+1}\, \frac{1}{2}\left(\mathrm{ch}_{lm}^{-\frac{1}{2}}\,\mathrm{ch}_{l'm'}^{-\frac{1}{2}\,*} + \widetilde{\mathrm{ch}}_{lm}^{-\frac{1}{2}}\,\widetilde{\mathrm{ch}}_{l'm'}^{-\frac{1}{2}\,*}\right)$$

$$+ \sum_{\substack{(lm)\in P_k \\ (l'm')\in P_k}} M_{\mathbf{j}(l,m+1);\,l',m'+1}\, \frac{1}{2}\left(\mathrm{ch}_{lm}^{-\frac{1}{2}}\,\mathrm{ch}_{l'm'}^{-\frac{1}{2}\,*} - \widetilde{\mathrm{ch}}_{lm}^{-\frac{1}{2}}\,\widetilde{\mathrm{ch}}_{l'm'}^{-\frac{1}{2}\,*}\right),$$

so the full Hilbert space of the R-R sector is

$$\mathbb{H}^{\mathrm{R}} = \bigoplus_{\substack{(lm)\in P_k \\ (l'm')\in P_k}} N^{\mathrm{R}}_{l,m;\,l',m'}\,\mathbb{H}^{-\frac{1}{2}}_{l,m}\,\overline{\mathbb{H}}^{-\frac{1}{2}}_{l',m'} \tag{2.31}$$

where

$$N^{\mathrm{R}}_{l,m;\,l',m'} = M_{l,m+1;\,l',m'+1} + M_{\mathbf{j}(l,m+1);\,l',m'+1}. \tag{2.32}$$

We will see in section 2.5.4 that there are two modular invariant subspaces associated to the R-R sector of the full Hilbert space of an $N = 2$ SCFT. In general we do know how to write the projections in a convenient form as we could for the NS-NS sector (2.30), but we give such an expression for the case when the level k is odd in section 3.5.

2.4 Simple currents and fusion rules

2.4.1 Definition of simple currents

In the study of conformal field theories, a rich symmetry structure arises out of the so-called *simple currents* [40, 59, 58]. A simple current is a primary field which upon fusion with any other field yields precisely one primary field (plus its descendants). The simple currents can therefore be found from the fusion coefficients $N^{\alpha}_{a,a'}$ defined by

$$[\phi_l] \times [\phi_{l'}] = \sum_{\alpha\in P} N^{\alpha}_{l,l'}[\phi_\alpha]$$

where ϕ_l are primary fields labelled by $l \in P$. $[\phi_l]$ represents a sum over the primary field ϕ_l and its descendants. $N^{\alpha}_{l,l'}$ counts the multiplicity of the field ϕ_α appearing in the OPE of ϕ_l and $\phi_{l'}$.

2.4.2 The Verlinde formula and fusion rules

The Verlinde formula [63] gives a surprising and elegant expression for fusion rules in terms of the S-matrix for (bosonic) CFTs. Inspired by this we define

$$N_{ac;\,a'c'}^{\alpha\gamma} := \sum_{(d,f)\in Q_k} \frac{S_{ac;df}S_{a'c';df}S_{\alpha\gamma;df}^*}{S_{00;df}}. \qquad (2.33)$$

Lemma 2.4.1. *Fix* $(ac),(a'c'),(\alpha,\gamma) \in Q_k$. *We have*

$$N_{ac;\,a'c'}^{\alpha\gamma} = \left\{ \begin{array}{ll} \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}\right)^{\gamma}_{c,c'}, & \text{if } [a+c][a'+c']=0 \\[2mm] \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{k-\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}\right)^{\gamma+\overline{k}}_{c,c'}, & \text{if } [a+c][a'+c']=1 \end{array} \right\}$$

Here $N^{\widehat{\mathfrak{su}(2)}_k}$ and $N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}$ are the fusion coefficients for the WZW models [66, 67] obtained from $\widehat{\mathfrak{su}(2)}$ at level k [32] and $\widehat{\mathfrak{u}(1)}$ at level $\overline{k}$ respectively. They read

$$\left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{l}_{a,a'} = \delta(|a-a'|\leq l \leq \min(a+a',2k-a-a'))\,\delta(a+a'\equiv l \mod 2)$$
$$\left(N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}\right)^{n}_{c,c'} = \delta(c+c'\equiv n \mod 2\overline{k}),$$

where $\delta(\text{condition}) = 1$ if and only if 'condition' is satisfied. In particular, we see that $N_{ac;\,a'c'}^{\alpha,\gamma}$ is only non-zero if $a+c+a'+c'+\alpha+\gamma\equiv 0 \mod 2$. If we can interpret the N as fusion coefficients of the minimal models then we obtain the following selection rules for the NS and R sectors:

$$NS \times NS \sim NS \qquad\qquad NS \times R \sim R$$
$$R \times NS \sim R \qquad\qquad R \times R \sim NS.$$

Proof. It is possible to expand the expression (2.33) into a sum of products of sines and exponentials which can be simplified at great length and tedium. We present here a very simple proof using simple currents of the S-matrices of the WZW models obtained from $\widehat{\mathfrak{su}(2)}$ and $\widehat{\mathfrak{u}(1)}$. Simple currents are explained in detail in the following section, but for now we will just use the fact that

$$\begin{aligned} S_{k-a,a'} &= (-1)^{a'} S_{a,a'}, \quad a,a' \in \{0,\ldots,k\} \\ \overline{S}_{c+\overline{k},c'} &= (-1)^{c'} \overline{S}_{c,c'}, \quad c,c' \in \mathbb{Z}_{2\overline{k}}, \end{aligned} \qquad (2.34)$$

where we have written S for the $\widehat{\mathfrak{su}(2)}$ S-matrix at level k and $\overline{S}$ for the $\widehat{\mathfrak{u}(1)}$ S-matrix at level $2\overline{k}$. This is easily checked from equations (2.16) and (2.17). Substituting in the definitions we find

$$N_{ac;\,a'c'}^{\alpha\gamma} = \sum_{(d,f)\in Q_k} \frac{S_{a,d}S_{a',d}S_{\alpha,d}^*}{S_{0,d}} e^{-\frac{\pi i}{2}([a+c]+[a'+c']-[\alpha+\gamma])} \frac{\overline{S}_{c,f}\overline{S}_{c',f}\overline{S}_{\gamma,f}^*}{\overline{S}_{0,f}}.$$

First suppose that $[a + c] + [a' + c'] - [\alpha + \gamma] = 0$. This corresponds to fusion $NS \times NS \sim NS$, $NS \times R \sim R$ or $R \times NS \sim R$. Then

$$N^{\alpha\gamma}_{ac;\,a'c'} = \sum_{d=0}^{k} \frac{S_{a,d}S_{a',d}S^*_{\alpha,d}}{S_{0,d}} \sum_{f \in \mathbb{Z}_{2\overline{k}}} \frac{\overline{S}_{c,f}\overline{S}_{c',f}\overline{S}^*_{\gamma,f}}{\overline{S}_{0,f}}$$

$$= \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}\right)^{\gamma}_{c,c'}.$$

Now suppose that $[a+c]+[a'+c']-[\alpha+\gamma] = 2$, corresponding to fusion $R \times R$. Then

$$N^{\alpha\gamma}_{ac;\,a'c'} = \sum_{(df) \in Q_k} \frac{S_{a,d}S_{a',d}S^*_{\alpha,d}}{S_{0,d}}(-1)^{d+f} \frac{\overline{S}_{c,f}\overline{S}_{c',f}\overline{S}^*_{\gamma,f}}{\overline{S}_{0,f}}$$

$$= \sum_{(df) \in Q_k} \frac{S_{a,d}S_{a',d}S^*_{k-\alpha,d}}{S_{0,d}} \sum_{f \in \mathbb{Z}_{2\overline{k}}} \frac{\overline{S}_{c,f}\overline{S}_{c',f}\overline{S}^*_{\gamma+\overline{k},f}}{\overline{S}_{0,f}}$$

$$= \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{k-\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}\right)^{\gamma+\overline{k}}_{c,c'}$$

where we used equations (2.34) in the middle step. Finally suppose that $[a + c] + [a' + c'] - [\alpha + \gamma]$ is odd. Then using equations (2.34) again, we see that under $(d, f) \mapsto (k - d, f + \overline{k})$ the summand

$$\frac{S_{a,d}S_{a',d}S^*_{\alpha,d}}{S_{0,d}}e^{-\frac{\pi i}{2}([a+c]+[a'+c']-[\alpha+\gamma])} \frac{\overline{S}_{c,f}\overline{S}_{c',f}\overline{S}^*_{\gamma,f}}{\overline{S}_{0,f}}$$

picks up a factor of $(-1)^{a+a'+\alpha+c+c'+\gamma} = -1$, and thus the sum vanishes. $\square$

We want to interpret N as the set of fusion coefficients for the $N = 2$ minimal models. Recall that the fusion in a (bosonic) CFT describes how the different conformal families interact under the operator product expansion (OPE). Let $\phi_a(z), \phi_b(z)$ be primary fields. Then the fusion of $\phi_a(z)$ with $\phi_b(w)$ is given by

$$\phi_a(z)\phi_b(w) = \sum_{z \in P} C^c_{ab}(z - w)^{h_c - h_a - h_b}\left[\phi_c(w) + \sum_{n>0}(z - w)^n\phi^{(n)}_c(w)\right] \quad (2.35)$$

where $C^c_{ab} \in \mathbb{C}$ are the *OPE coefficients*, $h_x \in \mathbb{C}$ is the conformal weight of the primary field ϕ_x and P labels the set of primary fields. $\phi^{(n)}_c(w)$ are descendent fields of $\phi_c(w)$, i.e. those built from linear combinations of fields of the form $(L_{-k_1} \ldots L_{-k_n}\phi)(w)$ for positive k_i.

The space of all descendent fields of a primary field $\phi_c(w)$ is the *conformal family* $[\phi_c]$ of $\phi_c(w)$. Under the state-field correspondence, the fields in a conformal family correspond precisely to vectors in the irreducible LWR built on the LWV $|\phi_c\rangle$. In equation (2.35) it is understood that more than one copy of each conformal family can appear in the sum on the right-hand side.

We record which conformal families appear in the fusion of $\phi_a(z)$ and $\phi_b(w)$ using the notation

$$[\phi_a] \times [\phi_b] \sim \sum_{c \in P} N^c_{a,b} \, [\phi_c]$$

where $N^c_{a,b} \in \mathbb{Z}$ counts the multiplicity of the family $[\phi_c]$ appearing on the right hand side. The integers $N^c_{a,b}$ are called the fusion rules of the theory.

In the $N = 2$ case, the fusion between the super primary fields is *a priori* again

$$\phi_a(z)\phi_b(w) = \sum_{z \in P} C^c_{ab}(z - w)^{h_c - h_a - h_b}\left[\phi_c(w) + \sum_{n > 0}(z - w)^n \phi^{(n)}_c(w)\right] \quad (2.36)$$

where the $\phi_c(w)$ are $N = 2$ descendent states (so in particular in the NS sector, the sum also runs over positive half integers). The fusion rules *a priori* are

$$[\phi_a] \times [\phi_b] \sim \sum_{c \in P} N^c_{a,b} \, [\phi_c].$$

We can view the OPE as a short-range expansion for fields inside a compatible system of n-point functions. Then J_0 invariance of the n-point functions constrains the form of the OPE in equation (2.36). It implies that the $U(1)$ charges of all the fields $\phi^{(n)}_c(w)$ must be equal. This allows us to refine the fusion rules. Descendants of $\phi_c(w)$ are of the form

$$(L_{-n_1} \ldots L_{-n_\alpha} J_{-m_1} \ldots J_{-m_\beta} G^+_{-l_1} \ldots G^+_{-l_\gamma} G^-_{-k_1} \ldots G^-_{-k_\delta} \phi_c)(w),$$

which has $U(1)$ charge $Q_c + \frac{1}{2}(\gamma - \delta)$, where Q_c is the $U(1)$ charge of $\phi_c(w)$. We split the superconformal family $[\phi_c]$ into two subfamilies: $[c, +]$ containing those descendants with $\gamma - \delta$ even and $[c, -]$ containing those descendants with $\gamma - \delta$ odd. We can then capture the interactions of the different even and odd superconformal 'half-families' in the super fusion rules

$$[a, \epsilon_a] \times [b, \epsilon_b] \sim \sum_{(c, \epsilon_c) \in P \times \{\pm\}} N^{(c, \epsilon_c)}_{(a, \epsilon_a),(b, \epsilon_b)} [c, \epsilon_c].$$

We now specialise to the case of the $N = 2$ minimal models. Recall that the super-primary fields of the $N = 2$ minimal models are labelled by the $(a, c) \in Q_k = \{0, \ldots, k\} \times \mathbb{Z}_{2\bar{k}}$ that satisfy $|c - [a + c]| \leq a$. According to the discussion after equation (2.15) we see that for the $N = 2$ minimal models, fields in $[(a, c), +]$ with $|c - [a + c]| \leq a$ correspond under the state-field correspondence precisely to states counted by the character χ_{ac}, and fields in $[(a, c), -]$ to states counted by $\chi_{k-a,c+\bar{k}}$. We will henceforth use the notation $[(a, c)]$ with $(a, c) \in Q_k$ to label the even and odd superconformal families for the $N = 2$ minimal models.

The integers $N^{\alpha\gamma}_{ac;a'c'}$ calculated in lemma 2.4.1 are the natural candidates for the super fusion rules. This result is confirmed by [48, 49] in the NS×NS and

R×R sectors, both through the Coulomb gas formalism and through the explicit construction of the unitary $N = 2$ minimal models via the parafermion-boson construction [55] (also see sections 3.6 and 3.7.1).

We can also read off the usual fusion between $N = 2$ primary fields by simply forgetting the distinction between $[ac]$ and $[k - a, c + \overline{k}]$. Then the fusion rules for the primary fields read

$$\widehat{N}^{\alpha,\gamma}_{ac;a'c'} = N^{\alpha\gamma}_{ac;a'c'} + N^{k-\alpha,\gamma+\overline{k}}_{ac;a'c'}$$

$$= \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}\right)^{\gamma}_{c,c'} + \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{k-\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}\right)^{\gamma+\overline{k}}_{c,c'}.$$

It is precisely this quantity that Wakimoto calculates in [64][13]. We summarise this section in the following theorem:

Theorem 2.4.2. *The fusion rules for the $N = 2$ minimal models are given by*

$$N^{\alpha\gamma}_{ac;\,a'c'} = \sum_{(d,f)\in Q_k} \frac{S_{ac;df} S_{a'c';df} S^*_{\alpha\gamma;df}}{S_{00;df}}$$

$$= \left\{ \begin{array}{ll} \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}\right)^{\gamma}_{c,c'}, & \text{if } [a+c][a'+c'] = 0 \\ \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{k-\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}\right)^{\gamma+\overline{k}}_{c,c'}, & \text{if } [a+c][a'+c'] = 1 \end{array} \right\}$$

$$\text{for } (ac),(a'c'),(\alpha\gamma) \in Q_k,$$

where we label fields in the superconformal family of the super-primary $\phi_{ac}(z)$ with the same $U(1)$ charge as $\phi_{ac}(z)$ by $[ac]$, and fields whose $U(1)$ charge differs by a half integer by $[k - a, c + \overline{k}]$ for $|c - [a + c]| \le a$.

If we simply wish to label fields in the same superconformal family as $\phi_{ac}(z)$ by $[ac]$ then the fusion rules are

$$\widehat{N}^{\alpha,\gamma}_{ac;a'c'} = N^{\alpha\gamma}_{ac;a'c'} + N^{k-\alpha,\gamma+\overline{k}}_{ac;a'c'}$$

$$= \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}\right)^{\gamma}_{c,c'} + \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{k-\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}(1)}_{\overline{k}}}\right)^{\gamma+\overline{k}}_{c,c'}$$

$$\text{for } (ac),(a'c'),(\alpha\gamma) \in \{(ln) \in Q_k \mid |n - [l + n]| \le l\}.$$

2.4.3 Simple currents of the minimal models

From the explicit formula for the fusion rules one can read off that the simple currents of the minimal models at level k are $\mathcal{J} = \{0, k\} \times \mathbb{Z}_{2\overline{k}}$. Each current acts naturally on the set of weights of the $N = 2$ minimal models: $\mathbf{j}$ maps the weight (a, c) to the weight labelling the field which appears in the OPE of $\phi_{\mathbf{j}}$ and $\phi_{a,c}$. Thus, writing J for the $\widehat{\mathfrak{su}(2)}_k$ current $J : a \mapsto k - a$,

$$(J^l 0, d) \cdot (a, c) = (J^{l+(lk+d)(a+c)} a, c + d + (lk + d)(a + c)\overline{k}).$$

---

[13] Actually there is a small error in the statement of the main theorem in [64]. On page 4, condition F2 should include the condition $m + (m - j - k) < (m - j' - k') + (m - j'' - k'')$.

We define a binary operation $\times$ on the set of currents by

$$((J^{l_1}0, d_1) \times (J^{l_1}0, d_1)) \cdot (a, c) := (J^{l_1}0, d_1) \cdot ((J^{l_1}0, d_1) \cdot (a, c)).$$

Then we see that

$$(J^{l_1}0, d_1) \times (J^{l_1}0, d_1) = (J^{l_1 + l_2 + (l_1 k + d_1)(l_2 k + d_2)}0,$$
$$d_1 + d_2 + (l_1 k + d_1)(l_2 k + d_2)\bar{k}) \tag{2.37}$$

and that $\times$ is a closed, associative and commutative binary operator on the set of currents $\mathcal{J}$. It is easy to check that $(0, 0)$ is an identity element and that

$$(J^l 0, d)^{-1} = (J^{l + lk + d}0, -d + (lk + d)\bar{k}).$$

So the set of simple currents form a commutative group isomorphic to

$$\mathcal{J} = \begin{cases} \mathbb{Z}_{4\bar{k}} & \text{if } k \text{ is odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_{2\bar{k}} & \text{if } k \text{ is even.} \end{cases}$$

The simple currents are of great use because the S-matrix behaves well under the action of the currents on the weights. In fact

$$S_{\mathbf{j} \cdot (a,c); \, a', c'} = \exp(2\pi i Q_{\mathbf{j}}(a', c')) S_{a,c; \, a', c'} \tag{2.38}$$

where $Q_{(J^l 0, d)}(a', c') = \frac{a'l}{2} + \frac{c'd}{2\bar{k}} - \frac{[kl+d][a'+c']}{4}$ and we have written $[b] \in \{0, 1\}$ for the value of b modulo 2, as before. $Q_{\mathbf{j}}$ is called the charge of the field $\phi_{a,c}$ with respect to the current $\mathbf{j}$. The charges satisfy

$$Q_{\mathbf{j}}(a, c) = h_{\mathbf{j}} + h_{(a,c)} - h_{\mathbf{j} \cdot (a,c)},$$

so $Q_{\mathbf{j}}(a, c)$ is also the monodromy of $\phi_{a,c}$ with $\phi_{\mathbf{j}}$, as expected [59].

Note that in particular, (2.38) applied to the simple current $\mathbf{j} := (J0, \bar{k})$ gives

$$S_{Ja, c + \bar{k}; \, a', c'} = (-1)^{a' + c'} S_{a, c; \, a', c'}. \tag{2.39}$$

2.4.4 Simple current invariants

It was observed [45] that in all then-known cases, almost all the rational CFTs that can be constructed are the so-called *simple current invariants* [28], leaving at worst a handful of "exceptional" models not of simple current type. By simple current invariant we mean a CFT with partition function $Z = \sum_{l,l'} M_{l,l'} \chi_l \chi_{l'}^*$ where χ_l are the characters of the representations of the $\mathcal{W}$-algebra such that

$$M_{l,l'} \neq 0 \quad \Rightarrow \quad l' = \mathbf{j}l \text{ for some } \mathbf{j} \in \mathcal{J}$$

where $\mathcal{J}$ is the set of simple currents of the CFT. This is a strong assumption indeed - see section 3 of [23] for a number of immediate consequences.

If we are interested in simple current invariants, then we are only concerned with those simple currents that could feasibly build a modular invariant partition function. T-invariance implies that we should only retain those current whose spin multiplied by their order is an even integer: this is the *effective centre*, $\mathcal{C}$ [45]. In the case of the $N = 2$ minimal models:

$$\mathcal{C}_k = \begin{cases} \{(J^l 0, d) \mid l + d \equiv 0 \mod 2\} \cong \mathbb{Z}_{2\overline{k}} & \text{if } k \text{ is odd}, \\ \{0, k\} \times \{2d \mid d \in \mathbb{Z}\} \cong \mathbb{Z}_2 \times \mathbb{Z}_{\overline{k}} & \text{if } 4|k+2, \\ \{0, k\} \times \mathbb{Z}_{2\overline{k}} \cong \mathbb{Z}_2 \times \mathbb{Z}_{2\overline{k}} & \text{if } 4|k, \end{cases}$$

which are groups under the group law inherited from (2.37).

2.5 Symmetries of the models

The rich simple current symmetries of the S and T matrices allow several ways of constructing new physical invariants from old, which we will investigate in this section.

2.5.1 Mirror symmetry

Define $C = S^2$, the so-called charge conjugation matrix. We see that

$$C_{ac;\, a'c'} = \delta(a' = J^{a+c}a)\delta(c' = -c + (a+c)\overline{k}).$$

C is a permutation matrix which acts on the primary fields. *Mirror symmetry* is realised by acting by the charge conjugation matrix $C = S^2$ on one of the chiral sectors. At the level of states of the conformal field theory, mirror symmetry acting on the left-hand representations maps states with $U(1)$ charges $(Q, \overline{Q})$ to states with charges $(-Q, \overline{Q})$. This implies that one model can be obtained from the other by relabelling the generators of the left $U(1)$ current and the generators of the charge-carrying super-currents:

$$\{L_n, J_n, G_r^{\pm}, \overline{L}_n, \overline{J}_n, \overline{G}_r^{\pm}\} \to \{L_n, -J_n, G_r^{\mp}, \overline{L}_n, \overline{J}_n, \overline{G}_r^{\pm}\}.$$

Thus the two mirror symmetric models describe identical physics, and we would normally consider them to be equivalent theories. However, since they give rise to different partition functions, it will be convenient to treat them as belonging to separate theories. The analogue is true for mirror symmetry acting on the right-hand states. We will discuss the case when mirror symmetry acts on both the left-hand and right-hand states in the next section.

At the level of primary fields, charge conjugation acts via

$$*: Q_k \to Q_k: (a, c) \mapsto (a, c)^* = \mathbf{j}^{a+c}(a, -c).$$

Note that we have

$$S_{(a,c)^*;\, a',c'} = S_{a,c;\, (a',c')^*} = S^*_{a,c;\, a',c'}$$

At the level of the (bosonic) partition function, charge conjugation on the left- and right-handed chiral sectors is given by

$$M_{a,c;\,a',c'} \mapsto M_{(a,c)*;\,a',c'} = (CM)_{a,c;\,a',c'},$$
$$M_{a,c;\,a',c'} \mapsto M_{a,c;\,(a',c')*} = (MC)_{a,c;\,a',c'},$$

respectively. In fact CM and MC are identical, since $C = S^2$ and S commutes with M by definition of physical invariant.

Lemma 2.5.1. *Let M be a physical invariant. Then $\widehat{M} = CM = MC$ is a physical invariant.*

Proof. Since C is a permutation matrix which leaves the vacuum invariant, $\widehat{M}$ has positive integer entries and the vacuum is unique. It is clear that S commutes with $\widehat{M}$, since it commutes with M and since $C = S^2$; and T commutes with both M and S^2, and hence with $\widehat{M}$. $\qquad\qquad\square$

2.5.2   Charge conjugation

We note that combining both left- and right- mirror symmetry transformations yields the *charge conjugation* transformation[14], which acts on charges of states via $(Q, \overline{Q}) \to (-Q, -\overline{Q})$. It follows that we obtain one model from its charge conjugate via the transformation

$$\{L_n, J_n, G_r^{\pm}, \overline{L}_n, \overline{J}_n, \overline{G}_r^{\pm}\} \to \{L_n, -J_n, G_r^{\mp}, \overline{L}_n, -\overline{J}_n, \overline{G}_r^{\mp}\}.$$

At the level of partition functions, since the charge conjugation matrix C satisfies $C^2 = S^4 = \text{Id}$, we see that the partition functions is left invariant under the charge conjugation transformation. We will therefore consider charge conjugate theories to be identical.

In particular, if M is a physical invariant, then acting with mirror symmetry on the left or the right yield identical physical invariants CM and MC belonging to identical SCFTs. We will refer to M and CM as 'mirror pairs'.

2.5.3   The symmetry $M_{a,c;\,a',c'} \leftrightarrow M_{a,-c;\,a',c'}$

Lemma 2.5.2. *Let M be a physical invariant. Then $\widehat{M}_{a,c;\,a',c'} = M_{a,-c;\,a',c'}$ is a physical invariant.*

Proof. Since $h_{ac} = h_{a,-c}$ the T-invariance of $\widehat{M}$ follows from equation (2.25) and the T-invariance of M. We then use the fact that $S_{a,-c;\,a',c'} = S_{a,c;\,a',-c'}$ for all $(ac), (a'c') \in Q_k$ and the S-invariance of M to show that $\widehat{M}$ is S-invariant:

$$(S\widehat{M}S^{\dagger})_{ac;\,a',c'} = \sum_{(rs),(uv)\in Q_k} S_{ac;\,rs} M_{r,-s;\,u,v} S^{\dagger}_{uv;\,a'c'}$$

---

[14] We emphasise that acting with the charge conjugation matrix C on one chiral halve yields the mirror symmetry transformation; acting on both halves simultaneously yields the charge conjugation transformation.

$$= \sum_{(rs),(uv)\in Q_k} S_{a,c;\,r,-s} M_{rs;\,uv} S^{\dagger}_{uv;\,a'c'}$$

$$= \sum_{(rs),(uv)\in Q_k} S_{a,-c;\,r,s} M_{rs;\,uv} S^{\dagger}_{uv;\,a'c'}$$

$$= M_{a,-c;\,a',c'}$$

$$= \widehat{M}_{ac;\,a'c'}. \qquad \qquad \square$$

Of course this symmetry applies equally well to the right-hand side giving us new physical invariants $M_{a,c;\,a',-c'}$ and $M_{a,-c;\,a',-c'}$.

2.5.4   The symmetry $M_{ac;\,a'c'} \leftrightarrow M_{\mathbf{j}^{a+c}(ac);\,a'c'}$

By combining mirror symmetry with the previous construction we find the following new physical invariants:

$$\widetilde{M}_{ac;\,a'c'} = M_{\mathbf{j}^{a+c}(ac);\,a'c'}. \tag{2.40}$$

We note that M and $\widetilde{M}$ have the same NS-NS sector content, and therefore belong to full theories whose full NS-NS sector Hilbert spaces are identical. Suppose for a moment that M (and therefore $\widetilde{M}$) belong to a modular invariant partition function of a theory with no states in the NS-R and R-NS sectors, as in section 2.3.3. Then from equation (2.32), the R-R sector of the Hilbert spaces of the associated full theories are also identical.

2.5.5   Transposing

Lemma 2.5.3. *Let M be a physical invariant. Then M^T is a physical invariant.*

Proof. M^T clearly has positive integer entries and a unique vacuum. Since S and T are symmetric, it follows that M^T commutes with S and T. $\qquad \square$

Chapter 3

Classification of the Partition Functions

3.1 Gannon's classification

Gannon's result [26] was to classify all the modular invariant partition functions Z of the form (2.23) with unique vacuum. Recall that the (non-negative integer) matrix of multiplicities M of such a partition function is called a physical invariant. We briefly describe here how this classification was achieved.

There are two key steps. The first is to observe that there is a connection between the minimal models, which as we mentioned earlier can be constructed via the coset representation $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{g} = \widehat{\mathfrak{su}(2)}_k \oplus \widehat{\mathfrak{u}(1)}_2$ and $\mathfrak{h} = \widehat{\mathfrak{u}(1)}_{\overline{k}}$, and the WZW model $\mathfrak{g} \oplus \mathfrak{h}$. Gannon had already shown [27] that the physical invariants of $\mathfrak{g}/\mathfrak{h}$ could be obtained from the physical invariants of $\mathfrak{g} \oplus \mathfrak{h}$ for various diagonal embeddings of $\mathfrak{h} \subset \mathfrak{g}$ at particular levels. This phenomenon occurs because of the similarity of the S-matrices of the two theories. In the case of the unitary $N = 2$ minimal models we have seen that the S-matrix is given by equation (2.20). The characters extend naturally to the indexing set $(a, b, c) \in \{0, \ldots, k\} \times \mathbb{Z}_4 \times \mathbb{Z}_{2\overline{k}} =: P_k''$ if we set

$$
\chi_{ac}^{(b)} := \chi_{ac} \quad \text{when } b = [a + c] \in \{0, 1\},
$$

$$
\chi_{k-a,c+\overline{k}}^{(b+2)} \equiv \chi_{ac}^{(b)} \quad \forall (a, b, c) \in P_k'', \tag{3.1}
$$

$$
\chi_{ac}^{(b)} = 0 \quad \text{when } a + b + c \not\equiv 0 \mod 2.
$$

With these definitions we find that the characters $\chi_{ac}^{(b)}$ transform under S with S-matrix $S(k) \otimes S'(2) \otimes S'(\overline{k})^*$. Meanwhile the WZW model $\widehat{\mathfrak{su}(2)}_k \oplus \widehat{\mathfrak{u}(1)}_2 \oplus \widehat{\mathfrak{u}(1)}_{\overline{k}}$ has characters $\chi_a \chi_b \chi_c$ with $(a, b, c) \in P_k''$, which transform under the action of S with S-matrix $S(k) \otimes S'(2) \otimes S'(\overline{k})$. The crucial observation is that $\chi_a \chi_b \chi_c^*$ transforms under S in exactly the same way as $\chi_{ac}^{(b)}$. Thus if $\sum M_{a,b,c; a',b',c'} \chi_{ac}^{(b)} \chi_{a'c'}^{(b')*}$ is a physical invariant of the coset $\mathfrak{g}/\mathfrak{h}$, then $\sum M_{a,b,c'; a',b',c} \chi_a \chi_b \chi_c \chi_{a'}^* \chi_{b'}^* \chi_{c'}^*$ is

a physical invariant of the WZW model $\mathfrak{g} \oplus \mathfrak{h}$ (note the interchange of c and c'). This correspondence is injective and thus every $\mathfrak{g}/\mathfrak{h}$ physical invariant is obtained from a $\mathfrak{g} \oplus \mathfrak{h}$ physical invariant, and the subset of $\mathfrak{g} \oplus \mathfrak{h}$ physical invariants corresponding to $\mathfrak{g}/\mathfrak{h}$ physical invariants are precisely those which respect the symmetry in (3.1), i.e.

$$M_{k-a,b+2,c;\,a',b',c'+\overline{k}} = M_{a,b,c+\overline{k};\,k-a',b'+2,c'} = M_{a,b,c;\,a',b',c'}.$$

Gannon showed in Lemma 3.1 of [23] that it is enough to check this condition when $a = b = c = a' = b' = c' = 0$:

$$M_{k,2,0;\,0,0,\overline{k}} = M_{0,0,\overline{k};\,k,2,0} = M_{0,0,0;0,0,0} = 1. \tag{3.2}$$

Thus the modular invariant partition functions of the minimal models at level k

$$Z(\tau, z) = \sum_{\substack{(ac) \in Q_k \\ (a'c') \in Q_k}} \widetilde{M}_{a,c;\,a',c'} \chi_{ac}(\tau, z) \chi_{a'c'}(\tau, z)^*$$

are obtained by

$$\widetilde{M}_{a,c;\,a',c'} = M_{a,[a+c],c';\,a',[a'+c'],c}$$

where M is a physical invariant of $\widehat{\mathfrak{su}(2)}_k \oplus \widehat{\mathfrak{u}(1)}_2 \oplus \widehat{\mathfrak{u}(1)}_{\overline{k}}$ satisfying equation (3.2), and where, as before, $[x]$ is 0 or 1 depending on whether x is even or odd.

