PIERRE E. CONNER

LARRY SMITH

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ON THE COMPLEX BORDISM OF FINITE COMPLEXES

by P. E. CONNER *and* LARRY SMITH

Let us denote by $\{\Omega_*^U(\cdot, \cdot), \varphi_*, \partial_*\}$ the homology theory determined by the Thom spectrum $\underset{\sim}{MU}$ ([9], [22]). The objective of this study is to examine the internal properties of this homology theory, which we refer to as complex bordism or U-bordism, and to indicate several applications of these results to related and allied areas. Let us recall that the coefficients, $\Omega_*^U = \Omega_*^U(\text{point})$ are a graded polynomial algebra over the integers $\mathbf{Z}$ with one generator in each even dimension, and for a pair of spaces (X, A), $\Omega_*^U(X, A)$ is an Ω_*^U-module ([22], VII). We will apply the techniques and results of homological algebra to examine the structure of these modules.

In the first section we review and extend the basic finiteness theorem of [20] (see also [2], V). With these in hand we will then proceed to our first subject of study, the Thom homomorphism

$$\mu : \Omega_*^U(\cdot) \to H_*(\cdot\,; \mathbf{Z}).$$

Our main technical tool is the notion, and subsequent construction, of a U-bordism resolution of a finite complex X. These are introduced in section 2 and employed in section 3 to study the Thom homomorphism. Among the results that we obtain is the following:

Theorem. — Let X *be a finite complex. Then the Thom homomorphism*

$$\mu : \Omega_*^U(X) \to H_*(X; \mathbf{Z})$$

is an epimorphism iff $\Omega_*^U(X)$ *has projective dimension* 0 *or* 1 *as an* Ω_*^U*-module.*

Much of the information contained in a U-bordism resolution of a finite complex X may be assembled into a spectral sequence $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ with

$$E^r\langle X\rangle \Rightarrow H_*(X; \mathbf{Z})$$

and

$$E^2_{p,q}\langle X\rangle = \mathrm{Tor}^{\Omega_*^U}_{p,q}(\mathbf{Z}, \Omega_*^U(X))$$

where $\mathbf{Z}$ is regarded as an Ω_*^U-module via the morphism

$$\Omega_*^U = \Omega_*^U(\text{point}) \xrightarrow{\mu} H_*(\text{point}; \mathbf{Z}) = \mathbf{Z},$$

which is the augmentation homomorphism. This spectral sequence is constructed in section 4 and section 5 and 6 are basically devoted to establishing the non-triviality of this spectral sequence. This is done by first providing examples of spaces E_n with

$\Omega_*^U(E_n)$ of projective dimension at least n as an Ω_*^U-module. We then move on to study the reduced Thom homomorphism

$$\tilde{\mu} : \mathbf{Z}\otimes_{\Omega_*^U}\Omega_*^U(\ ) \rightarrow H_*(\ ;\mathbf{Z}).$$

We are primarily concerned with determining whether $\tilde{\mu}$ can have a non-trivial kernel. This is shown to be so in two distinct manners. First by an explicit construction and computation of an example, and then by non-constructive methods. This section closes with an open problem.

As a large part of our study centers around the numerical invariant $\operatorname{hom.dim}_{\Omega_*^U}\Omega_*^U(X)$ where X is a finite CW-complex, we should devote some time to the study of the behavior of this invariant under cell attachement. This we take up for the first time in section 7. An outcome of our study is a determination of the meaning of the condition

$$\operatorname{hom.dim}_{\Omega_*^U}\Omega_*^U(X)\leq 2$$

for a finite complex X in terms of the reduced Thom homomorphism

$$\tilde{\mu} : \mathbf{Z}\otimes_{\Omega_*^U}\Omega_*^U(X_\alpha) \rightarrow H_*(X_\alpha;\mathbf{Z})$$

for a preferred family of subcomplexes $\{X_\alpha \mid X_\alpha \subseteq X\}$ of X.

In section 8 we apply our construction of U-bordism resolutions to obtain a spectral sequence of Künneth type for $\Omega_*^U(\cdot)$. More precisely we construct for each pair of finite CW-complexes X, Y, a spectral sequence $\{E^r(X, Y), d^r(X, Y)\}$ with

$$E^r(X, Y) \Rightarrow \Omega_*^U(X\times Y)$$

and

$$E^2_{p,q}(X, Y) = \operatorname{Tor}_{p,q}^{\Omega_*^U}(\Omega_*^U(X), \Omega_*^U(Y)).$$

Of particular interest we note the construction also of a finite CW-complex X such that the exterior product

$$\Omega_*^U(X)\otimes_{\Omega_*^U}\Omega_*^U(X) \rightarrow \Omega_*^U(X\times X)$$

has a non-trivial kernel. This implies that the Künneth spectral sequence

$$\{E^r(X, X), d^r(X, X)\}$$

is non-trivial.

In section 9 we take up the study of the relation between U-cobordism theory and K-theory initiated in [9]. With the aid of U-cobordism resolutions we easily rederive the results of ([9], § 10). A slight additional argument yields the following:

Theorem. — Let X *be a finite complex. Then there exists a natural exact sequence*

$$0 \rightarrow \operatorname{Ext}^{1,*}_{\Omega_*^U}(\Omega_*^U(X), \mathbf{Z}) \xrightarrow[\psi]{} K^*(X) \xrightarrow[\varphi]{} \operatorname{Hom}_{\Omega_*^U}(\Omega_*^U(X), \mathbf{Z}) \rightarrow 0.$$

Here we regard $\Omega_*^U(\cdot)$ as being $\mathbf{Z}_2 \equiv \mathbf{Z}/2\mathbf{Z}$ graded by its even and odd components and $\mathbf{Z}$ as being an Ω_*^U-module via the Todd genus. $K^*(\cdot)$ is $\mathbf{Z}_2$-graded in the usual manner.

The map φ is induced by the cap product pairing in a natural way. The sequence splits, although in a non-canonical way, and ψ maps $\mathrm{Ext}^{1,*}_{\Omega^U_*}(\Omega^U_*(X), \mathbf{Z})$ isomorphically onto the torsion subgroup of $K^*(X)$.

In sections 10, 11 we take up the study of the relation between U-bordism and the connective k-homology functor determined by the spectrum $\underset{\sim}{bu}$. We find analogs of the results of sections 3 and 4 for the natural transformation

$$\zeta : \Omega^U_*(\cdot) \rightarrow k_*(\cdot) \equiv H_*(\cdot\,; \underset{\sim}{bu})$$

derived from the K-theory orientation of $\underset{\sim}{MU}$. These results are applied in section 12 to the study of the numerical invariant

$$\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X)$$

and its behavior under cell attachement. With the aid of the results of sections 10 and 11 we are able to greatly simplify our example of a finite CW-complex W for which the reduced Thom homomorphism

$$\tilde{\mu} : \mathbf{Z}\otimes_{\Omega^U_*}\Omega^U_*(W) \rightarrow H_*(W;\mathbf{Z})$$

has a non-trivial kernel.

Our study closes with an examination of how our results may be applied to the study of U-framed cobordism. We will reprove in the final section the result of ([9], § 15) that a compact U-manifold with a compatible framing on its boundary, (M, ∂M), has the same Chern numbers as a closed U-manifold iff Todd [M, ∂M] is an integer.

The arrangement of the paper is as follows:

§ 1. Finiteness Theorems.
§ 2. U-Bordism Resolutions.
§ 3. The Thom Homomorphism.
§ 4. A Spectral Sequence.
§ 5. Bounds for the Projective Dimension of U-Bordism Modules.
§ 6. Generators for U-Bordism Modules.
§ 7. Attaching Cells. Some Special Results.
§ 8. The Spectral Künneth Theorem.
§ 9. The Relation of U-Bordism to K-Theory.
§ 10. The Relation of U-Bordism to Connective K-Theory.
§ 10 *bis*. The Relation Between $k_*(\cdot)$ and $H_*(\cdot\,;\mathbf{Z})$.
§ 11. More on the Relation of U-Bordism to Connective K-Theory.
§ 12. More on Attaching Cells.
§ 13. An Application to U-Framed Cobordism.

The notion of a U-bordism resolution, which is one of our main technical tools is an obvious extension of ideas of Atiyah [4] and Landweber [14], our essential contribution being the finiteness theorems of the first section. These ideas have also been applied by J. F. Adams in a very general setting to discuss the Universal Coefficient Theorems and Künneth Theorems. There is some overlap between our work and his lectures [2].

The study presented here arose from conversations and correspondences between the two authors. The present exposition is an amalgam of lectures given by the first author at L.S.U. and the second at I.H.E.S.

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§ 1. Finiteness Theorems.

In this section we will collect the finiteness theorems that we will require in the sequel. We will begin by recalling several elementary algebraic notions which may be found in [2], [6], [18].

Notation. — Ω always denotes a ring with 1. Module always means left unital Ω-module.

Definition. — *A presentation of an* Ω*-module* M *is an exact sequence of* Ω*-modules*

$$0 \to R \to F \to M \to 0$$

where F *is a free* Ω*-module. A presentation of* M *is called finite iff* F *and* R *are finitely generated* Ω*-modules.*

An Ω-module is *coherent* iff M and all of its finitly generated submodules are finitely presentable.

Note that a coherent Ω-module is always finitely generated.

Definition. — *A ring* Ω *is coherent iff* Ω *is coherent as an* Ω*-module.*

Note that any noetherian ring is coherent. The converse is false, for if **K** is a noetherian ring then the infinite polynomial ring $\mathbf{K}[x_1, \ldots, x_n, \ldots]$ is coherent, but not noetherian [2], [20].

From the point of view of homological algebra coherence serves as an adequate substitute for noetherian. The notion of coherence arose in algebraic geometry [18] and has recently found applications in algebraic topology [2], [6], [20].

The following elementary properties of coherence are readily established (see e.g. [2], [6], [20]).

Proposition (1.1). — *If*

$$0 \to M' \xrightarrow{f'} M \xrightarrow{f''} M'' \to 0$$

is an exact sequence of Ω*-modules and two of the modules,* M', M, M'' *are coherent then so is the third.* □

Proposition (**1.2**). — *If*

$$\begin{tikzcd} M'' \arrow[rr, "f"] & & M' \arrow[dl, "f'"] \\ & M \arrow[ul, "f''"] & \end{tikzcd}$$

is an exact triangle of Ω*-modules and two of the modules* M′, M, M″ *are coherent then so is the third.* □

Let $\Omega_*^U(\cdot)$ denote the singular weakly complex bordism functor [3], [9], [22]. This is a homology theory and is represented by the Thom spectrum $\underset{\sim}{MU}$. The cohomology theory represented by $\underset{\sim}{MU}$ will be denoted by $\Omega_U^*(\cdot)$. We will denote Ω_*^U(point) simply by Ω_*^U and similarly for Ω_U^*(point).

Recall that $\Omega_*^U = \mathbf{Z}[x_1, x_2, \ldots]$ [22] is a coherent ring. Hence a finitely generated free Ω_*^U-module is coherent. In particular $\Omega_*^U(S^n)$ is a coherent Ω_*^U-module. It is now an easy matter [20] to prove by induction on the number of cells the following:

Theorem $(\mathbf{1.3})_*$ — *If* X *is a finite complex then* $\Omega_*^U(X)$ *is a coherent* Ω_*^U*-module.* □

Theorem $(\mathbf{1.3})^*$ — *If* X *is a finite complex then* $\Omega_U^*(X)$ *is a coherent* Ω_U^**-module.* □

Proposition (**1.4**). — *Let* $\Omega = \mathbf{Z}[x_1, x_2, \ldots]$ *and suppose that* F *is a free* Ω*-module and* $M \subset F$ *is a finitely generated submodule. Then* $\operatorname{hom.dim}_\Omega M$ *is finite.*

Proof. — For each integer $n > 0$ let $\Omega(n) \subset \Omega$ be the sub-polynomial algebra $\mathbf{Z}[x_1, \ldots, x_n]$ of Ω. Note that Ω is a free $\Omega(n)$-module. Let $S \subset F$ be a free basis for F as an Ω-module. Let $F(n)$ denote the free $\Omega(n)$-module generated by S. Note that the natural map

$$\psi(n) : F(n) \otimes_{\Omega(n)} \Omega \to F$$

is an isomorphism.

The natural inclusion $\Omega(n) \subset \Omega$ extends to an inclusion (of abelian groups) $F(n) \subset F$. Let T be a finite set of generators for M. Since T is finite there exists an integer n such that $T \subset F(n)$. Let $M(n)$ be the $\Omega(n)$-submodule of $F(n)$ generated by T. The map $\psi(n)$ then provides us with an isomorphism

$$\varphi(n) : M(n) \otimes_{\Omega(n)} \Omega \to M.$$

By Hilbert's syzygy theorem ([5], VIII, (4.2); [15], VII) $M(n)$ admits a finite projective resolution

$$0 \leftarrow M(n) \leftarrow P_0 \leftarrow P_1 \leftarrow \ldots \leftarrow P_n \leftarrow P_{n+1} \leftarrow 0$$

as an $\Omega(n)$-module. Since Ω is a free $\Omega(n)$-module the sequence

$$\begin{tikzcd} 0 & M(n) \otimes_{\Omega(n)} \Omega \arrow[l] \arrow[d, equal] & P_0 \otimes_{\Omega(n)} \Omega \arrow[l] & \ldots \arrow[l] & P_{n+1} \otimes_{\Omega(n)} \Omega \arrow[l] & 0 \arrow[l] \\ & M & & & & \end{tikzcd}$$

provides a projective resolution of M as an Ω-module. Hence $\operatorname{hom.dim}_\Omega M$ is finite as claimed. □

Corollary **(1.5)**. — *Let* $\Omega=\mathbf{Z}[x_1, x_2, \ldots]$ *and suppose that* M *is a coherent* Ω-*module. Then* $\mathrm{hom.dim}_\Omega \mathrm{M}$ *is finite.*

Proof. — Since M is coherent there is an exact sequence of Ω-modules

$$0 \to \mathrm{R} \to \mathrm{F} \to \mathrm{M} \to 0$$

where F is a finitely generated free Ω-module and R is a finitely generated submodule. Thus by ([5], VIII, (2.2)) we have

$$\mathrm{hom.dim}_\Omega \mathrm{M} \leq 1 + \mathrm{hom.dim}_\Omega \mathrm{R}$$

with equality holding unless M is projective. The result now follows from Proposition (1.4). □

Applying Corollary 1.5 to Theorem 1.3 yields:

Theorem **(1.6)$_*$**. — *If* X *is a finite complex then* $\mathrm{hom.dim}_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{X})$ *is finite.* □

Theorem **(1.6)***. — *If* X *is a finite complex then* $\mathrm{hom.dim}_{\Omega^*_{\mathrm{U}}}\Omega^*_{\mathrm{U}}(\mathrm{X})$ *is finite.* □

Remark. — Finiteness theorems of the above type have also been obtained by Novikov (for Theorem (1.3)) and Adams [2] (for Theorem (1.6)).

§ 2. U-Bordism Resolutions.

In this section we introduce U-bordism resolutions for finite complexes X. These ideas go back to Atiyah [4] and Landweber [14]. A formulation of these ideas in the stable category may be found in [2].

Convention. — Throughout this paper the word complex will mean a *finite* CW-complex.

Definition. — *If* X *is a complex, a partial* U-*bordism resolution of* X *of length k and degree l consists of complexes*

$$\emptyset = \mathrm{A}_{-1} \subset \mathrm{A}_0 \subset \mathrm{A}_1 \subset \ldots \subset \mathrm{A}_{k-1} \subset \mathrm{A}_k$$

and a homotopy equivalence

$$\varphi : \Sigma^l \mathrm{X} \sim \mathrm{A}_k$$

($\Sigma^l\mathrm{X}$ *is the l-fold suspension of* X) *such that*

1) $\Omega^{\mathrm{U}}_*(\mathrm{A}_i, \mathrm{A}_{i-1})$ *is a projective* Ω^{U}_*-*module for* $i=0, 1, \ldots, k-1$,

and 2) $\Omega^{\mathrm{U}}_*(\mathrm{A}_i, \mathrm{A}_{i-1}) \to \Omega^{\mathrm{U}}_*(\mathrm{A}_k, \mathrm{A}_{i-1})$ *is an epimorphism for* $i=0, 1, \ldots, k$.