The second step is to classify the physical invariants of $\widehat{\mathfrak{su}(2)}_k \oplus \widehat{\mathfrak{u}(1)}_2 \oplus \widehat{\mathfrak{u}(1)}_{\overline{k}}$ subject to equation (3.2). The crucial step is to note that the Verlinde formula [63] implies that there is a Galois action on the S-matrix:

$$\sigma \cdot S_{a,b,c;\,a',b',c'} = \epsilon_\sigma(a,b,c) S_{(a,b,c)^\sigma;\,a',b',c'} \quad \forall (a,b,c), (a',b',c') \in P_k''$$

where $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ for some cyclotomic extension K of $\mathbb{Q}$, for some $\epsilon \colon P_k'' \to \{\pm 1\}$ and a permutation $\lambda \mapsto \lambda^\sigma$ of P_k''. From this we obtain a selection rule for the physical invariant M:

$$M_{a,b,c;\,a',b',c'} \neq 0 \Rightarrow \epsilon_\sigma(a,b,c) = \epsilon_\sigma(a',b',c').$$

This can be solved exactly: we find that either $k \in \{4, 8, 10, 28\}$ or that whenever $M_{0,0,0;\,a',b',c'} \neq 0$ we have $a' \in \{0, k\}$. The former case can be solved by brute force. The latter solutions comprise the so-called $\mathcal{A}$-$\mathcal{D}$-$\mathcal{E}_7$-invariants[1][23]. The $\mathcal{A}$-$\mathcal{D}$-$\mathcal{E}_7$-invariants are defined by the condition

$$M_{a,b,c;\,0,0,0} \neq 0 \Rightarrow (a,b,c) \in \mathcal{J}(0,0,0)$$
$$M_{0,0,0;\,a',b',c'} \neq 0 \Rightarrow (a',b',c') \in \mathcal{J}(0,0,0)$$

---

[1]So called because in the classification of the $\widehat{\mathfrak{su}(2)}_k$ WZW models [5], these are precisely the models $\mathcal{A}, \mathcal{D}$ and $\mathcal{E}_7$.

where $\mathcal{J}$ is the set of simple currents of the physical invariant (see section 2.4.3). This is a generalisation of the notion of simple current invariant (see section 2.4.4). The classification of the physical invariants of $\widehat{\mathfrak{su}(2)}_k \oplus \widehat{\mathfrak{u}(1)}_2 \oplus \widehat{\mathfrak{u}(1)}_{\overline{k}}$ thus reduces to the classification of the $\mathcal{A}$-$\mathcal{D}$-$\mathcal{E}_7$-invariants of $\widehat{\mathfrak{su}(2)}_k \oplus \widehat{\mathfrak{u}(1)}_2 \oplus \widehat{\mathfrak{u}(1)}_{\overline{k}}$, which is carried out using the ideas of [23].

3.2 Explicit classification of the minimal partition functions

We state the list of partition functions of the minimal models here for two reasons: firstly, it did not appear explicitly in Gannon's paper [26], and deserves to be accessible in the literature; and secondly because there were a few minor errors in the 'trivial' (read: beneath contempt[2]) application of the main theorem of that paper to the case of $\widehat{\mathfrak{su}(2)}_k \oplus \widehat{\mathfrak{u}(1)}_2 \oplus \widehat{\mathfrak{u}(1)}_{\overline{k}}$. The corrections are highlighted in footnotes.

Throughout this section and the rest of the paper J will denote the $\widehat{\mathfrak{su}(2)}_k$ simple current $J : a \mapsto k - a$ and we write $\overline{k} = k + 2$.

k **odd:**

- We have a physical invariant $\widetilde{M}^0$ for each triple (v, z, n) with $v|\overline{k}$, $\overline{k}|v^2$ and $\overline{k}(4z^2 - 1)/v^2 \in \mathbb{Z}$ where $z \in \{1, ..., v^2/\overline{k}\}$ and $n \in \{0, 1\}$. Its non-zero entries are

$$\widetilde{M}^0_{a,c\overline{k}/v;\, a',c'\overline{k}/v} = 1 \iff \left\{ \begin{array}{c} a' = J^{(a+c)n}a \\ c' \equiv c + (a+c)n \ (\mathrm{mod}\ 2) \\ c' \equiv 2cz \ (\mathrm{mod}\ v^2/\overline{k}) \end{array} \right\}. \quad (3.3)$$

4 **divides $\overline{k}$:**

- We have a physical invariant $\widetilde{M}^{2,0}$ for each triple (v, z, n) with $2v|\overline{k}$, $\overline{k}|v^2$ and $y := \overline{k}(z^2 - 1)/2v^2 \in \mathbb{Z}$ where $z \in \{1, ..., 2v^2/\overline{k}\}$ and $n \in \{0, 1\}$. Its non-zero entries are

$$\widetilde{M}^{2,0}_{a,c\overline{k}/v;\, a',c'\overline{k}/v} = 1 \iff \left\{ \begin{array}{c} a' = J^{an+cy}a \\ c' \equiv cz + ayv^2/\overline{k} \ (\mathrm{mod}\ 2v^2/\overline{k}) \end{array} \right\}. \quad (3.4)$$

- We have a physical invariant $\widetilde{M}^{2,1}$ for each triple (v, z, n) with $2v|\overline{k}$, $\frac{2v^2}{\overline{k}} \in 2\mathbb{Z} + 1$ and $\overline{k}(z^2 - 1)/2v^2 \in \mathbb{Z}$ where $z \in \{1, ..., 2v^2/\overline{k}\}$ and $n \in \{0, 1\}$. Its non-zero entries are

$$\widetilde{M}^{2,1}_{a,c\overline{k}/2v;\, a',c'\overline{k}/2v} = 1 \iff \left\{ \begin{array}{c} a \equiv a' \equiv c \equiv c' \ (\mathrm{mod}\ 2) \\ a' = J^{an+(c+c')/2}a \\ c' \equiv cz \ (\mathrm{mod}\ 2v^2/\overline{k}) \end{array} \right\}. \quad (3.5)$$

---

[2]Gannon's words!

- We have a physical invariant $\widetilde{M}^{2,2}$ for each quadruple (v,z,n,m) with $\bar{k}/v$ odd, $v^2/\bar{k} \in \mathbb{Z}$ and $\bar{k}(z^2-1)/4v^2 \in \mathbb{Z}$ where $z \in \{1,...,2v^2/\bar{k}\}$ and $n,m \in \{0,1\}$. Its non-zero entries are

$$\widetilde{M}^{2,2}_{a,c\bar{k}/v;\, a',c'\bar{k}/v} = 1 \iff \left\{ \begin{array}{c} a' = J^{an+cm}a \\ c' \equiv cz + (a+c)mv^2/\bar{k} \ (\mathrm{mod}\ 2v^2/\bar{k}) \end{array} \right\}.$$
(3.6)

4 divides k

- If $8|k+4$ then we have a physical invariant $\widetilde{M}^{4,0}$ for each quadruple (v,z,n,m) with $\bar{k}/2v \in \mathbb{Z}$, $x := (1/4 + v^2/2\bar{k}) \in \mathbb{Z}$ and $\bar{k}(z^2-1)/2v^2 \in \mathbb{Z}$ where $z \in \{1,...,2v^2/\bar{k}\}$ and $m,n \in \{0,1\}$. Its non-zero entries are

$$\widetilde{M}^{4,0}_{a,c\bar{k}/2v;\, a',c'\bar{k}/2v} = 1 \iff \left\{ \begin{array}{c} c+c' \equiv a \equiv a' \ (\mathrm{mod}\ 2) \\ a' = J^{ax+cn+c(1-c)/2}a \\ c' \equiv cz \ (\mathrm{mod}\ 2v^2/\bar{k}) \\ 2c'm + c'(1-c') \equiv 2cn + c(1-c) \ (\mathrm{mod}\ 4) \end{array} \right\}.$$
(3.7)

  Note that $\widetilde{M}^{4,0}$ is only symmetric when $m = n$. In fact $(\widetilde{M}^{[v,z,n,m]})^T = \widetilde{M}^{[v,z,m,n]}$. Note also that the condition that x be an integer follows directly from the conditions that $8|k+4$ and $\bar{k}|2v^2$.

- If $8|k$ then we have a physical invariant[3] $\widetilde{M}^{4,1}$ for each quadruple (v,z,x,y) with $v|\bar{k}$, $\bar{k}|v^2$, $2\bar{k}(4z^2-1)/v^2 \equiv 7 \ (\mathrm{mod}\ 8)$ where $z \in \{1,...,v^2/\bar{k}\}$ and $x,y \in \{1,3\}$. Its non-zero entries are

$$\widetilde{M}^{4,1}_{a,c\bar{k}/v;\, a',c'\bar{k}/v} = 1 + \delta_{a,k/2} \iff \left\{ \begin{array}{c} a \equiv a' \equiv 0 \ (\mathrm{mod}\ 2) \\ a' = J^l a \quad \text{for some } l \in \mathbb{Z} \\ c' \equiv 2cz \ (\mathrm{mod}\ v^2/2\bar{k}) \\ c(c-x) \equiv 2c'z \ (\mathrm{mod}\ 4) \\ c'(c'-y) \equiv 2cz \ (\mathrm{mod}\ 4) \end{array} \right\}.$$
(3.8)

  Note that $\widetilde{M}^{4,1}$ is only symmetric when $x = y$. In fact $(M^{[v,z,x,y]})^T = M^{[v,z,y,x]}$. Note also that the condition $2\bar{k}(4z^2-1)/v^2 \equiv 7 \ (\mathrm{mod}\ 8)$ is equivalent to $2\bar{k}(4z^2-1)/v^2 \in \mathbb{Z}$ and $k/8 \equiv z \ (\mathrm{mod}\ 2)$.

- We have a physical invariant $\widetilde{M}^{4,2}$ for each triple (v,z,x) with $2v|\bar{k}$, $\bar{k}|2v^2$ and $\bar{k}(z^2-1)/2v^2 \in \mathbb{Z}$ where $z \in \{1,...,2v^2/\bar{k}\}$ and $x \in \{1,3\}$. Its non-zero entries are

$$\widetilde{M}^{4,2}_{a,c\bar{k}/2v;\, a',c'\bar{k}/2v} = 1 + \delta_{a,k/2} \iff \left\{ \begin{array}{c} a \equiv a' \equiv 0 \ (\mathrm{mod}\ 2) \\ a' = J^l a \quad \text{for some } l \in \mathbb{Z} \\ c' \equiv cz \ (\mathrm{mod}\ 2v^2/\bar{k}) \\ c' \equiv cx \ (\mathrm{mod}\ 4) \end{array} \right\}.$$
(3.9)

---

[3]In the original classification the modulo 8 condition was only given modulo 1

- We have a physical invariant[4] $\widetilde{M}^{4,3}$ for each triple (v, z, n) with $2v|\overline{k}$, $\overline{k}|2v^2$ and $\overline{k}(z^2 - 1)/4v^2 \in \mathbb{Z}$ where $z \in \{1, ..., 8v^2/\overline{k}\}$ and $n \in \{0, 1\}$. Its non-zero entries are

$$\widetilde{M}^{4,3}_{a, c\overline{k}/2v; \, a', c'\overline{k}/2v} = 1 \iff \left\{ \begin{array}{c} a' = J^{(a+c)n}a \\ c' \equiv cz \ (\text{mod } 2v^2/\overline{k}) \\ c' \equiv cz + 2(a + c)n \ (\text{mod } 4) \end{array} \right\}. \quad (3.10)$$

Exceptional Invariants

- When $k = 10$ we have a physical invariant $\widetilde{E}^{10}_1$ for the 2 pairs $(v = 6, z)$ with $z \in \{1, 5\}$. $\widetilde{E}^{10}_1 = E^{10} \otimes \overline{M}$ where E^{10} is the $\widehat{\mathfrak{su}(2)}_{10}$ exceptional physical invariant and $\overline{M}$ is the projection onto the $\widehat{\mathfrak{u}(1)}$ part of $\widetilde{M}^{2,0}$: the non-zero entries of $\overline{M}$ are

$$\overline{M}_{2c; \, 2c'} = 1 \iff \left\{ c' \equiv cz \ (\text{mod } 6) \right\}. \quad (3.11)$$

- When $k = 10$ we have a physical invariant $\widetilde{E}^{10}_2$ for the 8 quadruples $(v = 12, z, n = 0, m)$ with $z \in \{1, 7, 17, 23\}$ and $m \in \{0, 1\}$. Let E^{10} be the $\widehat{\mathfrak{su}(2)}_{10}$ exceptional physical invariant. Then $\widetilde{M}$ is given by

$$(\widetilde{E}^{10}_2)_{a, c; \, a', c'} = 1 \iff \left\{ \begin{array}{c} E^{10}_{J^{cm}a, a'} = 1 \\ c' \equiv cz + 12(a + c)m \ (\text{mod } 24) \end{array} \right\}. \quad (3.12)$$

- When $k = 16$ we have a physical invariant $\widetilde{E}^{16}_1$ for the 12 quadruples (v, z, x, y) with either $v = 6, z = 2$ or $v = 18, z \in \{4, 5\}$, and $x, y \in \{1, 3\}$. $\widetilde{E}^{16}_1 = E^{16} \otimes \overline{M}$ where E^{16} is the $\widehat{\mathfrak{su}(2)}_{16}$ exceptional physical invariant and $\overline{M}$ is the projection onto the $\widehat{\mathfrak{u}(1)}$ part of $\widetilde{M}^{4,1}$: the non-zero entries of $\overline{M}$ are

$$\overline{M}_{18c/v; \, ,18c'/v} = 1 \iff \left\{ \begin{array}{c} c' \equiv 2cz \ (\text{mod } v^2/36) \\ c(c - x) \equiv 0 \ (\text{mod } 4) \\ c'(c' - y) \equiv 0 \ (\text{mod } 4) \end{array} \right\}. \quad (3.13)$$

- When $k = 16$ we have a physical invariant $\widetilde{E}^{16}_2$ for the 6 triples (v, z, x) with either $v = 3, z = 1$ or $v = 9, z \in \{1, 8\}$, and $x \in \{1, 3\}$. $\widetilde{E}^{16}_2 = E^{16} \otimes \overline{M}$ where E^{16} is the $\widehat{\mathfrak{su}(2)}_{16}$ exceptional physical invariant and $\overline{M}$ is the projection onto the $\widehat{\mathfrak{u}(1)}$ part of $\widetilde{M}^{4,2}$: the non-zero entries of $\overline{M}$ are:

$$\overline{M}_{9c/v; \, 9c'/v} = 1 \iff \left\{ \begin{array}{c} c' \equiv cz \ (\text{mod } v^2/9) \\ c' \equiv cx \ (\text{mod } 4) \end{array} \right\}. \quad (3.14)$$

---

[4]In the original classification of the $\widehat{\mathfrak{su}(2)}_k \oplus \widehat{\mathfrak{u}(1)}_2 \oplus \widehat{\mathfrak{u}(1)}_{\overline{k}}$ invariants, the non-zero entries of $M^{4,3}$ should have read $M_{a,b,c; \, J^l a, bx+2l, cv+2lv} = 1$ with $(c + bv - av)v/\overline{k} \in \mathbb{Z}$ and $l \in \mathbb{Z}$, and z should be allowed to run from 1 to $8v^2/\overline{k}$ rather than only up to $4v^2/\overline{k}$.

- When $k = 28$ we have a physical invariant[5] $\widetilde{E}^{28}$ for the 8 triples ($v = 15, z, x$) with $z \in \{1, 4, 11, 14\}$ and $x \in \{1, 3\}$. $\widetilde{E}^{28} = E^{28} \otimes \overline{M}$ where E^{28} is the $\widehat{\mathfrak{su}(2)}_{28}$ exceptional physical invariant and $\overline{M}$ is the projection onto the $\widehat{\mathfrak{u}(1)}$ part of $\widetilde{M}^{4,2}$: the non-zero entries of $\overline{M}$ are

$$\overline{M}_{c;\, c'} = 1 \iff \begin{cases} c' \equiv cz \pmod{15} \\ c' \equiv cx \pmod{4} \end{cases}. \tag{3.15}$$

3.3 Simple examples of partition functions

To illustrate the foregoing classification, and to demonstrate that, at least for the lowest levels, the partition functions turn out to be given in terms of familiar functions, we will calculate the partition functions explicitly for levels $k = 1$ and $k = 2$.

3.3.1 $k = 1$

Level $k = 1$ yields $N = 2$ superconformal unitary minimal models with central charge $c = 1$. Since the string functions $c_{0,0}^{(1)}(\tau) = c_{1,1}^{(1)}(\tau)$ are equal to the reciprocal of the Dedekind eta function $\frac{1}{\eta(\tau)}$, the characters in equation (2.9) reduce to

$$\chi_{a,c}(\tau, z) = \frac{1}{\eta(\tau)} \Theta_{2c-3[a+c],6}(\tau, z, 0)$$
$$= K_{2c-3[a+c]}^{(6)}(\tau, z)$$

where $K_x^{(6)}$ are the $\widehat{\mathfrak{u}(1)}_6$ characters[6] defined by

$$K_x^{(l)}(\tau, z) = \frac{1}{\eta(\tau)} \sum_{Q \in \Gamma_x^{(l)}} q^{lQ^2} y^Q, \quad x \in \mathbb{Z}_{2l}, \tag{3.16}$$

and the lattice $\Gamma_x^{(l)}$ is given by $\Gamma_x^{(l)} = \left\{ \left(n + \frac{x}{2l} \right) \middle| n \in \mathbb{Z} \right\}$. We can then read off the partition functions of the 4 minimal models with $c = 1$ from section 3.2.

We note here that the non-zero entries of the physical invariant with level k odd and parameters v, z, n given in equation (3.3) can be expressed by the formula

$$\widetilde{M}^0_{a, c\overline{k}/v;\, a', c'\overline{k}/v} = 1 \iff (a', c') \equiv \mathbf{j}^{(a+c)n} \left(a, \left(2z - \frac{v^2}{\overline{k}} \right) c \mod \frac{2v^2}{\overline{k}} \right), \tag{3.17}$$

---

[5] There are 16 physical invariants described as coming from $M^{4,0}$ in the original classification, but no such invariants in fact exist.

[6] The Kac-Moody algebra of $\widehat{\mathfrak{u}(1)}$ does not have levels as such, since the generators can always be rescaled. We borrowed the notation $\widehat{\mathfrak{u}(1)}_l$ from [8].

where, as before, $\mathbf{j}$ is the simple current given by $\mathbf{j}(a,c) = (k-a, c+\overline{k})$. Specialising to the case $k=1$ with parameters $v=3, z \in \{2,3\}, n \in \{0,1\}$ we read off

$$M_{a,c;a',c'} = 1 \iff (a',c') = \mathbf{j}^{(a+c)n}(a, (2z-3)c),$$

and so the partition functions $Z[v,z,n]$ read

$$Z[3,2,0](\tau,z) = \sum_{d\in\mathbb{Z}_{12}} K_d(\tau,z)K_d(\tau,z)^* = Z_{R=\sqrt{6}}(\tau,z);$$

$$Z[3,1,1](\tau,z) = \sum_{d\in\mathbb{Z}_{12}} K_d(\tau,z)K_{11d}(\tau,z)^* = Z_{R=\frac{1}{\sqrt{6}}}(\tau,z),$$

$$Z[3,2,1](\tau,z) = \sum_{d\in\mathbb{Z}_{12}} K_d(\tau,z)K_{7d}(\tau,z)^* = Z_{R=\sqrt{\frac{3}{2}}}(\tau,z);$$

$$Z[3,1,0](\tau,z) = \sum_{d\in\mathbb{Z}_{12}} K_d(\tau,z)K_{5d}(\tau,z)^* = Z_{R=\sqrt{\frac{2}{3}}}(\tau,z);$$

where Z_R is the partition function of the boson on the circle at radius R (see e.g [33]):[7]

$$Z_R(\tau,z) = \frac{1}{|\eta(\tau)|^2} \sum_{(Q,\overline{Q})\in\Gamma_R} q^{lQ^2} y^Q \overline{q}^{l\overline{Q}^2} \overline{y}^{\overline{Q}}, \tag{3.18}$$

$$\Gamma_R = \left\{ \frac{1}{\sqrt{2l}} \left(\frac{n}{R} + mR, \frac{n}{R} - mR \right) \Big| n, m \in \mathbb{Z} \right\}, \tag{3.19}$$

where here $l=6$. The pair $(Q,\overline{Q}) \in \Gamma_R$ labels a conformal primary state with $U(1)$ charges $(Q,\overline{Q})$ and conformal weights $(h,\overline{h}) = (6Q^2, 6\overline{Q}^2)$.[8]

The partition function with $[z,v,n] = [3,2,0]$ is that of the diagonal model. The first and second partition functions, and the third and fourth partition functions belong to mirror symmetry pairs. In the current case, we note that mirror symmetry acts by T-duality, interchanging Z_R and $Z_{\frac{1}{R}}$.

3.3.2 $k = 2$

The level $k=2$ models correspond to the $N=2$ superconformal unitary minimal models with central charge $c = \frac{3}{2}$. Again we can express the characters in terms of familiar functions:

$$\chi_{a,c}(\tau,z) = c^{(2)}_{a,c-[a+c]}(\tau) \sum_{j\in\mathbb{Z}_2} \Theta_{2c+4(4j-[a+c]),16}(\tau, 2z, 0)$$

---

[7]In our normalisation the self-dual radius is $R=1$. Some authors use $R=\sqrt{2}$.

[8]It is perhaps more usual to re-scale the $U(1)$ current for the boson on the circle by $\sqrt{12}$ to obtain $h = \frac{Q^2}{2}$. The price, of course, is that the $N=2$ algebra, which is a symmetry of these $c=1$ theories at the special radii $R, R^{-1} \in \{\sqrt{6}, \sqrt{\frac{3}{2}}\}$, will then differ from its usual form: e.g. we would find $[J_0, G_r^\pm] = \pm\sqrt{3}G_r^\pm$. See Waterson [65] for an explicit construction of the irreducible representations of the unitary $N=2$ minimal models at $c=1$.

$$= \eta(\tau) c_{a,c-[a+c]}^{(2)}(\tau) K_{c-2[a+c]}^{(4)}(\tau, z)$$

where $K_x^{(4)}$ are the $\widehat{\mathfrak{u}(1)}_l$ characters given in equation (3.16) for $l = 4$ and $x \in \mathbb{Z}_8$, and $c_{a,c}^{(2)}$ are the level 2 $\widehat{\mathfrak{su}(2)}$ string functions as before. The string functions can be written in terms of the Jacobi theta functions and the Dedekind eta function as follows:

$$\eta(\tau) c_{a,c}^{(2)}(\tau) = \begin{cases} \sqrt{\frac{\theta_2(\tau,0)}{2\eta(\tau)}} & \text{if } a = 1 \\ \frac{1}{2} \left(\sqrt{\frac{\theta_3(\tau,0)}{\eta(\tau)}} + (-1)^{\frac{a+c}{2}} \sqrt{\frac{\theta_4(\tau,0)}{\eta(\tau)}} \right) & \text{if } a \text{ is even.} \end{cases}$$

We can now evaluate the five modular invariant partition functions[9] using the labels $[0; v, z]$ for the unique $\widetilde{M}^{2,0}$ invariant (see equation (3.4) – we have dropped the label n since $n = 0$ or 1 give the same partition function when $k = 2$) and labels $[2; v, z, m]$ for the four partition functions in the family $\widetilde{M}^{2,2}$ (see equation (3.6) – again we have dropped the n label).

$$Z[0; 2, 1](\tau, z) = Z_{\text{Ising}}(\tau) Z_{R=1}(\tau, z);$$
$$Z[2; 4, 1, 0](\tau, z) = Z_{\text{Ising}}(\tau) Z_{R=2}(\tau, z);$$
$$Z[2; 4, 7, 1](\tau, z) = Z_{\text{Ising}}(\tau) Z_{R=\frac{1}{2}}(\tau, z);$$
$$Z[2; 4, 7, 0](\tau, z) = \frac{1}{2} \sum_{c \in \mathbb{Z}_8} \left(\left| \frac{\theta_3(\tau, 0)}{\eta(\tau)} \right| + (-1)^c \left| \frac{\theta_4(\tau, 0)}{\eta(\tau)} \right| \right) K_c^{(4)}(\tau, z) K_{3c}^{(4)}(\tau, z)^*$$
$$+ \frac{1}{2} \left| \frac{\theta_2(\tau, 0)}{\eta(\tau)} \right| \sum_{c \in \mathbb{Z}_8} K_c^{(4)}(\tau, z) K_{3c+4}^{(4)}(\tau, z)^*;$$
$$Z[2; 4, 1, 1](\tau, z) = \frac{1}{2} \sum_{c \in \mathbb{Z}_8} \left(\left| \frac{\theta_3(\tau, 0)}{\eta(\tau)} \right| + (-1)^c \left| \frac{\theta_4(\tau, 0)}{\eta(\tau)} \right| \right) K_c^{(4)}(\tau, z) K_{5c}^{(4)}(\tau, z)^*$$
$$+ \frac{1}{2} \left| \frac{\theta_2(\tau, 0)}{\eta(\tau)} \right| \sum_{c \in \mathbb{Z}_8} K_c^{(4)}(\tau, z) K_{5c+4}^{(4)}(\tau, z)^*,$$

where here

$$Z_{\text{Ising}} = \frac{1}{2} \left(\left| \frac{\theta_2(\tau, 0)}{\eta(\tau)} \right| + \left| \frac{\theta_3(\tau, 0)}{\eta(\tau)} \right| + \left| \frac{\theta_4(\tau, 0)}{\eta(\tau)} \right| \right)$$

is the partition function of the Ising model (see e.g. [33]), and Z_R is the partition function of the boson on the circle given in equation (3.18) with $l = 4$.

We note that the second partition function is that of the diagonal model. The first partition function belongs to a self–mirror-symmetric model, and the second and third, and the fourth and fifth partition functions belong to mirror

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[9] In a later section when we count the number of simple current invariants, we will see that there should be 10 partition functions at level 2. This discrepancy arises from the identity $\mathcal{A}_2 = \mathcal{D}_2$, which does not generalise to other levels k.

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symmetry pairs. Again the mirror symmetry is realised via T-duality, by interchanging Z_R and $Z_{\frac{1}{R}}$; on the level of primary states it acts on the left-hand representations by mapping the primary state $|\text{Ising}\rangle \otimes |Q, \overline{Q}\rangle$ to $|\text{Ising}\rangle \otimes |-Q, \overline{Q}\rangle$, and similarly on the right-hand representations.

3.4 Classification of theories with space-time supersymmetry

In this section we show that those partition functions belonging to space-time supersymmetric models fall into the well-known $\mathcal{A}$-$\mathcal{D}$-$\mathcal{E}$ pattern in accordance with [46, 62]. Specifically we will find which of the partition functions satisfy the following condition: the R-R sector of the theory should be obtained from NS-NS sector under simultaneous spectral flow by half a unit on both chiral halves of the theory, and the NS-R and R-NS sectors are similarly interchanged. The spectral flow is rather easy to describe in our notation: it simply maps between the NS sector and the R sector via $(a, c) \leftrightarrow (a, c+1)$ where $a + c$ is even. One can check using equations (2.6),(2.12) and (2.13) that for $a + c$ even we have

$$h_{ac} \to h_{a,c+1} = h_{ac} - Q_{ac} + \frac{\mathbf{c}}{24},$$

as expected from e.g. [38]. The constraint that a theory should be invariant under the interchange of NS-NS↔R-R and NS-R↔R-NS is a very strong one. In particular, since the vacuum representation must be present in any theory, the representation obtained from the vacuum by spectral flow should be present in the R-R sector; that is, $M_{0,1;0,1} \neq 0$. One can read off from the explicit list in section 3.2 that the only space-time supersymmetric theories have the following partition functions:

$$\widetilde{M}^0[v = \overline{k}, 2z = 1, n = 0] = \mathcal{A}_k \otimes I_{2\overline{k}}, \qquad\qquad k \text{ odd}$$

$$\widetilde{M}^{2,2}[v = \overline{k}, z = 1, n = 0, m = 0] = \mathcal{A}_k \otimes I_{2\overline{k}}, \qquad 4 \text{ divides } \overline{k}$$

$$\widetilde{M}^{2,2}[v = \overline{k}, z = 1, n = 1, m = 0] = \mathcal{D}_k \otimes I_{2\overline{k}}, \qquad 4 \text{ divides } \overline{k}$$

$$\widetilde{M}^{4,3}[v = \frac{\overline{k}}{2}, z = 1, n = 0] = \mathcal{A}_k \otimes I_{2\overline{k}}, \qquad 4 \text{ divides } k$$

$$\widetilde{M}^{4,2}[v = \frac{\overline{k}}{2}, z = 1, x = 1] = \mathcal{D}'_k \otimes I_{2\overline{k}}, \qquad 4 \text{ divides } k$$

$$\widetilde{E}^{10}_2[v = 12, z = 1, n = 0, m = 0] = \mathcal{E}_{10} \otimes I_{24}, \qquad k = 10$$

$$\widetilde{E}^{16}_2[v = 9, z = 1, x = 1] = \mathcal{E}_{16} \otimes I_{36}, \qquad k = 16$$

$$\widetilde{E}^{28}[v = 15, z = 1, x = 1] = \mathcal{E}_{28} \otimes I_{60}, \qquad k = 28$$

Here the $\mathcal{A}_k, \mathcal{D}_k, \mathcal{E}_k$ are the $\widehat{\mathfrak{su}(2)}_k$ physical invariants of [5] and the $I_{2\overline{k}}$ are $\widehat{\mathfrak{u}(1)}_{\overline{k}}$ diagonal invariants[10]. These theories have no NS-R or R-NS sectors, and the

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[10]We use the notation $I_{2\overline{k}}$ since they are $2\overline{k} \times 2\overline{k}$ matrices. Some authors use $I_{\overline{k}}$.