If in addition $\Omega^{\mathrm{U}}_*(\mathrm{A}_k, \mathrm{A}_{k-1})$ is a projective Ω^{U}_*-module then we say that

$$\emptyset = \mathrm{A}_{-1} \subset \mathrm{A}_0 \subset \mathrm{A}_1 \subset \ldots \subset \mathrm{A}_{k-1} \subset \mathrm{A}_k \overset{\varphi}{\sim} \Sigma^l \mathrm{X}$$

is a U-*bordism resolution* of X of length k and degree l.

Since the questions that will be of interest in the sequel are stable we will not always indicate the degree of a particular U-bordism resolution.

Our interest in U-bordism resolutions stems from:

Proposition (**2.1**). — *Let* X *be a complex and*

$$\emptyset = A_{-1} \subset A_0 \subset \ldots \subset A_{k-1} \subset A_k \overset{\varphi}{\sim} \Sigma^l X$$

a partial U*-bordism resolution of* X *of length k and degree l. Then the sequence*

$$0 \leftarrow \Omega^U_*(A_k) \xleftarrow{\varepsilon} \Omega^U_*(A_0, \emptyset) \xleftarrow{\partial_1} \Omega^U_*(A_1, A_0) \leftarrow \ldots \xleftarrow{\partial_k} \Omega^U_*(A_k, A_{k-1}) \leftarrow 0$$

is exact. Here ε *is induced by the inclusion* $A_0 \subset A_k$ *and* ∂_i *is the boundary operator of the triple* (A_i, A_{i-1}, A_{i-2}).

Proof. — For each $i = 0, \ldots, k-1$ we have from the definition of a partial U-bordism resolution the exact sequences

$$0 \leftarrow \Omega^U_*(A_k, A_{i-1}) \leftarrow \Omega^U_*(A_i, A_{i-1}) \leftarrow \Omega^U_*(A_k, A_i) \leftarrow 0.$$

These may be assembled into the diagram

$$\begin{tikzcd}
& & & 0 & & 0 \arrow[dl] & & \\
& & & & \Omega^U_*(A_k, A_i) \arrow[ul] \arrow[dl] & & & \\
& \cdots \arrow[rr, dash] & & \Omega^U_*(A_i, A_{i-1}) \arrow[dl] & & \Omega^U_*(A_{i+1}, A_i) \arrow[ll, "\partial_{i+1}"'] \arrow[ul] & & {} \arrow[ll, "\partial_{i+2}"'] \\
& & \Omega^U_*(A_k, A_{i-1}) \arrow[ul] \arrow[dl] & & & & \Omega^U_*(A_k, A_{i+1}) \arrow[ul] \arrow[dl] \arrow[ur] & \\
& 0 & & 0 \arrow[ul] & & 0 & & 0 \arrow[ul]
\end{tikzcd}$$

and the result follows from the exactness of the diagonal sequences. □

Corollary (**2.2**). — *Let* X *be a finite complex and*

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \subset \ldots \subset A_{k-1} \subset A_k \overset{\varphi}{\sim} \Sigma^l X$$

a U*-bordism resolution of* X *of length k and degree l. Then the sequence*

$$0 \leftarrow \Omega^U_*(A_k) \xleftarrow{\varepsilon} \Omega^U_*(A_0, \emptyset) \xleftarrow{\partial_1} \Omega^U_*(A_1, A_0) \leftarrow \ldots \leftarrow \Omega^U_*(A_k, A_{k-1}) \leftarrow 0$$

is an Ω^U_**-projective resolution of* $\Omega^U_*(A_k)$.

Proof. — This follows immediately from Proposition (1.1) and the definition of U-bordism resolution. □

Remark. — Note that it follows from the definition of a U-bordism resolution that $\Omega^U_*(A_k) \cong \Omega^U_*(\Sigma^l X)$ and by excision $\Omega^U_*(\Sigma^l X) \cong s^l \Omega^U_*(X)$. Thus the exact sequence of Corollary (2.2) is an Ω^U_*-projective resolution of $\Omega^U_*(X)$ of length k and degree l. Hence the terminology.

We turn next to the existence of partial U-bordism resolutions and U-bordism resolutions.

Lemma **(2.3)**. — *Suppose that* X *is a finite complex with* $H_*(X; \mathbf{Z})$ *free abelian. Then the* U*-bordism spectral sequence of* X *collapses and* $\Omega_*^U(X)$ *is a free* Ω_*^U*-module.*

Proof. — An elementary consequence of the lack of torsion in Ω_*^U and the triviality of the bordism spectral sequence mod torsion. □

Notation. — If X and Y are spaces [X, Y] denotes the set of homotopy classes of maps $X \to Y$.

If X is a finite complex we denote by D(X) a Spanier-Whitehead dual of X ([3], [21]). If $f: X \to Y$ is a map between finite complexes and DX and DY are duals of X and Y, then there is induced a dual map $Df: DY \to DX$ ([3], [21]).

Proposition **(2.4)**. — *If* X *is a finite complex, then there exists a complex* A *and a map* $f: A \to \Sigma^l X$ *such that*

1) $\Omega_*^U(A)$ *is a free* Ω_*^U*-module, and*
2) $f_*: \Omega_*^U(A) \to \Omega_*^U(\Sigma^l X)$ *is onto.*

Proof. — Let DX be a dual of X. By Theorem (1.3)* $\Omega_U^*(DX)$ is a finitely generated Ω_U^*-module. Thus we may choose maps

$$g_i: \Sigma^{n_i} DX \to MU(N_i) \qquad i = 1, \ldots, t$$

whose homotopy classes generate $\Omega_U^*(DX)$. Let $n = \max\{n_i\}$. For each integer $i = 1, \ldots, t$ choose a finite subcomplex $M_i \subset (N_i + n - n_i)$ such that

1) $H_*(M_i; \mathbf{Z})$ is free abelian, and
2) $$[\Sigma^n DX, M_i] \to [\Sigma^n DX, MU \subset (N_i + n - n_i)]$$

is an isomorphism.
Such choices are always possible [21]. Consider now the map

$$g: \Sigma^n DX \to M = M_1 \times \ldots \times M_t$$

obtained by choosing maps $\overline{g}_i: \Sigma^n DX \to M_i$ that correspond to

$$\Sigma^{n-n_i} g_i: \Sigma^n DX \to MU(N_i + n - n_i),$$

and setting

$$g(a) = (\overline{g}_1(a), \ldots, \overline{g}_t(a))$$

for all $a \in \Sigma^n DX$. By construction we then have

1) $H_*(M; \mathbf{Z})$ is free abelian, and
2) $g^*: \Omega_U^*(M) \to \Omega_U^*(\Sigma^n DX)$ is an epimorphism.

Since M is a finite CW-complex by construction, we may pass to duals, obtaining

$$Dg: DM \to D(\Sigma^n DX).$$

From Spanier-Whitehead duality it follows that

1) $H_*(DM; \mathbf{Z})$ is free abelian, and

2) $(Dg)_* : \Omega_*^U(DM) \rightarrow \Omega_*^U(D\Sigma^n DX)$ is an epimorphism.

Since our spaces are finite complexes we obtain, by setting $A = DM, f = Dg$, a map

$$f : A \rightarrow \Sigma^n X$$

where

$$f_* : \Omega_*^U(A) \rightarrow \Omega_*^U(\Sigma^n X)$$

is onto. Since $H_*(A; \mathbf{Z})$ is free abelian the result now follows from Lemma (2.3). □

Proposition **(2.5)**. — *Let* X *be a finite complex and k be a positive integer. Then there exists a partial* U*-bordism resolution of* X *of length k.*

Proof. — By induction on k. The case $k=1$ follows quickly from Proposition (2.4). For suppose that we have chosen

$$f : A \rightarrow \Sigma^l X$$

such that

1) $\Omega_*^U(A)$ is a free Ω_*^U-module, and

2) $f_* : \Omega_*^U(A) \rightarrow \Omega_*^U(\Sigma^l X)$ is onto.

Let A_1 be the mapping cylinder of f and $A_0 = A \subset A_1$. Then A_1 has the homotopy type of $\Sigma^l X$ and

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \sim \Sigma^l X$$

is a partial U-bordism resolution of X of length 1.

Proceeding inductively we may suppose $k > 1$, and that for any finite complex there exists a partial U-bordism resolution of X of length $k-1$.

Let X be a finite complex. Choose a partial U-bordism resolution of X of length $k-1$, say

$$\emptyset = A_{-1} \subset A_0 \subset \ldots \subset A_{k-1} \sim \Sigma^l X.$$

By our inductive assumption we may choose a map $f : A \rightarrow \Sigma^s(A_{k-1}/A_{k-2})$ such that

1) $\Omega_*^U(A)$ is a free Ω_*^U-module, and

2) $f_* : \Omega_*^U(A) \rightarrow \Omega_*^U(\Sigma^s(A_{k-1}/A_{k-2}))$ is onto.

Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "hf"] \arrow[d, "f"'] & \Sigma^{s+1}A_{k-2} \arrow[r, "\varphi"] \arrow[d, "1"] & B \arrow[d, "g"] \\
\Sigma^s(A_{k-1}/A_{k-2}) \arrow[r, "h"] & \Sigma^{s+1}A_{k-2} \arrow[r] & \Sigma^{s+1}A_{k-1}
\end{tikzcd}$$

where the horizontal sequences are cofibrations.

Define

$$\begin{aligned} B_i &= \Sigma^{s+1}A_i \quad 0 \leq i \leq k-2 \\ B_{k-1} &= B \\ B_k &= \Sigma^{s+1}A_{k-1}. \end{aligned}$$

By forming telescoping mapping cylinders we have

$$\emptyset = B_{-1} \subset B_0 \subset B_1 \subset \ldots \subset B_{k-2} \subset B_{k-1} \subset B_k \sim \Sigma^{l+s+1}X.$$

Note that

$$\begin{tikzcd}
\Omega^U_*(B_{k-1}, B_{k-2}) \arrow[r, phantom, "\cong"] \arrow[d] & \widetilde{\Omega}^U_*(\Sigma A) \arrow[d, "(\Sigma f)"] \\
\Omega^U_*(B_{k-1}, B_{k-2}) \arrow[r, phantom, "\cong"] & \Omega^U_*(\Sigma^{s+1}(A_{k-1}/A_{k-2})) \arrow[d] \\
& 0
\end{tikzcd}$$

and hence $\Omega^U_*(B_{k-1}, B_{k-2}) \to \Omega^U_*(B_k, B_{k-2})$ is onto.

Thus we have by construction that

1) $\Omega^U_*(B_i, B_{i-1})$ is a free Ω^U_*-module for $i = 0, 1, \ldots, k-1$, and
2) $\Omega^U_*(B_i, B_{i-1}) \to \Omega^U_*(B_k, B_{i-1})$ is an epimorphism for $i = 0, \ldots, k$.

Therefore

$$\emptyset = B_{-1} \subset B_0 \subset \ldots \subset B_{k-1} \subset B_k \sim \Sigma^{l+s+1}X$$

is a partial U-bordism resolution of X of length k.

This completes the inductive step and hence the proof. □

Theorem **(2.6)**. — *Let* X *be a finite complex. Then there exists a* U*-bordism resolution of* X.

Proof. — By Theorem (1.6) $\Omega^U_*(X)$ has finite projective dimension as an Ω^U_*-module. Let $\mathrm{hom.dim}_{\Omega_*}\Omega^U_*(X) = k$. By Proposition (2.5) we may choose a partial U-bordism resolution of X of length k, say

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \subset \ldots \subset A_k \sim \Sigma^l X.$$

By Proposition (2.1) we then have the exact sequence

$$0 \leftarrow \Omega^U_*(A_k) \leftarrow \Omega^U_*(A_0, \emptyset) \leftarrow \ldots \leftarrow \Omega^U_*(A_{k-1}, A_{k-2}) \leftarrow \Omega^U_*(A_k, A_{k-1}) \leftarrow 0$$

where $\Omega^U_*(A_i, A_{i-1})$ are projective Ω^U_*-modules for $i = 0, \ldots, k-1$.

Since $\Omega^U_*(A_k) \cong \Omega^U_*(\Sigma^l X)$ it follows that

$$\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(A_k) = \mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(\Sigma^l X) = \mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X) = k.$$

Therefore by ([5], VI, (2.1)), $\Omega^U_*(A_k, A_{k-1})$ is a projective Ω^U_*-module and the result follows from the definitions. □

Remark. — Let X be a finite complex. Then $\hom.\dim_{\Omega_*^U}\Omega_*^U(X)$ is finite, say equal to n. Suppose that

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \subset \ldots \subset A_k \sim \Sigma^l X$$

is a partial U-bordism resolution of X of length $k \geq n$. Then the argument employed above shows that

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \subset \ldots \subset A_k \sim \Sigma^l X$$

is actually a U-bordism resolution of X.

Hence if X admits a U-bordism resolution of length k, then any partial U-bordism resolution of X of length k is already a U-bordism resolution of X.

§ 3. The Thom Homomorphism.

The results of this section center around the Thom homomorphism whose definition we now recall.

Definition. — *Let* X *be a space and* $[M,f] \in \Omega_*^U(X)$. *Then* $\mu([M,f]) = f_*([M])$, *where* $[M] \in H_*(M; \mathbf{Z})$ *is the orientation class of* M.

It is easy to verify ([3], [8], [22]) that

$$\mu : \Omega_*^U(\cdot) \to H_*(\cdot\,; \mathbf{Z})$$

is a natural homomorphism of homology theories. We will be concerned with conditions that assure μ is an epimorphism.

Let $\varepsilon : \Omega_*^U \to \mathbf{Z}$ be the augmentation. If we regard $\mathbf{Z}$ as an Ω_*^U-module via ε then the Thom homomorphism is easily seen to induce a natural homomorphism of functors

$$\tilde{\mu} : \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(\cdot) \to H_*(\cdot\,; \mathbf{Z}).$$

As an elementary consequence of the lack of torsion in Ω_*^U and the triviality of the bordism spectral sequence mod torsion we have:

Lemma **(3.1)**. — *Suppose that* X *is a complex with* $H_*(X; \mathbf{Z})$ *free abelian. Then*

1) *the* U-*bordism spectral sequence for* X *collapses*;

2) $\Omega_*^U(X)$ *is a free* Ω_*^U-*module, and*

3) *the homomorphism induced by the Thom map*

$$\tilde{\mu} : \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(X) \to H_*(X; \mathbf{Z})$$

is an isomorphism. □

Comments on Gradings. — We will have occasions to consider various types of gradings in the sequel. Rather than dwell at length on the point we will adopt the following conventions. If **K** is a ring a graded **K**-module will mean a collection $\{M_i \mid i \in \mathbf{Z}\}$ of **K**-modules, where $M_i = 0$ for i large and negative. Morphisms are defined componentwise. We will follow the usual sign conventions governing raising and lowering indices [13]. All other terminology is as in [14].

Proposition (**3.2**). — *Let* **K** *be a ring and* Ω *a graded connected* **K**-*algebra. Suppose that projective* **K**-*modules are free modules. Then projective graded* Ω-*modules are free graded* Ω-*modules.*

Proof (1). — Let M be a graded **K**-module. Regard **K** as an Ω-module via the augmentation $\varepsilon : \Omega \to \mathbf{K}$. Let $Q(M) = \mathbf{K} \otimes_\Omega M$. If $f : M \to N$ is a morphism of graded **K**-modules let $Q(f) = \mathbf{K} \otimes_\Omega f : Q(M) \to Q(N)$. We shall need the following elementary lemma (see e.g. [17]).