NS-NS sector can be recovered from the R-R sector via spectral flow by half a unit in the opposite direction.

The familiar $\mathcal{A}$-$\mathcal{D}$-$\mathcal{E}$ pattern has emerged. It is quite remarkable that the $\mathcal{A}$-$\mathcal{D}$-$\mathcal{E}$ classification arises already at the level of partition functions.

We note here that there is (at least) one space-time supersymmetric minimal model in each "orbifold class" of the unitary $N = 2$ minimal models; that is, every partition function in Gannon's list can be mapped to one of the space-time supersymmetric partition functions by an orbifolding constructed in section 4.

3.5 Characterisation of the projection for odd k

Recall from section 2.3.2 that when we construct an $N = 2$ SCFT we need to choose a projection from the full Hilbert space of the theory to a subspace with modular invariant partition function. We now characterise this projection for minimal models at odd level k.

From Gannon's list of possible partition functions in section 3.2 we see that when k is odd the modular invariant partition function associated to a physical invariant M has no states in the NS-R or R-NS sectors. For such a theory we saw in section 2.5.4 that there are actually two choices of projection in the R-R sector, corresponding to the modular invariants M and $\widehat{M}$ (see equation (2.40)).

For concreteness we write $M[v, z, n]$ for the physical invariant with parameters v, z, n (see equation (3.3)). Then from the expression (3.17) we see that M and $\widehat{M}$ correspond to $M[v, z, 0]$ and $M[v, z, 1]$.

Theorem 3.5.1. *Let k be odd and fix v, z such that $\frac{\overline{k}}{v}, \frac{v^2}{k} \in \mathbb{Z}$ (see equation (3.3)). Then the modular invariant partition functions given by $M[v, z, 0]$ and $M[v, z, 1]$ give rise to the same Hilbert space $\mathbb{H}$. For $n \in \{0, 1\}$ the projection from $\mathbb{H}$ to the modular invariant subspace corresponding to $M[v, z, n]$ is given by*

$$\mathcal{P}_b = \frac{1}{2}\left(1 + (-1)^{(J_0, \overline{J}_0)\cdot(C, \overline{C})}\right)$$

where the scalar product is defined by

$$(Q, \overline{Q})\cdot(C, \overline{C}) := 8\overline{k}(QC - \overline{Q}\,\overline{C}) \tag{3.20}$$

and $(C, \overline{C}) \in \mathbb{C}^2$ are the left and right $U(1)$ charges of any fixed R-R state in the modular invariant subspace corresponding to $M[v, z, n]$.

In other words, the projection corresponds to choosing one of the two natural even sublattices of the integral charge lattice of the theory.

Proof. We first show that $(Q, \overline{Q})\cdot(Q', \overline{Q}') \equiv 0 \mod 2$ whenever $(Q, \overline{Q})$ and $(Q', \overline{Q}')$ are the charges of states in the modular invariant subspace of $\mathbb{H}$ given by the physical invariant $M = M[v, z, n]$. The states are counted by the partition

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function

$$Z(\tau, z) = \sum_{\substack{a+c \in Q_k \\ a'+c' \in Q_k}} M_{a,c;\,a',c'} \chi_{ac}(\tau, z) \chi_{a'c'}(\tau, z)^*.$$

Recall from equations (2.14) and (2.15) that for all $(ac) \in Q_k$ the character χ_{ac} counts only states with $U(1)$ charge equal to $Q_{ac} \mod 1$. So we need to check that for all $(a, c), (a', c'), (d, f), (d', f') \in Q_k$ we have

$$M_{a,c;\,a',c'} \neq 0, M_{d,f;\,d',f'} \neq 0 \implies (Q_{ac}, Q_{a'c'}) \cdot (Q_{df}, Q_{d'f'}) \in 2\mathbb{Z}.$$

So suppose $M_{a,c;\,a',c'} \neq 0, M_{d,f;\,d',f'} \neq 0$. Then from equation (3.17) we see that $cv, fv \in \bar{k}\mathbb{Z}$ and

$$c' = \left(2z - \frac{v^2}{\bar{k}}\right) c + 2lv + [a+c]n\bar{k}$$

$$f' = \left(2z - \frac{v^2}{\bar{k}}\right) f + 2mv + [d+f]n\bar{k}$$

for some $l, m \in \mathbb{Z}$. We calculate

$$(Q_{ac}, Q_{a'c'}) \cdot (Q_{df}, Q_{d'f'}) = \frac{2}{\bar{k}}(cf - c'f') + [d+f](c-c') - [a+c](f-f')$$

$$\equiv \frac{2}{\bar{k}}(cf - c'f') \mod 2$$

using equation (2.13). Applying $4z^2 \equiv 1 \mod \frac{v^2}{\bar{k}}$ (see equation (3.3)), we see that this expression is an even integer.

Next we show that $(-1)^{(J_0, \bar{J}_0) \cdot (C, \overline{C})}$ is well-defined on all of $\mathbb{H}$. Recall how we constructed $\mathbb{H}$ from the modular invariant subspace given by M: whenever $M_{a,c;\,a',c'} \neq 0$ (and hence by lemma 2.3.1 $M_{\mathbf{j}(a,c);\,\mathbf{j}(a',c')} \neq 0$) the states counted by $\chi_{ac}\chi^*_{a'c'}$ and $\chi_{\mathbf{j}(a,c)}\chi^*_{\mathbf{j}(a',c')}$ are present in the modular invariant subspace, and from equations (2.14) and (2.15) they have charges $(Q_{ac}, Q_{a'c'})$ and $(Q_{ac} + \frac{1}{2}, Q_{a'c'} + \frac{1}{2}) \mod 1$. According to equations (2.29) and (2.31), to get the full Hilbert space we should add in the states counted by $\chi_{ac}\chi^*_{\mathbf{j}(a'c')}$ and $\chi_{\mathbf{j}(a,c)}\chi^*_{a',c'}$ to obtain states with charges $(Q_{ac}, Q_{a'c'} + \frac{1}{2})$ and $(Q_{ac} + \frac{1}{2}, Q_{a'c'}) \mod 1$. But if $M_{a,c;\,a',c'} \neq 0, M_{d,f;\,d',f'} \neq 0$ where $(d, f), (d', f')$ are R-R labels then

$$\left(Q_{ac} + \frac{l}{2}, Q_{a'c'} + \frac{l'}{2}\right) \cdot (Q_{df}, Q_{d'f'}) \equiv l + l' \mod 2.$$

Thus $(-1)^{(J_0, \bar{J}_0) \cdot (C, \overline{C})}$ is well-defined on $\mathbb{H}^R$ and $\mathcal{P}_b$ projects to the invariant subspace given by M. □

3.6 Construction of minimal models

In this section we shall review how one should go about trying to construct the minimal models via the parafermion-boson construction [55]. In the first

section we will discuss the parafermion construction; in the next, we recall some pertinent facts about the free boson. Then in section 3.6.3 we describe how candidate fields for the minimal models may be found using the parafermion-boson construction.

3.6.1 Parafermions

Parafermions are non-local fields with a $\mathbb{Z}_k$ symmetry, generalising the $k = 2$ case of the usual fermions. They were first introduced by Zamolodchikov and Fateev [17, 69]. The review in this section is taken from an article by Gepner and Qiu [31], in which they calculated the characters of the representations of the parafermion algebra. In the rest of this section we will take advantage of the state-field correspondence, and so will be a little careless with the distinction between states and fields.

The state space of the parafermion theory is a direct sum

$$\mathbb{H} = \bigoplus_{m,\overline{m}} \mathbb{H}_{m,\overline{m}}$$

where the sum runs over charges in $\{(m,\overline{m}) \in \mathbb{Z}_{2k} \times \mathbb{Z}_{2k} | m - \overline{m} \in 2\mathbb{Z}\}$ modulo $(m,\overline{m}) \equiv (m + k, \overline{m} + k)$, and $\mathbb{H}_{m,\overline{m}}$ contains only those states with charge $(m,\overline{m})$.

$\mathbb{H}$ contains $2k - 1$ distinguished fields $\psi_l(z), \overline{\psi}_l(\overline{z})$ for $l = 0, \ldots, k - 1$. The fields ψ_0 and $\overline{\psi}_0$ correspond to the identity field. $\psi_l(z)$ and $\overline{\psi}_l(\overline{z})$ have $\mathbb{Z}_{2k} \times \mathbb{Z}_{2k}$ charges $(2l, 0)$ and $(0, 2l)$ respectively. We will introduce a scalar product on the state space with respect to which we obtain the hermiticity relation $\psi_l^\dagger = \psi_{k-l}$.

We now focus on the z-dependent fields, bearing in mind that the analogous statements are true for the $\overline{z}$-dependent fields. ψ_l has conformal weight $d_l = l - \frac{l^2}{k}$. The OPE of two parafermions is given by

$$\psi_l(z)\psi_m(w) = c_{l,m}(z - w)^{d_{l+m} - d_l - d_m} \sum_{j=0}^{\infty} (z - w)^j \Psi_{l+m}^{(j)}(w)$$

where $\Psi_{l+m}^j \in \mathbb{H}_{2(l+m),0}$ and $c_{l,m} \in \mathbb{C}$. We see that the mutual locality exponent of ψ_l and ψ_m is $d_{l+m} - d_l - d_m = -\frac{2lm}{k}$ (see below). The parafermion algebra is then defined by the following OPEs:

$$\psi_l(z)\psi_m(w) = c_{l,m}(z - w)^{-\frac{2lm}{k}}(\psi_{l+m}(w) + O(z - w)), \quad l + m < k;$$

$$\psi_l(z)\psi_m^\dagger(w) = c_{l,k-m}(z - w)^{-\frac{2l(k-m)}{k}}(\psi_{l-m}(w) + O(z - w)), \quad m < l;$$

$$\psi_l(z)\psi_l(w) = (z - w)^{-\frac{2l(k-l)}{k}}(I + \frac{2d_l}{c}(z - w)^2 T_{\text{pf}}(w) + O((z - w)^3)),$$

where $c \in \mathbb{C}$. One then finds that $T_{\text{pf}}(w)$ is a Virasoro field with central charge c (i.e. that it obeys the first equation in (2.1)) if and only if $c = \frac{2(k-1)}{k+2}$. Furthermore, the parafermion ψ_l is primary with respect to the Virasoro algebra with conformal weight $h = d_l$.

The mutual locality of any two fields $\phi_a(z,\overline{z}), \phi_b(w,\overline{w})$ is measured by the mutual locality exponent $\mu(a,b)$ (MLE). $\mu(a,b)$ is given by the factor picked up by circling z once anti-clockwise around w. For example, if $\phi_a(z,\overline{z})$ and $\phi_b(w,\overline{w})$ have OPE

$$\phi_a(z,\overline{z})\phi_b(w,\overline{w}) = (z-w)^\mu |z-w|^\alpha [\phi_c(w,\overline{w}) + \ldots]$$

then the mutual locality exponent is obtained by replacing $z \to w + (z-w)e^{2\pi i t}$ and letting $t \to 1$. We obtain a factor of $e^{2\pi i \mu}$ and so $\mu = \mu(a,b)$ is the mutual locality exponent.

The mutual locality exponent of any two fields in $\mathbb{H}$ with $\mathbb{Z}_{2k} \times \mathbb{Z}_{2k}$ charges $(p,\overline{p}), (q,\overline{q})$ is given by

$$\mu_{\mathrm{pf}}(p,\overline{p}; q,\overline{q}) = -\frac{pq - \overline{p}\,\overline{q}}{2k} \quad \mathrm{mod}\ \mathbb{Z}. \tag{3.21}$$

We see that all fields in $\mathbb{H}_{l,\overline{l}}$ and all fields in $\mathbb{H}_{l,-\overline{l}}$ are mutually local.

We can expand the basic parafermionic fields $\psi_1(z), \psi_1^\dagger(z)$ into their modes. Let $\phi_{l,\overline{l}} \in \mathbb{H}_{l,\overline{l}}$ with conformal weights $(h,\overline{h})$. Then

$$\psi_1(z)\phi_{l,\overline{l}}(0,0) = \sum_{m=-\infty}^{\infty} z^{-\frac{l}{k}+m-1} A_{\frac{l}{k}-m}\phi_{l,\overline{l}}(0,0),$$

$$\psi_1^\dagger(z)\phi_{l,\overline{l}}(0,0) = \sum_{m=-\infty}^{\infty} z^{\frac{l}{k}+m-1} A^\dagger_{-\frac{l}{k}-m}\phi_{l,\overline{l}}(0,0),$$

where the power of z is determined by the mutual locality exponent of the two fields. We obtain new fields $A_{\frac{l}{k}-m}\phi_{l,\overline{l}} \in \mathbb{H}_{l+2,\overline{l}}$ and $A^\dagger_{-\frac{l}{k}-m}\phi_{l,\overline{l}} \in \mathbb{H}_{l-2,\overline{l}}$ for each $m \in \mathbb{Z}$ with conformal weights

$$\left(h + m - \frac{l+1}{k}, \overline{h}\right) \quad \text{and} \quad \left(h + m - \frac{1-l}{k}, \overline{h}\right),$$

respectively. The analogous mode expansion on the right-hand side yields operators $\overline{A}_r, \overline{A}_r^\dagger$.

The Hilbert space $\mathbb{H}$ admits a description as being generated by the action of $A_r, A_r^\dagger, \overline{A}_r$ and $\overline{A}_r^\dagger$ on certain primary fields; in other words $\mathbb{H}$ is the sum of highest weight representations of the algebra generated by $A_r, A_r^\dagger, \overline{A}_r$ and $\overline{A}_r^\dagger$. The primary fields $\Phi_{l,\overline{l}}^{l,\overline{l}} \in \mathbb{H}_{l,\overline{l}}$ are defined to be those satisfying

$$A_{\frac{l}{k}+m}\Phi_{l,\overline{l}}^{l,\overline{l}} = A^\dagger_{-\frac{l}{k}+m+1}\Phi_{l,\overline{l}}^{l,\overline{l}} = 0,$$

$$\overline{A}_{\frac{\overline{l}}{k}+m}\Phi_{l,\overline{l}}^{l,\overline{l}} = \overline{A}^\dagger_{-\frac{\overline{l}}{k}+m+1}\Phi_{l,\overline{l}}^{l,\overline{l}} = 0$$

for all $m \geq 0$ and $l, \bar{l} = 0, \ldots k - 1$. Their dimensions are

$$h_l = \frac{l(k-l)}{2k(k+2)}, \qquad \overline{h}_{\bar{l}} = \frac{\bar{l}(k-\bar{l})}{2k(k+2)}.$$

Let us restrict our attention to just the left-handed fields once more. From the parafermionic primaries above we can generate Virasoro primary fields by acting several times with ψ_1 and $\psi_1^\dagger$:

$$\Phi^l_{l+2n} = A_{\frac{l+2n-2}{k}-1} A_{\frac{l+2n-4}{k}-1} \ldots A_{\frac{l}{k}-1} \Phi^l_l, \qquad n = 0, 1, \ldots k - l,$$

$$\Phi^l_{l-2n} = A^\dagger_{-\frac{l-2n+2}{k}} A^\dagger_{-\frac{l-2n+4}{k}} \ldots A^\dagger_{-\frac{l}{k}} \Phi^l_l, \qquad n = 0, 1, \ldots l.$$

Setting $\Phi^l_{m+2k} = \Phi^l_m$ we obtain Virasoro primary fields Φ^l_m for $l = 0, \ldots k - 1$, $-l \leq m \leq 2k - l$ with conformal weight

$$h^l_m = h_l + \frac{(l-m)(l+m)}{4k}, \qquad \text{for } -l \leq m \leq l;$$

$$h^l_m = h_l + \frac{(m-l)(2k-l-m)}{4k}, \qquad \text{for } l \leq m \leq 2k - l.$$

Returning to the full left-right viewpoint, we can now further decompose $\mathbb{H}$ according to both charge and parafermion algebra highest weight representation:

$$\mathbb{H} = \bigoplus \mathbb{H}^{l,\bar{l}}_{m,\overline{m}}$$

where $\mathbb{H}^{l,\bar{l}}_{m,\overline{m}}$ contains states which are descendents of $\Phi^{l,\bar{l}}_{l,\bar{l}}$ with charges $(m, \overline{m}) \equiv (m+k, \overline{m}+k)$.

Finally we mention that $\sigma_l := \Phi^{l,\bar{l}}_{l,\bar{l}}$ are called spin fields and $\mu_l := \Phi^{l,\bar{l}}_{-l,\bar{l}}$ are dual spin fields.

3.6.2 The free boson

The field content of the theory of the free boson on the circle is well-known (see e.g. [53]). The free boson is given by $\phi(z, \overline{z}) = \phi(z) + \overline{\phi}(\overline{z})$, where

$$\phi(z) = \phi_0 - ia_0 \log(z) + i \sum_{n \neq 0} \frac{1}{n} a_n z^{-n}$$

$$\overline{\phi}(\overline{z}) = \overline{\phi}_0 - i\overline{a}_0 \log(z) + i \sum_{n \neq 0} \frac{1}{n} \overline{a}_n z^{-n}.$$

The modes ϕ_0, a_n satisfy the commutator relations $[a_m, a_n] = m\delta_{m+n,0}$ and $[\phi_0, a_0] = i$, and similarly for the right-handed modes. All other commutators vanish. Due to the presence of the log, ϕ is not well-defined on the whole of $\mathbb{C}^*$, but only on simply connected subsets with a choice of branch of the log function.

The commutators can be equivalently expressed via the locally-defined OPE of $\phi(z, \overline{z})$ with itself:

$$\phi(z, \overline{z})\phi(w, \overline{w}) \sim -2 \log |z - w|.$$

Let us again restrict attention to the left-movers. The energy-momentum tensor is

$$T_{\mathrm{b}}(z) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) :,$$

where : denotes normal-ordering. $\partial\phi(z)$, and hence $T(z)$, is well defined on $\mathbb{C}^*$, and $T(z)$ satisfies the Virasoro condition with central charge $c = 1$. The primary fields are given by so-called *vertex operators*[11]

$$\begin{aligned} V_\alpha(z) &=: \exp(i\alpha\phi(z)) : \\ &\equiv \exp(i\alpha\phi^-(z))\exp(i\alpha\phi^+(z)) \end{aligned}$$

where we have decomposed $\phi(z)$ into creation and annihilation modes:

$$\phi^+(z) = -a_0 \log z + iz \sum_{n>0} \frac{1}{n} a_n z^{-n}$$

$$\phi^-(z) = \phi_0 - i \sum_{n>0} \frac{1}{n} a_{-n} z^n.$$

The creation modes mutually commute, as do the annihilation modes, allowing us to make sense of the exponential. Again, $\phi^+(z)$ is only defined on simply connected subsets of $\mathbb{C}^*$ because of the log function, but the vertex operator is well-defined on the whole of $\mathbb{C}^*$. The vertex operators satisfy the following equations:

$$\begin{aligned} [a_n, V_\alpha(z)] &= \alpha z^n V_\alpha(z), \\ [L_n, a_m] &= -m a_{m+n} \\ [L_n, V_\alpha(z)] &= \frac{\alpha^2}{2}(n+1)z^n V_\alpha(z) + z^{n+1}\partial V_\alpha(z), \end{aligned}$$

where, as usual, L_n are the modes of the energy-momentum tensor $T(z)$. The last equation tells us that $V_\alpha(z)$ is a primary field with conformal weight $h_\alpha = \frac{\alpha^2}{2}$.

The OPE of arbitrarily many vertex operators can be calculated explicitly:

$$V_{\alpha_1}(z_1)\dots V_{\alpha_n}(z_n) =: \exp(i\alpha_1\phi(z_1) + \dots + i\alpha_n\phi(z_n)) : \prod_{j<k} (z_j - z_k)^{\alpha_j \alpha_k}$$

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[11]Note here that cocycles [19, 35] are not required: we do not wish to impose on the vertex operators that they commute as in chapter 6 of [35]. Rather we want to allow bosonic and fermionic vertex operators.

where again the normal ordered product is defined by bringing all annihilation modes to the right of all the creation modes. The n-point functions are then seen to be

$$\langle V_{\alpha_1}(z_1) \ldots V_{\alpha_n}(z_n) \rangle = \prod_{j<k} (z_j - z_k)^{\alpha_j \alpha_k}$$

as long as $\sum_i \alpha_i = 0$, and zero otherwise.

In particular, the OPE of two vertex operators is

$$V_\alpha(z) V_\beta(w) \sim (z-w)^{\alpha\beta} V_{\alpha+\beta}(w) + \ldots .$$

It will also be useful to consider full vertex operators which contain both left-hand and right-hand modes:

$$V_{\alpha,\beta}(z,\overline{z}) =: \exp(i\alpha\phi(z) + i\beta\overline{\phi}(\overline{z})) : \qquad (3.22)$$

$$=: V_\alpha(z) \overline{V}_\beta(\overline{z}) :, \qquad (3.23)$$

using the fact that the left-hand and right-hand modes mutually commute. Then

$$\langle V_{\alpha_1,\beta_1}(z_1,\overline{z}_1) \ldots V_{\alpha_n,\beta_n}(z_n,\overline{z}_n) \rangle = \prod_{j<k} (z_j - z_k)^{\alpha_j \alpha_k} \prod_{j<k} (\overline{z}_j - \overline{z}_k)^{\beta_j \beta_k}$$

and we have the OPE

$$V_{\alpha_1,\beta_1}(z,\overline{z}) V_{\alpha_2,\beta_2}(w,\overline{w}) \sim (z-w)^{\alpha_1\alpha_2} (\overline{z}-\overline{w})^{\beta_1\beta_2} V_{\alpha_1+\alpha_2,\beta_1+\beta_2}(w,\overline{w}) + \ldots$$
$$= (z-w)^{\alpha_1\alpha_2 - \beta_1\beta_2} |z-w|^{\beta_1\beta_2} V_{\alpha_1+\alpha_2,\beta_1+\beta_2}(w,\overline{w}) + \ldots$$

from which we read of the mutual locality exponent of the fields $V_{\alpha_1,\beta_1}(z,\overline{z})$ and $V_{\alpha_2,\beta_2}(w,\overline{w})$ as

$$\mu_{\mathrm{b}}(\alpha_1,\beta_1;\alpha_2,\beta_2) = \alpha_1\alpha_2 - \beta_1\beta_2 \mod \mathbb{Z}. \qquad (3.24)$$

3.6.3 Parafermion construction of the minimal models

This construction is due to Qiu [55]. Candidate fields for the $N=2$ minimal models at level k can be assembled out of the normal ordered product of the parafermion theory with $\mathbb{Z}_k$ symmetry and the theory of the free boson. The left-movers are given by

$$T(z) = T_{\mathrm{pf}}(z) + T_{\mathrm{b}}(z)$$

$$J(z) = \frac{i}{2}\sqrt{\frac{k}{k}} \partial\phi(z)$$

$$G^+(z) = \sqrt{\frac{2k}{\overline{k}}} : \psi_1(z) V_{\sqrt{\frac{k}{k}}}(z) :$$

$$G^-(z) = \sqrt{\frac{2k}{\overline{k}}} : \psi_1^\dagger(z) V_{-\sqrt{\frac{k}{k}}}(z) :$$

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and analogously for the right-movers. The primary fields are simply the normal ordered product of the parafermion primaries and an appropriate choice of vertex operator:

$$\phi_{a,c;\,a',c'}(z,\overline{z}) =: \Phi^{a,a'}_{c-[a+c],c'-[a'+c']}(z,\overline{z})V_{\alpha_{a,c},\alpha_{a',c'}}(z,\overline{z}):, \qquad (3.25)$$

where $V_{\alpha,\alpha'}(z,\overline{z})$ is the vertex operator given in equation (3.23) and

$$\alpha_{a,c} = \frac{1}{\sqrt{k\overline{k}}}\left(c - \frac{[a+c]\overline{k}}{2}\right).$$

The parafermionic fields $\Phi^{l,l'}_{m,m'}(z,\overline{z})$ were given in section 3.6.1. We emphasise here that equation (3.25) defines fields for all $a = 0,\ldots,k$ and $c \in \mathbb{Z}$. For the range $|c - [a + c]| \leq a$ we obtain genuine SVA primaries. We note that the superconformal half-families $[a, c]$ of section 2.4.2 are well defined for $(a, c) \in Q_k$: for example, using $G^+(z) \sim: \psi_1(z)V_{\sqrt{\frac{\overline{k}}{k}}}(z):$ we have

$$\phi_{a,c+2\overline{k};a',c'}(w) =: \Phi^{a,a'}_{c-b+4,c'-b'}(w)V_{\alpha_{a,c+2\overline{k}},\alpha_{a',c'}}(w):$$
$$\sim G^+(x)G^+(z)\phi_{a,c;a',c'}(w) \in [a,c;a',c'].$$

For convenience, in what follows we will choose a representative of $[a, c]$ with $-\overline{k} + 1 \leq c \leq \overline{k}$.

It also follows immediately from equation (3.25) that the conformal weights of $\phi_{a,c;\,a',c'}$ are given by equation (2.12) and its right-handed analogue[12]. Similarly the $U(1)$ charges are given by equation (2.13).

3.7 Some necessary conditions

At this point, we will prove two consistency checks of the minimal models, one pertaining to the fusion rules and one to the locality of the theory.

3.7.1 Fusion rules

In section 2.4.2 we derived the chiral fusion rules of the minimal models. The fusion rules enforce harsh restrictions on the OPE of a SCFT, so if a physical invariant M really corresponds to the partition function of a minimal model, it must pass a consistency test imposed by the fusion rules. This consistency test was performed in the case of $N = 0$ minimal models by Gepner [29].

Consider a possible theory with partition function corresponding to some physical invariant M. If fields $\phi_{a,c;a',c'} \in [a,c] \otimes [a'c']$ and $\phi_{d,f;d',f'} \in [d,f] \otimes [d'f']$ are present then the fusion rules restrict the fusion between $\phi_{a,c;a',c'}$ and

---

[12]Recall that this equation is exact for primary fields (when $|c - [a + c]| \leq a$) and modulo $\mathbb{Z}$ otherwise.

$\phi_{d,f;d',f'}$ to lie in

$$\sum_{\substack{(\alpha,\gamma)\in Q_k \\ (\alpha',\gamma')\in Q_k}} N^{\alpha\gamma}_{ac,df} N^{\alpha'\gamma'}_{a'c',d'f'} [\alpha,\gamma] \otimes [\alpha',\gamma'].$$

This expression is further constrained since only fields that show up in the partition function can be present[13]. If our theory is to be consistent, then we require that the fusion between any two fields is non-zero. We confirm that the $N = 2$ minimal models conform to this requirement in the following theorem:

Theorem 3.7.1. *For any physical invariant M in the list of Gannon (see section 3.2) we have*

$$M_{a,c;a',c'} \neq 0, \ M_{d,f;d',f'} \neq 0 \implies N^{\alpha,\gamma}_{ac,df} M_{\alpha,\gamma;\alpha',\gamma'} N^{\alpha'\gamma'}_{a'c',d'f'} \neq 0$$
$$\text{for some } (\alpha,\gamma),(\alpha',\gamma') \in Q_k.$$

This proves that the fusion rules do not preclude the existence of the $N = 2$ minimal models.

Proof. The fusion coefficients N were given in lemma 2.4.1. One must work through the list of physical invariants (3.3)-(3.15) checking the condition each time by hand. The calculations are tedious and unenlightening, so they are not presented here. □

3.7.2 Semi-locality

In this section we will prove the following result:

Theorem 3.7.2. *The fields appearing in the possible theories given by Gannon's list are semi-local to one another; that is, cycling one field around any other inside an n-point function at worst introduces a square-root branch cut.*

Proof. We first show that all fields counted by any possible bosonic partition function are semi-local. We use the construction of the minimal models given in section 3.6.3.

The mutual locality exponent of the field (3.25) is simply the sum of those of its constituents. The MLE of the parafermionic part was given in equation (3.21), and that of the bosonic part was given in equation (3.24). Putting these together we find that the MLE of the fields $\phi_{a,c;a',c'}$ and $\phi_{d,f;d',f'}$ is

$$\mu((ac,a'c');(df,d'f')) = \frac{c'f' - cf}{2\bar{k}} + \frac{[a+c][d+f] - [a'+c'][d'+f']}{4}.$$

---

[13]We remind the reader that the fusion rules give only an upper bound to the number fields produced under fusion of two fields – it can easily happen that fewer fields appear than are allowed by the fusion rules.

One then trawls through the classification of section 3.2, checking that for each M in equations (3.3)-(3.15):

$$M_{a,c;a',c'} \neq 0 \text{ and } M_{d,f;d',f'} \neq 0 \implies \mu((ac, a'c'); (df, d'f')) \equiv 0 \mod \frac{1}{2}.$$

The calculation are tedious but straightforward, and thus will not be presented here. It follows that all fields counted by a bosonic partition are mutually semi-local. Finally we note the fields in the full Hilbert space are obtained from these by OPE with the operators $T, J, G^\pm$, which have MEL $\mu \in \frac{1}{2}\mathbb{Z}$ and thus all fields are at worst semi-local. □

3.8 Examples of minimal models

We describe here the full field content of the minimal models for the first two levels, which we can deduce from the well-understood field content of the free boson and the Ising model.

3.8.1 $k = 1$

As we saw in section 3.3.1 the modular invariant partition functions of the $N = 2$ minimal models at level $k = 1$ are those of the theory of the boson compactified on the circle of radius $R = \frac{p}{\sqrt{6}}$ where $p \in \{1, 2, 3, 6\}$. The primary fields are labelled by $(a, c) \in Q_1 = \{0, 1\} \times \mathbb{Z}_6$. Let us use the more convenient label $d := 2c - 3[a + c] \in \mathbb{Z}_{12}$ (so in particular d even labels the NS sector and d odd labels those in the R sector). Then the four partition functions correspond to the physical invariants $M_{d,d'} = \delta(d' = pd)$ with $p = 1, 11, 7, 5$ respectively, as seen above.

The Virasoro primary fields are the vertex operators

$$V_{d,d'}(z, \bar{z}) =: \exp\left(\frac{id}{\sqrt{12}}\phi(z) + \frac{id'}{\sqrt{12}}\bar{\phi}(\bar{z})\right):, \quad d, d' \in \mathbb{Z}. \quad (3.26)$$

and the $N = 2$ SVA primaries are those fields with $-3 \leq d, d' \leq 2$. $V_{d,d'}(z, \bar{z})$ has conformal weights and $U(1)$ charges

$$h_{d,d'} = \frac{d^2}{24}, \qquad\qquad \bar{h}_{d,d'} = \frac{d'^2}{24},$$
$$Q_{d,d'} = \frac{d}{12}, \qquad\qquad \bar{Q}_{d,d'} = \frac{d'}{12},$$

which agree with equations (2.12) and (2.13) when $-3 \leq d \leq 2$, and with the right-handed analogues when $-3 \leq d' \leq 2$. The n-point functions for the vertex operators are

$$\langle V_{d_1,d_1'}(z_1, \bar{z}_1) \dots V_{d_n,d_n'}(z_n, \bar{z}_n)\rangle = \prod_{j<k}(z_j - z_k)^{\frac{d_j d_k}{12}}(\bar{z}_j - \bar{z}_k)^{\frac{d_j' d_k'}{12}},$$

as long as $\sum_j d_j = \sum_j d'_j = 0$, and vanish otherwise. The OPE is

$$V_{d_1,d'_1}(z,\overline{z})V_{d_2,d'_2}(w,\overline{w}) = (z-w)^{\frac{d_1 d_2}{12}}(\overline{z}-\overline{w})^{\frac{d'_1 d'_2}{12}}$$
$$\times \left(V_{d_1+d_2,d'_1+d'_2}(w,\overline{w}) + O(|z-w|)\right), \tag{3.27}$$

which yields additive fusion rules:

$$N^{(d_3,d'_3)}_{(d_1,d'_1),(d_2,d'_2)} = \delta(d_3 \equiv d_1 + d_2 \mod 12)\,\delta(d'_3 \equiv d'_1 + d'_2 \mod 12).$$

The fusion rules for each chiral half are therefore $N^{d_3}_{d_1,d_2} = \delta(d_3 \equiv d_1 + d_2 \mod 12)$, which agrees with the result of lemma 2.4.1. Finally, the fields given by any of the partition functions are mutually local, as expected:

$$V_{d,pd}(z,\overline{z})V_{d',pd'}(w,\overline{w}) = |z-w|^{\frac{p^2 dd'}{12}}(z-w)^{\frac{1-p^2}{12}dd'}$$
$$\times (V_{d+d',p(d+d')}(w,\overline{w}) + O(|z-w|)),$$

since $p^2 \equiv 1 \mod 24$. The supersymmetry operators are realised via

$$G^{\pm}(z) = V_{\pm 6,0}(z)$$

and similarly for the right chiral half. The super partners for the primaries $V_{d,pd}(z,\overline{z})$ are therefore of the form $V_{d\pm 6,pd'}(z,\overline{z})$. As one can see from equation (3.27), this introduces a square-root branch cut into the OPE, so the fields are at worst fermionic, as we expect for a SCFT.

3.8.2 $k = 2$

As we saw in section 3.3.2, the five possible partition functions are expressed in terms of those of the Ising model and the free boson. The field content and OPEs of the Ising model is well-known. The fields, given in table 3.1, admit a closed associative OPE (see for example chapter 12 of [8]).

Table 3.1: Fields of the Ising model

Field	h	$\overline{h}$
I	0	0
$\psi(z)$	$\frac{1}{2}$	0
$\overline{\psi}(\overline{z})$	0	$\frac{1}{2}$
$\epsilon(z,\overline{z})$	$\frac{1}{2}$	$\frac{1}{2}$
$\sigma(z,\overline{z})$	$\frac{1}{16}$	$\frac{1}{16}$
$\mu(z,\overline{z})$	$\frac{1}{16}$	$\frac{1}{16}$

In terms of the parafermions of section 3.6.1, we have

$$I = \Phi^{0,0}_{0,0} = \Phi^{2,2}_{2,2} = \Phi^{0,2}_{0,2} = \Phi^{2,0}_{2,0}$$

$$\epsilon(z,\overline{z}) = \Phi_{2,2}^{0,0} = \Phi_{0,0}^{2,2} = \Phi_{2,0}^{0,2} = \Phi_{0,2}^{2,0}$$

$$\psi(z) = \Phi_{2,0}^{0,0} = \Phi_{0,0}^{2,0} = \Phi_{2,2}^{0,2} = \Phi_{0,2}^{2,2}$$

$$\overline{\psi}(\overline{z}) = \Phi_{0,2}^{0,0} = \Phi_{0,0}^{0,2} = \Phi_{2,2}^{2,0} = \Phi_{2,0}^{2,2}$$

$$\sigma(z,\overline{z}) = \Phi_{1,1}^{1,1} = \Phi_{-1,-1}^{1,1}$$

$$\mu(z,\overline{z}) = \Phi_{1,-1}^{1,1} = \Phi_{-1,1}^{1,1}.$$

We can now explicitly write out representative fields for each superconformal half-family $\phi_{a,c;a',c'}$ appearing in the five minimal models at $k = 2$ in terms of the Ising model fields and vertex operators. For convenience we will renormalise the vertex operators so that

$$V_{c,c'}(z,\overline{z}) =: \exp\left(\frac{ic}{\sqrt{12}}\phi(z) + \frac{ic'}{\sqrt{12}}\overline{\phi}(\overline{z})\right) : .$$

We first consider the four partition functions $M^{2,2}[v,z,n]$ (see section 3.3.2). The corresponding physical invariants M are in fact automorphism invariants (i.e. permutation matrices). We can thus label a representative of the superconformal half family $[a,c] \otimes [a',c']$ unambiguously by $|a,c\rangle$, whenever $M_{a,c;a',c'} = 1$. The fields $|a,c\rangle$ are defined for $a = 0,1,2$ and $c = -3,\ldots,4$. They are listed in tables 3.2, 3.3, 3.4 and 3.5, for the models $M^{2,2}[v,z,n]$ with $[v,z,n] = [4,1,0]$, $[4,1,1]$, $[4,7,0]$, $[4,7,1]$, respectively. (See sections 3.3.2 and 3.2 for notation.)

Table 3.2: Fields of the model $M^{2,2}[4,1,0]$

	0	1	2
-3	$IV_{3,3}$	$\sigma V_{-3,-3}$	$\epsilon V_{3,3}$
-2	$\epsilon V_{-2,-2}$	$\sigma V_{4,4}$	$IV_{-2,-2}$
-1	$\epsilon V_{-3,-3}$	$\sigma V_{-1,-1}$	$IV_{-3,-3}$
0	$IV_{0,0}$	$\sigma V_{-2,-2}$	$\epsilon V_{0,0}$
1	$IV_{-1,-1}$	$\sigma V_{1,1}$	$\epsilon V_{-1,-1}$
2	$\epsilon V_{2,2}$	$\sigma V_{0,0}$	$IV_{2,2}$
3	$\epsilon V_{1,1}$	$\sigma V_{3,3}$	$IV_{1,1}$
4	$IV_{4,4}$	$\sigma V_{2,2}$	$\epsilon V_{4,4}$

We will next read off the primary fields in the full Hilbert space $\mathbb{H}$ (see section 2.3), but first we observe that for fixed z, the models $M^{2,2}[4,z,n]$ with $n = 0,1$ are related by the symmetry of section 2.5.4. As we noted there, models related by that symmetry and contain no states in the NS-R or R-NS sectors (such as the ones currently under consideration) have identical full Hilbert spaces.

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Table 3.3: Fields of the model $M^{2,2}[4,1,1]$

	0	1	2
-3	$\overline{\psi}V_{3,-1}$	$\sigma V_{-3,-3}$	$\psi V_{3,-1}$
-2	$\epsilon V_{-2,-2}$	$\mu V_{4,0}$	$IV_{-2,-2}$
-1	$\psi V_{-3,1}$	$\sigma V_{-1,-1}$	$\overline{\psi}V_{-3,1}$
0	$IV_{0,0}$	$\mu V_{-2,2}$	$\epsilon V_{0,0}$
1	$\overline{\psi}V_{-1,3}$	$\sigma V_{1,1}$	$\psi V_{-1,3}$
2	$\epsilon V_{2,2}$	$\mu V_{0,4}$	$IV_{2,2}$
3	$\psi V_{1,-3}$	$\sigma V_{3,3}$	$\overline{\psi}V_{1,-3}$
4	$IV_{4,4}$	$\mu V_{2,-2}$	$\epsilon V_{4,4}$

For the models $M^{2,2}[4,1,0]$ and $M^{2,2}[4,1,1]$ the NS-NS primary states are

$$IV_{0,0},\ \epsilon V_{0,0},\ \sigma V_{1,1},\ \sigma V_{-1,-1},\ IV_{2,2},\ IV_{-2,-2}.$$

In the R-R sector there are three states of the form (R ground state)$\otimes$(R ground state) (see section 2.2.1). They are $IV_{1,1}, IV_{-1,-1}, \sigma V_{0,0}$. The remaining Ramond primaries are[14]

$$\{IV_{-3,-3},\ \psi V_{1,-3},\ \overline{\psi}V_{-3,1},\ \epsilon V_{1,1}\}$$
$$\{IV_{3,3},\ \psi V_{-1,3},\ \overline{\psi}V_{3,-1},\ \epsilon V_{-1,-1}\}$$
$$\{\sigma V_{2,2},\ \mu V_{-2,2},\ \mu V_{2,-2},\ \sigma V_{-2,-2}\}$$

For the models $M^{2,2}[4,7,0]$ and $M^{2,2}[4,7,1]$ the NS primary fields are

$$IV_{0,0},\ \epsilon V_{0,0},\ \mu V_{1,1},\ \mu V_{-1,-1},\ IV_{2,2},\ IV_{-2,-2}.$$

In the R-R sector we again have three primaries of the form (R ground state)$\otimes$(R ground state): they are $IV_{1,-1}, IV_{-1,1}, \mu V_{0,0}$, and the remaining primaries are

$$\{IV_{-3,3},\ \psi V_{1,3},\ \overline{\psi}V_{-3,-1},\ \epsilon V_{1,-1}\}$$
$$\{IV_{3,-3},\ \psi V_{-1,-3},\ \overline{\psi}V_{3,1},\ \epsilon V_{-1,1}\}$$
$$\{\sigma V_{2,2},\ \mu V_{-2,2},\ \mu V_{2,-2},\ \sigma V_{-2,-2}\}$$

It is clear that the models $M^{2,2}[4,1,0]$ and $M^{2,2}[4,1,1]$ and the models $M^{2,2}[4,7,0]$ and $M^{2,2}[4,7,1]$ are identical: they have the same field content and the same OPE. But we can go further and show that all four models are

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[14]Recall that Ramond primaries that are not Ramond ground states are degenerate with multiplicity two in each chiral half. Thus the full left-right states have a four-fold degeneracy.

Table 3.4: Fields of the model $M^{2,2}[4,7,0]$

	0	1	2
-3	$\overline{\psi}V_{3,1}$	$\mu V_{-3,3}$	$\psi V_{3,1}$
-2	$\epsilon V_{-2,2}$	$\sigma V_{4,0}$	$I V_{-2,2}$
-1	$\psi V_{-3,-1}$	$\mu V_{-1,1}$	$\overline{\psi}V_{-3,-1}$
0	$I V_{0,0}$	$\sigma V_{-2,-2}$	$\epsilon V_{0,0}$
1	$\overline{\psi}V_{-1,-3}$	$\mu V_{1,-1}$	$\psi V_{-1,-3}$
2	$\epsilon V_{2,-2}$	$\sigma V_{0,4}$	$I V_{2,-2}$
3	$\psi V_{1,3}$	$\mu V_{3,-3}$	$\overline{\psi}V_{1,3}$
4	$I V_{4,4}$	$\sigma V_{2,2}$	$\epsilon V_{4,4}$

Table 3.5: Fields of the model $M^{2,2}[4,7,1]$

	0	1	2
-3	$I V_{3,-3}$	$\mu V_{-3,3}$	$\epsilon V_{3,-3}$
-2	$\epsilon V_{-2,2}$	$\mu V_{4,4}$	$I V_{-2,2}$
-1	$\epsilon V_{-3,3}$	$\mu V_{-1,1}$	$I V_{-3,3}$
0	$I V_{0,0}$	$\mu V_{-2,2}$	$\epsilon V_{0,0}$
1	$I V_{-1,1}$	$\mu V_{1,-1}$	$\epsilon V_{-1,1}$
2	$\epsilon V_{2,-2}$	$\mu V_{0,0}$	$I V_{2,-2}$
3	$\epsilon V_{1,-1}$	$\mu V_{3,-3}$	$I V_{1,-1}$
4	$I V_{4,4}$	$\mu V_{2,-2}$	$\epsilon V_{4,4}$

equivalent. We note that there is a $\mathbb{Z}_2$ symmetry λ of the Ising model (see e.g. [8]) which sends

$$\sigma \leftrightarrow \mu$$
$$\psi \leftrightarrow \overline{\psi}$$
$$\epsilon \leftrightarrow -\epsilon$$

This map is an equivalence of CFTs; in other words the OPE is preserved under this transformation of the fields. We can extend λ to an equivalence of SCFTs: map a field $A V_{c,c} \to \lambda(A) V_{c,-c}$ for $A \in \{I, \epsilon, \psi, \overline{\psi}, \sigma, \mu\}$. This mapping takes the field content of the first pair of models into the field content of the second, and furthermore preserves the OPE of the SCFT, demonstrating the equivalence of the four minimal models labelled by $M^{2,2}$. In fact this mapping is nothing other

than a realisation of mirror symmetry (see section 2.5.1).

Finally we turn to the model coming from the partition function $M^{2,0}[2,1]$. Precisely two representatives of $[a,c] \otimes [a',c']$ exist whenever c is even. Again we will arrange them according to their (a,c) values: see table 3.6.

Table 3.6: Fields of the model $M^{2,0}[2,1]$

	0	1	2
-2	$\epsilon V_{-2,-2}, \epsilon V_{-2,2}$	$\sigma V_{4,0}, \sigma V_{4,4}$	$IV_{-2,2}, IV_{-2,-2}$
0	$IV_{0,0}, IV_{0,4}$	$\sigma V_{-2,-2}, \sigma V_{-2,2}$	$\epsilon V_{0,0}, \epsilon V_{0,4}$
2	$\epsilon V_{2,2}, \epsilon V_{2,-2}$	$\sigma V_{0,0}, \sigma V_{0,4}$	$IV_{2,2}, IV_{2,-2}$
4	$IV_{4,0}, IV_{4,4}$	$\sigma V_{2,2}, \sigma V_{2,-2}$	$\epsilon V_{4,0}, \epsilon V_{4,4}$

The NS-NS primaries are

$$IV_{0,0}, \ \epsilon V_{0,0}, \ IV_{2,2}, \ IV_{2,-2}, \ IV_{-2,2}, \ IV_{-2,-2}, \ \psi V_{0,0}, \ \overline{\psi} V_{0,0}.$$

The R-R primary fields are

$$\sigma V_{0,0}$$
$$\mu V_{0,0}$$
$$\{\sigma V_{2,2}, \ \mu V_{2,-2}, \ \mu V_{-2,2}, \ \sigma V_{-2,-2}\}$$
$$\{\mu V_{2,2}, \ \sigma V_{2,-2}, \ \sigma V_{-2,2}, \ \mu V_{-2,-2}\}.$$

We note that all fields in each of the five models are semi-local with respect to one another, and the full fusion rules for each are given by

$$N^{(\alpha\gamma,\alpha'\gamma')}_{(ac,a'c');(df,d'f')} = N^{(\alpha\gamma)}_{(ac),(df)} M_{\alpha,\gamma} N^{(\alpha'\gamma')}_{(a'c'),(d'f')}$$

in accordance with the results of section 3.7.

Chapter 4

Orbifolds of the $N = 2$ Unitary Minimal Models

The aim of this chapter is the construction of a unitary $N = 2$ minimal model for each possible partition function. The main step is to prove the following theorem:

Main Theorem.

- *Every non-exceptional partition function of a unitary $N = 2$ minimal model at level k can be obtained by orbifoldings of the diagonal partition function at level k.*

- *Every exceptional partition function of a unitary $N = 2$ minimal model with level $k = 10$, 16 or 28 can be obtained by orbifoldings of the $\mathcal{E}_6 \otimes I_{24}$, $\mathcal{E}_7 \otimes I_{36}$ or $\mathcal{E}_8 \otimes I_{60}$ partition functions, respectively, where $\mathcal{E}_{6,7,8}$ are the $\widehat{\mathfrak{su}(2)}_k$ exceptional physical invariants, and $I_{2\overline{k}}$ is the $\widehat{\mathfrak{u}(1)}_{\overline{k}}$ diagonal invariant.*

The proof is constructive: given any non-exceptional physical invariant M at level k in Gannon's list, we construct a chain of orbifoldings (by cyclic groups) mapping M to a particular level k physical invariant. Since this also applies to $\mathcal{A}_k$, and since an orbifolding by a solvable group always has an orbifolding inverse (see e.g. [33]), we see that any non-exceptional partition function belongs to a model that can be obtained as the result of a chain of orbifoldings beginning at the diagonal invariant. Similarly, given an exceptional physical invariant at level $k = 10$, 16 or 28, we will construct a chain of orbifoldings to a particular level k physical invariant.

The proof will be broken down into several sections. We must first explain what we mean by *orbifolding*. This is done in section 4.1, and a simple example is given in section 4.2. In section 4.3, we realise the symmetries of the minimal models found in section 2.5 via orbifoldings.

In section 4.4 we generalise a well-known $\mathbb{Z}_2$ orbifold from the $\widehat{\mathfrak{su}(2)}_k$ models to the minimal models, and observe that we can construct an orbifolding between the minimal "families" listed in section 3.2.

In section 4.5 we state and prove a proposition that every physical invariant M can be mapped into either $\widetilde{M}^0, \widetilde{M}^{4,2}, \widetilde{M}^{2,0}, \widetilde{E}_1^{10}, \widetilde{E}_2^{16}$ or $\widetilde{E}^{28}$ depending on the level k and whether M is exceptional or not.

We then attempt to control the parameter v – we find an orbifolding to map any given physical invariant in one of the above families to the physical invariant with the lowest possible value of v. This is section 4.6.

Lastly, in section 4.7 we try to control the parameter z. We summarise these results in subsection 4.8, finally completing the proof.

4.1 The orbifold construction

We first describe the orbifolding procedure [13, 14, 12] in the case of a bosonic CFT, i.e. when no fermionic modes are present. Let $\mathbb{H}$ be the underlying pre-Hilbert space of a CFT $\mathcal{C}$ and let $\rho : G \rightarrow \text{End}(\mathbb{H})$ be an action of a discrete group on $\mathbb{H}$ such that

1. $\mathbb{H}$ is simultaneously diagonalisable with respect to $L_0, \overline{L}_0$ and $\rho(g)$ for every $g \in G$, where $L_0, \overline{L}_0$ are viewed as linear operators on $\mathbb{H}$;

2. $\rho(g)$ commutes with L_n and $\overline{L}_n$ for every n, where $L_n, \overline{L}_n$ are viewed as linear operators on $\mathbb{H}$.

3. The action of G preserves the n-point functions of $\mathcal{C}$.

Decomposing $\mathbb{H} = \bigoplus_{a,b \in P} \mathbb{H}_a \otimes \overline{\mathbb{H}}_b$ into a direct sum of irreducible components, we see that the above conditions imply that $\rho(g)$ must act by multiplication by a root of unity $\xi_{a,b}(g)$ on the lowest weight vector of $\mathbb{H}_a \otimes \overline{\mathbb{H}}_b$, and therefore by multiplication by $\xi_{a,b}(g)$ on the whole of $\mathbb{H}_a \otimes \overline{\mathbb{H}}_b$. It follows that the action of G on the states of $\mathbb{H}$ is entirely described by its action on the characters $\rho(g)(\chi_{ac}\chi^*_{a'c'}) = \xi_{a,b}(g)\chi_{ac}\chi^*_{a'c'}$. For notational simplicity we shall now simply write g in place of $\rho(g)$.

We want to construct a G-invariant CFT from $\mathcal{C}$, the *G-orbifold* of $\mathcal{C}$, denoted $\mathcal{C}/G$. We will restrict our attention to an abelian group G for ease of notation, but one can generalise to non-abelian groups with a little care (see e.g. [33]).

We begin by projecting onto the G-invariant states of $\mathcal{C}$:

$$\mathbb{H}^{\text{inv}} := \mathcal{P} \cdot \mathbb{H}$$

where the projector $\mathcal{P}$ is given by $\frac{1}{|G|} \sum_{g \in G} g\cdot$. We use a notational shorthand

$$g\,\boxed{}_{\,1} := \text{Tr}_{\mathbb{H}}(g q^{L_0 - \frac{\mathbf{c}}{24}} \overline{q}^{\overline{L}_0 - \frac{\mathbf{c}}{24}})$$

for the trace with g inserted, which makes sense because of condition 1 above. This allows us to write the partition function of the G-invariant sector as

$$Z^{\mathrm{inv}}(\tau) = \mathrm{Tr}_{\mathbb{H}}(\mathcal{P}q^{L_0 - \frac{\mathbf{c}}{24}}\bar{q}^{\bar{L}_0 - \frac{\mathbf{c}}{24}}) = \frac{1}{|G|}\sum_{g \in G} g\boxed{}_{1}.$$

Unless G is trivial, $Z^{\mathrm{inv}}(\tau)$ will not be modular invariant. In order to restore modular invariance we need to add in extra G-invariant states, the so called *twisted states*. Two problems arise here: how do we go about constructing the twisted sector? And how do we extend the action of G to the twisted states?

The first question is difficult to answer in general, but we will only be interested in the case of the unitary $N = 2$ minimal models. In this case we can construct the twisted sector out of known representations, using the following arguments: by condition 2, the $L_n, \bar{L}_n$ modes commute with the G-action and so the central charge $\mathbf{c}$ is left invariant, and since the action of $\mathrm{SL}(2,\mathbb{Z})$ leaves $\mathbf{c}$ invariant, the twisted sector should also be composed of irreducible representations at central charge $\mathbf{c}$. But in the situation of interest to us, the collection of irreducible representations are explicitly known for fixed $\mathbf{c}$. Thus the twisted sector can be constructed from these known representations. It is therefore sufficient to find the partition function of the twisted sector using standard tricks below.

The answer to the second question is that there may be no unique way to extend the action of G to the twisted sector. The freedom we have in choosing an extension is called *discrete torsion* and is classified by the second group cohomology class $H^2(G, U(1))$ [61]. Here we will need to consider only the cases $G = \mathbb{Z}_k$ with discrete torsion $\mathbb{Z}_1$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$ with discrete torsion $\mathbb{Z}_2$.

We now return to the construction of the partition function of the twisted sector. For each $h \in G$ we denote by $\mathbb{H}_h$ the sector of states 'twisted by h' in the space direction; in the language of fields we make a cut from 0 to τ along the world-sheet torus $T = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ and require that a field crossing the cut is acted on by h:

$$\phi(z + 1) = h\phi(z).$$

Since we want to keep only G-invariant states, we project the partition function of $\mathbb{H}_h$ with $\mathcal{P}$:

$$\mathrm{Tr}_{\mathbb{H}_h}(\mathcal{P}q^{L_0 - \frac{\mathbf{c}}{24}}\bar{q}^{\bar{L}_0 - \frac{\mathbf{c}}{24}}) = \frac{1}{|G|}\sum_{g \in G} g\boxed{}_{h},$$

where we have introduced the notational shorthand

$$g\boxed{}_{h} := \mathrm{Tr}_{\mathbb{H}_h}(gq^{L_0 - \frac{\mathbf{c}}{24}}\bar{q}^{\bar{L}_0 - \frac{\mathbf{c}}{24}}).$$

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Then the partition function of the orbifold theory is the sum of the contributions from each of the twisted sectors:

$$Z^{\text{orb}} = \frac{1}{|G|} \sum_{g,h \in G} \, {}_g\boxed{}_h.$$

We interpret the box ${}_g\boxed{}_h$ as counting states whose fields live on the world-sheet torus with a cut along each cycle, such that cycling around once in the space-direction yields a factor of h and cycling around once in the time-direction yields a factor of g:

$$\phi(z + 1) = h\phi(z),$$
$$\phi(z + \tau) = g\phi(z).$$

Then we find that the S and T-transformations act to permute the 'boundary conditions' in the following way:

$$S\left({}_g\boxed{}_h \right) = {}_{h^{-1}}\boxed{}_g,$$

$$T\left({}_g\boxed{}_h \right) = {}_{gh}\boxed{}_h,$$

thus ensuring modular invariance of the orbifold partition function.

This completes the construction for bosonic CFTs. In order to extend the prescription to the SCFT case, we just replace the space of states $\mathbb{H}$ with the bosonic states, and add the z-dependence (via y^{J_0}) into the traces in the obvious manner.

4.2   A simple example

As an illustration we present a well-known example: let $\mathbb{H} = \bigoplus_{l=0,\dots,k} \mathbb{H}_l \otimes \overline{\mathbb{H}}_l$ be the diagonal combination of irreducible representations of affine $\widehat{\mathfrak{su}(2)}$ at level k, i.e. the A-model of the $\widehat{\mathfrak{su}(2)}_k$ WZW-models ([5], or see [26] for review of the classification of $\widehat{\mathfrak{su}(2)}$ WZW models). Writing $\chi_l(\tau)$ for the character of $\mathbb{H}_l$, the partition function is

$$\, {}_1\boxed{}_1 = \sum_{l=0,\dots,k} \chi_l(\tau)\chi_l(\tau)^*.$$

Define an action of $G = \mathbb{Z}_2 = \langle g \rangle$ on $\mathbb{H}$ by $g \cdot |x\rangle = (-1)^l|x\rangle$ for $|x\rangle \in \mathbb{H}_l \otimes \overline{\mathbb{H}}_l$. Then the invariant-sector of $\mathbb{H}$ is $\bigoplus_{l \text{ even}} \mathbb{H}_l \otimes \overline{\mathbb{H}}_l$ with partition function

$$Z^{\text{inv}} = \frac{1}{2}\left({}_1\boxed{}_1 + {}_g\boxed{}_1 \right)$$

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$$= \sum_{l \text{ even}} \chi_l(\tau)\chi_l(\tau)^*.$$

We use the S-matrix to calculate ${}_1\boxed{}_g$:

$$ {}_1\boxed{}_g = S\left({}_g\boxed{}_1 \right)$$

$$= \sum_{l=0,\ldots,k} (-1)^l \chi_l(S\cdot\tau)\chi_l(S\cdot\tau)^*$$

$$= \sum_{\substack{l=0,\ldots,k \\ a,a'=0,\ldots,k}} (-1)^l S_{l,a} S_{l,a'}^* \chi_a(\tau)\chi_{a'}(\tau)^*$$

$$= \sum_{\substack{l=0,\ldots,k \\ a,a'=0,\ldots,k}} S_{l,Ja} S_{l,a'}^* \chi_a(\tau)\chi_{a'}(\tau)^*$$

$$= \sum_{a,a'=0,\ldots,k} \delta_{a'=Ja} \chi_a(\tau)\chi_{a'}(\tau)^*$$

$$= \sum_{a=0,\ldots,k} \chi_a(\tau)\chi_{Ja}(\tau)^*,$$

where we have used the fact that $J : a \mapsto k - a$ is a simple current and the S-matrix is unitary and symmetric. Finally we calculate

$$ {}_g\boxed{}_g = T\left({}_1\boxed{}_g \right)$$

$$= \sum_{a=0,\ldots,k} \chi_a(\tau+1)\chi_{Ja}(\tau+1)^*$$

$$= \sum_{a=0,\ldots,k} \exp\left[\frac{2\pi i}{4\overline{k}}\left((a+1)^2 - (Ja+1)^2\right)\right] \chi_a(\tau)\chi_{Ja}(\tau)^*$$

$$= \sum_{a=0,\ldots,k} \exp\left[\frac{\pi i(2a-k)}{2}\right] \chi_a(\tau)\chi_{Ja}(\tau)^*$$

using the expression for T from (2.19). Thus the twisted sector has partition function

$$Z^{\text{twist}} = \frac{1}{2}\left({}_1\boxed{}_g + {}_g\boxed{}_g \right)$$

$$= \begin{cases} \text{non-real coefficients} & \text{if } k \text{ is odd} \\ \sum_{a \text{ odd}} \chi_a(\tau)\chi_{k-a}(\tau)^* & \text{if 4 divides } \overline{k} \\ \sum_{a \text{ even}} \chi_a(\tau)\chi_{k-a}(\tau)^* & \text{if 4 divides } k \end{cases}$$

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and we find

$$
Z^{\mathrm{orb}} =
\begin{cases}
\text{no orbifolding} & \text{if } k \text{ is odd} \\
\sum_{a=0,\ldots,k} \chi_{J^a a}(\tau)\chi_a(\tau)^* & \text{if 4 divides } \overline{k} \\
\sum_{a \text{ even}} (\chi_a(\tau) + \chi_{Ja}(\tau))\chi_a(\tau)^* & \text{if 4 divides } k.
\end{cases}
$$

When k is odd, Z^{twist} has non-real coefficients, but Z^{inv} has real coefficients. Therefore $Z^{\mathrm{orb}} = Z^{\mathrm{twist}} + Z^{\mathrm{inv}}$ cannot have real coefficients, and cannot be the partition function of an orbifold. When $4|k+2$ we obtain a $\mathbb{Z}_2$ orbifolding $\mathcal{A}_k \mapsto \mathcal{D}_k$ and when $4|k$ we obtain a $\mathbb{Z}_2$ orbifolding $\mathcal{A}_k \mapsto \mathcal{D}'_k$.

4.3 Symmetries of the minimal models as orbifoldings

The aim of this section is to demonstrate that the mappings between possible partition functions enumerated in section 2.5 can be realised as orbifolds.