Lemma. — *Let* $f : M \to N$ *be a morphism of graded* Ω-*modules. Then f is an epimorphism of* Ω-*modules iff* $Qf : QM \to QN$ *is an epimorphism of* **K**-*modules.* ☐

Now suppose that M is a projective Ω-module. Then there exists an Ω-module N such that $M \oplus N$ is a free Ω-module. For any free Ω-module F, one easily sees that QF is a free **K**-module. Thus $Q(M \oplus N) = QM \oplus QN$ is a free **K**-module. Hence QM is a projective **K**-module. Our hypothesis on **K** yields that QM is a free **K**-module. Let $F = QM \otimes_{\mathbf{K}} \Omega$. If $\{a_i \mid i \in I\}$ is a free **K**-basis for QM then one checks that $\{\alpha_i = a_i \otimes 1\}$ is a free Ω-basis for F. Choose elements $b_i \in M$ with $1 \otimes b_i = a_i \in QM$. Define a map of Ω-modules $\varphi : F \to M$ by setting $\varphi(\alpha_i) = b_i$ and requiring φ to be a map of Ω-modules. Then $Q(\varphi) : QF \to QM$ is an isomorphism. Thus $\varphi : F \to M$ is an epimorphism. Since M is a projective Ω-module there exists $\psi : M \to F$ such that $\varphi \circ \psi = 1_M$. Thus $1 = Q(\varphi)Q(\psi) : QM$ and hence $Q(\varphi)$ is an isomorphism. Thus $\psi : M \to F$ is an epimorphism, and since $\psi\varphi\psi = \psi$ we may conclude $\psi\varphi = 1_F$, by right cancellation. Thus φ and ψ are inverse isomorphisms and hence M is a free Ω-module. ☐

Proposition (**3.3**). — *Let* X *be a finite complex. Then the following conditions are equivalent*:

1) $\Omega^U_*(X)$ *is a projective* Ω^U_*-*module*;

2) $\Omega^U_*(X)$ *is a free* Ω^U_*-*module*;

3) $H_*(X; \mathbf{Z})$ *is a free abelian group.*

Proof. — The equivalence of 1) and 2) follows instantly from Proposition (3.2) and the structure of Ω^U_* as determined by Milnor and Novikov. The implication 3) ⇒ 2) was recorded in Lemma (2.3) and so it remains for us to show that 2) ⇒ 3).

So let us suppose that X is a finite complex with $\Omega^U_*(X)$ a free Ω^U_*-module. Suppose to the contrary that $H_*(X; \mathbf{Z})$ is not free abelian. Then $H_*(X; \mathbf{Z})$ has torsion. We denote by $v \in H_*(X; \mathbf{Z})$ a torsion element of lowest dimension. Consider the bordism spectral sequence ([8], [2])

$$E^r \Rightarrow \Omega^U_*(X)$$
$$E^2 = H_*(X; \Omega^U_*) \cong H_*(X; \mathbf{Z}) \otimes \Omega^U_*,$$

as Ω^U_* is a torsion free **Z**-module. Recall [8] that the differentials of this spectral sequence are torsion valued. A simple degree check therefore shows that $v \in E^2_{*,0}$ is an infinite cycle and hence v is in the image of the edge map

$$\Omega^U_*(X) \to H_*(X; \mathbf{Z}).$$

---

(1) This proof was suggested by J. F. Adams.

Thus there exists a class

$$\alpha \in \mathbf{Z} \otimes_{\Omega^U_*} \Omega^U_*(X)$$

with

$$\tilde{\mu}(\alpha) = v.$$

Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{Z} \otimes_{\Omega^U_*} \Omega^U_*(X) \arrow[r, "\tilde{\mu}"] \arrow[d, "\xi"] & H_*(X; \mathbf{Z}) \arrow[d, "\zeta"] \\
\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z} \otimes_{\Omega^U_*} \Omega^U_*(X) \arrow[r, "\cong", "\hat{\mu}"'] & H_*(X; \mathbf{Q}).
\end{tikzcd}$$

Since $\Omega^U_*(X)$ is a free Ω^U_*-module it follows that $\mathbf{Z} \otimes_{\Omega^U_*} \Omega^U_*(X)$ is a free abelian group and hence ξ is a monomorphism. However

$$\hat{\mu}\xi(\alpha) = \zeta\tilde{\mu}(\alpha) = \zeta(v) = 0.$$

Since ξ and $\hat{\mu}$ are monic this implies $\alpha = 0$. Hence $v = \tilde{\mu}(\alpha) = 0$ contrary to the choice of v. Therefore our original supposition must be false and hence $H_*(X; \mathbf{Z})$ is free abelian as required. □

Remark. — The equivalence of 1) and 2) of Corollary (3.3) is quite useful and will be used often in the sequel without explicit reference.

Proposition **(3.4)**. — *Let* X *be a finite complex. Suppose that the Thom homomorphism*

$$\mu : \Omega^U_*(X) \to H_*(X; \mathbf{Z})$$

is an epimorphism. Let

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \sim \Sigma^l X$$

be a partial U*-bordism resolution of* X *of length* 1. *Then*

$$0 \to \Omega^U_*(A_1, A_0) \to \Omega^U_*(A_0) \to \Omega^U_*(A_1) \to 0$$

is a free resolution of $\Omega^U_*(A_1)$ *as an* Ω^U_**-module. Hence*

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \sim \Sigma^l X$$

is a U*-bordism resolution of* X.

Proof. — Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^U_*(A_1, A_0) \arrow[r] \arrow[d, "\mu_{0,1}"] & \Omega^U_*(A_0) \arrow[r, "j_*"] \arrow[d, "\mu_0"] & \Omega^U_*(A_1) \arrow[r] \arrow[d, "\mu_1"] & 0 \\
& H_*(A_1, A_0; \mathbf{Z}) \arrow[r, "\partial_*"] & H_*(A_0; \mathbf{Z}) \arrow[r, "i_*"] & H_*(A_1; \mathbf{Z}) &
\end{tikzcd}$$

By hypothesis the Thom map $\mu : \Omega_*^{\mathrm{U}}(\mathrm{X}) \rightarrow \mathrm{H}_*(\mathrm{X}; \mathbf{Z})$ is onto. Since the Thom homomorphism is stable it follows that $\mu : \Omega_*^{\mathrm{U}}(\Sigma^l \mathrm{X}) \rightarrow \mathrm{H}_*(\Sigma^l \mathrm{X}; \mathbf{Z})$ is onto. Since A_1 has the same homotopy type as $\Sigma^l \mathrm{X}$ it follows that μ_1 is also onto. Thus $\mu_1 j_*$ is an epimorphism and commutativity of the right hand square implies that i_* is onto. The exact homology triangle of $(\mathrm{A}_1, \mathrm{A}_0)$ then yields that ∂_* is a monomorphism. By Lemma (3.2) $\mathrm{H}_*(\mathrm{A}_0; \mathbf{Z})$ is free abelian. Thus we find that $\mathrm{H}_*(\mathrm{A}_1, \mathrm{A}_0; \mathbf{Z})$ is free abelian. Hence by Lemma (3.1) $\Omega_*^{\mathrm{U}}(\mathrm{A}_1, \mathrm{A}_0)$ is a free Ω_*^{U}-module and the result follows from the definition of a partial U-bordism resolution of X. ☐

Corollary **(3.5)**. — *Let* X *be a finite complex. Suppose that the Thom homomorphism*

$$\mu : \Omega_*^{\mathrm{U}}(\mathrm{X}) \rightarrow \mathrm{H}_*(\mathrm{X}; \mathbf{Z})$$

is onto. Then the projective dimension of $\Omega_*^{\mathrm{U}}(\mathrm{X})$ *as an* Ω_*^{U}*-module is either* 0 *or* 1. ☐

Proposition **(3.6)**. — *Let* X *be a finite complex. Suppose that the Thom homomorphism*

$$\mu : \Omega_*^{\mathrm{U}}(\mathrm{X}) \rightarrow \mathrm{H}_*(\mathrm{X}; \mathbf{Z})$$

is onto. Then the bordism spectral sequence of X *collapses.*

Proof. — Since the bordism spectral sequence is stable under suspension it suffices to show that the bordism spectral sequence of some suspension of X collapses.

Choose a U-bordism resolution (Proposition (3.4)) of X

$$\emptyset = \mathrm{A}_1 \subset \mathrm{A}_0 \subset \mathrm{A}_1 \sim \Sigma^l \mathrm{X}.$$

For any complex Y let $\{\mathrm{E}^r(\mathrm{Y}), d^r(\mathrm{Y})\}$ denote the bordism spectral sequence of Y, i.e.,

$$\mathrm{E}^r(\mathrm{Y}) \Rightarrow \Omega_*^{\mathrm{U}}(\mathrm{Y})$$
$$\mathrm{E}_{pq}^2(\mathrm{Y}) = \mathrm{H}_p(\mathrm{Y}; \Omega_q^{\mathrm{U}}).$$

Note that since Ω_*^{U} is torsion free we have $\mathrm{H}_*(\mathrm{Y}; \Omega_*^{\mathrm{U}}) \cong \mathrm{H}_*(\mathrm{Y}; \mathbf{Z}) \otimes_{\mathbf{Z}} \Omega_*^{\mathrm{U}}$.

Consider the map of spectral sequences

$$\{\mathrm{E}^r(\mathrm{A}_0), d^r(\mathrm{A}_0)\} \rightarrow \{\mathrm{E}^r(\mathrm{A}_1), d^r(\mathrm{A}_1)\}.$$

Since $\Omega_*^{\mathrm{U}}(\mathrm{A}_0)$ is a free Ω_*^{U}-module $\mathrm{H}_*(\mathrm{A}_0; \mathbf{Z})$ is free abelian (Proposition (3.2)). Hence by Lemma (3.1) $\mathrm{E}^2(\mathrm{A}_0) = \mathrm{E}^\infty(\mathrm{A}_0)$. As in the proof of Proposition (3.4) the commutative diagram

$$\begin{tikzcd}
\Omega_*^{\mathrm{U}}(\mathrm{A}_0) \arrow[r] \arrow[d, "\mu_0"] & \Omega_*^{\mathrm{U}}(\mathrm{A}_1) \arrow[r] \arrow[d, "\mu_1"] & 0 \\
\mathrm{H}_*(\mathrm{A}_0; \mathbf{Z}) \arrow[r] & \mathrm{H}_*(\mathrm{A}_1; \mathbf{Z}) \arrow[d] & \\
 & 0 &
\end{tikzcd}$$

shows that $H_*(A_0;\mathbf{Z}) \to H_*(A_1;\mathbf{Z})$ is onto. Hence $E^2(A_0) \to E^2(A_1)$ is onto. Since $E^2(A_0)=E^\infty(A_0)$ the result now follows. □

Theorem **(3.7)**. — *Let* X *be a finite complex. Suppose that the Thom homomorphism*

$$\mu : \Omega^U_*(X) \to H_*(X;\mathbf{Z})$$

is onto. Then
$$\mathrm{Tor}^{\Omega^U_*}_{1,*}(\mathbf{Z}, \Omega^U_*(X)) = 0.$$

Proof. — Choose a partial U-bordism resolution of X

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \sim \Sigma^l X.$$

Then $\Omega^U_*(A_1) \cong \Omega^U_*(\Sigma^l X) \cong s^l \Omega^U_*(X)$. Hence $\mathrm{Tor}^{\Omega^U_*}_{1,*}(\mathbf{Z}, \Omega^U_*(X))$ and $\mathrm{Tor}^{\Omega^U_*}_{1,*}(\mathbf{Z}, \Omega^U_*(A_1))$ are isomorphic with a dimension shift. Thus it suffices to show $\mathrm{Tor}^{\Omega^U_*}_{1,*}(\mathbf{Z}, \Omega^U_*(A_1)) = 0$.

Since
$$\emptyset = A_{-1} \subset A_0 \subset A_1 \sim \Sigma^l X$$

is a partial U-bordism resolution we have the exact sequence

$$0 \to \Omega^U_*(A_1, A_0) \to \Omega^U_*(A_0) \to \Omega^U_*(A_1) \to 0.$$

Thus we obtain an exact sequence

$$\begin{tikzcd}
0 = \mathrm{Tor}^{\Omega^U_*}_{1,*}(\mathbf{Z}, \Omega^U_*(A_0)) \arrow[r] & \mathrm{Tor}^{\Omega^U_*}_{1,*}(\mathbf{Z}, \Omega^U_*(A_1)) \arrow[dll] & & \\
\mathbf{Z}\otimes_{\Omega^U_*} \Omega^U_*(A_1, A_0) \arrow[r, "\theta"] & \mathbf{Z}\otimes_{\Omega^U_*} \Omega^U_*(A_0) \arrow[r] & \mathbf{Z}\otimes_{\Omega^U_*} \Omega^U_*(A_1) \arrow[r] & 0
\end{tikzcd}$$

and hence
$$\mathrm{Tor}^{\Omega^U_*}_{1,*}(\mathbf{Z}, \Omega^U_*(A_1)) \cong \ker \theta.$$

Consider next the commutative diagram

$$\begin{tikzcd}
\mathbf{Z}\otimes_{\Omega^U_*} \Omega^U_*(A_1, A_0) \arrow[r, "\theta"] \arrow[d, "\tilde\mu_{0,1}"] & \mathbf{Z}\otimes_{\Omega^U_*} \Omega^U_*(A_0) \arrow[r] \arrow[d, "\tilde\mu_0"] & \mathbf{Z}\otimes_{\Omega^U_*} \Omega^U_*(A_1) \arrow[r] \arrow[d, "\tilde\mu_1"] & 0 \\
H_*(A_1, A_0;\mathbf{Z}) \arrow[r, "\partial_*"] & H_*(A_0;\mathbf{Z}) \arrow[r, "i_*"] & H_*(A_1;\mathbf{Z}) &
\end{tikzcd}$$

By hypothesis $\mu : \Omega^U_*(X) \to H_*(X;\mathbf{Z})$ is onto and hence, as we have seen before, so is $\tilde\mu_1$. Thus commutativity of the right hand square shows that i_* is onto. The exact homology triangle of (A_1, A_0) then yields that ∂_* is monic. By Proposition (3.4) $\Omega^U_*(A_1, A_0)$ is a free Ω^U_*-module. Hence by Proposition (3.2) and Lemma (3.1) $\tilde\mu_{0,1}$ is an isomorphism. Thus the commutativity of the left hand square shows that θ is a monomorphism. Hence $\ker\theta = 0$ and the result follows. □

Theorem **(3.8)**. — *Let* X *be a finite complex. Suppose that n is the lowest dimension in which the Thom map*

$$\mu : \Omega^U_*(X) \to H_*(X;\mathbf{Z})$$

fails to be onto. Then

$$\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{1,m}(\mathbf{Z},\Omega^{\mathrm{U}}_*(\mathrm{X}))\begin{cases}=0 & \text{if } 0\leq m\leq n-2\\ \neq 0 & \text{if } m=n-1.\end{cases}$$

Proof. — Choose a partial U-bordism resolution of X of length 1:

$$\emptyset=\mathrm{A}_{-1}\subset\mathrm{A}_0\subset\mathrm{A}_1\sim\Sigma^l\mathrm{X}.$$

Note that the lowest dimension in which the Thom map

$$\mu:\Omega^{\mathrm{U}}_*(\mathrm{A}_1)\to\mathrm{H}_*(\mathrm{A}_1;\mathbf{Z})$$

fails to be onto is $n+l$. Note also that

$$\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{1,p}(\mathbf{Z},\Omega^{\mathrm{U}}_*(\mathrm{X}))\cong\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{1,p+l}(\mathbf{Z},\Omega^{\mathrm{U}}_*(\mathrm{A}_1)).$$

We shall need the following:

Lemma. — The Thom map

$$\mu:\Omega^{\mathrm{U}}(\mathrm{A}_1,\mathrm{A}_0)\to\mathrm{H}(\mathrm{A}_1,\mathrm{A}_0;\mathbf{Z})$$

is onto in dimensions $\leq n+l+1$.

Proof. — Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{\mathrm{U}}_*(\mathrm{A}_1,\mathrm{A}_0) \arrow[r] \arrow[d, "\mu_{1,0}"] & \Omega^{\mathrm{U}}_*(\mathrm{A}_0) \arrow[r] \arrow[d, "\mu_0"] & \Omega^{\mathrm{U}}_*(\mathrm{A}_1) \arrow[r] \arrow[d, "\mu_1"] & 0 \\
 & \mathrm{H}_*(\mathrm{A}_1,\mathrm{A}_0;\mathbf{Z}) \arrow[r, "\partial_*"] & \mathrm{H}_*(\mathrm{A}_0;\mathbf{Z}) \arrow[r, "i_*"] & \mathrm{H}_*(\mathrm{A}_1;\mathbf{Z}) &
\end{tikzcd}$$

We have seen μ_1 is onto in dimensions $\leq n+l-1$. Hence by commutativity of the right hand square i_* is onto in dimensions $\leq n+l-1$. The exact homology triangle of $(\mathrm{A}_1,\mathrm{A}_0)$ therefore yields that ∂_* is monic in dimensions $\leq n+l-1$. By Proposition (3.2) $\mathrm{H}_*(\mathrm{A}_0;\mathbf{Z})$ is free abelian in dimensions $\leq n+l-1$. Thus the lowest possible dimensional torsion element in $\mathrm{H}_*(\mathrm{A}_1,\mathrm{A}_0;\mathbf{Z})$ occurs in a dimension $\geq n+l$.