4.3.1 The orbifoldings $\mathcal{O}^1_L, \mathcal{O}^1_R$

Let $\mathbb{H}$ be a minimal model with partition function $Z = Z(\tau, z)$ from the list in section 3.2. Write

$$
Z = {}_1\boxed{}_1 = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'} \chi_{ac}\chi^*_{a'c'}
$$

and let $\mathbb{Z}_2 = \langle g \rangle$ act on the states via $g \cdot \chi_{ac}\chi^*_{a'c'} = (-1)^{a+c}\chi_{ac}\chi^*_{a'c'}$, i.e. leaving the left-handed NS-sector invariant and twisting the left-handed R-states in the time-direction. Then we find

$$
{}_g\boxed{}_1 = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} (-1)^{a+c} M_{a,c;\,a',c'} \chi_{ac}\chi^*_{a'c'},
$$

$$
{}_1\boxed{}_g = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} (-1)^{a+c} S_{r,s;\,a,c} M_{a,c;\,a',c'} S^*_{a',c';\,t,u} \chi_{rs}\chi^*_{tu}
$$

which, using equation (2.39),

$$
= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} S_{Jr,s+\overline{k};\,a,c} M_{a,c;\,a',c'} S^*_{a',c';\,t,u} \chi_{r,s}\chi^*_{t,u}
$$

$$
= \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} M_{Jr,s+\overline{k};\,t,u} \chi_{rs}\chi^*_{tu}
$$

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}\chi_{Ja,c+\overline{k}}\chi^*_{a'c'},$$

where in the penultimate line we used the fact that M and S commute and that S is unitary and symmetric. Finally we calculate

$$g\boxed{}_{g} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} T_{r,s;\,Ja,c+\overline{k}}M_{a,c;\,a',c'}T^*_{a',c';\,t,u}\chi_{rs}\chi^*_{tu},$$

which, by equation (2.18),

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}e^{2\pi i(h_{Ja,c+\overline{k}}-h_{a',c'})}\chi_{Ja,c+\overline{k}}\chi^*_{a'c'},$$

which, by equation (2.12),

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}e^{2\pi i(h_{a,c}-h_{a',c'})}(-1)^{a+c+1}\chi_{Ja,c+\overline{k}}\chi^*_{a'c'},$$

which, by equation (2.26),

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}(-1)^{a+c+1}\chi_{Ja,c+\overline{k}}\chi^*_{a'c'}$$

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{Ja,c+\overline{k};\,a',c'}(-1)^{a+c+1}\chi_{ac}\chi^*_{a'c'}.$$

This gives

$$Z^{\mathrm{inv}} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k \\ a+c\text{ even}}} M_{a,c;\,a',c'}\chi_{ac}\chi^*_{a'c'},$$

$$Z^{\mathrm{twist}} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k \\ a+c\text{ odd}}} M_{Ja,c+\overline{k};\,a',c'}\chi_{ac}\chi^*_{a'c'},$$

$$Z^{\mathrm{orb}} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k \\ a+c\text{ even}}} M_{a,c;\,a',c'}\chi_{ac}\chi^*_{a'c'} + \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k \\ a+c\text{ odd}}} M_{Ja,c+\overline{k};\,a',c'}\chi_{ac}\chi^*_{a'c'}.$$

This orbifolding defines an involution on the set of physical invariants. We will refer to this orbifolding as $\mathcal{O}^1_L$ (where the L stands for left). Since S and T

are symmetric, it is clear that we could equally well have let $\mathbb{Z}_2$ act on the right-hand representations, $g \cdot \chi_{ac}\chi^*_{a'c'} = (-1)^{a'+c'}\chi_{ac}\chi^*_{a'c'}$. The result would be

$$Z^{\mathrm{orb}} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k \\ a'+c' \text{ even}}} M_{a,c;\,a',c'}\chi_{ac}\chi^*_{a'c'} + \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k \\ a'+c' \text{ odd}}} M_{a,c;\,Ja',c'+\bar{k}}\chi_{ac}\chi^*_{a'c'}.$$

We will refer to this orbifolding as $\mathcal{O}^1_R$. Note that if there are no NS×R or R×NS contributions then the action of $\mathbb{Z}_2$ on the left- and right-handed states is the same. That the resulting partition functions are equal in this case follows from Lemma 2.3.1.

4.3.2 The orbifoldings $\mathcal{O}^2_L, \mathcal{O}^2_R$

Again we start with a minimal model with partition function

$$Z = {}_1\boxed{}_1 = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}\chi_{ac}\chi^*_{a'c'}$$

and define a group action by $g \cdot \chi_{ac}\chi^*_{a'c'} = e^{\frac{2\pi i c}{k}}\chi_{ac}\chi^*_{a'c'}$. This defines a $\mathbb{Z}_{\bar{k}}$-action. We claim that the general box is given by

$$g^m\boxed{}_{g^n} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}e^{\frac{2\pi i m(c-n)}{k}}\chi_{a,c-2n}\chi^*_{a',c'}.$$

One easily checks that this is correct when $n = 0$. It remains to check that it transforms correctly under the S and T transformations. For the T transformation we find, using equations (2.18), (2.12) and then (2.26),

$$T \cdot g^m\boxed{}_{g^n} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}e^{\frac{2\pi i m(c-n)}{k}}e^{2\pi i(h_{a,c-2n}-h_{a',c'})}\chi_{a,c-2n}\chi^*_{a',c'}$$

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}e^{2\pi i\frac{m(c-n)}{k}}e^{2\pi i\frac{n(c-n)}{k}}e^{2\pi i(h_{a,c}-h_{a',c'})}\chi_{a,c-2n}\chi^*_{a',c'}$$

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}e^{2\pi i\frac{(m+n)(c-n)}{k}}\chi_{a,c-2n}\chi^*_{a',c'}$$

$$= g^{m+n}\boxed{}_{g^n}.$$

For the S-transformation we find, using equation (2.38) in the second and third lines, and equation (2.24) in the next,

$$
S \cdot g^m \boxed{}_{g^n} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} S_{r,s;\,a,c-2n} M_{a,c;\,a'c'} S^*_{a',c';\,t,u} e^{\frac{2\pi i m(c-n)}{k}} \chi_{rs}\chi^*_{tu}
$$

$$
= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} S_{r,s;\,a,c} M_{a,c;\,a'c'} S^*_{a',c';\,t,u} e^{\frac{2\pi i(m(c-n)-ns)}{k}} \chi_{rs}\chi^*_{tu}
$$

$$
= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} S_{r,s+2m;\,a,c} M_{a,c;\,a'c'} S^*_{a',c';\,t,u} e^{\frac{2\pi i(-n(m+s))}{k}} \chi_{rs}\chi^*_{tu}
$$

$$
= \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} M_{r,s+2m;\,t,u} e^{\frac{2\pi i(-n(m+s))}{k}} \chi_{rs}\chi^*_{tu}
$$

$$
= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'} e^{\frac{2\pi i(-n(c-m))}{k}} \chi_{a,c-2m}\chi^*_{a'c'}
$$

$$
= g^{-n} \boxed{}_{g^m}.
$$

Thus the boxes span a representation of $\mathrm{SL}_2(\mathbb{Z})$. To calculate the resulting orbifolding we need

$$
Z_{g^n} := \frac{1}{\bar{k}} \sum_{m=0,\ldots,\bar{k}-1} g^m \boxed{}_{g^n}
$$

$$
= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'} \left[\frac{1}{\bar{k}} \sum_{m=0,\ldots,\bar{k}-1} e^{2\pi i \frac{m(c-n)}{\bar{k}}} \right] \chi_{a,c-2n}\chi^*_{a',c'}
$$

$$
= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}\, \delta(c \equiv n \mod \bar{k}) \chi_{a,c-2n}\chi^*_{a',c'}
$$

$$
= \sum_{\substack{a=0,\ldots,k \\ l=0,1}} \sum_{(a',c')\in Q_k} M_{a,n+l\bar{k};\,a',c'} \chi_{a,-n+l\bar{k}}\chi^*_{a'c'}.
$$

Thus we see that

$$
Z^{\mathrm{orb}} = \sum_{n=0,\ldots,\bar{k}-1} Z_{g^n}
$$

$$
= \sum_{\substack{a=0,\ldots,k \\ l=0,1}} \sum_{n=0,\ldots,\bar{k}-1} \sum_{(a',c')\in Q_k} M_{a,n+l\bar{k};\,a',c'} \chi_{a,-n+l\bar{k}}\chi^*_{a'c'}
$$

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,-c;\,a',c'} \chi_{ac} \chi^*_{a'c'}.$$

This orbifolding is well-defined on all physical invariants. We will refer to it by $\mathcal{O}^2_L$. Again, the group $\mathbb{Z}_{\overline{k}}$ could equally as well have acted upon the right-hand representations. In that case we would obtain

$$Z^{\mathrm{orb}} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',-c'} \chi_{ac} \chi^*_{a'c'}.$$

We will refer to this orbifolding as $\mathcal{O}^2_R$. Clearly these orbifoldings give the same result if the initial physical invariant is symmetric.

4.3.3 Symmetries generated by $\mathcal{O}^1_{L,R}$ and $\mathcal{O}^2_{L,R}$

Note that these orbifoldings are self-inverse, they are mutually commuting, and the effect of concatenating $\mathcal{O}^1_L \mathcal{O}^2_L$ or $\mathcal{O}^1_R \mathcal{O}^2_R$ is to perform the left- or right-hand mirror symmetry transformation of section 2.5.1, respectively. The orbifoldings $\mathcal{O}^2_{L,R}$ realise the symmetry given in section 2.5.3.

Performing charge conjugation on both sides simultaneously amounts to performing all 4 orbifoldings $\mathcal{O}^1_L \mathcal{O}^2_L \mathcal{O}^1_R \mathcal{O}^2_R$ in succession, recovering the charge conjugation symmetry of section 2.5.2. As discussed there, we consider two charge conjugate models to be equivalent; and indeed they have the same partition function.

The results of applying $\mathcal{O}^1_{L,R}$ and $\mathcal{O}^2_{L,R}$ to the minimal partition functions listed in section 3.2 are given in table 4.1.[1] The third column lists the values of the defining parameters before any orbifolding is applied.

We can immediately read off from the table that performing the orbifoldings $\mathcal{O}^1_L, \mathcal{O}^1_R$ in succession leaves all of the partition functions invariant except for $\widetilde{M}^{4,0}[v,z,n,m]$ when $n \neq m$, $\widetilde{M}^{4,1}[v,z,x,y]$ when $x \neq y$ and $\widetilde{E}^{16}_1[v,z,x,y]$ when $x \neq y$. These are precisely the physical invariants which are not symmetric[2]. The effect of performing these two orbifoldings in succession is to transpose these physical invariants, realising the symmetry of section 2.5.5. Thus we have realised all the symmetries of section 2.5 via orbifolding.

We also observe that any two physical invariants in the same family with parameters $v_1 = v_2$ and $z_1 = \pm z_2$ can be mapped into one another by some combination of the orbifoldings $\mathcal{O}^1_L, \mathcal{O}^1_R, \mathcal{O}^2_L$ and $\mathcal{O}^2_R$, with the exception of the family $\widetilde{M}^{2,2}$. In the next section we shall find an additional $\mathbb{Z}_2$ orbifolding which allows to extend this observation to all families of minimal partition functions.

---

[1] The parameter z is defined modulo some number α in each case. $-z$ is to be understood as $-z \mod \alpha$.

[2] The transpose of the physical invariants was given in the classification of the partition functions in section 3.2

Table 4.1: Action of $\mathcal{O}^1_{L,R}, \mathcal{O}^2_{L,R}$ on minimal partition functions

		Id	$\mathcal{O}^1_L$	$\mathcal{O}^1_R$	$\mathcal{O}^2_L$	$\mathcal{O}^2_R$
k odd	$\widetilde{M}^0$	$[v,z,n]$	$[v,z,n+1]$	$[v,z,n+1]$	$[v,-z,n]$	$[v,-z,n]$
4 divides $\overline{k}$	$\widetilde{M}^{2,0}$	$[v,z,n]$	$[v,z,n+1]$	$[v,z,n+1]$	$[v,-z,n]$	$[v,-z,n]$
	$\widetilde{M}^{2,1}$	$[v,z,n]$	$[v,z,n+1]$	$[v,z,n+1]$	$[v,-z,n+1]$	$[v,-z,n+1]$
	$\widetilde{M}^{2,2}$	$[v,z,n,m]$	$[v,z,n+1,m+1]$	$[v,z,n+1,m+1]$	$[v,-z,n,m]$	$[v,-z,n,m]$
4 divides k	$\widetilde{M}^{4,0}$	$[v,z,n,m]$	$[v,z,n,m+1]$	$[v,z,n+1,m]$	$[v,-z,n+1,m]$	$[v,-z,n,m+1]$
	$\widetilde{M}^{4,1}$	$[v,z,x,y]$	$[v,z,x+2,y]$	$[v,z,x,y+2]$	$[v,-z,x+2,y]$	$[v,-z,x,y+2]$
	$\widetilde{M}^{4,2}$	$[v,z,x]$	$[v,z,x+2]$	$[v,z,x+2]$	$[v,-z,x+2]$	$[v,-z,x+2]$
	$\widetilde{M}^{4,3}$	$[v,z,n]$	$[v,z,n+1]$	$[v,z,n+1]$	$[v,-z,n]$	$[v,-z,n]$
$k=10$	$\widetilde{E}^{10}_1$	$[6,z]$	$[6,z]$	$[6,z]$	$[6,-z]$	$[6,-z]$
	$\widetilde{E}^{10}_2$	$[12,z,0,m]$	$[12,z,0,m+1]$	$[12,z,0,m+1]$	$[12,-z,0,m]$	$[12,-z,0,m]$
$k=16$	$\widetilde{E}^{16}_1$	$[v,z,x,y]$	$[v,z,x+2,y]$	$[v,z,x,y+2]$	$[v,-z,x+2,y]$	$[v,-z,x,y+2]$
	$\widetilde{E}^{16}_2$	$[v,z,x]$	$[v,z,x+2]$	$[v,z,x+2]$	$[v,-z,x+2]$	$[v,-z,x+2]$
$k=28$	$\widetilde{E}^{28}$	$[15,z,x]$	$[15,z,x+2]$	$[15,z,x+2]$	$[15,-z,x+2]$	$[15,-z,x+2]$

4.4   The generalised $\mathcal{A}_k \leftrightarrow \mathcal{D}_k$ orbifolding

The family $\widetilde{M}^{2,2}$ exists for any k with $4|\bar{k}$. Given such a k, we can always choose $v = \bar{k}$ and $z = 1$. Then, from equation (3.6), we obtain a physical invariant M with $M_{a,c;\,a',c'} = \delta(a' = J^{an}a)\delta(c' = c)$. Thus

$$M = \begin{cases} \mathcal{A}_k \otimes \mathcal{I}_{\bar{k}} & \text{if } n = 0 \\ \mathcal{D}_k \otimes \mathcal{I}_{\bar{k}} & \text{if } n = 1, \end{cases}$$

where $\mathcal{A}$ and $\mathcal{D}$ are the $\widehat{\mathfrak{su}(2)}_k$ physical invariants encountered in 4.2 and $\mathcal{I}_k$ is the diagonal $\widehat{\mathfrak{u}(1)}_{\bar{k}}$ invariant. Inspired by section 4.2, we define a $\mathbb{Z}_2$ action on the states of an arbitrary physical invariant with $4|\bar{k}$ by

$$g \cdot \chi_{ac}\chi_{a'c'}^* := (-1)^a \chi_{ac}\chi_{a'c'}^*.$$

Then we find

$$1\,\boxed{}_1 = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}\chi_{ac}\chi_{a'c'}^*,$$

$$g\,\boxed{}_1 = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} (-1)^a M_{a,c;\,a',c'}\chi_{ac}\chi_{a'c'}^*,$$

$$1\,\boxed{}_g = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} (-1)^a S_{r,s;\,a,c} M_{a,c;\,a',c'} S_{a',c';\,t,u}^* \chi_{rs}\chi_{tu}^*$$

Using equations (2.38) and (2.24) we have

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} S_{Jr,s;\,a,c} M_{a,c;\,a',c'} S_{a',c';\,t,u}^* \chi_{rs}\chi_{tu}^*$$

$$= \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} M_{Jr,s;\,t,u}\chi_{rs}\chi_{tu}^*$$

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c';\,a',c'}\chi_{Ja,c}\chi_{a'c'}^*,$$

and using equation (2.18), (2.12) and then (2.26) we find

$$g\,\boxed{}_g = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c';\,a',c'}\,e^{2\pi i(h_{Ja,c}-h_{a'c'})}\chi_{Ja,c}\chi_{a'c'}^*,$$

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c';\,a',c'}\,e^{2\pi i(h_{a,c}-h_{a'c'})}(-1)^{a+1}\chi_{Ja,c}\chi^*_{a'c'}$$

$$= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c';\,a',c'}(-1)^{a+1}\chi_{Ja,c}\chi^*_{a'c'}.$$

Thus

$$Z^{\text{inv}} = \frac{1}{2}\left(1\,\boxed{}_{1} + g\,\boxed{}_{1}\right)$$

$$= \sum_{\substack{(a,c)\in Q_k \\ a\equiv 0\ \text{mod}\ 2}} \sum_{(a',c')\in Q_k} M_{a,c;\,a',c'}\chi_{ac}\chi^*_{a'c'},$$

$$Z^{\text{twist}} = \frac{1}{2}\left(1\,\boxed{}_{g} + g\,\boxed{}_{g}\right)$$

$$= \sum_{\substack{(a,c)\in Q_k \\ a\equiv 1\ \text{mod}\ 2}} \sum_{(a',c')\in Q_k} M_{Ja,c;\,a',c'}\chi_{ac}\chi^*_{a'c'},$$

$$Z^{\text{orb}} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{J^a a,c';\,a',c'}\chi_{a,c}\chi^*_{a'c'}.$$

The action on the minimal partition functions $4|\overline{k}$ is given by table 4.2.

Table 4.2:

$\widetilde{M}^{2,0}$	$[v,z,n] \leftrightarrow [v,z,n+1]$
$\widetilde{M}^{2,1}$	$[v,z,n] \leftrightarrow [v,z,n+1]$
$\widetilde{M}^{2,2}$	$[v,z,n,m] \leftrightarrow [v,z,n+1,m]$
$\widetilde{E}^{10}_1$	$[6,z] \leftrightarrow [6,z]$
$\widetilde{E}^{10}_2$	$[12,z,0,m] \leftrightarrow [12,z,0,m]$

For $\widetilde{M}^{2,0}$ and $\widetilde{M}^{2,1}$ the action coincides with that of $\mathcal{O}^1$ (as we would expect since if $\widetilde{M}_{a,c;\,a'c'}\neq 0$ then c is even for these families). For $\widetilde{M}^{2,2}$ we have obtained an additional $\mathbb{Z}_2$ symmetry, which along with $\mathcal{O}^1$ and $\mathcal{O}^2$ from the previous section allows us to construct an orbifolding between any two $\widetilde{M}^{2,2}$ physical invariants with $v_1 = v_2$ and $z_1 = \pm z_2$. As one might expect, for the special case $v = \overline{k}$ and $z = 1$ this orbifolding manifests itself as $\mathcal{A}_k \otimes \mathcal{I}_{\overline{k}} \leftrightarrow \mathcal{D}_k \otimes \mathcal{I}_{\overline{k}}$. The exceptional physical invariants $\widetilde{E}^{10}_{1,2}$ are left invariant.

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One might naturally ask about the case $4|k$: can we generalise the $\mathbb{Z}_2$ orbifolding of section 4.2 to an orbifolding of minimal physical invariants? The answer is yes; applying the $\mathbb{Z}_2$ group action $g \cdot \chi_{ac}\chi^*_{a'c'} = (-1)^a \chi_{ac}\chi^*_{a'c'}$ to $\sum M_{a,c;\,a',c'}\chi_{ac}\chi^*_{a'c'}$, we find[3]

$$g^m \boxed{}_{g^n} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} (-1)^{am} M_{J^n a,c;\,a',c'}\chi_{ac}\chi^*_{a'c'}, \quad m,n \in \mathbb{Z}_2.$$

One checks that this transforms correctly under S, T. This furnishes us with an orbifold $\sum_{a\ \text{even}}(M_{a,c;\,a',c'} + M_{Ja,c;\,a',c'})\chi_{ac}\chi^*_{a'c'}$. We will refer to the generalised $\mathcal{A} \leftrightarrow \mathcal{D}$ orbifolding as $\mathcal{O}^3$. This effect on the minimal partition functions with $4|k$ is given in table 4.3.

Table 4.3:

$\widetilde{M}^{4,0}[v,z,n,m]$	$\rightarrow$	$\widetilde{M}^{4,2}[v,z,2m+2n+1]$
$\widetilde{M}^{4,1}[v,z,x,y]$	$\rightarrow$	$\widetilde{M}^{4,1}[v,z,x,y]$
$\widetilde{M}^{4,2}[v,z,x]$	$\rightarrow$	$\widetilde{M}^{4,2}[v,z,x]$
$\widetilde{M}^{4,3}[v,z,n]$	$\rightarrow$	$\widetilde{M}^{4,2}[v,z,2n+z]$
$\widetilde{E}^{16}_1[v,z,x,y]$	$\rightarrow$	$\widetilde{E}^{16}_1[v,z,x,y]$
$\widetilde{E}^{16}_2[v,z,x]$	$\rightarrow$	$\widetilde{E}^{16}_2[v,z,x]$
$\widetilde{E}^{28}[15,z,x]$	$\rightarrow$	$\widetilde{E}^{28}[15,z,x]$

In particular, $\widetilde{M}^{4,3}[\frac{\bar{k}}{2},1,0] = \mathcal{A}_k$ is mapped to $\widetilde{M}^{4,2}[\frac{\bar{k}}{2},1,1] = \mathcal{D}'_k$ as we might expect. The physical invariants in the families $\widetilde{M}^{4,1}$ and $\widetilde{M}^{4,2}$ and the exceptionals are left invariant.[4] We note that physical invariants in $\widetilde{M}^{4,0}$ and $\widetilde{M}^{4,3}$ are sent to $\widetilde{M}^{4,2}$ under this orbifolding. This demonstrates that orbifoldings can map between, as well as within, families of minimal partition functions. In the next section we will show that in fact all the non-exceptional families at a given level k can be mapped into one another via orbifoldings, and that the same holds true for the exceptional families.

4.5   Orbifoldings between minimal families

We shall prove the following proposition:

---

[3]We could equally as well have acted on the right-hand side: $g = (-1)^{a'}$. But one sees in section 3.2 that if k is even and $M_{a,c;\,a',c'} \neq 0$ then $a \equiv a' \mod 2$.

[4]Actually the formula given above for the $\mathbb{Z}_2$ orbifolding has to be divided through by 2 in order to get $\widetilde{M}_{0,0;\,0,0} = 1$. This factor of 2 appears because $\mathbb{Z}_2$ acts trivially on all the states so $Z = Z^{\text{inv}} = Z^{\text{twist}}$ and so $Z^{\text{orb}} = 2Z$.

Proposition 4.5.1. *1. Let k be odd. Then all simple current invariants at level k can be mapped by an orbifolding to the family $\widetilde{M}^0$.*

2. Let $4|k$. Then all simple current invariants at level k can be mapped by an orbifolding to the family $\widetilde{M}^{4,2}$.

3. Let $4|\overline{k}$. Then all simple current invariants at level k can be mapped by an orbifolding to the family $\widetilde{M}^{2,0}$.

4. Let $k = 10$. Then all exceptional invariants at level k can be mapped by an orbifolding to $\widetilde{E}_1^{10}$.

5. Let $k = 16$. Then all exceptional invariants at level k can be mapped by an orbifolding to $\widetilde{E}_2^{16}$.

6. Let $k = 28$. Then all exceptional invariants at level k can be mapped by an orbifolding to $\widetilde{E}^{28}$.

When k is odd there is only one family of partition functions of minimal models and when $k = 28$ there is only one family of exceptionals, so parts 1 and 6 are trivial, but we include these statements for completeness.

4.5.1   Orbifoldings between minimal families: $4|k$

In section 4.4 we saw that the generalised $\mathcal{A} \leftrightarrow \mathcal{D}$ orbifolding $\mathcal{O}^3$ mapped members of the family $\widetilde{M}^{4,0}$ and $\widetilde{M}^{4,3}$ into the family $\widetilde{M}^{4,2}$. We will now show that $\widetilde{M}^{4,2}$ contains an orbifold of every member of the family $\widetilde{M}^{4,1}$, and that $\widetilde{E}_2^{16}$ contains an orbifold of every member of $\widetilde{E}_1^{16}$. This will prove parts 2 and 5.

Fix some $k \in 4\mathbb{Z}$. We want to construct an orbifolding which in particular sends $\widetilde{M}^{4,1}$ to $\widetilde{M}^{4,2}$. The latter only has entries in the NS×NS and R×R sectors, but the former has entries in all 4 sectors. So we define a $\mathbb{Z}_2$ action by $g \cdot \chi_{ac}\chi^*_{a'c'} = (-1)^{a+c+a'+c'}\chi_{ac}\chi^*_{a'c'}$ in order to preserve the NS×NS and R×R sectors and remove the NS×R and R×NS sectors. For $m, n \in \{0, 1\}$ we find

$$g^m \boxed{}_{g^n} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} (-1)^{(a+c+a'+c')m} M_{J^n a, c+n\overline{k};\, J^n a', c'+n\overline{k}} \chi_{ac}\chi^*_{a'c'}.$$

This transforms correctly under the S- and T-transformations, resulting in an orbifold

$$Z^{\mathrm{orb}} = \sum_{a+c+a'+c'\equiv 0 \bmod 2} (M_{a,c;\, a',c'} + M_{Ja,c+\overline{k};\, Ja',c'+\overline{k}})\chi_{ac}\chi^*_{a'c'}.$$

We call this orbifolding $\mathcal{O}^4$. It acts trivially on those physical invariants which only have NS×NS and R×R sectors: $\widetilde{M}^{4,2}$, $\widetilde{M}^{4,3}$, $\widetilde{E}_2^{16}$ and $\widetilde{E}^{28}$. The action of $\mathcal{O}^4$ on the other minimal partition functions that occur when $4|k$ is given in table 4.4[5]:

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[5]Note that in the RHS of the second and fifth lines the parameter $2z$ is to be understood

<div align="center">Table 4.4:</div>

$\widetilde{M}^{4,0}[v,z,n,m]$	$\rightarrow$	$\widetilde{M}^{4,2}[v,z,2m+2n+1]$
$\widetilde{M}^{4,1}[v,z,x,y]$	$\rightarrow$	$\widetilde{M}^{4,2}[\frac{v}{2},2z,y-x+1]$
$\widetilde{M}^{4,2}[v,z,x]$	$\rightarrow$	$\widetilde{M}^{4,2}[v,z,x]$
$\widetilde{M}^{4,3}[v,z,n]$	$\rightarrow$	$\widetilde{M}^{4,3}[v,z,2n+z]$
$\widetilde{E}^{16}_1[v,z,x,y]$	$\rightarrow$	$\widetilde{E}^{16}_2[\frac{v}{2},2z,y-x+1]$
$\widetilde{E}^{16}_2[v,z,x]$	$\rightarrow$	$\widetilde{E}^{16}_2[v,z,x]$
$\widetilde{E}^{28}[15,z,x]$	$\rightarrow$	$\widetilde{E}^{28}[15,z,x]$

4.5.2 Orbifoldings between minimal families: $4|\overline{k}$

In this section we shall show that all non-exceptional invariants with $4|\overline{k}$ can be mapped by an orbifolding into $\widetilde{M}^{2,0}$ and all exceptional invariants can be mapped by an orbifolding into $\widetilde{E}^{10}_1$, proving parts 3 and 4 of Proposition (4.5.1).

Orbifoldings between minimal families: $\widetilde{M}^{2,1} \rightarrow \widetilde{M}^{2,0}$

First we shall construct an orbifolding $\mathcal{O}^5$ from $\widetilde{M}^{2,1}$ to $\widetilde{M}^{2,0}$. Fix a k with $4|\overline{k}$ and fix (v,z,n) satisfying $\frac{\overline{k}}{2v} \in \mathbb{Z}$, $\frac{2v^2}{\overline{k}} \in 2\mathbb{Z}+1$ and $\frac{\overline{k}(z^2-1)}{2v^2} \in \mathbb{Z}$ where $z \in \{1,\ldots,\frac{2v^2}{\overline{k}}\}$ and $n \in \{0,1\}$. Then from section 3.2 there is a physical invariant $\widetilde{M}^{2,1}[v,z,n]$. We need to define a group action on the states of $M \equiv \widetilde{M}^{2,1}[v,z,n]$. Note that $M_{a,d;a',d'} \neq 0 \Rightarrow d = \frac{c\overline{k}}{2v}$, $d' = \frac{c'\overline{k}}{2v}$ and $c + c' \equiv 0 \bmod 2$; thus there is a $\mathbb{Z}_2$ action on the states given by $g \cdot \chi_{a,\frac{c\overline{k}}{2v}}\chi^*_{a,\frac{c'\overline{k}}{2v}} = (-1)^{\frac{c+c'}{2}}\chi_{a,\frac{c\overline{k}}{2v}}\chi^*_{a,\frac{c'\overline{k}}{2v}}$ and so we have

$$1\;\boxed{}_1 = \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{4v}}} M_{a,\frac{c\overline{k}}{2v};a',\frac{c'\overline{k}}{2v}}\;\chi_{a,\frac{c\overline{k}}{2v}}\chi^*_{a,\frac{c'\overline{k}}{2v}},$$

$$g\;\boxed{}_1 = \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{4v}}} (-1)^{\frac{c+c'}{2}} M_{a,\frac{c\overline{k}}{2v};a',\frac{c'\overline{k}}{2v}}\;\chi_{a,\frac{c\overline{k}}{2v}}\chi^*_{a,\frac{c'\overline{k}}{2v}}.$$

Using equation (2.38) in the second line and equation (2.24) in the third,

$$1\;\boxed{}_g = \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{4v}}}\sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} (-1)^{\frac{c+c'}{2}} S_{r,s;a,\frac{c\overline{k}}{2v}} M_{a,\frac{c\overline{k}}{2v};a',\frac{c'\overline{k}}{2v}} S^*_{a',\frac{c'\overline{k}}{2v};t,u}\chi_{rs}\chi^*_{tu}$$

---

modulo $\frac{v^2}{2\overline{k}}$. Recall that the z parameter in each of the minimal partition functions given in section 3.2 is defined modulo some integer.

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$$
= \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{4v}}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} S_{r,s+v;\,a,\frac{c\overline{k}}{2v}} M_{a,\frac{c\overline{k}}{2v};\,a',\frac{c'\overline{k}}{2v}} S^*_{a',\frac{c'\overline{k}}{2v};\,t,u-v} \chi_{rs}\chi^*_{tu}
$$

$$
= \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} M_{r,s+v;\,t,u-v}\ \chi_{rs}\chi^*_{tu}
$$

$$
= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}\ \chi_{a,c-v}\chi^*_{a',c'+v}.
$$

Finally, using equation (2.18), (2.12) and then (2.26),

$$
g\,\boxed{}_{g} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}\, e^{2\pi i(h_{a,c-v}-h_{a',c'+v})}\chi_{a,c-v}\chi^*_{a',c'+v},
$$

$$
= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}\, e^{\frac{2\pi i v}{k}\left(\frac{c+c'}{2}\right)} e^{2\pi i(h_{a,c}-h_{a',c'})}\chi_{a,c-v}\chi^*_{a',c'+v},
$$

$$
= \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} M_{a,c;\,a',c'}\, e^{\frac{2\pi i v}{k}\left(\frac{c+c'}{2}\right)}\chi_{a,c-v}\chi^*_{a',c'+v}.
$$

Since $c\equiv c'\bmod 2$ whenever $M_{a,\frac{c\overline{k}}{2v};\,a',\frac{c'\overline{k}}{2v}}\neq 0$ we can read off

$$
Z^{\mathrm{inv}} = \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{4v}}} M_{a,\frac{c\overline{k}}{2v};\,a',\frac{c'\overline{k}}{2v}}\delta(c+c'\equiv 0\ \mathrm{mod}\ 4)\ \chi_{a,\frac{c\overline{k}}{2v}}\chi^*_{a',\frac{c'\overline{k}}{2v}},
$$

$$
Z^{\mathrm{twist}} = \sum_{\substack{(a,c)\in Q_k \\ (a',c')\in Q_k}} \frac{1}{2}\left(1+e^{\frac{2\pi i v}{k}\left(\frac{c+c'}{2}\right)}\right) M_{a,c+v;\,a',c'-v}\ \chi_{ac}\chi^*_{a'c'}
$$

$$
= \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{4v}}} M_{a,\left(c+\frac{2v^2}{k}\right)\frac{\overline{k}}{2v};\,a',\left(c'-\frac{2v^2}{k}\right)\frac{\overline{k}}{2v}}\delta(c+c'\equiv 0\ \mathrm{mod}\ 4)\ \chi_{a,\frac{c\overline{k}}{2v}}\chi^*_{a',\frac{c'\overline{k}}{2v}},
$$

and

$$
Z^{\mathrm{orb}} = \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{4v}}} \left(M_{a,\frac{c\overline{k}}{2v};\,a',\frac{c'\overline{k}}{2v}} + M_{a,\left(c+\frac{2v^2}{k}\right)\frac{\overline{k}}{2v};\,a',\left(c'-\frac{2v^2}{k}\right)\frac{\overline{k}}{2v}} \right)
$$

$$
\times\,\delta(c+c'\equiv 0\quad\mathrm{mod}\ 4)\,\chi_{a,\frac{c\overline{k}}{2v}}\chi^*_{a',\frac{c'\overline{k}}{2v}}
$$

$$
= \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{4v}}} \left[\delta(a\equiv c\ \mathrm{mod}\ 2)\,\delta(a'=J^{an+\frac{c+c'}{2}}a)\,\delta\left(c'\equiv cz\ \mathrm{mod}\ \frac{2v^2}{k}\right)\right.
$$

$$+ \delta(a \equiv c + 1 \mod 2)\,\delta(a' = J^{an + \frac{c+c'}{2}} a)\,\delta\left(c' \equiv cz \mod \frac{2v^2}{\overline{k}}\right)\Bigg]$$

$$\times\, \delta(c + c' \equiv 0 \mod 4)\, \chi_{a, \frac{c\overline{k}}{2v}} \chi^*_{a', \frac{c'\overline{k}}{2v}}$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{4v}}} \delta(a' = J^{an} a)\,\delta\left(c' \equiv cz \mod \frac{2v^2}{\overline{k}}\right)$$

$$\times\, \delta(c + c' \equiv 0 \mod 4)\chi_{a, \frac{c\overline{k}}{2v}} \chi^*_{a', \frac{c'\overline{k}}{2v}},$$

where in the second line we used the explicit expression for $\widetilde{M}^{2,1}$ given in equation (3.5).

Define $z' = z + \left(\frac{2v^2}{\overline{k}}\right)^2 (3 - z) \in \mathbb{Z}_{\frac{8v^2}{\overline{k}}}$. Then, since $z' \equiv z \mod \frac{2v^2}{\overline{k}}$ and $z' \equiv -1 \mod 4$, we see that the conditions $c' \equiv cz \mod \frac{2v^2}{\overline{k}}$ and $c + c' \equiv 0 \mod 4$ are equivalent to $c' \equiv cz' \mod \frac{8v^2}{\overline{k}}$. Thus

$$Z^{\mathrm{orb}} = \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{4v}}} \delta(a' = J^{an} a)\,\delta\left(c' \equiv cz' \mod \frac{8v^2}{\overline{k}}\right) \chi_{a, \frac{c\overline{k}}{2v}} \chi^*_{a', \frac{c'\overline{k}}{2v}}$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{2v'}}} \delta(a' = J^{an} a)\,\delta\left(c' \equiv cz' \mod \frac{2v'^2}{\overline{k}}\right) \chi_{a, \frac{c\overline{k}}{v'}} \chi^*_{a', \frac{c'\overline{k}}{v'}}.$$

where we have written $v' = 2v$. It remains to check that $\frac{\overline{k}}{2v'}, \frac{v'^2}{\overline{k}} \in \mathbb{Z}$ and that, since z' is odd, $\frac{\overline{k}(z'^2 - 1)}{2v'^2}$ is even. Thus we see from (3.4) that

$$Z^{\mathrm{orb}} = \widetilde{M}^{2,0}[v', z', n], \quad \text{where } v' = 2v, \ z' = \left(\frac{2v^2}{\overline{k}}\right)^2 (3 - z).$$

Orbifoldings between minimal families: $\widetilde{M}^{2,2} \to \widetilde{M}^{2,0}$

Constructing an orbifolding $\mathcal{O}^6$ from $\widetilde{M}^{2,2}$ to $\widetilde{M}^{2,0}$ is a little more straightforward. Fixing some k such that $4|\overline{k}$, we define a $\mathbb{Z}_2$ action by $g \cdot \chi_{ac}\chi^*_{a'c'} = (-1)^c \chi_{ac}\chi^*_{a'c'}$. We claim that for $m, n \in \{0, 1\}$

$$g^m \boxed{}_{g^n} = \sum_{\substack{(a,c) \in Q_k \\ (a',c') \in Q_k}} (-1)^{cm} M_{a,c+n\overline{k};\,a',c'} \chi_{ac}\chi^*_{a'c'}.$$

This is evidently correct when $n = 0$ and it is not hard to check that it transforms correctly under the S and T transformations. It yields

$$Z^{\mathrm{orb}} = \sum_{\substack{(a,c) \in Q_k \\ (a',c') \in Q_k}} \left[M_{a,c;\,a',c'} + M_{a,c+\overline{k};\,a',c'}\right] \delta(c \equiv 0 \mod 2)\chi_{ac}\chi^*_{a'c'}. \qquad (4.1)$$

Choose some v, z such that $\frac{\bar{k}}{v}$ is odd and $\frac{v^2}{\bar{k}}, \frac{\bar{k}(z^2-1)}{4v^2} \in \mathbb{Z}$. Then we can apply $\mathcal{O}^6$ to the physical invariant $M \equiv \widetilde{M}^{2,2}[v, z, n, m]$ by plugging equation (3.6) into equation (4.1):

$$Z^{\text{orb}} = \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{2v}}} \left[\delta(a' = J^{an+cm}a)\delta\left(c' \equiv cz + \frac{(a+c)mv^2}{\bar{k}} \mod \frac{2v^2}{\bar{k}}\right) \right.$$

$$+ \delta(a' = J^{an+cm}a)\delta\left(c' \equiv (c+v)z + \frac{(a+c)mv^2}{\bar{k}} \mod \frac{2v^2}{\bar{k}}\right) \Bigg]$$

$$\times \delta(c \equiv 0 \mod 2)\chi_{a,\frac{c\bar{k}}{v}}\chi^*_{a,\frac{c'\bar{k}}{v}}$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{2v}}} \delta(a' = J^{an}a)\delta\left(c' \equiv cz \mod \frac{v^2}{\bar{k}}\right) \delta(c \equiv 0 \mod 2)\chi_{a,\frac{c\bar{k}}{v}}\chi^*_{a,\frac{c'\bar{k}}{v}} .$$

Since z is odd and $\frac{v^2}{\bar{k}}$ is even, we can absorb the condition $c \equiv c' \equiv 0 \mod 2$ into the range of definition of c and c':

$$Z^{\text{orb}} = \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_v}} \delta(a' = J^{an}a)\, \delta\left(c' \equiv cz \mod \frac{v^2}{2\bar{k}}\right) \chi_{a,\frac{2c\bar{k}}{v}}\chi^*_{a,\frac{2c'\bar{k}}{v}}$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{2v'}}} \delta(a' = J^{an}a)\, \delta\left(c' \equiv cz \mod \frac{2v'^2}{\bar{k}}\right) \chi_{a,\frac{c\bar{k}}{v'}}\chi^*_{a,\frac{c'\bar{k}}{v'}}$$

$$= \widetilde{M}^{2,0}[v', z, n]$$

(see equation (3.4)), where we have set $2v' = v$ and we understand z to be defined modulo $\frac{2v'^2}{\bar{k}}$. This completes the proof of the assertion that all simple current invariants with $4|\bar{k}$ can be mapped via an orbifolding into the family $\widetilde{M}^{2,0}$.

4.5.3   The exceptional case $k = 10$

It remains to show that the family $\widetilde{E}_2^{10}$ can be mapped via an orbifolding into the family $\widetilde{E}_1^{10}$. We simply apply the orbifolding $\mathcal{O}^6$ from the previous subsection to the exceptional invariant $\widetilde{E}_2^{10}[12, v, 0, m]$: substituting (3.12) into (4.1) we obtain

$$Z^{\text{orb}} = \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{24}}} \delta(E^{10}_{J^{cm}a,a'} = 1)\left[\delta(c' \equiv cz + 12(a+c)m \mod 12)\right.$$

$$+ \delta(c' \equiv (c+12)z + 12(a+c)m \mod 12)\Big]\,\delta(c \equiv 0 \mod 2)\chi_{ac}\chi^*_{a'c'}$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{24}}} \delta(E^{10}_{a,a'} = 1)\delta(c' \equiv cz \mod 12)\delta(c \equiv 0 \mod 2)\chi_{ac}\chi^*_{a'c'}$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{12}}} \delta(E_{a,a'}^{10} = 1)\delta(c' \equiv cz \mod 6)\chi_{a,2c}\chi_{a',2c'}^*$$

$$= \widetilde{E}_1^{10}[6,z].$$

This completes the proof of Proposition (4.5.1).

4.6 Orbifoldings within minimal families – controlling the v parameter

Our overall aim is to find orbifoldings within the families $\widetilde{M}^0$, $\widetilde{M}^{4,2}$, $\widetilde{M}^{2,0}$, $\widetilde{E}_1^{10}$, $\widetilde{E}_2^{16}$ and $\widetilde{E}^{28}$ which map all members down to a specific physical invariant. Since we already have control of the $\mathbb{Z}_2$ parameters (labelled by n or x) via the orbifoldings $\mathcal{O}^1$ and $\mathcal{O}^2$, in this section we concentrate on trying to control the parameter v.

4.6.1 A useful formula

We begin by considering a general orbifolding by a group $\mathbb{Z}_\beta$, acting on the $\widehat{\mathfrak{u}(1)}$ label c on the left-hand side.[6] Fix a physical invariant M in one of the above families and take the largest integer α such that

$$M_{a,c;\,a'c'} \neq 0 \quad \Rightarrow \quad c,c' \in \alpha\mathbb{Z}.$$

For these families, $\frac{\overline{k}}{\alpha^2} \in \mathbb{Z}$. We will define a $\mathbb{Z}_\beta$-orbifolding $\mathcal{O}^7$ for some integer β satisfying $\beta|\frac{\overline{k}}{\alpha^2}$. Let $\mathbb{Z}_\beta = \langle g \rangle$ act on the states of M via $g \cdot \chi_{a,\alpha c}\chi_{a',\alpha c'}^* = e^{\frac{2\pi ic}{\beta}}\chi_{a,\alpha c}\chi_{a',\alpha c'}^*$. We claim that the result is

$$g^m \boxed{}_{g^n} = \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha c;\,a',\alpha c'}\; e^{\frac{2\pi im}{\beta}\left(c-\frac{n\overline{k}}{\alpha^2\beta}\right)}\chi_{a,\alpha\left(c-\frac{2n\overline{k}}{\alpha^2\beta}\right)}\chi_{a',\alpha c'}^*.$$

It is easy to see this is correct when $n = 0$. We must check that it behaves correctly under the action of the S- and T-transformations:

$$S \cdot g^m \boxed{}_{g^n} = \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} S_{r,s;\,a,\alpha\left(c-\frac{2n\overline{k}}{\alpha^2\beta}\right)}M_{a,\alpha c;\,a',\alpha c'}S_{a',c';\,tu}^*$$

$$\times\; e^{\frac{2\pi im}{\beta}\left(c-\frac{n\overline{k}}{\alpha^2\beta}\right)}\chi_{rs}\chi_{tu}^*$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} S_{r,s;\,a,\alpha c}M_{a,\alpha c;\,a',\alpha c'}S_{a',\alpha c';\,tu}^*$$

---

[6]Note that the remaining families are all symmetric, so it doesn't matter whether we act on the left- or right-hand sides

$$\times\, e^{\frac{2\pi imc}{\beta}} e^{-\frac{2\pi ins}{\alpha\beta}} e^{-\frac{2\pi imn\overline{k}}{\alpha^2\beta^2}} \chi_{rs}\chi_{tu}^*$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} S_{r,s+\frac{2m\overline{k}}{\alpha\beta};\,a,\alpha c}\, M_{a,\alpha c;\,a',\alpha c'}\, S^*_{a',\alpha c';\,tu}$$

$$\times\, e^{-\frac{2\pi ins}{\alpha\beta}} e^{-\frac{2\pi imn\overline{k}}{\alpha^2\beta^2}} \chi_{rs}\chi_{tu}^*$$

$$= \sum_{\substack{(r,s)\in Q_k \\ (t,u)\in Q_k}} M_{r,s+\frac{2m\overline{k}}{\alpha\beta};\,tu}\, e^{-\frac{2\pi in}{\beta}\left(\frac{s}{\alpha}+\frac{m\overline{k}}{\alpha^2\beta}\right)} \chi_{rs}\chi_{tu}^*$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha\left(c+\frac{2m\overline{k}}{\alpha^2\beta}\right);\,a',\alpha c'}\, e^{-\frac{2\pi in}{\beta}\left(c+\frac{m\overline{k}}{\alpha^2\beta}\right)} \chi_{a,\alpha c}\chi_{a',\alpha c'}^*$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha c;\,a',\alpha c'}\, e^{-\frac{2\pi in}{\beta}\left(c-\frac{m\overline{k}}{\alpha^2\beta}\right)} \chi_{a,\alpha\left(c+\frac{2m\overline{k}}{\alpha^2\beta}\right)}\chi_{a',\alpha c'}^*$$

$$= g^{-n}\;\boxed{}_{\,g^m},$$

where, as usual, we used equations (2.38) in the second and third lines and equation (2.24) in the fourth. Just as in previous calculations, we use equation (2.18), (2.12) and then (2.26) to compute

$$T\cdot g^m\,\boxed{}_{\,g^n} = \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha c;\,a',\alpha c'}\, e^{\frac{2\pi im}{\beta}\left(c-\frac{n\overline{k}}{\alpha^2\beta}\right)}$$

$$\times\, e^{2\pi i\left(h_{a,\alpha\left(c-\frac{2n\overline{k}}{\alpha^2\beta}\right)}-h_{a',\alpha c'}\right)} \chi_{a,\alpha\left(c-\frac{2n\overline{k}}{\alpha^2\beta}\right)}\chi_{a',\alpha c'}^*$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha c;\,a',\alpha c'}\, e^{\frac{2\pi im}{\beta}\left(c-\frac{n\overline{k}}{\alpha^2\beta}\right)}$$

$$\times\, e^{\frac{2\pi in}{\beta}\left(c-\frac{n\overline{k}}{\alpha^2\beta}\right)} e^{2\pi i(h_{a,\alpha c}-h_{a',\alpha c'})} \chi_{a,\alpha\left(c-\frac{2n\overline{k}}{\alpha^2\beta}\right)}\chi_{a',\alpha c'}^*$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha c;\,a',\alpha c'}\, e^{\frac{2\pi i(m+n)}{\beta}\left(c-\frac{n\overline{k}}{\alpha^2\beta}\right)} \chi_{a,\alpha\left(c-\frac{2n\overline{k}}{\alpha^2\beta}\right)}\chi_{a',\alpha c'}^*$$

$$= g^{m+n}\,\boxed{}_{\,g^n},$$

as required, and thus the boxes span a representation of $\mathrm{SL}_2(\mathbb{Z})$. We can now calculate the $\mathbb{Z}_\beta$-invariant g^N-twisted sectors for $N = 0, \ldots, \beta - 1$:

$$Z^N = \frac{1}{\beta} \sum_{M=0,\ldots,\beta-1} g^M \boxed{}_{g^N}$$

$$= \frac{1}{\beta} \sum_{\substack{M=0,\ldots,\beta-1 \\ a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha c;\, a',\alpha c'}\; e^{\frac{2\pi i M}{\beta}\left(c - \frac{N\overline{k}}{\alpha^2\beta}\right)} \chi_{a,\alpha\left(c - \frac{2N\overline{k}}{\alpha^2\beta}\right)} \chi^*_{a',\alpha c'}$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha c;\, a',\alpha c'} \left[\frac{1}{\beta} \sum_{M=0,\ldots,\beta-1} e^{\frac{2\pi i M}{\beta}\left(c - \frac{N\overline{k}}{\alpha^2\beta}\right)}\right] \chi_{a,\alpha\left(c - \frac{2N\overline{k}}{\alpha^2\beta}\right)} \chi^*_{a',\alpha c'}$$

$$= \sum_{\substack{a,a'=0,\ldots,k \\ c,c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha c;\, a',\alpha c'}\; \delta\left(c \equiv \frac{N\overline{k}}{\alpha^2\beta} \mod \beta\right) \chi_{a,\alpha\left(c - \frac{2N\overline{k}}{\alpha^2\beta}\right)} \chi^*_{a',\alpha c'}$$

$$= \sum_{\substack{a=0,\ldots,k \\ s \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}}}} \sum_{\substack{a'=0,\ldots,k \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha\left(\beta s + \frac{N\overline{k}}{\alpha^2\beta}\right);\, a',\alpha c'} \chi_{a,\alpha\left(\beta s - \frac{N\overline{k}}{\alpha^2\beta}\right)} \chi^*_{a',\alpha c'},$$

where in the last line we wrote $c = \beta s + \frac{N\overline{k}}{\alpha^2\beta}$ where s is defined modulo $\frac{2\overline{k}}{\alpha\beta}$. The partition function of the orbifolding $\mathcal{O}^7$ is then given by the sum over the twisted sectors:

$$Z^{\mathrm{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{s \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha\left(s\beta + \frac{N\overline{k}}{\alpha^2\beta}\right);\, a',\alpha c'} \chi_{a,\alpha\left(s\beta - \frac{N\overline{k}}{\alpha^2\beta}\right)} \chi^*_{a',\alpha c'}. \tag{4.2}$$

If it happens that $\frac{\overline{k}}{\alpha^2\beta^2} \in \mathbb{Z}$ then the above simplifies to

$$Z^{\mathrm{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{s \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha\beta\left(s + \frac{N\overline{k}}{\alpha^2\beta^2}\right);\, a',\alpha c'} \chi_{a,\alpha\beta\left(s - \frac{N\overline{k}}{\alpha^2\beta^2}\right)} \chi^*_{a',\alpha c'}$$

$$= \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha\beta\left(c + \frac{2N\overline{k}}{\alpha^2\beta^2}\right);\, a',\alpha c'}\; \chi_{a,\alpha\beta c}\; \chi^*_{a',\alpha c'}. \tag{4.3}$$

4.6.2 Controlling the parameter v

The aim of this subsection is to find an orbifolding which sends the parameter v to the smallest possible value it can take:

Proposition 4.6.1. *Fix k and let M be a level k physical invariant in one of the families $\widetilde{M}^0$, $\widetilde{M}^{4,2}$, $\widetilde{M}^{2,0}$, $\widetilde{E}_1^{10}$, $\widetilde{E}_2^{16}$ or $\widetilde{E}^{28}$ with parameters $(v, z, *)$ where $*$ is either n or x. Then we can map M via an orbifolding to a minimal partition function within the same family with parameters $(v', z, *)$ where v' is smallest possible value of v allowed.*

In the exceptional cases $\widetilde{E}_1^{10}$ and $\widetilde{E}^{28}$ there is only one allowed value of v, so the proposition is trivial in these cases; they are included for completeness.

We shall prove the claim using the orbifoldings constructed in section 4.6.1. The idea is to map by the orbifolding with the largest possible value of β that satisfies $\frac{\overline{k}}{\alpha^2\beta^2} \in \mathbb{Z}$.

k odd

Let k be an odd integer and let M be a physical invariant at level k with parameters (v, z, n) (see (3.3)). Write $\overline{k} = \prod_{i=1}^{l} p_i^{2a_i + \delta_i}$ where the p_i are distinct odd primes and $\delta_i \in \{0, 1\}$ for each $i = 1, \ldots, l$. Similarly write $v = \prod_{i=1}^{l} p_i^{b_i}$ for some integers b_i. The conditions $\frac{\overline{k}}{v}, \frac{v^2}{\overline{k}} \in \mathbb{Z}$ are equivalent to $a_i + \delta_i \le b_i \le 2a_i + \delta_i$, so we can define an integer $\beta = \prod_{i=1}^{l} p_i^{b_i - a_i - \delta_i}$.

As in the previous section we find the biggest integer α such that $M_{a,c;\,a'c'} \neq 0 \Rightarrow c, c' \in \alpha\mathbb{Z}$; here, $\alpha = \frac{\overline{k}}{v} = \prod_{i=1}^{l} p_i^{2a_i - b_i + \delta_i}$. With these values we see that $\frac{\overline{k}}{\alpha^2\beta^2} = \prod_{i=1}^{l} p_i^{\delta_i} \in \mathbb{Z}$, so we can perform $\mathcal{O}^7$, the $\mathbb{Z}_\beta$ orbifolding from the previous subsection, on M using the simplified formula in equation (4.3).

$$
Z^{\mathrm{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha\beta\left(c + \frac{2N\overline{k}}{\alpha^2\beta^2}\right);\,a',\alpha c'}\, \chi_{a,\alpha\beta c}\, \chi^*_{a',\alpha c'}
$$

$$
= \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \delta(a' = J^{n(a+c)}a)\, \delta(c' \equiv c + n(a+c) \mod 2)
$$

$$
\times\, \delta\left(c' \equiv 2z\beta\left(c + \frac{2N\overline{k}}{\alpha^2\beta^2}\right) \mod \frac{\overline{k}}{\alpha^2}\right) \chi_{a,\alpha\beta c}\, \chi^*_{a',\alpha c'}
$$

$$
= \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}}}} \delta(a' = J^{n(a+c)}a)\, \delta(c' \equiv c + n(a+c) \mod 2)
$$

$$
\times\, \delta\left(c' \equiv 2z\left(c + \frac{2N\overline{k}}{\alpha^2\beta^2}\right) \mod \frac{\overline{k}}{\alpha^2\beta}\right) \chi_{a,\alpha\beta c}\, \chi^*_{a',\alpha\beta c'},
$$

where in the last line we implement the fact that the summand vanishes unless $c' \equiv 0 \mod \beta$. Now let us evaluate $\sum_{N\in\mathbb{Z}_\beta} \delta(x \equiv 4zN\frac{\overline{k}}{\alpha^2\beta^2} \mod \frac{\overline{k}}{\alpha^2\beta})$. From the condition $(2z+1)(2z-1) \equiv 0 \mod \frac{\overline{k}}{\alpha^2}$ and the fact that β divides $\frac{\overline{k}}{\alpha^2}$ we

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see that $\mathrm{hcf}(2z, \beta) = 1$. β is odd, so in fact $\mathrm{hcf}(4z, \beta) = 1$. It follows that $4zN$ mod β cycles over the values $1, \ldots, \beta$ as N runs over $1, \ldots, \beta$. Thus

$$
\sum_{N \in \mathbb{Z}_\beta} \delta \left(x \equiv 4zN \frac{\overline{k}}{\alpha^2 \beta^2} \quad \mathrm{mod} \ \frac{\overline{k}}{\alpha^2 \beta} \right) = \sum_{N \in \mathbb{Z}_\beta} \delta \left(x \equiv N \frac{\overline{k}}{\alpha^2 \beta^2} \quad \mathrm{mod} \ \frac{\overline{k}}{\alpha^2 \beta} \right)
$$

$$
= \delta \left(x \equiv 0 \quad \mathrm{mod} \ \frac{\overline{k}}{\alpha^2 \beta^2} \right).
$$

Plugging this with $x = c' - 2zc$ into the main calculation gives

$$
Z^{\mathrm{orb}} = \sum_{\substack{a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}}}} \delta(a' = J^{n(a+c)}a) \, \delta(c' \equiv c + n(a+c) \ \mathrm{mod} \ 2)
$$

$$
\times \delta \left(c' \equiv 2zc \ \mathrm{mod} \ \frac{\overline{k}}{\alpha^2 \beta^2} \right) \chi_{a,\alpha\beta c} \chi^*_{a',\alpha\beta c'}
$$

$$
= \sum_{\substack{a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c \in \mathbb{Z}_{2v'} \\ c' \in \mathbb{Z}_{2v'}}} \delta(a' = J^{n(a+c)}a) \, \delta(c' \equiv c + n(a+c) \ \mathrm{mod} \ 2)
$$

$$
\times \delta \left(c' \equiv 2zc \ \mathrm{mod} \ \frac{v'^2}{\overline{k}} \right) \chi_{a,\frac{c\overline{k}}{v'}} \chi^*_{a',\frac{c'\overline{k}}{v'}}
$$

$$
= \widetilde{M}^0[v', z, n]
$$

where we have defined $v' = \frac{\overline{k}}{\alpha\beta} = \prod_{i=1}^l p_i^{a_i + \delta_i}$. Note that this is the smallest divisor v' of $\overline{k}$ satisfying $\frac{v'^2}{\overline{k}} \in \mathbb{Z}$. Thus we have successfully minimised the parameter v.

4 divides k

The $\widetilde{M}^{4,2}$ case is similar. Fix k such that $4|k$ and choose a minimal $\widetilde{M}^{4,2}$ invariant with parameters (v, z, x) (see equation (3.9)). We write $\overline{k} = 2 \prod_{i=1}^l p_i^{2a_i + \delta_i}$ with p_i distinct odd primes and $\delta_i \in \{0, 1\}$ and write $v = \prod_{i=1}^l p_i^{b_i}$ for some integers b_i. This time $\alpha = \frac{\overline{k}}{2v} = \prod_{i=1}^l p_i^{2a_i - b_i + \delta_i}$ and we set $\beta = \prod_{i=1}^l p_i^{b_i - a_i - \delta_i}$. Again we find that $\frac{\overline{k}}{\alpha^2 \beta^2} = \prod_{i=1}^l p_i^{\delta_i} \in \mathbb{Z}$ so we can use equations (4.3) and (3.9) to calculate the $\mathbb{Z}_\beta$ orbifolding:

$$
Z^{\mathrm{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha\beta \left(c + \frac{2N\overline{k}}{\alpha^2 \beta^2} \right); a', \alpha c'} \, \chi_{a,\alpha\beta c} \, \chi^*_{a',\alpha c'}
$$

$$
= \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \delta(a \equiv a' \equiv 0 \ \mathrm{mod} \ 2) \left(\delta(a' = a) + \delta(a' = Ja) \right)
$$

$$\times \delta \left(c' \equiv \beta x \left(c + \frac{2N\overline{k}}{\alpha^2 \beta^2} \right) \mod 4 \right)$$

$$\times \delta \left(c' \equiv \beta z \left(c + \frac{2N\overline{k}}{\alpha^2 \beta^2} \right) \mod \frac{\overline{k}}{2\alpha^2} \right) \chi_{a,\alpha\beta c} \, \chi^*_{a',\alpha c'}$$

$$= \sum_{\substack{a=0,\ldots,k \\ a'=0,\ldots,k}} \delta(a \equiv a' \equiv 0 \mod 2) \left(\delta(a' = a) + \delta(a' = Ja) \right)$$

$$\times \sum_{c,c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}}} \delta \left(c' \equiv cx \mod 4 \right)$$

$$\times \sum_{N=0,\ldots,\beta-1} \delta \left(c' \equiv z \left(c + \frac{2N\overline{k}}{\alpha^2 \beta^2} \right) \mod \frac{\overline{k}}{2\alpha^2 \beta} \right) \chi_{a,\alpha\beta c} \, \chi^*_{a',\alpha\beta c'}.$$

We use the condition $z^2 - 1 \equiv 0 \mod \frac{\overline{k}}{2\alpha^2}$ and the fact that β divides $\frac{\overline{k}}{2\alpha^2}$ to deduce that $\mathrm{hcf}(z,\beta) = 1$. Since β is odd, $\mathrm{hcf}(4z,\beta) = 1$, and so

$$\sum_{N=0,\ldots,\beta-1} \delta \left(x \equiv 4z \frac{N\overline{k}}{\alpha^2 \beta^2} \mod \frac{\overline{k}}{2\alpha^2 \beta} \right) = \sum_{N=0,\ldots,\beta-1} \delta \left(x \equiv \frac{N\overline{k}}{\alpha^2 \beta^2} \mod \frac{\overline{k}}{2\alpha^2 \beta} \right)$$

$$= \delta \left(x \equiv 0 \mod \frac{\overline{k}}{2\alpha^2 \beta^2} \right).$$

Setting $x = c' - cz$ we find

$$Z^{\mathrm{orb}} = \sum_{\substack{a=0,\ldots,k \\ a'=0,\ldots,k}} \delta(a \equiv a' \equiv 0 \mod 2) \left(\delta(a' = a) + \delta(a' = Ja) \right)$$

$$\times \sum_{\substack{c \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}}}} \delta \left(c' \equiv cx \mod 4 \right) \times \delta \left(c' \equiv cz \mod \frac{\overline{k}}{2\alpha^2 \beta^2} \right) \chi_{a,\alpha\beta c} \, \chi^*_{a',\alpha\beta c'}$$

$$= \sum_{\substack{a=0,\ldots,k \\ a'=0,\ldots,k}} \delta(a \equiv a' \equiv 0 \mod 2) \left(\delta(a' = a) + \delta(a' = Ja) \right)$$

$$\times \sum_{\substack{c \in \mathbb{Z}_{4v'} \\ c' \in \mathbb{Z}_{4v'}}} \delta \left(c' \equiv cx \mod 4 \right) \times \delta \left(c' \equiv cz \mod \frac{2v'^2}{\overline{k}} \right) \chi_{a,\frac{c\overline{k}}{2v'}} \, \chi^*_{a',\frac{c'\overline{k}}{2v'}}$$

$$= \widetilde{M}^{4,2}[v', z, x],$$

where we have defined $v' = \frac{\overline{k}}{2\alpha\beta} = \prod_{i=1}^{l} p_i^{a_i + \delta_i}$. This shows that for a fixed k we can always send v to its smallest possible value in the family $\widetilde{M}^{4,2}$.