Consider the bordism spectral sequence of $(\mathrm{A}_1,\mathrm{A}_0)$. Recall that the differentials are torsion valued. Note that the torsion occurs in $\mathrm{E}^2_{p,q}$ with $p\geq n+l$. Since $\deg(d^r)=(-r,r-1)$ a simple degree check shows that any element of $\mathrm{E}^2_{p,0}$ for $p\leq n+l+1$ is an infinite cycle. Since the Thom homomorphism is the edge map in the spectral sequence the lemma follows.**

We return now to the proof of Theorem (3.8).

Consider the exact sequence

$$\begin{tikzcd}
0=\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{1,*}(\mathbf{Z},\Omega^{\mathrm{U}}_*(\mathrm{A}_0)) \arrow[r] & \mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{1,*}(\mathbf{Z},\Omega^{\mathrm{U}}_*(\mathrm{A}_1)) \arrow[dl] & \\
\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_1,\mathrm{A}_0) \arrow[r, "\theta"] & \mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_0) \arrow[r] & \mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_1) \arrow[r] & 0
\end{tikzcd}$$

and observe (since θ has degree -1)

$$\mathrm{Tor}_{1,*}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}, \Omega_*^{\mathrm{U}}(\mathrm{A}_1)) \cong \ker \theta$$

as graded abelian groups by an isomorphism of degree 1.

Consider the diagram

$$\begin{tikzcd}
\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{A}_1, \mathrm{A}_0) \arrow[r, "\theta"] \arrow[d, "\tilde{\mu}_{1,0}"] & \mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{A}_0) \arrow[r] \arrow[d, "\tilde{\mu}_0", "\cong"'] & \mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{A}_1) \arrow[r] \arrow[d, "\tilde{\mu}_1"] & 0 \\
\mathrm{H}_*(\mathrm{A}_1, \mathrm{A}_0; \mathbf{Z}) \arrow[r, "\partial_*"] & \mathrm{H}_*(\mathrm{A}_0; \mathbf{Z}) \arrow[r, "i_*"] & \mathrm{H}_*(\mathrm{A}_1; \mathbf{Z}) \arrow[ll, bend left, "j_*"] &
\end{tikzcd}$$

Let $u \in \mathrm{H}_{n+l}(\mathrm{A}_1; \mathbf{Z})$ be a non-representable class (such a class exists by hypothesis). Then $j_*(u) \neq 0$. For if $j_*(u) = 0$ then $u = i_*(v)$ and commutativity of the right hand square shows that u is representable. Thus $j_*(u) \neq 0 \in \mathrm{H}_*(\mathrm{A}_1, \mathrm{A}_0; \mathbf{Z})$. By the lemma there exists $\alpha \in \mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{A}_1, \mathrm{A}_0)$ with $\tilde{\mu}_{1,0}(\alpha) = j_*(u) \neq 0$. Commutativity of the left hand square yields

$$\tilde{\mu}_0 \theta(\alpha) = \partial_* \tilde{\mu}_{1,0}(\alpha) = \partial_* j_*(u) = 0.$$

Since $\tilde{\mu}_0$ is monic this implies $\theta(\alpha) = 0$ and thus $\alpha \neq 0 \in \mathrm{Tor}_{1,n+l-1}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}, \Omega_*^{\mathrm{U}}(\mathrm{A}_1))$.

To complete the proof note that as shown in the lemma $\mathrm{H}_*(\mathrm{A}_1, \mathrm{A}_0; \mathbf{Z})$ is free abelian and ∂_* is monic, in dimensions $\leq n+l-1$. Commutativity of the left hand square fields shows that θ is monic in dimensions $\leq n+l-1$. Hence we have

$$\mathrm{Tor}_{1,m}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}, \Omega_*^{\mathrm{U}}(\mathrm{A}_1)) \begin{cases} = 0 & \text{if} \quad m \leq n+l-2 \\ \neq 0 & \text{if} \quad m = n+l-1 \end{cases}$$

and the theorem follows by stability. □

Corollary (3.9). — *Let* X *be a finite complex. Then the Thom map*

$$\mu : \Omega_*^{\mathrm{U}}(\mathrm{X}) \to \mathrm{H}_*(\mathrm{X}; \mathbf{Z})$$

is onto iff

$$\mathrm{Tor}_{1,*}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}, \Omega_*^{\mathrm{U}}(\mathrm{X})) = 0.$$

Proof. — An instant consequence of Theorem (3.7) and Theorem (3.8). □

Remark. — Actually we have shown more. Namely if X is a finite complex then the Thom map

$$\mu : \Omega_m^{\mathrm{U}}(\mathrm{X}) \to \mathrm{H}_m(\mathrm{X}; \mathbf{Z})$$

is onto for $m<n$ iff

$$\mathrm{Tor}_{1,m}^{\Omega_*^U}(\mathbf{Z}, \Omega_*^U(X))=0$$

for $m\leq n$.

We summarize the results of this section in the following two results.

Corollary **(3.10)**. — *Let* X *be a finite complex. Then the following conditions are equivalent*:

1) $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)=0$;
2) $\Omega_*^U(X)$ *is a projective* Ω_*^U*-module*;
3) $\Omega_*^U(X)$ *is a free* Ω_*^U*-module*;
4) $H_*(X;\mathbf{Z})$ *is free abelian.*

Proof. — The equivalence of 1) and 2) may be found in ([18], Corollary (VI.2.2)) while the rest follows from Proposition (3.2) and Corollary (3.3). □

Corollary **(3.11)**. — *Let* X *be a finite complex. Then the following conditions are equivalent*:

1) *the Thom homomorphism*

$$\mu : \Omega_*^U(X) \to H_*(X;\mathbf{Z})$$

is onto;

2) *the reduced Thom Homomorphism*

$$\tilde{\mu} : \mathbf{Z}\otimes\Omega_*^U(X) \to H_*(X;\mathbf{Z})$$

is an isomorphism;

3) *the bordism spectral sequence of* X *collapses*;
4) $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)\leq 1$;
5) $\mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}, \Omega_*^U(X))=0$;
6) $\mathrm{Tor}_{p,*}^{\Omega_*^U}(\mathbf{Z}, \Omega_*^U(X))=0$ *for all* $p\geq 1$.

Proof. — The equivalence of 1), 2) and 3) is an elementary consequence of Proposition (3.6). The equivalence of 1) and 5) is Corollary (3.9). Notice that 6) follows from 4) and 5) and that 1) follows from 6). Thus it suffices to prove the equivalence of 1) and 4).

Note first that 1) implies 4) by Corollary (3.5). Thus all that remains to be shown is that 4) implies 1).

Let $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)\leq 1$. Let

$$\emptyset=A_{-1}\subset A_0\subset A_1\sim\Sigma^l X.$$

be a partial U-bordism resolution of X of length 1. We then have an exact sequence

$$0\to\Omega_*^U(A_1, A_0)\to\Omega_*^U(A_0)\to\Omega_*^U(A_1)\to 0.$$

Since $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(A_1)=\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)=1$ it follows from the definition of a partial U-bordism resolution and ([5], VI, (2.1), *d)*) that $\Omega_*^U(A_1, A_0)$ is a projective Ω_*^U-module. Hence by Corollary (3.10) $\Omega_*^U(A_1, A_0)$ is a free Ω_*^U-module.

Thus by Lemma (3.1) and Proposition (3.2) we have a commutative diagram wherein $\mu_{1,0}$ and μ_0 are epimorphisms:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^U_*(A_1, A_0) \arrow[r] \arrow[d, "\text{epic}"', "\mu_{1,0}"] & \Omega^U_*(A_0) \arrow[r] \arrow[d, "\text{epic}"', "\mu_0"] & \Omega^U_*(A_1) \arrow[r] \arrow[d, "\mu_1"] & 0 \\
& H_*(A_1, A_0; \mathbf{Z}) \arrow[r] & H_*(A_0; \mathbf{Z}) \arrow[r] & H_*(A_1; \mathbf{Z}) \arrow[ll, bend left=60, "j_*"] &
\end{tikzcd}$$

We assert that μ_1 is onto. For suppose that $u \in H_*(A_1; \mathbf{Z})$ is a non-representable class. Then commutativity of the right hand square shows that $u \notin \operatorname{im} i_*$. Hence $j_*(u) \neq 0 \in H_*(A_1, A_0; \mathbf{Z})$. Since some multiple of u is representable it follows from commutativity of the right hand square that there is an integer $n \neq 0$ such that $n.u \in \operatorname{im} i_*$. Hence $n.j_*(u) = 0 \in H_*(A_1, A_0; \mathbf{Z})$ and thus $j_*(u) \in H_*(A_1, A_0; \mathbf{Z})$ is a non-zero torsion element. However by Proposition (3.2) $H_*(A_1, A_0; \mathbf{Z})$ is free abelian. This is a contradiction. Therefore $u = 0$ and hence μ_1 is onto. Since A_1 has the homotopy type of $\Sigma^l X$ it follows by stability that $\mu : \Omega^U_*(X) \to H_*(X; \mathbf{Z})$ is onto as required. ☐

The properties of the Thom homomorphism discussed in the introduction follow easily from Corollary (3.11).

As an example of how our results on U-bordism representability may be applied we will consider an application to the study of periodic transformations of prime order on closed U-manifolds. The standard references for background results are [8] and [7].

Let p denote a fixed prime. Recall that a periodic transformation of order p on a U-manifold V, is a homeomorphism

$$T : V \to V$$

that preserves the U-structure of V and such that $T^p = \mathrm{id}$. Such a periodic transformation is usually denoted by (T, V). Proceeding in the naive manner we obtain the module $\Omega^U_*(\mathbf{Z}_P)$ of weakly complex bordism classes of fixed point free periodic transformations of order p on closed U-manifolds. If (T, V) is a fixed point free periodic transformation of order p on a closed U-manifold V, then the orbit map

$$\pi_T : V \to V/T$$

is a principal $\mathbf{Z}_p$-bundle and hence is classified by a mapping

$$\zeta_T : V/T \to B\mathbf{Z}_p$$

where $B\mathbf{Z}_p$ denotes a classifying space for the group $\mathbf{Z}_p$ which for definiteness we may take to be the infinite dimensional lens space $L(p, \infty) = S^\infty/\mathbf{Z}_p$. As V/T is again a closed U-manifold there results a map

$$\varphi : \Omega_*^U(\mathbf{Z}_p) \to \Omega_*^U(B\mathbf{Z}_p)$$

which is easily seen to be an isomorphism.

Next we form $\mathcal{O}_*^U(\mathbf{Z}_p)$, the bordism algebra of all (i.e., no restriction on the fixed point set) periodic transformations of period p on closed U-manifolds. Finally we form $\mathcal{M}_*^U(\mathbf{Z}_p)$, the bordism algebra of pairs (T, W), where T is a period p transformation on the compact U-manifold W with no fixed points in ∂W.

As one would naively expect there is an exact triangle

$$\begin{tikzcd} \Omega_*^U(\mathbf{Z}_p) \arrow[rr, "\beta"] & & \mathcal{O}_*^U(\mathbf{Z}_p) \arrow[dl, "\alpha"] \\ & \mathcal{M}_*^U(\mathbf{Z}_p) \arrow[ul, "\partial"] & \end{tikzcd}$$

wherein β and α are forgetful morphisms and ∂ is restriction to the boundary.

Now if (T, V) is a period p transformation on the U-manifold V and M is a closed U-manifold then

$$1 \times T : M \times V \to M \times V$$

is a period p transformation on the U-manifold $M \times V$. If moreover T is fixed point free (has no fixed points in ∂V) then $(1 \times T, M \times V)$ is fixed point free (has no fixed points in $\partial(M \times V) = M \times \partial V$). If we therefore write

$$M \times (T, V) = (1 \times T, M \times V)$$

we find this compatible with the cobordism relation, and passing to cobordism classes we obtain an Ω_*^U-module structure on $\mathcal{M}_*^U(\mathbf{Z}_p)$, $\mathcal{O}_*^U(\mathbf{Z}_p)$ and $\Omega_*^U(\mathbf{Z}_p)$. Under the identification $\varphi : \Omega_*^U(\mathbf{Z}_p) \cong \Omega_*^U(B\mathbf{Z}_p)$, we find that the Ω_*^U-module structures coincide.

We recall [8] that the Thom homomorphism

$$\mu : \Omega_*^U(B\mathbf{Z}_p) \to H_*(B\mathbf{Z}_p; \mathbf{Z})$$

is onto, and that the lens spaces $[L(p, 2n+1)]$ form a generating set for $\widetilde{\Omega}_*^U(B\mathbf{Z}_p)$. Thus the natural period p-transformations on the spheres (T_p, S^{2n+1}) form a generating set for $\widetilde{\Omega}_*^U(\mathbf{Z}_p)$ as an Ω_*^U-module. As the natural period p transformation on the closed $2n$-cell I^{2n}, (T_p, I^{2n}) bounds (T_p, S^{2n-1}) we find that

$$\partial : \mathcal{M}_*^U(\mathbf{Z}_p) \to \widetilde{\Omega}_*^U(B\mathbf{Z}_p)$$

is surjective. Thus the above exact triangle reduces to the following fundamental exact sequence:

$$(*) \qquad 0 \to \Omega_*^{\mathrm{U}} \xrightarrow{\alpha} \mathcal{O}_*^{\mathrm{U}}(\mathbf{Z}_p) \xrightarrow{\beta} \mathcal{M}_*^{\mathrm{U}}(\mathbf{Z}_p) \xrightarrow{\partial} \tilde{\Omega}_*^{\mathrm{U}}(\mathbf{Z}_p) \to 0$$

of Ω_*^{U}-modules [8].

Proposition **(3.12).** — *For each prime p the bordism algebra of arbitrary period p transformations on closed* U*-manifolds is a free Ω_*^{U}-module.*

Proof. — Let us define $\tilde{\mathcal{O}}_*^{\mathrm{U}}(\mathbf{Z}_p)$ via the exact sequence

$$0 \to \Omega_*^{\mathrm{U}} \xrightarrow{\alpha} \mathcal{O}_*^{\mathrm{U}}(\mathbf{Z}_p) \to \tilde{\mathcal{O}}_*^{\mathrm{U}}(\mathbf{Z}_p) \to 0.$$

Clearly it will suffice to show that $\tilde{\mathcal{O}}_*^{\mathrm{U}}(\mathbf{Z}_p)$ is a free Ω_*^{U}-module.

Let us next recall [8] that an analaysis of the fixed point set leads to an isomorphism of Ω_*^{U}-modules

$$\mathcal{M}_*^{\mathrm{U}}(\mathbf{Z}_p) \cong \Omega_*^{\mathrm{U}}(\underbrace{\mathrm{BU} \times \mathrm{BU} \times \ldots \times \mathrm{BU}}_{q\text{-times}})$$

where $q = (p-1)/2$. Thus since $\mathrm{H}_*(\mathrm{BU}; \mathbf{Z})$ is a free $\mathbf{Z}$-module, we find in view of (3.10) that $\mathcal{M}_*^{\mathrm{U}}(\mathbf{Z}_p)$ is a free Ω_*^{U}-module.

Next recall [8], as noted above, that the Thom homomorphism

$$\mu : \Omega_*^{\mathrm{U}}(\mathrm{B}\mathbf{Z}_p) \to \mathrm{H}_*(\mathrm{B}\mathbf{Z}_p; \mathbf{Z})$$

is onto. Hence by Corollary (3.11) we deduce that $\tilde{\Omega}_*^{\mathrm{U}}(\mathbf{Z}_p) \cong \tilde{\Omega}_*^{\mathrm{U}}(\mathrm{B}\mathbf{Z}_p)$ has projective dimension 1 as an Ω_*^{U}-module.

Consider now the exact sequence

$$0 \to \tilde{\mathcal{O}}_*^{\mathrm{U}}(\mathbf{Z}_p) \to \mathcal{M}_*^{\mathrm{U}}(\mathbf{Z}_p) \to \tilde{\Omega}_*^{\mathrm{U}}(\mathbf{Z}_p) \to 0$$

of Ω_*^{U}-modules. We have seen:

1) $\mathcal{M}_*^{\mathrm{U}}(\mathbf{Z}_p)$ is a free Ω_*^{U}-module;

2) $\tilde{\Omega}_*^{\mathrm{U}}(\mathbf{Z}_p)$ has projective dimension 1 as an Ω_*^{U}-module.

Therefore by ([5], VI, (2.1)) $\mathcal{O}_*^{\mathrm{U}}(\mathbf{Z}_p)$ is a projective Ω_*^{U}-module and therefore by (3.3) is a free Ω_*^{U}-module. ☐

Remark. — Proposition (3.12) has also been found by R. E. Stong by completely different methods.

§ 4. A Spectral Sequence.

Let X be a finite complex and let

$$0 = \mathrm{A}_{-1} \subset \mathrm{A}_0 \subset \mathrm{A}_1 \subset \ldots \subset \mathrm{A}_k \sim \Sigma^l \mathrm{X}$$

be a U-bordism resolution of X. By forming the homology exact couple of the filtered space

$$0 = \mathrm{A}_{-1} \subset \mathrm{A}_0 \subset \mathrm{A}_1 \subset \ldots \subset \mathrm{A}_k$$

we will obtain a spectral sequence with target $H_*(A_k; \mathbf{Z}) \cong s^l H_*(X; \mathbf{Z})$. We will begin by studying this spectral sequence and then passing to some applications and related results. We begin with:

Theorem (**4.1**). — *Let* X *be a finite complex. Then there exists a natural spectral sequence* $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ *with*

$$E^r\langle X\rangle \Rightarrow H_*(X; \mathbf{Z})$$

and

$$E^2_{p,q}\langle X\rangle = \operatorname{Tor}^{\Omega^U_*}_{p,q}(\mathbf{Z}, \Omega^U_*(X)).$$

Proof. — Since X is a finite complex we may, according to Theorem (2.6), choose a U-bordism resolution of X, say

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \subset \ldots \subset A_k \sim \Sigma^l X.$$

Associated to the filtered space

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \subset \ldots \subset A_k$$

we have the exact couple

$$\begin{tikzcd} D \arrow[rr] & & D \arrow[dl] \\ & E \arrow[ul] & \end{tikzcd}$$

where

$$D_{p,q} = H_{p+q}(A_p; \mathbf{Z})$$
$$E_{p,q} = H_{p+q}(A_p, A_{p-1}; \mathbf{Z})$$

the maps of the couple being the maps in the exact homology triangle of the pairs (A_p, A_{p-1}) for $p = 0, \ldots, k$.

Let $\{E^r, d^r\}$ be the spectral sequence of this exact couple. We define $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ by

$$E^r_{p,q}\langle X\rangle = E^r_{p,q+l}$$
$$d^r_{p,q}\langle X\rangle = d^r_{p,q+l}.$$

It is evident that $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ is a first quadrant homology spectral sequence.

Convergence. — From the construction of the spectral sequence $\{E^r, d^r\}$ as the spectral sequence of the *finitely* filtered space

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \subset \ldots \subset A_k$$

it is immediately obvious that $\{E^r, d^r\}$ converges in the naive sense to $H_*(A_k; \mathbf{Z})$. Taking into account the dimension shifts in the definition of $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ and the suspension isomorphism we see that the spectral sequence $\{E^r\langle X\rangle d^r\langle X\rangle\}$ converges in the naive sense to $H_*(X; \mathbf{Z})$.

Identification of $E^2\langle X\rangle$. — We turn now to the identification of $E^2\langle X\rangle$.

From the construction of the spectral sequence $\{E^r, d^r\}$ it follows that E^2 is the homology of the complex

$$0 \leftarrow H_*(A_0; \mathbf{Z}) \leftarrow H_*(A_1, A_0; \mathbf{Z}) \leftarrow \ldots \leftarrow H_*(A_k, A_{k-1}; \mathbf{Z}) \leftarrow 0.$$

For each integer $p=0, 1, \ldots, k$, $\Omega_*^U(A_p, A_{p-1})$ is a free Ω_*^U-module, and thus we have the isomorphism

$$H_*(A_p, A_{p-1}; \mathbf{Z}) \cong \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(A_p, A_{p-1}),$$

by Lemma (3.1).

From the definition of a U-bordism resolution of X it follows that

$$0 \leftarrow \Omega_*^U(A_k) \overset{\varepsilon}{\leftarrow} \Omega_*^U(A_0) \leftarrow \Omega_*^U(A_1, A_0) \leftarrow \ldots \leftarrow \Omega_*^U(A_k, A_{k-1}) \leftarrow 0$$

is a free resolution of $\Omega_*^U(A_k)$ as an Ω_*^U-module.

Since E^2 is the homology of the complex

$$0 \leftarrow \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(A_0) \leftarrow \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(A_1, A_0) \leftarrow \ldots \leftarrow \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(A_k, A_{k-1}) \leftarrow 0$$

it follows from the definition of the functor $\mathrm{Tor}^{\Omega_*^U}(\mathbf{Z}, -)$ that

$$E_{p,q}^2 \cong \mathrm{Tor}_{p,q}^{\Omega_*^U}(\mathbf{Z}, \Omega_*^U(A_k)).$$

Taking into account the dimension shifts in the definition of $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ and the suspension isomorphism

$$\mathrm{Tor}_{p,q}^{\Omega_*^U}(\mathbf{Z}, \Omega_*^U(X)) \cong \mathrm{Tor}_{p,q+l}^{\Omega_*^U}(\mathbf{Z}, \Omega_*^U(A_k))$$

we obtain that

$$E_{p,q}^2\langle X\rangle \cong \mathrm{Tor}_{p,q}^{\Omega_*^U}(\mathbf{Z}, \Omega_*^U(X)).$$

Naturality. — We turn now to the question of the naturality of our spectral sequence. Note that at present the construction of $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ depends on the choice of a particular U-bordism resolution of X. We discuss this dependence first.

Let X be a finite complex and

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \subset \ldots \subset A_k \sim \Sigma^l X$$

a U-bordism resolution of X. Then

$$\emptyset = \Sigma^t A_{-1} \subset \Sigma^t A_0 \subset \Sigma^t A_1 \subset \ldots \subset \Sigma^t A_k \sim \Sigma^{l+t} X$$

is also a bordism resolution of X. It is obvious that these two bordism resolutions lead to the same spectral sequence $\{E^r\langle X\rangle, d^r\langle X\rangle\}$.

We may also obtain a bordism resolution

$$\emptyset = B_{-1} \subset B_1 \subset \ldots \subset B_{k+s} \sim \Sigma^l X$$

by setting

$$B_i = \begin{cases} A_i & \text{for } i \leq k \\ A_k & \text{for } k+1 \leq i \leq k+s. \end{cases}$$

The map $1 : A_k \to B_k \to B_{k+s}$ is then a map of filtered spaces

$$\{A_{-1} \subset A_0 \subset \ldots \subset A_k\} \to \{B_{-1} \subset B_0 \subset \ldots \subset B_k \subset \ldots \subset B_{k+s}\}$$

that induces an isomorphism of the associated spectral sequences from the term E^2 on.

Hence the spectral sequence $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ does not depend on this operation of adding trivial factors.

We are now ready to deal with the naturality of the spectral sequence.

Let $f: X \to Y$ be a map between finite complexes. Let

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \subset \dots \subset A_k \sim \Sigma^l X$$

and

$$\emptyset = B_{-1} \subset B_0 \subset B_1 \subset \dots \subset B_t \sim \Sigma^s Y$$

be U-bordism resolutions of X and Y respectively. Denote by $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ and $\{E^r\langle Y\rangle, d^r\langle Y\rangle\}$ the spectral sequences associated to these U-bordism resolutions. Our discussion above shows that there is no loss of generality in assuming that $k=t$ and $l=s$.

We begin by constructing a U-bordism resolution of Y

$$\emptyset = C_{-1} \subset C_0 \subset \dots C_k \sim \Sigma^{r+s} Y$$

and a homotopy commutative diagram

$$\begin{tikzcd}
\emptyset = \Sigma^r A_{-1} \arrow[d, "\varphi_{-1}"] & \subset & \Sigma^r A_0 \arrow[d, "\varphi_0"] & \subset \dots \subset & \Sigma^r A_i \arrow[d, "\varphi_i"] & \subset & \Sigma^r A_k \arrow[d, "\varphi_k"] & \sim & \Sigma^{r+s} X \arrow[d, "\Sigma^{r+s} f"] \\
\emptyset = C_{-1} \arrow[d, "\psi_{-1}"] & \subset & C_0 \arrow[d, "\psi_0"] & \subset \dots \subset & C_i \arrow[d, "\psi_i"] & \subset & C_k \arrow[d, "\psi_k"] & \sim & \Sigma^{r+s} Y \arrow[d] \\
\emptyset = \Sigma^r B_{-1} & \subset & \Sigma^r B_0 & \subset \dots \subset & \Sigma^r B_i & \subset & \Sigma^r B_k & \sim & \Sigma^{r+s} Y
\end{tikzcd}$$

The construction is by a tedious induction process.

Let $\varphi : A_k \to \Sigma^s X$ and $\psi : B_k \to \Sigma^s Y$ be homotopy equivalences. Denote the composites

$$A_i \hookrightarrow A_k \to \Sigma^s X$$

$$B_i \hookrightarrow B_k \to \Sigma^s Y$$

by φ_i and ψ_i respectively.

We then have the map

$$\theta : A_0 + B_0 \to \Sigma^s Y \quad (+ = \text{disjoint union})$$

defined by

$$\theta(a) = (\Sigma^s f)\varphi_0(a) \quad \text{for all } a \in A_0$$

$$\theta(b) = \psi_0(b) \quad \text{for all } b \in B_0.$$

Since $\Omega_*^U(B_0) \to \Omega_*^U(B_k)$ is onto it is clear that $\Omega_*^U(A_0 + B_0) \to \Omega_*^U(\Sigma^s Y)$ is also onto.

Let $M_{0,0} = A_0 + B_0$ and $M_{1,0}$ be the mapping cylinder of θ. We then have a partial U-bordism resolution

$$\emptyset = M_{-1,0} \subset M_{0,0} \subset M_{1,0} \sim \Sigma^s Y$$

and a homotopy commutative diagram

$$\begin{tikzcd}
\emptyset = A_{-1} \arrow[r, phantom, "\subset"] \arrow[d, "\alpha(-1,0)"] & A_0 \arrow[r, phantom, "\subset"] \arrow[d, "\alpha(0,0)"] & A_1 \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & A_k \arrow[r, phantom, "\sim"] \arrow[d, "\alpha(k,0)"] & \Sigma^s X \arrow[d, "\Sigma^s f"] \\
\emptyset = M_{-1,0} \arrow[r, phantom, "\subset"] & M_{0,0} \arrow[rrr, phantom, "\subset"] & & & M_{1,0} \arrow[r, phantom, "\sim"] & \Sigma^s Y \\
\emptyset = B_{-1} \arrow[r, phantom, "\subset"] \arrow[u, "\beta(-1,0)"] & B_0 \arrow[r, phantom, "\subset"] \arrow[u, "\beta(0,0)"] & B_1 \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & B_k \arrow[r, phantom, "\sim"] \arrow[u, "\beta(k,0)"] & \Sigma^s Y \arrow[u, equal]
\end{tikzcd}$$

We now proceed inductively and assume that we have constructed a partial U-bordism resolution

$$\emptyset = M_{-1,j} \subset M_{0,j} \subset \ldots \subset M_{j,j} \sim \Sigma^{r_j+s} Y$$

and a homotopy commutative diagram

$$\begin{tikzcd}
\emptyset = \Sigma^{r_j} A_{-1} \arrow[r, phantom, "\subset"] \arrow[d, "\alpha(-1,j)"] & \Sigma^{r_j} A_0 \arrow[r, phantom, "\subset"] \arrow[d, "\alpha(0,j)"] & \cdots \arrow[r, phantom, "\subset"] & \Sigma^{r_j} A_{j-1} \arrow[r, phantom, "\subset"] \arrow[d, "\alpha(j-1,j)"] & \Sigma^{r_j} A_j \arrow[r, phantom, "\subset"] & \cdots & \Sigma^{r_j} A_k \arrow[r, phantom, "\sim"] \arrow[d, "\alpha(k,j)"] & \Sigma^{r_j+s} X \arrow[d, "\Sigma^{r_j+s} f"] \\
\emptyset = M_{-1,j} \arrow[r, phantom, "\subset"] & M_{0,j} \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & M_{j-1,j} \arrow[rrr, phantom, "\subset"] & & & M_{j,j} \arrow[r, phantom, "\sim"] & \Sigma^{r_j+s} Y \\
\emptyset = \Sigma^{r_j} B_{-1} \arrow[r, phantom, "\subset"] \arrow[u, "\beta(-1,j)"] & \Sigma^{r_j} B_0 \arrow[r, phantom, "\subset"] \arrow[u, "\beta(0,j)"] & \cdots \arrow[r, phantom, "\subset"] & \Sigma^{r_j} B_{j-1} \arrow[r, phantom, "\subset"] \arrow[u, "\beta(j-1,j)"] & \Sigma^{r_j} B_j \arrow[r, phantom, "\subset"] & \cdots & \Sigma^{r_j} B_k \arrow[r, phantom, "\sim"] \arrow[u, "\beta(k,j)"] & \Sigma^{r_j+s} Y \arrow[u]
\end{tikzcd}$$

We then have a natural map

$$(\Sigma^{r_j} A_j + \Sigma^{r_j} B_j, \Sigma^{r_j} A_{j-1} + \Sigma^{r_j} B_{j-1}) \to (M_{j,j}, M_{j-1,j})$$

but unfortunately

$$\Omega_*^U(\Sigma^{r_j} A_j + \Sigma^{r_j} B_j, \Sigma^{r_j} A_{j-1} + \Sigma^{r_j} B_{j-1}) \to \Omega_*^U(M_{j,j}, M_{j-1,j})$$

need not be an epimorphism. However it follows from Proposition (2.4) that we may choose a complex D, with $\Omega_*^U(D)$ a free Ω_*^U-module, and a map $g : D \to \Sigma^q[M_{j,j}/M_{j-1,j}]$ such that the natural map

$$\Sigma^q \left[\frac{\Sigma^{r_j} A_j + \Sigma^{r_j} B_j}{\Sigma^{r_j} A_{j-1} + \Sigma^{r_j} B_{j-1}} \right] + D \to \Sigma^q[M_{j,j}/M_{j-1,j}]$$

induces an epimorphism

$$\Omega_*^U \left(\Sigma^q \left[\frac{\Sigma^{r_j} A_j + \Sigma^{r_j} B_j}{\Sigma^{r_j} A_{j-1} + \Sigma^{r_j} B_{j-1}} \right] + D \right) \to \Omega_*^U(\Sigma^q[M_{j,j}/M_{j-1,j}]).$$

Consider the following diagram

$$\begin{tikzcd}
\Sigma^q\left[\dfrac{\Sigma^{r_j}A_j+\Sigma^{r_j}B_j}{\Sigma^{r_j}A_{j-1}+\Sigma^{r_j}B_{j-1}}\right] \arrow[r] \arrow[d] & \Sigma^{q+1}[\Sigma^{r_j}A_{j-1}+\Sigma^{r_j}B_{j-1}] \arrow[r] \arrow[d, "\Sigma^{q+1}{[\alpha(j-1,j)+\beta(j-1,j)]}"] & \Sigma^{q+1}[\Sigma^{r_j}A_j+\Sigma^{r_j}B_j] \arrow[d, "\eta"] \\
\Sigma^q\left[\dfrac{\Sigma^{r_j}A_j+\Sigma^{r_j}B_j}{\Sigma^{r_j}A_{j-1}+\Sigma^{r_j}B_{j-1}}\right]+D \arrow[r] \arrow[d] & \Sigma^{q+1}M_{j-1,j} \arrow[r] \arrow[d] & N \arrow[d, "\theta"] \\
\Sigma^q[M_{j,j}/M_{j-1,j}] \arrow[r] & \Sigma^{q+1}M_{j-1,j} \arrow[r] & \Sigma^{q+1}M_{j,j}
\end{tikzcd}$$

in which the horizontal sequences are cofibrations.