4 divides $\overline{k}$

Finally we address the case when k satisfies $4|\overline{k}$. Fix such a k and a $\widetilde{M}^{2,0}$ physical invariant M with parameters (v, z, n) (see equation (3.4)). As before write $\overline{k} = \prod_{i=0}^{l} p_i^{2a_i + \delta_i}$ where $p_0 = 2$ and the p_i are distinct odd primes for $i \geq 1$, $\delta_i \in \{0, 1\}$ for each $i = 0, \ldots, l$ and $a_0 \geq 1$. For this physical invariant $\alpha = \frac{\overline{k}}{v} = \prod_{i=0}^{l} p_i^{2a_i + \delta_i - b_i}$ and we set $\beta = \prod_{i=0}^{l} p_i^{b_i - a_i - \delta_i}$, which is bound to be an integer by the condition $\frac{v^2}{\overline{k}} \in \mathbb{Z}$. We find once again that $\frac{\overline{k}}{\alpha^2 \beta^2} = \prod_{i=0}^{l} p_i^{\delta_i} \in \mathbb{Z}$ and so we can use the formula (4.3) to calculate the $\mathbb{Z}_\beta$ orbifold of M. Substituting in equation (3.4) we find

$$
Z^{\text{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \delta(a' = J^{an + c\beta y} a)
$$

$$
\times \delta \left(c' \equiv \beta z \left(c + \frac{2N\overline{k}}{\alpha^2 \beta^2} \right) + \frac{ay\overline{k}}{\alpha^2} \mod \frac{2\overline{k}}{\alpha^2} \right) \chi_{a,\alpha\beta c} \, \chi^*_{a',\alpha c'}
$$

$$
= \sum_{\substack{a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}}}} \delta(a' = J^{an + c\beta y} a)
$$

$$
\times \delta \left(c' \equiv cz + a\beta y \frac{\overline{k}}{\alpha^2 \beta^2} \mod \frac{2\overline{k}}{\alpha^2 \beta^2} \right) \chi_{a,\alpha\beta c} \, \chi^*_{a',\alpha\beta c'}
$$

by the same arguments as in the previous two cases. The only subtlety here is why βy is even: clearly if β is even, we are done. If β is odd then $b_0 - a_0 = \delta_0 \in \{0, 1\}$. Now $\beta y = \frac{z^2 - 1}{2 \prod_{i=0}^{l} p_i^{b_i - a_i}}$, so the denominator is even, but not a multiple of 8. That forces z to be odd, and so $z^2 - 1 \equiv 0 \mod 8$ and βy is even. Thus

$$
Z^{\text{orb}} = \sum_{\substack{a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}}}} \delta(a' = J^{an} a) \, \delta \left(c' \equiv cz \mod \frac{2\overline{k}}{\alpha^2 \beta^2} \right) \chi_{a,\alpha\beta c} \, \chi^*_{a',\alpha\beta c'}
$$

$$
= \sum_{\substack{a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c \in \mathbb{Z}_{2v'} \\ c' \in \mathbb{Z}_{2v'}}} \delta(a' = J^{an} a) \, \delta \left(c' \equiv cz \mod \frac{2v'^2}{\overline{k}} \right) \chi_{a, \frac{c\overline{k}}{v'}} \, \chi^*_{a', \frac{c'\overline{k}}{v'}}
$$

$$
= \widetilde{M}^{2,0}[v', z, n]
$$

where we have defined $v' = \frac{\overline{k}}{\alpha\beta} = \prod_{i=0}^{l} p_i^{a_i + \delta_i}$. This completes the proof of proposition 4.6.1 for the simple current invariants.

It remains to check the case $\widetilde{E}_2^{16}$. Let M be the physical invariant $\widetilde{E}_2^{16}$ with parameters $(v = 9, z, x)$. Then $\alpha = 1$ and we choose $\beta = 3$ so that $\frac{\overline{k}}{\alpha^2 \beta^2} = 2 \in \mathbb{Z}$.

It is then straight-forward to apply equation (4.3) to find

$$
\begin{aligned}
Z^{\mathrm{orb}} &= \sum_{\substack{N=0,1,2 \\ a,a'=0,\dots,16}} \sum_{\substack{c\in\mathbb{Z}_{12} \\ c'\in\mathbb{Z}_{36}}} M_{a,3(c+4N);\,a'c'} \; \chi_{a,3c} \; \chi^*_{a'c'} \\
&= \sum_{\substack{N=0,1,2 \\ a,a'=0,\dots,16}} \sum_{\substack{c\in\mathbb{Z}_{12} \\ c'\in\mathbb{Z}_{36}}} \delta(E^{16}_{a,a'}=1)\delta(c'\equiv 3z(c+4N)\ \mathrm{mod}\ 9) \\
&\qquad \times \delta(c'\equiv 3cx\ \mathrm{mod}\ 4)\ \chi_{a,3c}\ \chi^*_{a'c'} \\
&= \sum_{\substack{a,a'=0,\dots,16}} \sum_{\substack{c\in\mathbb{Z}_{12} \\ c'\in\mathbb{Z}_{12}}} \delta(E^{16}_{a,a'}=1)\delta(c'\equiv cx\ \mathrm{mod}\ 4)\ \chi_{a,3c}\ \chi^*_{a'3c'} \\
&= \widetilde{E}^{16}_2[3,1,x].
\end{aligned}
$$

4.7 Orbifoldings within minimal families – controlling the z parameter

Now that we can map via orbifoldings any minimal partition function into a particular family with a particular value of v, it remains to find an orbifolding which lets us control the parameter z. We will prove

Proposition 4.7.1. *Fix k and let M be a level k physical invariant in one of the families $\widetilde{M}^0$, $\widetilde{M}^{4,2}$, $\widetilde{M}^{2,0}$, $\widetilde{E}^{10}_1$, $\widetilde{E}^{16}_2$ or $\widetilde{E}^{28}$ with parameters $(v,z,*)$ where v is as small as possible and $*$ is either n or x. Then we can map M via orbifoldings to a physical invariant within the same family with parameters $(v,z',*)$ where*

$$
\begin{cases}
2z \equiv 1 \mod \frac{v^2}{k} & \text{for odd } k, \\
z \equiv 1 \mod \frac{2v^2}{k} & \text{otherwise.}
\end{cases}
$$

When v is minimised in the family $\widetilde{E}^{16}_2$ then z is forced to be 1, so the statement is trivial is this case; it is included in the proposition only for completeness.

The proof is similar for each family of simple current invariants, so we do the odd k case in detail and then go through the other two simple current cases a little more quickly. Finally we will tackle the exceptional cases.

k odd

Let k be odd and let M be a level k physical invariant with parameters (v,z,n) where v is as small as possible (see (3.3)). Write $\overline{k} = \prod_{i=1}^{l} p_i^{2a_i+1} \prod_{j=1}^{m} q_j^{2b_j}$ where the p_i and q_j are mutually distinct odd primes. Then since v is the smallest solution to $\frac{\overline{k}}{v}, \frac{v^2}{\overline{k}} \in \mathbb{Z}$, we must have $v = \prod_{i=1}^{l} p_i^{a_i+1} \prod_{j=1}^{m} q_j^{b_j}$ and therefore $\frac{v^2}{\overline{k}} = \prod_{i=1}^{l} p_i$. Now z is defined to be a solution to $4z^2 - 1 \equiv 0$

mod $\frac{v^2}{k}$. So we have

$$(2z + 1)(2z - 1) \equiv 0 \mod \prod_{i=1}^{l} p_i.$$

But since a given odd prime cannot divide both $2z+1$ and $2z-1$, it is equivalent to say that there must exist a partition $\{p_{i_1}, \ldots, p_{i_t}\} \cup \{p_{j_1}, \ldots, p_{j_u}\}$ of the p_i such that

$$\begin{cases} 2z + 1 \equiv 0 \mod \prod_{k=1}^{t} p_{i_k}, \\ 2z - 1 \equiv 0 \mod \prod_{k=1}^{u} p_{j_k}. \end{cases} \tag{4.4}$$

We are trying to map this physical invariant via an orbifolding to one where z is given by the choice of partition $\{\} \cup \{p_1, \ldots, p_l\}$. So we set $\beta = \prod_{k=1}^{t} p_{i_k}$ and try to make a $\mathbb{Z}_\beta$ orbifold. Recall that the largest integer α satisfying the condition

$$M_{a,c; a',c'} \neq 0 \quad \Rightarrow \quad c, c' \in \alpha \mathbb{Z}$$

is $\alpha = \frac{\overline{k}}{v} = \prod_{i=1}^{l} p_i^{a_i} \prod_{j=1}^{m} q_j^{b_j}$. Thus $\frac{\overline{k}}{\alpha^2 \beta} = \prod_{k=1}^{u} p_{j_k} \in \mathbb{Z}$ and we can apply the orbifolding in equation (4.2). We obtain

$$Z^{\mathrm{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{s \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha\left(s\beta + \frac{N\overline{k}}{\alpha^2\beta}\right); a',\alpha c'} \chi_{a,\alpha\left(s\beta - \frac{N\overline{k}}{\alpha^2\beta}\right)} \chi^*_{a',\alpha c'}.$$

Note that $\mathrm{hcf}\left(\beta, \frac{\overline{k}}{\alpha^2\beta}\right) = 1$ so we cannot pull out any common factor in the 'c' label as we did in equation (4.3). This is as it should be, as it was that mechanism that was used to change the value of v in the previous proposition. We now substitute in equation (3.3), the defining equation of the physical invariant M, to find

$$Z^{\mathrm{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{s \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c' \in \mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \delta(a' = J^{n(a+s+N)}a)\, \delta(c' \equiv s+N+n(a+s+N) \mod 2)$$

$$\times \delta\left(c' \equiv 2z\left(s\beta + \frac{N\overline{k}}{\alpha^2\beta}\right) \mod \frac{\overline{k}}{\alpha^2}\right) \chi_{a,\alpha\left(s\beta - \frac{N\overline{k}}{\alpha^2\beta}\right)} \chi^*_{a',\alpha c'}.$$

We claim that $2z\left(s\beta + \frac{N\overline{k}}{\alpha^2\beta}\right) \equiv s\beta - \frac{N\overline{k}}{\alpha^2\beta} \mod \frac{\overline{k}}{\alpha^2}$. To prove this note that we have

$$2z\left(s\beta + \frac{N\overline{k}}{\alpha^2\beta}\right) - \left(s\beta - \frac{N\overline{k}}{\alpha^2\beta}\right) \equiv (2z+1)N\frac{\overline{k}}{\alpha^2\beta} + (2z-1)s\beta$$

$$\equiv (2z+1)N \prod_{k=1}^{u} p_{j_k} + (2z-1)s \prod_{k=1}^{t} p_{i_k}$$

$$\equiv 0 \mod \prod_{k=1}^{l} p_k$$

by equation (4.4). Substituting this back in allows us to make a simple change of variables:

$$Z^{\mathrm{orb}} = \sum_{\substack{N=0,\dots,\beta-1 \\ a=0,\dots,k \\ a'=0,\dots,k}} \sum_{\substack{s\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \delta(a' = J^{n(a+s+N)}a)\,\delta(c' \equiv s+N+n(a+s+N) \bmod 2)$$

$$\times \delta\!\left(c' \equiv \left(s\beta - \frac{N\overline{k}}{\alpha^2\beta}\right) \bmod \frac{\overline{k}}{\alpha^2}\right) \chi_{a,\alpha\left(s\beta - \frac{N\overline{k}}{\alpha^2\beta}\right)} \chi^{*}_{a',\alpha c'}$$

$$= \sum_{\substack{a=0,\dots,k \\ a'=0,\dots,k}} \sum_{\substack{c\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \delta(a' = J^{n(a+c)}a)\,\delta(c' \equiv c + n(a+c) \bmod 2)$$

$$\times \delta\!\left(c' \equiv c \bmod \frac{\overline{k}}{\alpha^2}\right)\chi_{a,\alpha c}\,\chi^{*}_{a',\alpha c'}$$

$$= \widetilde{M}^{0}[v, z', n]$$

where z' is the unique solution to $2z \equiv 1$ modulo $\frac{v^2}{k}$ as required.

4 divides k

The proof of proposition 4.7.1 in the case where $4|k$ proceeds in a very similar way to the case where k is odd. Fix a physical invariant $M \equiv \widetilde{M}^{4,2}$ with parameters (v,z,x) where v is minimal. We write $\overline{k} = 2\prod_{i=1}^{l} p_i^{2a_i+1} \prod_{j=1}^{m} q_j^{2b_j}$ with p_i, q_j mutually distinct odd primes and note that since v is minimal (see (3.9)) we must have $v = \prod_{i=1}^{l} p_i^{a_i+1} \prod_{j=1}^{m} q_j^{b_j}$ and $\frac{2v^2}{k} = \prod_{i=1}^{l} p_i$. The equation for z for $\widetilde{M}^{4,2}$ is $z^2 - 1 \equiv 0 \bmod \frac{2v^2}{k}$ so we have $(z+1)(z-1) \equiv 0 \bmod \prod_{i=1}^{l} p_i$. Equivalently, there exists a t such that, after relabelling the p_i,

$$\begin{cases} z+1 \equiv 0 \mod \prod_{i=1}^{t} p_i, \\ z-1 \equiv 0 \mod \prod_{i=t+1}^{l} p_i. \end{cases} \tag{4.5}$$

This time $\alpha = \frac{\overline{k}}{2v} = \prod_{i=1}^{l} p_i^{a_i} \prod_{j=1}^{m} q_j^{b_j}$ and again we set $\beta = \prod_{i=1}^{t} p_i$. Then we can perform the $\mathbb{Z}_\beta$ orbifolding given in equation (4.2) on M by substituting in equation (3.9):

$$Z^{\mathrm{orb}} = \sum_{\substack{N=0,\dots,\beta-1 \\ a=0,\dots,k \\ a'=0,\dots,k}} \sum_{\substack{s\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha\left(s\beta + \frac{N\overline{k}}{\alpha^2\beta}\right);\,a',\alpha c'}\,\chi_{a,\alpha\left(s\beta - \frac{N\overline{k}}{\alpha^2\beta}\right)}\chi^{*}_{a',\alpha c'}$$

$$= \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{s\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \mathcal{D}'_{a,a'}\delta(c' = (s\beta + 2N)x \mod 4)$$

$$\times \delta\left(c' \equiv \left(s\beta + \frac{N\overline{k}}{\alpha^2\beta}\right)z \mod \frac{\overline{k}}{2\alpha^2}\right)\chi_{a,\alpha\left(s\beta-\frac{N\overline{k}}{\alpha^2\beta}\right)}\chi^*_{a',\alpha c'}.$$

But equation (4.5) implies $\left(s\beta + \frac{N\overline{k}}{\alpha^2\beta}\right)z \equiv s\beta - \frac{N\overline{k}}{\alpha^2\beta} \mod \frac{\overline{k}}{2\alpha^2}$ and so we find

$$Z^{\mathrm{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{s\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \mathcal{D}'_{a,a'}\delta(c' = (s\beta + 2N)x \mod 4)$$

$$\times \delta\left(c' \equiv s\beta - \frac{N\overline{k}}{\alpha^2\beta} \mod \frac{\overline{k}}{2\alpha^2}\right)\chi_{a,\alpha\left(s\beta-\frac{N\overline{k}}{\alpha^2\beta}\right)}\chi^*_{a',\alpha c'}$$

$$= \sum_{\substack{a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \mathcal{D}'_{a,a'}\delta(c' = c \mod 4)\,\delta\left(c' \equiv c \mod \frac{\overline{k}}{2\alpha^2}\right)\chi_{a,\alpha c}\,\chi^*_{a',\alpha c'}$$

$$= \widetilde{M}^{4,2}[v,1,x]$$

as required.

4 divides $\overline{k}$

The case where 4 divides $\overline{k}$ is again very similar. Fix a physical invariant $M \equiv \widetilde{M}[v,z,n]$ where v is minimal. We write $\overline{k} = 2^{2r+\epsilon}\prod_{i=1}^{l}p_i^{2a_i+1}\prod_{j=1}^{m}q_j^{2b_j}$ with p_i, q_j mutually distinct odd primes, $r \geq 1$ and $\epsilon \in \{0,1\}$. Note that since v is minimal (see (3.4)) we must have $v = 2^{r+\epsilon}\prod_{i=1}^{l}p_i^{a_i+1}\prod_{j=1}^{m}q_j^{b_j}$ and $\frac{2v^2}{\overline{k}} = 2^{1+\epsilon}\prod_{i=1}^{l}p_i$. Since z satisfies $z^2-1 \equiv 0 \mod \frac{2v^2}{\overline{k}}$ we must have $(z+1)(z-1) \equiv 0 \mod 2^{1+\epsilon}\prod_{i=1}^{l}p_i$. Equivalently, there exists a t such that, after relabelling the p_i,

$$\begin{cases} z + 1 \equiv 0 \mod 2\prod_{i=1}^{t}p_i, \\ z - 1 \equiv 0 \mod 2\prod_{i=t+1}^{l}p_i. \end{cases} \tag{4.6}$$

We have $\alpha = \frac{\overline{k}}{v} = 2^r\prod_{i=1}^{l}p_i^{a_i}\prod_{j=1}^{m}q_j^{b_j}$ and we set $\beta = 2^{x\epsilon}\prod_{i=1}^{t}p_i$ where x is either 0 or 1 and will be specified later. Then $\frac{\overline{k}}{\alpha^2\beta} = 2^{\epsilon(1-x)}\prod_{i=t+1}^{l}p_i$ is an integer, so we may perform the $\mathbb{Z}_\beta$ orbifolding given in equation (4.2) on M by substituting in equation (3.4):

$$Z^{\mathrm{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{s\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} M_{a,\alpha\left(s\beta+\frac{N\overline{k}}{\alpha^2\beta}\right);a',\alpha c'}\,\chi_{a,\alpha\left(s\beta-\frac{N\overline{k}}{\alpha^2\beta}\right)}\chi^*_{a',\alpha c'}$$

$$= \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{s\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \delta(a'=J^{an}a)\,\delta\left(c'\equiv\left(s\beta+\frac{N\overline{k}}{\alpha^2\beta}\right)z \mod \frac{2\overline{k}}{\alpha^2}\right)$$

$$\times\,\chi_{a,\alpha}\!\left(s\beta-\tfrac{N\overline{k}}{\alpha^2\beta}\right)\chi^*_{a',\alpha c'}$$

where we have used the fact that the parameter y must be even when v is minimal. In analogy with the previous two cases, we wish to conclude from the equation (4.6) that $\left(s\beta+\frac{N\overline{k}}{\alpha^2\beta}\right)z \equiv s\beta-\frac{N\overline{k}}{\alpha^2\beta} \mod \frac{2\overline{k}}{\alpha^2}$. We have to be a little careful with the powers of 2: since z is odd, either $z-1$ or $z+1$ must be a multiple of 4. If the former we set $x=0$ and if the latter, $x=1$. With this definition, it is easy to check that the desired conclusion holds and we have

$$Z^{\mathrm{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{s\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha\beta}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \delta(a'=J^{an}a)\,\delta\left(c'\equiv s\beta-\frac{N\overline{k}}{\alpha^2\beta} \mod \frac{2\overline{k}}{\alpha^2}\right)$$

$$\times\,\chi_{a,\alpha}\!\left(s\beta-\tfrac{N\overline{k}}{\alpha^2\beta}\right)\chi^*_{a',\alpha c'}$$

$$= \sum_{\substack{a=0,\ldots,k \\ a'=0,\ldots,k}} \sum_{\substack{c\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}} \\ c'\in\mathbb{Z}_{\frac{2\overline{k}}{\alpha}}}} \delta(a'=J^{an}a)\,\delta\left(c'\equiv c \mod \frac{2\overline{k}}{\alpha^2}\right)\chi_{a,\alpha c}\,\chi^*_{a',\alpha c'}$$

$$= \widetilde{M}^{2,0}[v,1,n]$$

which completes the proof of proposition 4.7.1 for the simple current invariants.

The exceptional cases

When $k=10$ we need to show that there is an orbifolding connecting the $\widetilde{E}_1^{10}$ invariants with those with parameters $(v=6,z=5)$ and $(v=6,z=1)$. But we have already seen in table 4.1 that the orbifolding $\mathcal{O}^2$ acts on $\widetilde{E}_1^{10}[6,z]$ by $z\leftrightarrow -z \mod 6$.

When $k=28$ we follow exactly the method we used for the simple current invariants for when $4|k$: we have $\overline{k}=30=2\cdot3\cdot5$ and $v=15$. The solutions to $z^2-1\equiv 0 \mod 15$ are $z\in\{1,4,11,14\}$ (see equation (3.15)), corresponding respectively to the situations

$$z=1, \qquad \left\{\begin{array}{l} z+1\equiv 0 \mod 1 \\ z-1\equiv 0 \mod 15 \end{array}\right\}, \qquad \beta=1$$

$$z=4, \qquad \left\{\begin{array}{l} z+1\equiv 0 \mod 5 \\ z-1\equiv 0 \mod 3 \end{array}\right\}, \qquad \beta=5$$

$$z=11, \qquad \left\{\begin{array}{l} z+1\equiv 0 \mod 3 \\ z-1\equiv 0 \mod 5 \end{array}\right\}, \qquad \beta=3$$

$$z = 14, \qquad \left\{ \begin{array}{l} z + 1 \equiv 0 \mod 15 \\ z - 1 \equiv 0 \mod 1 \end{array} \right\}, \qquad \beta = 15.$$

In each case $\alpha = 1$ and so we apply orbifolding $\mathcal{O}^7$ to the physical invariant $M \equiv \widetilde{E}^{28}[15, z, x]$ using equations (4.2) and (3.15):

$$Z^{\mathrm{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,28 \\ a'=0,\ldots,28}} \sum_{\substack{s \in \mathbb{Z}_{\frac{60}{\beta}} \\ c' \in \mathbb{Z}_{60}}} M_{a,\left(s\beta+\frac{30N}{\beta}\right);\, a',c'} \, \chi_{a,\left(s\beta-\frac{30N}{\beta}\right)} \, \chi^*_{a'c'}$$

$$= \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,28 \\ a'=0,\ldots,28}} \sum_{\substack{s \in \mathbb{Z}_{\frac{60}{\beta}} \\ c' \in \mathbb{Z}_{60}}} E^{28}_{a,a'} \delta(c' \equiv (s\beta + 2N)x \mod 4)$$

$$\times \, \delta\left(c' \equiv \left(s\beta + \frac{30N}{\beta}\right)z \mod 15\right) \chi_{a,\left(s\beta - \frac{30N}{\beta}\right)} \chi^*_{a'c'}.$$

Exactly as in the simple current case, the β is carefully chosen so as to satisfy $\left(s\beta + \frac{30N}{\beta}\right) z \equiv s\beta - \frac{30N}{\beta} \mod 15$. Thus we find

$$Z^{\mathrm{orb}} = \sum_{\substack{N=0,\ldots,\beta-1 \\ a=0,\ldots,28 \\ a'=0,\ldots,28}} \sum_{\substack{s \in \mathbb{Z}_{\frac{60}{\beta}} \\ c' \in \mathbb{Z}_{60}}} E^{28}_{a,a'} \delta(c' \equiv (s\beta - 2N)x \mod 4)$$

$$\times \, \delta\left(c' \equiv s\beta - \frac{30N}{\beta} \mod 15\right) \chi_{a,\left(s\beta - \frac{30N}{\beta}\right)} \chi^*_{a'c'}$$

$$= \sum_{\substack{a=0,\ldots,28 \\ a'=0,\ldots,28}} \sum_{\substack{c \in \mathbb{Z}_{60} \\ c' \in \mathbb{Z}_{60}}} E^{28}_{a,a'} \delta(c' \equiv cx \mod 4)\, \delta(c' \equiv c \mod 15) \, \chi_{ac} \, \chi^*_{a'c'}$$

$$= \widetilde{E}^{28}[15, 1, x].$$

This completes the proof of proposition 4.7.1. We summarise the result in the next section.

4.8 Proof of the main theorem

We are now ready to prove the main theorem. We restate the theorem here in a little more detail. For notation, see section 3.2.

Theorem 4.8.1. • *Let k be odd and let M be a simple current invariant at level k. Then there exists a chain of orbifoldings mapping M to $\mathcal{A}_k \otimes \overline{M}$ where $\mathcal{A}_k$ is the diagonal $\widehat{\mathfrak{su}(2)}$ invariant at level k and the non-zero values of $\overline{M}$ are given by*

$$\overline{M}_{\frac{c\overline{k}}{v},\,\frac{c'\overline{k}}{v}} = 1 \iff c' \equiv c \mod \frac{2v^2}{\overline{k}}$$

where v is the smallest divisor of $\overline{k}$ satisfying $\frac{v^2}{\overline{k}} \in \mathbb{Z}$.

- Let $4|\overline{k}$ and let M be a simple current invariant at level k. Then there exists a chain of orbifoldings mapping M to $\mathcal{A}_k \otimes \overline{M}$ where $\mathcal{A}_k$ is the diagonal $\widehat{\mathfrak{su}(2)}$ invariant at level k and the non-zero values of $\overline{M}$ are given by

$$\overline{M}_{\frac{c\overline{k}}{v}, \frac{c'\overline{k}}{v}} = 1 \iff c' \equiv c \mod \frac{2v^2}{\overline{k}}$$

where v is the smallest divisor of $\frac{\overline{k}}{2}$ satisfying $\frac{v^2}{\overline{k}} \in \mathbb{Z}$.

- Let $4|k$ and let M be a simple current invariant at level k. Then there exists a chain of orbifoldings mapping M to $\mathcal{D}'_k \otimes \overline{M}$ where $\mathcal{D}'_k$ is the level k $\mathcal{D}$ invariant in the $\widehat{\mathfrak{su}(2)}$ $\mathcal{A}$-$\mathcal{D}$-$\mathcal{E}$ classification, and the non-zero values of $\overline{M}$ are given by

$$\overline{M}_{\frac{c\overline{k}}{2v}, \frac{c'\overline{k}}{2v}} = 1 \iff c' \equiv c \mod \frac{8v^2}{\overline{k}}$$

where v is the smallest divisor of $\frac{\overline{k}}{2}$ satisfying $\frac{2v^2}{\overline{k}} \in \mathbb{Z}$.

- Let M be an exceptional invariant at level $k = 10$. Then there exists a chain of orbifoldings mapping M to $\mathcal{E}^{10} \otimes \overline{M}$ where $\mathcal{E}^{10}$ is the exceptional $\widehat{\mathfrak{su}(2)}$ invariant at level 10 and the non-zero values of $\overline{M}$ are given by

$$\overline{M}_{2c,2c'} = 1 \iff c' \equiv c \mod 6.$$

- Let M be an exceptional invariant at level $k = 16$. Then there exists a chain of orbifoldings mapping M to $\mathcal{E}^{16} \otimes \overline{M}$ where $\mathcal{E}^{16}$ is the exceptional $\widehat{\mathfrak{su}(2)}$ invariant at level 16 and the non-zero values of $\overline{M}$ are given by

$$\overline{M}_{3c,3c'} = 1 \iff c' \equiv c \mod 4.$$

- Let M be an exceptional invariant at level $k = 28$. Then there exists a chain of orbifoldings mapping M to $\mathcal{E}^{28} \otimes \overline{M}$ where $\mathcal{E}^{28}$ is the exceptional $\widehat{\mathfrak{su}(2)}$ invariant at level 28 and $\overline{M}$ is given by

$$\overline{M}_{c,c'} = 1 \iff c' \equiv c \mod 60.$$

Proof. The requisite orbifoldings were constructed in the previous sections. Given a physical invariant M we use proposition 4.5.1 to map M into one of the families $\widetilde{M}^0, \widetilde{M}^{2,0}, \widetilde{M}^{4,2}, \widetilde{E}_1^{10}, \widetilde{E}_2^{16}$ or $\widetilde{E}^{28}$ depending on the value of k and whether M is a simple current invariant. We can then apply proposition 4.6.1 to map v to the smallest possible value it can take in that family, while leaving the other parameters unchanged. Proposition 4.7.1 sends z to 1 if k is even or $2z \equiv 1$ if k is odd. Finally, if necessary, we use the orbifolding $\mathcal{O}^1$ of subsection 4.3.1 to fix $n = 0$ when k is odd or $4|\overline{k}$; or to fix $x = 1$ when $4|k$. The resulting partition functions are given explicitly above using equations (3.3)–(3.15). $\qquad\square$

4.9 Existence of the $N = 2$ minimal models

We would like to conclude that our main theorem implies the existence of a unitary $N = 2$ minimal model for every possible partition function in Gannon's list (see section 3.2). We caution the reader that we cannot yet make this argument fully rigorous: it depends on ideas in the physics literature that are derived from string theory and might not be rigorously applicable to conformal field theory. We therefore present our conclusions in the form of a 'physics theorem'.

'Physics Theorem' 4.9.1. *To each of the candidate partition functions given in Gannon's list (see section 3.2) there corresponds a fully-fledged superconformal field theory.*

Proof. The theorem proved in the last section showed that every partition function is obtained from one of a handful of possible partition functions by a chain of orbifoldings by cyclic groups. Since orbifoldings by solvable groups can be inverted (see e.g. [33]) it follows that we can obtain by a chain of orbifoldings any given partition function from the $\mathcal{A}$ model (if it is a simple current invariant), or from the $\mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8$ model (if it is an exceptional invariant with $k = 10, 16, 28$ respectively).

Up to now, the orbifoldings we have constructed have been given entirely in terms of the partition function. In order to have chance of getting an orbifold SCFT we must impose the *level-matching conditions* [61, 14]; that is, we must check that the spin $h - \bar{h}$ of the fields in the orbifold theory remain at worst half-integral and also that we do not destroy semi-locality. We saw in sections 4.3-4.7 that in all the orbifolds we consider, we obtain another partition function from Gannon's list. But we know from equation (2.26) that all states counted by the partition functions have integral spin, and that the spins of states in the full Hilbert space differ at worst by a half-integer. We also checked in theorem 3.7.2 that states appearing in the full Hilbert space of a theory are at worst mutually semi-local.