We now define

$$M_{i,j+1}=\begin{cases} \Sigma^{q+1}M_{j,j} & i<j \\ N & i=j \\ \Sigma^{q+1}M_{j,j} & i=j+1. \end{cases}$$

From the construction it follows that (forming telescoping mapping cylinders)

$$\emptyset=M_{-1,j+1}\subset M_{0,j+1}\subset\dots\subset M_{j+1,j+1}\sim\Sigma^{q+1+s}Y$$

is a partial U-bordism resolution of Y. Setting $q+1=r_{j+1}$ we obtain a commutative diagram (up to homotopy)

$$\begin{tikzcd}
\emptyset=\Sigma^{r_{j+1}}A_{-1} \arrow[r, phantom, "\subset"] \arrow[d] & \Sigma^{r_{j+1}}A_0 \arrow[r, phantom, "\subset\dots\subset"] \arrow[d, "\alpha(0,j+1)"] & \Sigma^{r_{j+1}}A_j \arrow[r, phantom, "\subset"] \arrow[d, "\alpha(j,j+1)"] & \Sigma^{r_{j+1}}A_{j+1} \arrow[r, phantom, "\subset\dots\subset"] & \Sigma^{r_{j+1}}A_k \arrow[r, phantom, "\sim"] \arrow[d, "\alpha(k,j+1)"] & \Sigma^{r_{j+1}+s}X \arrow[d, "\Sigma^{r_{j+1}+s}f"] \\
\emptyset=M_{-1,j+1} \arrow[r, phantom, "\subset"] & M_{0,j+1} \arrow[r, phantom, "\subset\dots\subset"] & M_{j,j+1} \arrow[rr, hook, equal] & & M_{j+1,j+1} \arrow[r, phantom, "\sim"] & \Sigma^{r_{j+1}+s}Y \arrow[d, equal] \\
\emptyset=\Sigma^{r_{j+1}}B_{-1} \arrow[r, phantom, "\subset"] \arrow[u] & \Sigma^{r_{j+1}}B_0 \arrow[r, phantom, "\subset\dots\subset"] \arrow[u, "\beta(0,j+1)"] & \Sigma^{r_{j+1}}B_j \arrow[r, phantom, "\subset"] \arrow[u, "\beta(j,j+1)"] & \Sigma^{r_{j+1}}B_{j+1} \arrow[r, phantom, "\subset\dots\subset"] & \Sigma^{r_{j+1}}B_k \arrow[r, phantom, "\sim"] \arrow[u, "\beta(k,j+1)"] & \Sigma^{r_{j+1}+s}Y.
\end{tikzcd}$$

where the maps $\alpha(i,j+1)$ and $\beta(i,j+1)$, $i\neq j$, are suspensions of $\alpha(i,j)$ and $\beta(i,j)$ and $\alpha(j,j+1)$ and $\beta(j,j+1)$ are derived from the map η in the obvious manner.

Proceeding in this inductive fashion we may construct a partial U-bordism resolution

$$\emptyset=M_{-1,k}\subset M_{0,k}\subset\dots\subset M_{k,k}\sim\Sigma^{r+s}Y.$$

By the remark following Theorem (2.6) this is already a U-bordism resolution. Let $C_i = M_{i,k}$. By our inductive procedure we have obtained a diagram

$$\begin{tikzcd}
\emptyset=\Sigma^rA_{-1} \arrow[r, phantom, "\subset"] \arrow[d, "\varphi_{-1}"] & \Sigma^rA_0 \arrow[r, phantom, "\subset"] \arrow[d, "\varphi_0"] & \cdots \arrow[r, phantom, "\subset"] & \Sigma^rA_k \arrow[r, phantom, "\sim"] \arrow[d, "\varphi_k"] & \Sigma^{r+s}X \arrow[d, "\Sigma^{r+s}f"] \\
\emptyset= C_{-1} \arrow[r, phantom, "\subset"] & C_0 \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & C_k \arrow[r, phantom, "\sim"] & \Sigma^{r+s}Y \\
\emptyset=\Sigma^rB_{-1} \arrow[r, phantom, "\subset"] \arrow[u, "\psi_{-1}"] & \Sigma^rB_0 \arrow[r, phantom, "\subset"] \arrow[u, "\psi_0"] & \cdots \arrow[r, phantom, "\subset"] & \Sigma^rB_k \arrow[r, phantom, "\sim"] \arrow[u, "\psi_k"] & \Sigma^{r+s}Y. \arrow[u, equal]
\end{tikzcd}$$

From the bottom two U-bordism resolutions we obtain spectral sequences $\{E^r\langle Y, \mathbf{B}\rangle, d^r\langle Y, \mathbf{B}\rangle\}$ and $\{E^r\langle Y, C\rangle, d^r\langle Y, C\rangle\}$. The morphism of filtered spaces induced by $\{\psi_j\}$ induces a morphism of spectral sequences

$$\psi_* = \{E^r\langle Y, \mathbf{B}\rangle, d^r\langle Y, \mathbf{B}\rangle\} \rightarrow \{E^r\langle Y, C\rangle, d^r\langle Y, C\rangle\}.$$

From our identification of the term $E^2\langle Y\rangle$ it readily follows that ψ_* induces an isomorphism

$$E^2\langle Y, \mathbf{B}\rangle \rightarrow E^2\langle Y, C\rangle$$

and hence the two spectral sequences are isomorphic from the term E^2 on. Note that the map $H_*(Y; \mathbf{Z}) \rightarrow H_*(Y; \mathbf{Z})$ induced by ψ_* is the identity map. We denote either spectral sequence by $\{E^r\langle Y\rangle, d^r\langle Y\rangle\}$.

The morphism of filtered spaces induced by $\{\varphi_j\}$ induces a morphism of spectral sequences

$$\varphi_* : \{E^r\langle X\rangle, d^r\langle X\rangle\} \rightarrow \{E^r\langle Y\rangle, d^r\langle Y\rangle\}.$$

From our identification of the E^2 terms we see that this map coincides, on this term, with the map

$$\mathrm{Tor}^{\Omega^U_*}(\mathbf{Z}, f_*) : \mathrm{Tor}^{\Omega^U_*}(\mathbf{Z}, \Omega^U_*(X)) \rightarrow \mathrm{Tor}^{\Omega^U_*}(\mathbf{Z}, \Omega^U_*(Y)).$$

The map $H_*(X; \mathbf{Z}) \rightarrow H_*(Y; \mathbf{Z})$ induced by φ_* coincides with the map

$$f_* : H_*(X; \mathbf{Z}) \rightarrow H_*(Y; \mathbf{Z}).$$

Thus the spectral sequence $\{E^r\langle W\rangle, d^r\langle W\rangle\}$ is independent of the particular resolution used to construct it and is functorial in W. □

Remarks. — 1) Let X be a finite complex. Then the edge map of the spectral sequence $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ is the composite

$$\mathbf{Z}\otimes_{\Omega^U_*}\Omega^U_*(X) = E^2_{0,*}\langle X\rangle \xrightarrow{e} E^\infty_{0,*}\langle X\rangle \subset H_*(X; \mathbf{Z})$$

and clearly coincides with the reduced Thom map

$$\tilde{\mu} : \mathbf{Z}\otimes_{\Omega^U_*}\Omega^U_*(X) \rightarrow H_*(X; \mathbf{Z}).$$

2) Let $f: X \to Y$ be a map between finite complexes. Then f induces a map of spectral sequences

$$\{\langle f\rangle\} : \{E^r\langle X\rangle, d^r\langle X\rangle\} \to \{E^r\langle Y\rangle, d^r\langle Y\rangle\}.$$

As noted in the proof of Theorem (4.1) $\langle f\rangle_2$ may be identified with

$$\begin{tikzcd}
\mathrm{Tor}^{\Omega^U_*}(\mathbf{Z}, f_*) : \mathrm{Tor}^{\Omega^U_*}(\mathbf{Z}, \Omega^U_*(X)) \arrow[r] \arrow[d, equal] & \mathrm{Tor}^{\Omega^U_*}(\mathbf{Z}, \Omega^U_*(Y)) \arrow[d, equal] \\
E^2\langle X\rangle & E^2\langle Y\rangle
\end{tikzcd}$$

and $\langle f\rangle_\infty$ with

$$\begin{tikzcd}
E^0 f_* : E^0 H_*(X;\mathbf{Z}) \arrow[r] \arrow[d, equal] & E^0 H_*(Y;\mathbf{Z}) \arrow[d, equal] \\
E^\infty\langle X\rangle & E^\infty\langle Y\rangle.
\end{tikzcd}$$

3) Let X be a finite complex. Then it follows that $H_*(X;\mathbf{Z})$ is naturally filtered by

$$0 \subset F_0(X) \subset F_1(X) \subset \ldots \subset H_*(X;\mathbf{Z})$$

where $$F_0(X) = \mathrm{Im}\{\mu : \Omega^U_*(X) \to H_*(X;\mathbf{Z})\}.$$

and $$E^0 H_*(X;\mathbf{Z}) \cong E^\infty\langle X\rangle.$$

The filtration $\{F_i(X)\}$ of $H_*(X;\mathbf{Z})$ is finite, an upper bound for its length being given by $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X)$.

The following results provide a useful link between $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X)$ and $\{E^r\langle X\rangle, d^r\langle X\rangle\}$.

Proposition **(4.2)**. — *Let* X *be a finite complex. Then for any positive integer* n

$$\mathrm{Tor}^{\Omega^U_*}_{n,*}(\mathbf{Z}, \Omega^U_*(X)) = 0$$

iff $$\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X) \leq n.$$

Proof. — By induction on n. The case $n=1$ has already been dealt with in Corollary (3.11). Suppose the result established for all n with $1 \leq n < m$. Let X be a space with $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X) = m$. Choose a partial U-bordism resolution of X

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \sim \Sigma^l X$$

of length 1. We then have the exact sequence

$$0 \leftarrow \Omega^U_*(A_1) \leftarrow \Omega^U_*(A_0) \leftarrow \tilde{\Omega}^U_*(A_1/A_0) \leftarrow 0.$$

Since $\Omega^U_*(A_0)$ is a free Ω^U_*-module, $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(\ )$ is stable under suspension, and $m \geq 1$, it follows from ([5], VI, (2.3)) that

$$1 + \mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(A_1/A_0) = \mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(A_1) = m.$$

This exact sequence also yields a natural isomorphism

$$\Delta : \mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{p+1,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{A}_1)) \cong \mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{p,*}(\mathbf{Z}, \widetilde{\Omega}^{\mathrm{U}}_*(\mathrm{A}_1/\mathrm{A}_0))$$

for all $p>0$.

The inclusion $\widetilde{\Omega}^{\mathrm{U}}_*(\mathrm{A}_1/\mathrm{A}_0) \to \Omega^{\mathrm{U}}_*(\mathrm{A}_1/\mathrm{A}_0)$ also yields an isomorphism

$$\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{p,*}(\mathbf{Z}, \widetilde{\Omega}^{\mathrm{U}}_*(\mathrm{A}_1/\mathrm{A}_0)) \to \mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{p,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{A}_1/\mathrm{A}_0)).$$

By our inductive hypothesis

$$\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{n,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{A}_1/\mathrm{A}_0)) = 0$$

iff $$m-1 = \mathrm{hom.dim}_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_1/\mathrm{A}_0) \leq n.$$

Hence we deduce

$$\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{n+1,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{A}_1)) = 0$$

iff $$m = \mathrm{hom.dim}_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_1) \leq n+1.$$

Since

$\mathrm{hom.dim}_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_1) = \mathrm{hom.dim}_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{X})$ and $\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{p,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{A}_1))$ and $\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{q,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{X}))$ are isomorphic with a dimension shift for all $q \geq 0$, we deduce

$$\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{n,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{X})) = 0$$

iff $$\mathrm{hom.dim}_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{X}) \leq n.$$

This completes the inductive step and hence the result follows by induction. ☐

Remark. — On general grounds [5] we have:

$$\mathrm{hom.dim}_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{X}) \leq n \Rightarrow \mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{n+1,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{X})) = 0.$$

Proposition (4.2) improves this result by 1.

Corollary **(4.3)**. — *Let* X *be a finite complex. Then the following conditions are equivalent:*

1) $\mathrm{hom.dim}_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{X}) \leq n$;
2) $\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{n,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{X})) = 0$;
3) $\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{p,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{X})) = 0$ *for all* $p \geq n$;
4) X *admits a* U-*bordism resolution of length* n.

Proof. — By Proposition (4.2), 1) and 2) are equivalent. Clearly 1) and 2) imply 3) while 3) implies 1). The equivalence of 1) and 4) follows from the remarks after Theorem (2.6). ☐

Theorem **(4.4)**. — *Let* X *be a finite complex. Suppose that one of the following equivalent conditions holds:*

1) $\mathrm{hom.dim}_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{X}) \leq 2$;
2) $\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{2,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{X})) = 0$;
3) $\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{p,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{X})) = 0$ *for all* $p \geq 2$;
4) X *admits a* U-*bordism resolution of length* 2.

Then there is a natural exact sequence

$$0 \to \mathbf{Z}\otimes_{\Omega_*^U}\Omega_*^U(X) \xrightarrow{\tilde{\mu}} H_*(X;\mathbf{Z}) \xrightarrow{\tilde{\lambda}} \mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}, \Omega_*^U(X)) \to 0.$$

Proof. — The conditions 1)-4) imply that $E^2_{p,*}\langle X\rangle = 0$ for all $p \geq 2$. The result now follows by standard spectral sequence arguments. □

Theorem (4.4) perhaps sheds a bit more light on Corollary (3.9) and the succeeding remarks.

We close this section with some remarks on the extension of Theorem (4.1) to non-finite CW-complexes.

Definition. — *A* CW-*complex* X *is called skeleton-finite iff all its skeletons are finite* CW-*complexes.*

If X is a skeleton finite CW-complex then

$$\Omega_*^U(X) = \varinjlim \Omega_*^U(X^n)$$

where $X^n \subset X$ is the n-skeleton of X. A similar remark applies to $H_*(X;\mathbf{Z})$.

Since each of the complexes X^n is finite we may obtain spectral sequences $\{E^r\langle X^n\rangle, d^r\langle X^n\rangle\}$. The inclusions $X^n \subset X^{n+1}$ induce maps of spectral sequences $\{E^r\langle X^n\rangle, d^r\langle X^n\rangle\} \to \{E^r\langle X^{n+1}\rangle, d^r\langle X^{n+1}\rangle\}$. Since direct limits preserve exactness we may *define* a spectral sequence $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ by

$$\{E^r\langle X\rangle, d^r\langle X\rangle\} = \varinjlim \{E^r\langle X^n\rangle, d^r\langle X^n\rangle\}.$$

A simple use of the cellular approximation theorem shows that $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ is independent of the particular CW-structure employed and is functorial in X.

We thus obtain:

Theorem **(4.5)**. — *Let* X *be a skeleton finite* CW-*complex. Then there exists a natural spectral sequence* $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ *with*

$$E^r\langle X\rangle \Rightarrow H_*(X;\mathbf{Z})$$

and

$$E^2_{p,q}\langle X\rangle = \mathrm{Tor}_{p,q}^{\Omega_*^U}(\mathbf{Z}, \Omega_*^U(X)).$$

Moreover the convergence is in the naive sense. □

For a more general treatment of universal coefficient type theorems similar to those dealt with in this section we refer to the lectures of J. F. Adams [2].

Remarks. — Let X be a finite complex and

$$\emptyset = A_{-1} \subset A_0 \subset A_1 \subset \dots \subset A_k \sim \Sigma^l X$$

a U-bordism resolution of X. Applying the functor $H^*(\ ;\mathbf{Z})$ to the filtered space

$$\emptyset = A_{-1} \subset A_0 \subset \dots \subset A_k$$

leads, after a suitable reindexing, to a spectral sequence $\{E_r\langle X\rangle, d_r\langle X\rangle\}$. We leave to the reader the detailed proof that $\{E_r\langle X\rangle, d_r\langle X\rangle\}$ is natural and that

$$E_r\langle X\rangle \Rightarrow H^*(X;\mathbf{Z})$$

and
$$E_2^{p,q}\langle X\rangle = \mathrm{Ext}^{p,q}_{\Omega_*^U}(\Omega_*^U(X), \mathbf{Z});$$

the convergence being in the naive sense.