Two points remain to be shown: firstly that A model and exceptional models $E_{6,7,8}$ are fully-fledged minimal models; and secondly, that the level-matching conditions are sufficient to ensure that an orbifold theory of an SCFT is again a fully-fledged SCFT. To address the first point, the OPE coefficients of the A model were calculated in [49] using the relation between the parafermion fields with those of the $\widehat{\mathfrak{su}(2)}$ WZW models [70]. The OPE coefficients of the exceptional models should in principal be calculable using the free field construction of e.g. [18].[7]

On the second point, we turn to the string theory literature. The theory of orbifolds was first described in [13, 14]. In [12] a method for calculating the n-point (correlation) functions of the fields of the twisted sector is given, providing the level-matching conditions are satisfied. $\square$

---

[7]The question then arises: can the OPE coefficients of all minimal models be calculated using a free field realisation, or perhaps directly from the parafermion construction? In principle, either is possible.

Chapter 5

Analysis of the simple current invariants

5.1 The Kreuzer-Schellekens construction

In [45] it is shown that all simple current invariants that obey both 1-loop and higher-genus modular invariance can be obtained as orbifolds of the diagonal physical invariant by a subgroup of the centre. It is conjectured that all simple current physical invariants can be obtained in this way; that is, it is conjectured that the constraint of higher-genus modular invariance is in fact superfluous. We will analyse the solutions of Gannon's classification to show that this is indeed the case for the unitary $N = 2$ minimal models.

5.1.1 k odd

One reads off immediately from Gannon's classification that every physical invariant with k odd is a simple current invariant. Furthermore, following [45], precisely one physical invariant can be constructed as an orbifold for each subgroup of the effective centre $\mathcal{C} \cong \mathbb{Z}_{2\bar{k}}$ (there is no discrete torsion in this case, since subgroups of $\mathbb{Z}_{2\bar{k}}$ are cyclic).

One can check using induction on the number of prime factors that the number of subgroups of $\mathbb{Z}_q$, equal to the number of divisors of q, is $d(q) := \prod_{i=1}^{l}(1 + n_i)$ where q is written $q = \prod_{i=1}^{l} p_i^{n_i}$ for distinct primes p_i. The following lemma establishes that the number of physical invariants at each odd level k (see equation (3.3)) is precisely the number of subgroups of $\mathbb{Z}_{2\bar{k}}$, showing that the Schellekens-Kreuzer orbifold construction does indeed give all physical invariants when the level k is odd.

Lemma 5.1.1. *Let k be odd. Then the number of solutions $(v, z, n) \in \{1, \ldots, \bar{k}\} \times \{1, \ldots, \frac{v^2}{\bar{k}}\} \times \{0, 1\}$ to the equations*

$$\frac{v^2}{\bar{k}}, \frac{\bar{k}}{v} \in \mathbb{Z}, \quad 4z^2 \equiv 1 \mod \frac{v^2}{\bar{k}}$$

is equal to $d(2\overline{k})$.

Proof. Write $\overline{k} = \prod_{i=1}^{l} p_i^{2a_i+\delta_i}$ where p_i are distinct odd primes and $\delta_i \in \{0, 1\}$. Write $v = \prod_{i=1}^{l} p_i^{b_i}$. Then $\frac{v^2}{k} \in \mathbb{Z}$ and $\frac{k}{v} \in \mathbb{Z}$ imply that each b_i can take any value in $\{a_i + \delta_i, \ldots, 2a_i + \delta_i\}$. So $\frac{v^2}{k}$ can take on any value of the form $\prod_{i=1}^{l} p_i^{2c_i+\delta_i}$ where $c_i \in \{0, \ldots, a_i\}$. So the number of solutions N for (v, z, n) is given by

$$N = 2 \sum_{c_1=0}^{a_1} \cdots \sum_{c_l=0}^{a_l} \sigma \left(\prod_{i}^{l} p_i^{2c_i+\delta_i} \right)$$

where $\sigma(q)$ is the number of solutions to $4z^2 \equiv 1 \bmod q$ with $z \in \{1, \ldots, q\}$.

We now calculate $\sigma(q)$ for $q = \prod_{i=1}^{m} p_i^{d_i}$ where the p_i are distinct odd primes and $d_i \geq 1$. Suppose z is a solution of $(2z+1)(2z-1) \equiv 0 \bmod \prod_{i=1}^{m} p_i^{d_i}$. Since $2z+1$ and $2z-1$ cannot both vanish modulo p for any odd prime, there must exist a partition of the m primes p_i such that (possibly after relabelling)

$$2z + 1 \equiv 0 \bmod \prod_{i=1}^{t} p_i^{d_i}, \qquad 2z - 1 \equiv 0 \bmod \prod_{i=t+1}^{m} p_i^{d_i}.$$

Writing $A = \prod_{i=1}^{t} p_i^{d_i}$ and $B = \prod_{i=t+1}^{m} p_i^{d_i}$ we see that $z = \frac{\overline{k}-1}{2} + rA = \frac{\overline{k}+1}{2} - sB$ for some $r, s \in \mathbb{Z}$. Thus $rA + sB = 1$, which has a unique solution for $r \bmod B$ and $s \bmod A$ by Euclid's algorithm. Since there are 2^m choices for the partition, $\sigma \left(\prod_{i=1}^{m} p_i^{d_i} \right) = 2^m$.

Taking care to note when $2c_i + \delta_i$ vanishes, we find that

$$N = 2 \sum_{c_1=0}^{a_1} \cdots \sum_{c_l=0}^{a_l} \left(\prod_{i=1}^{l} 2^{\delta(2c_i+\delta_i>0)} \right)$$

$$= 2 \left[\sum_{c_1=0}^{a_1} 2^{\delta(2c_1+\delta_1>0)} \right] \cdots \left[\sum_{c_l=0}^{a_l} 2^{\delta(2c_l+\delta_l>0)} \right]$$

$$= 2(2a_1 + \delta_1 + 1) \ldots (2a_1 + \delta_1 + 1)$$

$$= 2 \prod_{i=1}^{l} (1 + 2a_i + \delta_i)$$

$$= d(2\overline{k}). \qquad \square$$

5.1.2   4 divides k

We now turn our attention to the case when $4|k$. Again we can immediately read off from Gannon's classification that $\widetilde{M}^{4,0}$, $\widetilde{M}^{4,1}$, $\widetilde{M}^{4,2}$ and $\widetilde{M}^{4,3}$ are all simple current invariants.

The subgroups of the effective centre $\mathcal{C}_k \cong \mathbb{Z}_2 \times \mathbb{Z}_{2\overline{k}}$ are given by

$$
\begin{aligned}
\mathbb{Z}_2 \times \mathbb{Z}_l &\cong \mathbb{Z}_{2l}, & 2l|\overline{k} \\
\mathbb{Z}_2 \times \mathbb{Z}_{2l}, & & l|\overline{k} \\
\{0\} \times \mathbb{Z}_l &\cong \mathbb{Z}_l, & l|2\overline{k} \\
\langle (J, \tfrac{\overline{k}}{l}) \rangle &\cong \mathbb{Z}_{2l}, & l|\overline{k}.
\end{aligned}
$$

We can define an orbifold for each subgroup of the centre and for each choice of discrete torsion associated to that subgroup. For a cyclic group $\mathbb{Z}_q$ there is no choice to make; for a group $\mathbb{Z}_2 \times \mathbb{Z}_{2q}$ there are two degrees of freedom. Writing $\tau(G)$ for the number of degrees of freedom coming from discrete torsion associated to the group G, we find the number of simple current invariants obtained via an orbifold of the diagonal invariant when $4|k$ is

$$
\begin{aligned}
N &= \sum_{G \leq \mathbb{Z}_2 \times \mathbb{Z}_{2\overline{k}}} \tau(G) \\
&= d\left(\frac{\overline{k}}{2}\right) + 2d(\overline{k}) + d(2\overline{k}) + d(\overline{k}) \\
&= 5d(\overline{k})
\end{aligned}
$$

where $d(q)$, as above, is the number of divisors of q.

The following lemma shows that if $4|k$ then the number of simple current physical invariants is equal to $N = 5d(\overline{k})$, the number of orbifolds of the diagonal invariant, so the Schellekens-Kreuzer construction does find all simple currents invariants when $4|k$.

Lemma 5.1.2. *Let $8|k+4$. Then the number of solutions $(v, z, n, m) \in \{1, \ldots, \frac{\overline{k}}{2}\}$ $\times \{1, \ldots, \frac{2v^2}{k}\} \times \{0,1\}^2$ to the equations*

$$
\frac{2v^2}{k}, \ \frac{\overline{k}}{2v} \in \mathbb{Z}, \quad z^2 \equiv 1 \ \text{mod} \ \frac{2v^2}{k}
$$

is equal to $2d(\overline{k})$.

Let $8|k$. Then the number of solutions $(v, z, x, y) \in \{1, \ldots, \overline{k}\} \times \{1, \ldots, \frac{v^2}{k}\} \times \{1,3\}^2$ to the equations

$$
\frac{v^2}{k}, \ \frac{\overline{k}}{v} \in \mathbb{Z}, \quad z \equiv \frac{k}{8} \ \text{mod} \ 2, \quad 4z^2 \equiv 1 \ \text{mod} \ \frac{v^2}{2\overline{k}}
$$

is equal to $2d(\overline{k})$.

Let $4|k$. Then the number of solutions $(v, z, x) \in \{1, \ldots, \frac{\overline{k}}{2}\} \times \{1, \ldots, \frac{2v^2}{k}\} \times \{1,3\}$ to the equations

$$
\frac{2v^2}{k}, \ \frac{\overline{k}}{2v} \in \mathbb{Z}, \quad z^2 \equiv 1 \ \text{mod} \ \frac{2v^2}{k}
$$

is equal to $d(\overline{k})$.

Let $4|k$. Then the number of solutions $(v, z, n) \in \{1, \ldots, \frac{\overline{k}}{2}\} \times \{1, \ldots, \frac{8v^2}{\overline{k}}\} \times \{0, 1\}$ to the equations

$$\frac{2v^2}{\overline{k}} \quad, \frac{\overline{k}}{2v} \in \mathbb{Z}, \quad z^2 \equiv 1 \mod \frac{4v^2}{\overline{k}}$$

is equal to $2d(\overline{k})$.

Proof. We first tackle the third and fourth cases. Write $\overline{k} = 2\prod_{i=1}^{l} p_i^{2a_i+\delta_i}$ where p_i are distinct odd primes and $\delta_i \in \{0, 1\}$. The values of v are the same in both cases: just as in the previous lemma, we write $v = \prod_{i=1}^{l} p_i^{b_i}$ where $b_i \in \{a_i + \delta_i, \ldots, 2a_i + \delta_i\}$. So $\frac{2v^2}{\overline{k}}$ can take on any value of the form $\prod_{i=1}^{l} p_i^{2c_i+\delta_i}$ where $c_i \in \{0, \ldots, a_i\}$. So the number of solutions N_3, N_4 in each case are given by

$$N_j = 2 \sum_{c_1=0}^{a_1} \cdots \sum_{c_l=0}^{a_l} \sigma_j \left(\prod_i^{l} p_i^{2c_i+\delta_i} \right), \; j = 3, 4.$$

where $\sigma_3(q)$ is the number of solutions to $z^2 \equiv 1 \mod q$, $z \in \{1, \ldots, q\}$ and $\sigma_4(q)$ is the number of solutions to $z^2 \equiv 1 \mod 2q$, $z \in \{1, \ldots, 4q\}$.

Since q is odd, we have $z^2 \equiv 1 \mod 2q \iff z^2 \equiv 1 \mod q$, z odd. If $z^2 \equiv 1$ mod q then precisely one of z and $z + q$ are odd. So there are $\sigma_3(q)$ solutions to $z^2 \equiv 1 \mod 2q$ in $z \in \{1, \ldots, 2q\}$. Also if z satisfies the equation for $\sigma_4(q)$ then so does $z \pm 2q$. Thus $\sigma_4(q) = 2\sigma_3(q)$.

It remains to calculate $\sigma_3(q)$. Since q is odd, $z \mapsto 2z \mod 2$ is a bijection on $\mathbb{Z}_q$. Thus we can read off from the proof of the previous lemma that for $q = \prod_{i=1}^{m} p_i^{d_i}$ with p_i distinct odd primes and $d_i \geq 1$ we have $\sigma_3(q) = 2^m$. Thus, taking care to note when $2c_i + \delta_i$ vanishes, we have

$$N_3 = 2 \sum_{c_1=0}^{a_i} \cdots \sum_{c_l=0}^{a_l} 2^{\delta(2c_i+\delta_i>0)}$$
$$= 2(2a_1 + \delta_1 + 1) \ldots (2a_l + \delta_l + 1)$$
$$= d(\overline{k}),$$
$$N_4 = 2d(\overline{k}).$$

Next we look at the first case. Supposing $8|k + 4$, exactly the same values of v, z occur as in the third case, and there are 4 choices of $n, m \in \mathbb{Z}_2$. So $N_1 = 2d(\overline{k})$.

Finally, the second case. Let $8|k$. As before we write $\overline{k} = 2\prod_{i=1}^{l} p_i^{2a_1+\delta_i}$. Then $\frac{v^2}{2\overline{k}}$ is of the form $\prod_{i=1}^{l} p_i^{2c_i+\delta_i}$ where $c_i \in \{0, \ldots, a_i\}$. Thus

$$N_2 = 4 \sum_{c_1=0}^{a_1} \cdots \sum_{c_l=0}^{a_l} \sigma_2 \left(\prod_i^{l} p_i^{2c_i+\delta_i} \right)$$

where $\sigma_2(q)$ is the number of solutions to $4z^2 \equiv 1 \mod q$ with $z \equiv \frac{k}{8} \mod 2$ and $z \in \{1, \ldots, 2q\}$. It is clear that $\sigma_2(q)$ is also equal to the number of solutions to $4z^2 \equiv 1 \mod q$ with $z \in \{1, \ldots, q\}$ since precisely one of z and $z + q$ has the same parity as $\frac{k}{8}$. In the proof of the first lemma, this number was found to be 2^m where $q = \prod_{i=1}^{m} p_i^{d_i}$ for $d_i \geq 1$. Thus

$$N_2 = 4 \sum_{c_1=0}^{a_i} \cdots \sum_{c_l=0}^{a_l} \prod_{i=1}^{l} 2^{\delta(2c_i + \delta_i > 0)}$$
$$= 2d(\overline{k}). \qquad \square$$

5.1.3   4 divides $k + 2$

As in the previous cases, every physical invariant with $4|k+2$ is a simple current invariant.

Write $\overline{k} = 2^m p$ where p is odd and $m \geq 2$. Then the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_{\overline{k}}$ are given by

$$\begin{array}{ll}
\mathbb{Z}_2 \times \mathbb{Z}_l \cong \mathbb{Z}_{2l}, & l|p \\
\mathbb{Z}_2 \times \mathbb{Z}_{2l}, & 2l|\overline{k} \\
\{0\} \times \mathbb{Z}_l \cong \mathbb{Z}_l, & l|\overline{k} \\
\langle (J, \frac{\overline{k}}{2l}) \rangle \cong \mathbb{Z}_{2l}, & 2l|\overline{k}.
\end{array}$$

Writing $\tau(G)$ for the number of degrees of freedom coming from discrete torsion of a subgroup G of $\mathbb{Z}_2 \times \mathbb{Z}_{\overline{k}}$ we find that the number of possible orbifolds of the diagonal partition function is

$$N = \sum_{G \leq \mathbb{Z}_2 \times \mathbb{Z}_{\overline{k}}} \tau(G)$$
$$= d(p) + 2d\left(\frac{\overline{k}}{2}\right) + d(\overline{k}) + d\left(\frac{\overline{k}}{2}\right)$$
$$= (1 + 2m + (m+1) + m)d(p)$$
$$= 2(2m+1)d(p)$$
$$= 2\left(d(\overline{k}) + d\left(\frac{\overline{k}}{2}\right)\right)$$

The following lemma shows that this is precisely the number of simple current invariants when the level k satisfies $4|k+2$, proving that the Schellekens-Kreuzer orbifolds do indeed find all the physical invariants at these levels.

Lemma 5.1.3. *Let $4|k+2$ and write $\overline{k} = 2^{2r+\epsilon}p$ where $\epsilon \in \{0,1\}, r > 0$ and p is odd.*

The number of solutions $(v, z, n) \in \{1, \ldots, \frac{\overline{k}}{2}\} \times \{1, \ldots, \frac{2v^2}{\overline{k}}\} \times \{0,1\}$ to the equations

$$\frac{v^2}{\overline{k}}, \quad \frac{\overline{k}}{2v} \in \mathbb{Z}, \quad z^2 \equiv 1 \mod \frac{2v^2}{\overline{k}}$$

is equal to $2(4r - 3 + \epsilon)d(p)$.

The number of solutions $(v, z, n) \in \{1, \ldots, \frac{\overline{k}}{2}\} \times \{1, \ldots, \frac{2v^2}{k}\} \times \{0, 1\}$ to the equations

$$\frac{2v^2}{k} \in 2\mathbb{Z} + 1, \quad \frac{\overline{k}}{2v} \in \mathbb{Z}, \quad z^2 \equiv 1 \mod \frac{2v^2}{k}$$

is equal to $2\epsilon d(p)$.

The number of solutions $(v, z, n, m) \in \{1, \ldots, \overline{k}\} \times \{1, \ldots, \frac{2v^2}{k}\} \times \{0, 1\}^2$ to the equations

$$\frac{v^2}{k} \in \mathbb{Z}, \quad \frac{\overline{k}}{v} \in 2\mathbb{Z} + 1, \quad z^2 \equiv 1 \mod \frac{4v^2}{k}$$

is equal to $8d(p)$.

Proof. We tackle the second claim first: note that $\frac{2v^2}{k}$ is odd if and only if ϵ is 1. If ϵ is equal to 1 then simply follow the proof of the third case in lemma 2 to find $2d(p)$ solutions.

Next we prove the third claim. Since $\frac{\overline{k}}{v}$ is odd, $\frac{v^2}{k}$ must be of the form $2^{2r+\epsilon} \prod_{i=1}^{l} p_i^{2c_i + \delta_i}$ where as usual $c_i \in \{0, \ldots, a_i\}$. We need to count the number of solutions to $z^2 \equiv 1 \mod 2^{2r+2+\epsilon} \prod_{i=1}^{m} p_i^{d_i}$ where $d_i \geq 1$ and $z \in \{1, \ldots, 2^{2r+1+\epsilon} \prod_{i=1}^{m} p_i^{d_i}\}$. After possibly relabelling the p_i's, solving this equation is equivalent to solving

$$\left\{ \begin{array}{c} z \equiv 1 \mod \prod_{i=1}^{t} p_i^{d_i} \\ \text{and} \\ z \equiv -1 \mod \prod_{i=t+1}^{m} p_i^{d_i} \end{array} \right\}, \left\{ \begin{array}{c} z \equiv 1 \mod 2^{2r+1+\epsilon} \\ \text{or} \\ z \equiv -1 \mod 2^{2r+1+\epsilon} \end{array} \right\}.$$

Since $r > 0$, there are 2^{m+1} such sets of distinct equations, and one uses the standard arguments to show that each gives a unique solution in the range $z \in \{1, \ldots, 2^{2r+1+\epsilon} \prod_{i=1}^{m} p_i^{d_i}\}$. Thus the total number of solutions is

$$N = 4 \sum_{c_1=0}^{a_i} \cdots \sum_{c_l=0}^{a_l} 2 \prod_{i=1}^{l} 2^{\delta(2c_i + \delta_i > 0)}$$
$$= 8d(p).$$

Finally we tackle the first claim. $\frac{v^2}{k}$ can take on the values $2^{2c_0+\epsilon} \prod_{i=1}^{l} p_i^{2c_i+\delta_i}$ where $c_0 \in \{0, \ldots, r-1\}$ and $c_i \in \{0, \ldots, a_i\}$. We need to find the number of solutions $\sigma(q)$ to $z^2 \equiv 1 \mod q$ with $z \in \{1, \ldots, q\}$ for various values of q. As we saw in the last claim, if $q = 2^s \prod_{i=1}^{m} p_i^{d_i}$ for $s \geq 4$ and $d_i \geq 1$ then there are 2^{m+1} solutions for z in the range $\{1, \ldots, \frac{q}{2}\}$. So for z in the range $\{1, \ldots, q\}$ there are 2^{m+2} solutions and $\sigma(q) = 2^{m+2}$. Now suppose $q = 2^s \prod_{i=1}^{m} p_i^{d_i}$ where $s \in \{1, 2, 3\}$ and $d_i \geq 1$. Then

$$z^2 \equiv 1 \mod q \iff z \text{ odd and } z^2 \equiv 1 \mod \prod_{i=1}^{m} p_i^{d_i},$$

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which has precisely 2^m solutions in $\{1, \ldots, 2\prod_{i=1}^{m} p_i^{d_i}\}$. So $\sigma(q) = 2^{m+s-1}$. Thus taking account of when $2c_i + \delta_i$ vanishes and when $2c_0 + \epsilon$ is less than 3 we have

$$
\begin{aligned}
N &= 2 \sum_{c_0=0}^{r-1} \sum_{c_1=0}^{a_1} \cdots \sum_{c_l=0}^{a_l} \sigma \left(2^{2c_0+\epsilon+1} \prod_{i=1}^{l} p_i^{2c_i+\delta_i} \right) \\
&= 2 \left[\sum_{c_0=0}^{r-1} 2^{2\delta(2c_0+\epsilon>1)+\delta(2c_0+\epsilon=1)} \right] \sum_{c_1=0}^{a_i} \cdots \sum_{c_l=0}^{a_l} \prod_{i=1}^{l} 2^{\delta(2c_i+\delta_i>0)} \\
&= 2(4(r-1)+\epsilon+1) \prod_{i=1}^{l} (2a_i + \delta_i + 1) \\
&= 2(4r-3+\epsilon)d(p).
\end{aligned}
$$
$\qquad\square$

5.1.4   Simple current classification

These counting results coupled with the explicit orbifolds given by Schellekens and Kreuzer [45] can be summarised in the following theorem:

Theorem 5.1.4. *Set $\overline{k} := k + 2$. Then every simple current unitary $N = 2$ minimal partition function at level k is realised via an orbifold (possibly with discrete torsion) of the diagonal partition function by a subgroup of the effective centre*

$$
\mathcal{C} \cong \begin{cases} \mathbb{Z}_{2\overline{k}} & \text{if } k \text{ is odd,} \\ \mathbb{Z}_2 \times \mathbb{Z}_{2\overline{k}} & \text{if } 4 \text{ divides } k, \\ \mathbb{Z}_2 \times \mathbb{Z}_{\overline{k}} & \text{if } 4 \text{ divides } k+2. \end{cases}
$$

The number of simple current invariants at each level k is given by

$$
N(k) = \begin{cases} 2d(\overline{k}) & \text{if } k \text{ is odd,} \\ 5d(\overline{k}) & \text{if } 4 \text{ divides } k, \\ 2d(\overline{k}) + 2d\left(\frac{\overline{k}}{2}\right) & \text{if } 4 \text{ divides } \overline{k}. \end{cases}
$$

where $d(n)$ is the number of divisors of n.

5.2   Generating functions for the simple current invariants

In theorem 5.1.4 we found the number of simple current invariants for each level k. For convenience, we define $N(-2) = 1, N(-1) = 2$ and $N(0) = 10$. This information can then be encoded in the generating function

$$
G(q) := \sum_{k=-2}^{\infty} N(k) q^{\overline{k}}
$$

$$= 2\sum_k d(\overline{k})q^{\overline{k}} + 2\sum_{2|k} d\left(\frac{\overline{k}}{2}\right)q^{\overline{k}} + 2\sum_{4|k} d(\overline{k})q^{\overline{k}}$$

where the sums are over $k \geq -2$. This is equal to

$$\frac{1}{2}\left(E_1(q) + 3E_1(q^2) - 4E_1(q^4) + 2E_1(q^8)\right)$$

where $E_1(q)$ is defined[1] for $q \in \mathbb{C}$ by

$$E_1(q) = 1 + 4\sum_{n=1}^{\infty} d(n)q^n.$$

It is called E_1 by analogy to the normalised Eisenstein series which, for even k, is defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

where τ lies in the upper half plane and $q = e^{2\pi i\tau}$. Here the $\sigma_k(n)$ function is the sum of the k-th powers of the divisors of n and B_k is the k-th Bernoulli number. Note that $\sigma_0(n) = d(n)$ and $B_1 = -\frac{1}{2}$. The Eisenstein series E_{2k} with $k \geq 2$ are modular forms. We have no explanation as to why such a neat generating function should exist for the simple current invariants in the minimal model series, and why it should be related to modular forms.

A further curiosity is motivated by an observation by Hecke [39]. He found a connection between modular forms and the Dirichlet series which shares the same coefficients. We are thus motived to consider

$$F(s) := \sum_{n=1}^{\infty} \frac{N(n-2)}{n^s}$$
$$= 2\zeta(s)^2\left(1 + 3\cdot 2^{-s} - 4\cdot 4^{-s} + 2\cdot 8^{-s}\right)$$

where we used the identity

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta(s)^2,$$

and ζ is the Riemann zeta function. Again, we can offer no explanation of why the Riemann zeta function should show up at all. The next logical step of investigation would be to see if a similar phenomenon occurs for other classifications of conformal fields theories. Unfortunately, not many of these are completely known, but the two best known examples are the $A - D - E$ classifications of the $\widehat{\mathfrak{su}(2)}_k$ WZW models and the $N = 0$ minimal models [5]. In these cases, the

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[1] We will soon consider the substitution $q = e^{2\pi i\tau}$ for τ in the upper-half plane, so one should really think of $G(q)$ and $E_1(\tau)$ as formal power series.

A and D models are simple current invariants, so there is one model when k is odd and two when k is even. So the generating function for the simple current invariants in both cases is

$$G(q) := \sum_{k=0}^{\infty} N(k)q^k$$
$$= \frac{1}{1-q} + \frac{1}{1-q^2}$$

and the associated Dirichlet series is

$$F(s) := \sum_{n=1}^{\infty} \frac{N(n)}{n^s}$$
$$= \zeta(s)(1+2^{-s}).$$

The appearance of the Riemann zeta function in this context perhaps merits further attention.

Chapter 6

Conclusions

The main result of this thesis is theorem 4.9.1, in which we show the existence of a unitary $N = 2$ minimal model associated to each of the candidate partition functions in Gannon's list. The bulk of the proof is the construction of a chain of orbifoldings from any given partition function to one of the known A, E_6, E_7 or E_8 models (theorem 4.8.1). We point out once more that the final step of the proof entails the application of results from the string theory literature to deduce that the orbifold of a fully fledged SCFT is again a fully-fledged SCFT, which might not meet the strict mathematical standards of rigour.

Along the way to this result, we proved some other results regarding the minimal models: firstly, we found a simple proof of the superconformal version of Verlinde's formula (theorem 2.4.2) using simple current techniques, circumventing the technical and long-winded proof of Wakimoto [64].

Next we found an interpretation of the two possible projectors from the full Hilbert space of a theory to a modular invariant subspace in the case where the level k is odd (theorem 3.5.1). The projections are defined by taking the states corresponding to one of the two even sublattices of the charge lattice.

We presented the explicit list of Gannon's partition functions in section 3.2, correcting a couple of minor errors. Armed with this list, we could perform nontrivial checks on all the candidate theories. Two of these were carried out in section 3.7. The first test was that the fusion rules did not preclude the existence of a non-trivial OPE à la Gepner [32]. This was theorem 3.7.1. In theorem 3.7.2 we checked that the fields in all the candidate theories were mutually semi-local.

In section 5.1 we counted the simple current invariants. The results enabled us to verify the hypothesis of Kreuzer and Schellekens [45], that all simple current invariants are obtained by orbifolds of the A-model by subgroups of the effective centre, without any recourse to extra assumptions regarding higher-genus modular invariance.

Finally in section 5.2 we noted a strange appearance of modular-like functions and the Riemann zeta function related to the generating function of the simple current invariants in the unitary $N = 2$ minimal models and others, which merits further attention.

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Research Publications

1. On the complete classification of the unitary $N = 2$ minimal superconformal field theories, 2008, arXiv:0812.1318

Papers presented

- I will give a short talk entitled *A complete classification of the $N = 2$ Virasoro unitary minimal models* explaining the results of my thesis at the upcoming First Cuban Congress on Symmetries in Geometry and Physics at Havana in December 2008.

- A short talk entitled *A complete classification of the $N = 2$ Virasoro unitary minimal models*, presented at the LMS 2008 AGM pre-meeting in November 2008.

- Gualtieri, M., *Generalised complex geometry*, presented in the joint Augsburg-Regensburg seminar "Generalised Complex Geometry" in October 2007.

- Kirillov, A., Jr., *On G-equivariant modular categories* and Müger, M., *Conformal orbifold theories and braided crossed G-categories*, presented in the Arbeits-gemeinschaft "Algebraic Structures in Conformal Field Theory" at Oberwolfach in April 2007.

- A talk on the subject of my thesis, presented in the Analysis und Geometry group seminar at Augsburg University in November 2006.

- A poster on the subject of my thesis, presented in the conference "Geometry, Conformal Field Theory and String Theory" in Durham in July 2005.

- Borcherds, R.E., *Quantum field theory*, presented in the Algebraic Geometry group seminar at Warwick University in January 2005.

Invitations and Visits

In the spring of 2006, I visited the University of North Carolina at Chapel Hill for 4 months to work with my supervisor Katrin Wendland. Whilst there, I attended the Duke/UNC string theory seminars.

Teaching Experience

- Summer semester 2008: Teaching assistant and exercise setter for the course Modular Forms at Augsburg University.

- Winter semester 2007: Seminar organiser for the course Representation Theory and Particle Physics at Augsburg University.

- Summer semester 2007: Teaching assistant and exercise setter for the course Lie Algebras and Super Lie Algebras and teaching assistant for Linear Algebra II at Augsburg University. Two replacement lectures for Professor Katrin Wendland.

- Winter semester 2006: Seminar organiser for Algebraic Geometry and teaching assistant for the course Linear Algebra I at the Augsburg University.

- October 2005-December 2006: Supervisor for the courses Analysis III and Algebra II at the University of Warwick.

- October 2004-June 2005: Teaching assistant for the course Analysis III and supervisor for the courses Linear Algebra I and 3D Geometry in Motion at the University of Warwick.

- October 2003-June 2004: Supervisor for the course Linear Algebra I and 3D Geometry in Motion at the University of Warwick.

Other Professional Experiences & Skills

1. Completed courses in the use of the Java and Fortran languages and used Fortran programs to assist in my research.

2. Attended three semesters of German lessons, ran a German language mathematics and physics seminar and produced exercise sheets in German.

3. Helped to organise the Analysis und Geometrie Oberseminar and weekly Colloquium in Augsburg.

Major areas of research interest

My doctoral thesis is on the topic of conformal field theory. While this is my main area of interest, I have further interests in the direction of modular forms, complex geometry and mathematical physics.

Extra-Curricula Achievements

- President of the Warwick Mountains Club.

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