§ 5. Bounds for the Projective Dimension of U-Bordism Modules.

As we have seen in the past two sections the integer $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)$, for a finite complex X, may provide us with information about the Thom homomorphism $\mu : \Omega_*^U(X) \to H_*(X;\mathbf{Z})$. We are thus led to inquire into the possible values the integer $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)$ may assume for finite complexes X. We will introduce an integer associated to each $\alpha\in\Omega_*^U(X)$ which is a lower bound for $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)$. Some results of E. E. Floyd will then provide us with examples to show that $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)$ can be arbitrarily large.

Since we are dealing with cell complexes we should devote some time to the operation of attaching a cell. We will obtain an upper bound on how attaching a cell might effect the projective dimension. Further results on attaching cells may be found in § 7 and § 11.

Notation. — Let X be a finite complex. For any $\alpha\in\Omega_*^U(X)$ we will denote by $A(\alpha)$ the annhilator ideal of α, i.e., $A(\alpha)$ is the ideal in Ω_*^U defined by $x\in A(\alpha) \Leftrightarrow x.\alpha=0\in\Omega_*^U(X)$.

Let X be a finite complex and $\alpha\in\Omega_*^U(X)$. Suppose that $\alpha\neq 0\in\mathbf{Q}\otimes_{\mathbf{Z}}\Omega_*^U(X)$. Recall that $\mathbf{Q}\otimes_{\mathbf{Z}}\Omega_*^U$ has no zero divisors. Thus since the natural map $\Omega_*^U\to\mathbf{Q}\otimes_{\mathbf{Z}}\Omega_*^U$ is a monomorphism we deduce that $A(\alpha)=0$. Thus $A(\alpha)=0$ unless α has finite order. If α has finite order m then clearly $m\in A(\alpha)$. That $A(\alpha)$ may contain elements not in the ideal generated by m will soon be apparent.

Proposition **(5.1)**. — *Let* X *be a finite complex and* $\alpha\in\Omega_*^U(X)$. *Then* $A(\alpha)\subset\Omega_*^U$ *is a finitely generated ideal. As an* Ω_*^U*-module* $A(\alpha)$ *is coherent and has finite projective dimension.*

Proof. — This may be proved quite easily by employing the ideas of the first section.

Let Λ be a coherent ring, M a coherent Λ-module and $\alpha\in M$. Let N denote the submodule of M generated by α. Then N is a finitely generated submodule of the coherent module M and hence N is also coherent. Let $I\subset\Lambda$ be the annihilator ideal of α. We have the exact sequence of Λ-modules

$$0\to I\xrightarrow[\eta]{}\Lambda\xrightarrow[\mu]{}N\to 0$$

where
$$\mu(\lambda)=\lambda.\alpha.$$

Since N and Λ are coherent Λ-modules it follows that I is also (Proposition (1.1)).

Since Ω_*^U is a coherent ring it follows from Theorem (1.3) that $A(\alpha)$ is a coherent Ω_*^U-module. From Corollary (1.5) we deduce that $\mathrm{hom.dim}_{\Omega_*^U}A(\alpha)$ is finite. □

Definition. — *Let* $\mathbf{\Lambda}$ *be a commutative ring with* 1. *A sequence of elements* $\lambda_1, \dots, \lambda_n \in \mathbf{\Lambda}$ *is called an* ESP-*sequence iff* λ_1 *is not a zero divisor in* $\mathbf{\Lambda}$ *and* λ_{i+1} *is not a zero divisor in* $\mathbf{\Lambda}/(\lambda_1, \dots, \lambda_i)$ *for* $i=1, \dots, n-1$.

For basic properties of ESP-sequences we refer the reader to [5], [15], [18], [19], [25], where he will find them called variously E—, S—, M— and primary-sequences.

A companion notion to that of an ESP-sequence is the Koszul complex construction ([5], [15], [18], [19], [25]). We shall have occasion to use only elementary properties of Koszul complexes and we recall these now.

Let $\mathbf{K}$ be a commutative ring with 1 and $\mathbf{\Lambda}$ a commutative $\mathbf{K}$-algebra. Assume that $\mathbf{\Lambda}$ is projective as a $\mathbf{K}$-module. Let $\lambda_1, \dots, \lambda_n \in \mathbf{\Lambda}$ be a sequence of elements. Let $E[u_1, \dots, u_n]$ denote an exterior algebra over $\mathbf{K}$ on generators $u_1, \dots, u_n$. We may then form the algebra

$$\mathscr{E}(\lambda_1, \dots, \lambda_n) = \mathbf{\Lambda} \otimes E[u_1, \dots, u_n]$$

upon which we impose a derivation by the requirement

$$d(\lambda) = 0 \quad \text{for all } \lambda \in \mathbf{\Lambda}$$
$$d(u_i) = \lambda_i.$$

Let $I = (\lambda_1, \dots, \lambda_n) \subset \mathbf{\Lambda}$ denote the ideal generated by the elements $\lambda_1, \dots, \lambda_n$. There is a natural algebra map $\varepsilon : \mathscr{E}(\lambda_1, \dots, \lambda_n) \to \mathbf{\Lambda}/I$ given by requiring $\varepsilon(u_i) = 0$ and $\varepsilon(\lambda_i) = \lambda_i I \in \mathbf{\Lambda}/I$. If the sequence $\lambda_1, \dots, \lambda_n \in \mathbf{\Lambda}$ is an ESP-sequence then

$$\mathbf{\Lambda}/I \xleftarrow{\varepsilon} \mathscr{E}(\lambda_1, \dots, \lambda_n)$$

is a $\mathbf{\Lambda}$-projective resolution of the $\mathbf{\Lambda}$-module $\mathbf{\Lambda}/I$.

Definition. — *Let* $\mathbf{\Lambda}$ *be a commutative ring with* 1 *and* $I \subset \mathbf{\Lambda}$ *an ideal. We define the girth of* I, *denoted* $\gamma(I)$, *to be the integer given by*

$$\gamma(I) = \sup\{r \in \mathbf{Z} \mid \textit{there exists an } \text{ESP-}\textit{sequence } \lambda_1, \dots, \lambda_r \in I\}$$

Proposition (**5.2**). — *Let* $\mathbf{K}$ *be a commutative ring with* 1 *and* $\mathbf{\Lambda}$ *a supplemented* $\mathbf{K}$-*algebra that is projective as a* $\mathbf{K}$-*module. Suppose that* $I \subset \mathbf{\Lambda}$ *is an ideal. Then* $1 + \text{hom.dim}_{\mathbf{\Lambda}} I \geq \gamma(I)$.

Proof. — Let $\gamma(I) = s$. Choose an ESP-sequence $\lambda_1, \dots, \lambda_s \in I$. Consider $\text{Tor}^{\mathbf{\Lambda}}(\mathbf{\Lambda}/I, \mathbf{\Lambda}/(\lambda_1, \dots, \lambda_s))$. By use of the Koszul resolution $(\lambda_1, \dots, \lambda_s)$ we have

$$\text{Tor}^{\mathbf{\Lambda}}(\mathbf{\Lambda}/I, \mathbf{\Lambda}/(\lambda_1, \dots, \lambda_s)) = H((\mathbf{\Lambda}/I) \otimes E[u_1, \dots, u_s]; d)$$

where d is the derivation given by

$$du_i = \lambda_i I \in \mathbf{\Lambda}/I$$
$$d(\alpha) = 0 \quad \text{for } \alpha \in \mathbf{\Lambda}/I.$$

Since $\lambda_i \in I$, $d(u_i) = 0$ and hence $d = 0$. Therefore

$$\text{Tor}^{\mathbf{\Lambda}}(\mathbf{\Lambda}/I, \mathbf{\Lambda}/(\lambda_1, \dots, \lambda_s)) = (\mathbf{\Lambda}/I) \otimes E[u_1, \dots, u_s]$$

Hence $$u_1 \dots u_s \neq 0 \in \text{Tor}_s^{\mathbf{\Lambda}}(\mathbf{\Lambda}/I, \mathbf{\Lambda}/(\lambda_1, \dots, \lambda_s)).$$

Thus [5] $\text{hom.dim}_\Lambda \Lambda/I \geq s$. Since ([5], VI, (2.3)) $\text{hom.dim}_\Lambda \Lambda/I \leq 1 + \text{hom.dim}_\Lambda I$, with equality holding unless I is a direct summand, the result follows. □

From the Koszul complex and our previous work we now obtain the following fundamental inequality:

Theorem **(5.3)**. — *Let* X *be a finite complex and* $\alpha \in \Omega_*^U(X)$. *Then*

$$\text{hom.dim}_{\Omega_*^U} \Omega_*^U(X) \geq \gamma(A(\alpha)).$$

Proof. — Let $\gamma(A(\alpha)) = s$. Choose an ESP-sequence $\lambda_1, \ldots, \lambda_s \in A(\alpha)$. Consider $\text{Tor}^{\Omega_*^U}(\Omega_*^U(X), \Omega_*^U/(\lambda_1, \ldots, \lambda_s))$. By use of the Koszul resolution $\mathscr{E}(\lambda_1, \ldots, \lambda_s)$ we have

$$\text{Tor}^{\Omega_*^U}(\Omega_*^U(X), \Omega_*^U/(\lambda_1, \ldots, \lambda_s)) = H(\Omega_*^U(X) \otimes E[u_1, \ldots, u_s]; d)$$

where d is determined by the formulas

$$d(\zeta \otimes 1) = 0 \quad \text{for all} \quad \zeta \in \Omega_*^U(X)$$
$$d(\zeta \otimes u_{i_1} \ldots u_{i_r}) = \sum_t \lambda_{i_t} \zeta \otimes u_{i_1} \ldots \hat{u}_{i_t} \ldots u_{i_r}$$

where as usual ^ denotes omission. Now quite evidently

$$d(\alpha \otimes u_1 \ldots u_s) = 0$$

since $\lambda_i \in A(\alpha)$ for $i = 1, \ldots, s$. Quite obviously $\alpha \otimes u_1 \ldots u_s$ cannot be a boundry and hence

$$\text{class}\{\alpha \otimes u_1 \ldots u_s\} \neq 0 \in \text{Tor}_{s,*}^{\Omega_*^U}(\Omega_*^U(X), \Omega_*^U/(\lambda_1, \ldots, \lambda_s))$$

and hence $\text{hom.dim}_{\Omega_*^U} \Omega_*^U(X) \geq s$ and the result follows. □

Let $\mathbf{R}P(n_1, \ldots, n_s)$ denote the product of real projective spaces

$$\mathbf{R}P(n_1) \times \ldots \times \mathbf{R}P(n_s).$$

There is a natural inclusion $\mathbf{R}P(1, \ldots, 1) \hookrightarrow \mathbf{R}P(n_1, \ldots, n_s)$. The manifold $\mathbf{R}P(1, \ldots, 1)$ has a natural weakly complex structure and hence the inclusion

$$\mathbf{R}P(1, \ldots, 1) \hookrightarrow \mathbf{R}P(n_1, \ldots, n_s)$$

represents an element $\gamma(n_1, \ldots, n_s) \in \Omega_s^U(\mathbf{R}P(n_1, \ldots, n_s))$. The case

$$n_1 = n_2 = \ldots = n_s = \infty$$

will be of particular interest to us and we will write

$$\otimes^n \gamma = \gamma(\overbrace{\infty, \ldots, \infty}^{n\text{-times}}) \in \Omega_n^U(\mathbf{R}P(\infty, \ldots, \infty)),$$

when $n = 1$ we abreviate $\otimes^n \gamma$ to γ.

Following [10] let us also introduce the ideals $I_n \subset \Omega_*^U$, defined inductively by

$$I_0 = (0)$$
$$I_n = \{\alpha \in \Omega_*^U \mid \alpha . \gamma \in I_{n-1} \Omega_*^U(\mathbf{R}P(\infty))\}.$$

The ideals $I_n \subset \Omega_*^U$ have been completely determined by E. E. Floyd [10]. His result on their structure may be stated as follows:

Theorem **(5.4)** (E. E. Floyd). — *There exist closed weakly complex manifolds* V^{2i} *of dimension* $2i$ *such that*

$$\Omega_*^U \cong \mathbf{Z}[[V^2], [V^4], \ldots].$$

and for each positive integer n

$$I_n = (2, [V^2], \ldots, [V^{2^n-2}]).$$

Hence $\gamma(I_n) \geq n$. $\square$

With the aid of Theorem (5.3) and Theorem (5.4) we may now obtain our first example.

Theorem **(5.5)**. — *Let* n *be a positive integer and*

$$E_n \subset \mathbf{RP}(\underbrace{\infty, \ldots, \infty}_{n\text{-times}})$$

a giant skeleton. Then $$\text{hom.dim}_{\Omega_*^U} \Omega_*^U(E_n) \geq n.$$

Proof. — For each positive integer n let us introduce the ideal $A_n \subset \Omega_*^U$ by

$$A_n = A(\otimes^n \gamma).$$

We shall need the following result, which may be of some independent interest.

Lemma. — *For each positive integer* n, $I_n \subseteq A_n$.

Proof. — We will proceed by induction on n. Consider first the case $n=1$. By definition we have

$$I_1 = \{\alpha \in \Omega_*^U \mid \alpha . \gamma = 0 \in \Omega_*^U(\mathbf{RP}(\infty))\} = A_1.$$

Thus clearly $I_1 \subseteq A_1$. Let us therefore proceed to the inductive step and suppose that $I_m \subseteq A_m$ for all $m < n$. We wish to conclude that $I_n \subseteq A_n$. We may as well assume that $n > 1$.

Consider the exterior product

$$\Omega_*^U(\mathbf{RP}(\overbrace{\infty, \ldots, \infty}^{n-1\text{ times}})) \otimes_{\Omega_*^U} \Omega_*^U(\mathbf{RP}(\infty)) \to \Omega_*^U(\mathbf{RP}(\overbrace{\infty, \ldots, \infty}^{n\text{-times}})).$$

One readily observes that, with the obvious abuse of notation

$$\otimes^{n-1}\gamma \otimes \gamma = \otimes^n \gamma$$

in $$\Omega_*^U(\mathbf{RP}(\underbrace{\infty, \ldots, \infty}_{n\text{-times}})).$$

Now suppose that $\alpha \in I_n$. Then

$$\alpha(\otimes^n \gamma) = \alpha(\otimes^{n-1}\gamma \otimes \gamma) = \otimes^{n-1}\gamma \otimes \alpha\gamma.$$

Since $\alpha \in I_n$ we have

$$\alpha\gamma \in I_{n-1}\Omega_*^U(\mathbf{RP}(\infty)).$$

Thus $$\alpha\gamma = \sum_{i=1}^{t} \omega_i \lambda_i$$

where $$\omega_i \in \mathrm{I}_{n-1}, \quad \lambda_i \in \Omega_*^{\mathrm{U}}(\mathbf{RP}(\infty)).$$

Thus we find $$\alpha(\otimes^n \gamma) = \otimes^{n-1}\gamma \otimes \alpha\gamma = \otimes^{n-1}\gamma \otimes \sum_i \omega_i \lambda_i$$
$$= \sum_i (\omega_i \otimes^{n-1}\gamma) \otimes \lambda_i.$$

By our inductive assumption $\mathrm{I}_{n-1} \subseteq \mathrm{A}_{n-1}$ and hence

$$\omega_i \otimes^{n-1}\gamma = 0 \in \Omega_*^{\mathrm{U}}(\mathbf{RP}(\overbrace{\infty, \ldots, \infty}^{n-1 \text{ times}})) : i = 1, \ldots, t.$$

Therefore we find $$\alpha(\otimes^n \gamma) = \sum_i 0 \otimes \lambda_i = 0.$$

Hence $\alpha \in \mathrm{A}_n$.

This completes the inductive step and hence the lemma follows. ☐

Returning now to the proof of Theorem (5.5) we find in view of (5.4) that for each $n > 0$

$$2, [\mathrm{V}^2], \ldots, [\mathrm{V}^{2^n-2}] \in \mathrm{A}_n.$$

Now recall that by choosing E_n to be a large enough skeleton of $\mathbf{RP}(\underbrace{\infty, \ldots, \infty}_{n\text{-times}})$, we will have

$$2, [\mathrm{V}^2], \ldots [\mathrm{V}^{2^n-2}] \in \mathrm{A}(\gamma(\mathrm{E}_n))$$

where $\gamma(\mathrm{E}_n) \in \Omega_n^{\mathrm{U}}(\mathrm{E}_n)$ is the class represented by the inclusion

$$\mathbf{RP}(1, \ldots, 1) \hookrightarrow \mathrm{E}_n.$$

Therefore, in view of (5.4) we find

$$\gamma(\mathrm{A}(\gamma(\mathrm{E}_n))) \geq n,$$

and the result now follows from (5.3). ☐

We note in passing that the relation between the ideals I_n, A_n and other ideals of [10] and [7] is perhaps of interest in its own right. A more thorough discussion of these relations will appear on another occasion.

Before moving on to our next example we note that there are numerous interesting algebraic questions relating girth to homological dimension, Borel ideals (= complete intersections in algebraic geometry) and several other numerical invariants of ideals. As succeeding paragraphs will show new relations between these invariants would be welcome.

We move on now to explore further applications of the notion of girth. We will turn to the question of how the integer $\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{X})$ may be changed by attaching a cell. We will return to this question often in the succeeding sections.

Let X be a finite complex and $f : \mathrm{S}^n \to \mathrm{X}$ a continuous map. The sphere S^n admits a natural framing and hence a natural U-structure. Thus $f : \mathrm{S}^n \to \mathrm{X}$ represents an element $[S^n, f] \in \Omega^U_n(X)$. We will denote by $X \cup_f e^{n+1}$ the space obtained from X by attaching an $(n+1)$-cell via the map f.

Theorem **(5.6).** — *Let* X *be a finite complex and* $f: S^n \to X$ *a continuous map. Then*

$$\text{hom.dim}_{\Omega^U_*} \Omega^U_*(X \cup_f e^{n+1}) \leq 1 + \max\{\text{hom.dim}_{\Omega^U_*} \Omega^U_*(X),\ 1 + \text{hom.dim}_{\Omega^U_*} A([S^n, f])\}$$

Proof. — We have a cofibration sequence

$$S^n \xrightarrow{f} X \to X \cup_f e^{n+1}.$$

Applying the functor $\Omega^U_*(\ )$ yields an exact triangle

$$\begin{tikzcd}
\Omega^U_*(S^n) \arrow[rr, "f_*"] & & \Omega^U_*(X) \arrow[dl] \\
& \widetilde{\Omega}^U_*(X \cup_f e^{n+1}) \arrow[ul] &
\end{tikzcd}$$

Let M be the submodule of $\Omega^U_*(X)$ defined by $M = f_* \Omega^U_*(S^n) \subset \Omega^U_*(X)$. We then obtain from the exact triangle, exact sequences

$$0 \to A([S^n, f]) \to \Omega^U_*(S^n) \to M \to 0$$

$$0 \to M \to \Omega^U_*(X) \to \Omega^U_*(X)/M \to 0$$

$$0 \to \Omega^U_*(X)/M \to \Omega^U_*(X \cup_f e^{n+1}) \to A([S^n, f]) \to 0.$$

From the first exact sequence we obtain

$$\text{hom.dim}_{\Omega^U_*} M \leq 1 + \text{hom.dim}_{\Omega^U_*} A([S^n, f])$$

while the second exact sequence yields

$$\text{hom.dim}_{\Omega^U_*} \Omega^U_*(X)/M \leq \max\{\text{hom.dim}_{\Omega^U_*} \Omega^U_*(X),\ 1 + \text{hom.dim}_{\Omega^U_*} M\}$$

and thus the last exact sequence provides the inequality

$$\text{hom.dim}_{\Omega^U_*} \Omega^U_*(X \cup_f e^{n+1}) \leq \max\{\text{hom.dim}_{\Omega^U_*} \Omega^U_*(X)/M,\ \text{hom.dim}_{\Omega^U_*} A([S^n, f])\}$$

and the result follows quite readily. ☐

Thus we see that the structure of the ideal $A([S^n, f]) \subset \Omega^U_*$ is important in computing $\text{hom.dim}_{\Omega^U_*} \Omega^U_*(X \cup_f e^{n+1})$. In certain favorable cases we can determine the structure of $A([S^n, f])$ from Theorem (5.3). We turn to this now.

Proposition **(5.7).** — *Let* X *be a finite complex and assume that* $\text{hom.dim}_{\Omega^U_*} \Omega^U_*(X) = 0$. *Then, for any* $\alpha \in \Omega^U_*(X)$, $A(\alpha) = 0$.

Proof. — By Corollary (3.3), $\Omega^U_*(X)$ is a free Ω^U_*-module. Since Ω^U_* has no zero divisors the result follows. ☐

Corollary **(5.8).** — *Let* X *be a finite complex and* $f: S^n \to X$ *a continuous map. If* $\text{hom.dim}_{\Omega^U_*} \Omega^U_*(X) = 0$ *then* $\text{hom.dim}_{\Omega^U_*} \Omega^U_*(X \cup_f e^{n+1}) \leq 1$.

Proof. — This is obvious from Theorem (5.6) and Proposition (5.7). ☐

Proposition (**5.9**). — *Let* X *be a finite complex and assume that* $\text{hom.dim}_{\Omega_*^U}\Omega_*^U(X)=1$. *Then for any* $\alpha\in\Omega_*^U(X)$ *either*:

1) α *has infinite order and* $A(\alpha)=0$, *or*

2) α *has finite order* m *and* $A(\alpha)=(m)$.

Proof. — The behavior in case 1) has been noted previously. So suppose that α has finite order m.

Consider first the case when the order of α is a prime p. Clearly $p\in A(\alpha)$. We contend that $(p)=A(\alpha)$. For suppose $\lambda\in A(\alpha)$. Then since $\gamma(A(\alpha))=1$ (by Theorem (5.3)) we must have that $\lambda\in\Omega_*^U/(p)$ is a zero divisor. However $(p)\subset\Omega_*^U$ is a prime ideal and hence there are no non-trivial zero divisors in $\Omega_*^U/(p)$. Thus $\lambda=0\in\Omega_*^U/(p)$ and hence $\lambda\in(p)$. Therefore we have $(p)\subset A(\alpha)\subset(p)$ and hence $A(\alpha)=(p)$ as required.

We now proceed inductively on the size of the prime factorization of m. We therefore assume the result true for all $\beta\in\Omega_*^U(X)$ whose order m' has a smaller prime factorization than m. Write $m=pm'$, $p\neq 1$, $p\neq m$, p a prime. Consider the element $m'\alpha\in\Omega_*^U(X)$. Clearly $m'\alpha\neq 0\in\Omega_*^U(X)$ and $p\in A(m'\alpha)$, for

$$p(m'\alpha)=(pm')\alpha=m\alpha=0.$$

Thus by the inductive assumption $A(m'\alpha)=(p)$.

A symmetric argument shows $A(p\alpha)=(m')$, for $p\alpha$ is easily seen to have order m'.

Now we wish to show that $A(\alpha)=(m)$. So suppose that $\lambda\in A(\alpha)$. Then

$$\lambda(m'\alpha)=m'\lambda\alpha=m'0=0.$$

Hence $\lambda\in A(m'\alpha)$. Therefore $\lambda=p\eta$ for some $\eta\in\Omega_*^U$. Now we have

$$\eta(p\alpha)=p\eta\alpha=\lambda\alpha=0.$$

Therefore $\eta\in A(p\alpha)=(m')$. Hence $\eta=m'\zeta$ for some $\zeta\in\Omega_*^U$. Thus

$$\lambda=p\eta=pm'\zeta=m\zeta,$$

and hence $\lambda\in(m)$. Thus we have $(m)\subset A(\alpha)\subset(m)$ and therefore $A(\alpha)=(m)$ as required.

This completes the inductive step and hence the proof of the Proposition. □

Corollary (**5.10**). — *Let* X *be a finite complex and* $f:S^n\to X$ *a continuous map. If* $\text{hom.dim}_{\Omega_*^U}\Omega_*^U(X)=1$ *then* $\text{hom.dim}_{\Omega_*^U}\Omega_*^U(X\cup_f e^{n+1})\leq 2$.

Proof. — From Proposition (5.9) it follows that either $A(\alpha)=0$ or is a free Ω_*^U-module on one generator. The result now follows from Theorem (5.6). □

Proposition (**5.11**). — *Let* X *be a finite complex and assume that* $\text{hom.dim}_{\Omega_*^U}\Omega_*^U(X)=2$. *Then for any* $\alpha\in\Omega_*^U(X)$ *of prime order* p, $A(\alpha)=(p, x)$ *where* $p, x\in\Omega_*^U$ *is an* ESP-*sequence, or* $A(\alpha)=(p)$.

Proof. — Note that by Theorem (5.3) $\gamma(\mathrm{A}(\alpha))\leq 2$. Suppose that $\mathrm{A}(\alpha)\neq(p)$. Note that

$$\Omega_*^{\mathrm{U}}/(p)\cong\mathbf{Z}/p\mathbf{Z}[x_1,\ \ldots]$$

is a unique factorization ring.

Choose an element $x\in\mathrm{A}(\alpha)$ of minimal degree such that $x\notin(p)$. Then since $\Omega_*^{\mathrm{U}}/(p)$ has no zero divisors, $p, x\in\Omega_*^{\mathrm{U}}$ is an ESP-sequence. We contend that $\mathrm{A}(\alpha)=(p, x)$. Our proof will be by induction on the size of the prime factorization of x regarded as an element of $\Omega_*^{\mathrm{U}}/(p)$.

We begin with the case when x is a prime in the ring $\Omega_*^{\mathrm{U}}/(p)$. We then have that $\Omega_*^{\mathrm{U}}/(p, x)=(\Omega_*^{\mathrm{U}}/(p))/(x)$ is an integral domain, and hence has no zero divisors. Let $\lambda\in\mathrm{A}(\alpha)$. Then since $\gamma(\mathrm{A}(\alpha))\leq 2$, λ must be a zero divisor in $\Omega_*^{\mathrm{U}}/(x, p)$. Thus $\lambda=0\in\Omega_*^{\mathrm{U}}/(p, x)$ and hence $\lambda\in(p, x)$. Therefore $(p, x)\subset\mathrm{A}(\alpha)\subset(p, x)$ and hence $\mathrm{A}(\alpha)=(p, x)$ as required.

Suppose inductively that the result has been established for all $\beta\in\Omega_*^{\mathrm{U}}(\mathrm{X})$ such that $\mathrm{A}(\beta)$ contains an element y, of minimal degree such that $y\notin(p)$, and with y having a smaller factorization into primes in $\Omega_*^{\mathrm{U}}/(p)$ than does x. We may also suppose that x is not a prime in $\Omega_*^{\mathrm{U}}/(p)$. Thus in the ring $\Omega_*^{\mathrm{U}}/(p)$ we have $x=x_1x_2$, where $\deg x_1, \deg x_2>0$ (recall that the primes in $\Omega_*^{\mathrm{U}}/(p)$ all have positive degree). Choose elements $y_1, y_2\in\Omega_*^{\mathrm{U}}$ such that y_i projects to $x_i\in\Omega_*^{\mathrm{U}}/(p)$, $i=1, 2$.

Consider $y_1\alpha\in\Omega_*^{\mathrm{U}}(\mathrm{X})$. We assert that $y_1\alpha\neq 0$. For if $y_1\alpha=0$ then $y_1\in\mathrm{A}(\alpha)$, $\deg y_1<\deg x$ and $y_1\notin(p)$, contrary to the choice of x. Thus $y_1\alpha\neq 0\in\Omega_*^{\mathrm{U}}(\mathrm{X})$. Now note that

$$x-y_1y_2=pv$$

and
$$y_2(y_1\alpha)=(x-pv)\alpha=x\alpha-vp\alpha=0-0=0.$$

Thus $y_2\in\mathrm{A}(y_1\alpha)$. Hence $\mathrm{A}(y_1\alpha)\ni p, y_2$. Moreover the prime factorization of

$$x_2=y_2\in\Omega_*^{\mathrm{U}}/(p)$$

is smaller than that of x. To apply our inductive hypothesis we must also show that $y_2\in\mathrm{A}(y_1\alpha)$ is an element of minimal degree not contained in (p). This is easily seen as follows.

Suppose $y\in\mathrm{A}(y_1\alpha), y\notin(p)$. Then

$$(yy_1)\alpha=y(y_1\alpha)=0.$$

Thus $yy_1\in\mathrm{A}(\alpha)$. Since $y\notin(p)$ and $y_1\notin(p)$, and $(p)\subset\Omega_*^{\mathrm{U}}$ is a prime ideal, it follows that $yy_1\notin(p)$. Therefore by the choice of x, $\deg yy_1\geq\deg x$. But

$$\deg yy_1=\deg y+\deg y_1$$
$$\deg x\ =\deg y_2+\deg y_1$$

and hence $\deg y\geq\deg y_2$ as claimed.

Thus by the inductive hypothesis $\mathrm{A}(y_1\alpha)=(p, y_2)$. By a symmetric argument it is also seen that $\mathrm{A}(y_2\alpha)=(p, y_1)$.

Now suppose that $\eta \in A(\alpha)$. Then

$$\eta(y_1\alpha) = \eta y_1 \alpha = y_1 \eta\alpha = y_1 0 = 0.$$

Thus $\eta \in A(y_1\alpha)$. Hence

$$\eta = p\eta_1' + y_2\eta_1''.$$

Next we have

$$\begin{aligned}\eta_1'' y_2 \alpha &= y_2 \eta_1''\alpha = (\eta - p\eta_1')\alpha \\ &= \eta\alpha - \eta_1 p\alpha = 0 - 0 = 0.\end{aligned}$$

Hence $\eta_1'' \in A(y_2\alpha) = (p, y_1)$. Therefore

$$\eta_1'' = p\eta_2' + y_1\eta_2''.$$

Thus we have

$$\begin{aligned}\eta &= p\eta_1' + y_2\eta_1'' = p\eta_1' + y_2(p\eta_2' + y_1\eta_2'') \\ &= p(\eta_1' + \eta_2') + y_1y_2\eta_2'' \\ &= p(\eta_1' + \eta_2') + (x - pv)\eta_2'' \\ &= p(\eta_1' + \eta_2' - v) + x\eta_2'' \\ &= p\eta' + x\eta''\end{aligned}$$

and thus $\eta \in (p, x)$. Hence $(p, x) \subset A(\alpha) \subset (p, x)$ and therefore $A(\alpha) = (p, x)$ as required.

This completes the induction on the size of the factorization of x into primes in $\Omega_*^U/(p)$ and hence the result follows. □

Corollary **(5.12)**. — *Let* X *be a finite complex and* $f: S^n \to X$ *a continuous map. If* $\text{hom.dim}_{\Omega_*^U} \Omega_*^U(X) = 2$ *and* $[S^n, f] \in \Omega_*^U(X)$ *is of prime order, then*

$$\text{hom.dim}_{\Omega_*^U} \Omega_*^U(X \cup_f e^{n+1}) \leq 3.$$

Proof. — Immediate from Theorem (5.6) and the structure of $A([S^n, f])$ described in Proposition (5.11). □

§ 6. Generators for U-Bordism Modules.

In the previous sections we have constructed the spectral sequence $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ for skeleton finite CW-complexes. We now inquire into the non-triviality of this spectral sequence.

Recall that the edge homomorphism

$$\mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(X) = E^2_{0,*}\langle X\rangle \to E^\infty_{0,*}\langle X\rangle \to H_*(X; \mathbf{Z})$$

may be identified with the reduced Thom homomorphism

$$\tilde{\mu} : \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(X) \to H_*(X; \mathbf{Z}).$$

Thus if the spectral sequence $\{E^r\langle X\rangle, d^r\langle X\rangle\}$ was always trivial then $\tilde{\mu}$ would always be a monomorphism. Now note that a system of elements in $\Omega_*^U(X)$ that projects to a set of generators for $\mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(X)$ as an abelian group, generates $\Omega_*^U(X)$ as an Ω_*^U-module.

Hence if $\tilde{\mu}$ was always monic it would follow that $\Omega_*^U(X)$ is always generated by elements whose dimension does not exceed that of X.

Thus we are led to the question as to whether or not it is always possible to choose generators for $\Omega_*^U(X)$ so that their dimensions do not exceed that of X. The major portion of this section is given over to the construction of a specific example that answers this question in the negative. We then show by non-constructive methods that there is no universal bound on the difference between the dimension of X and the largest dimension in which a generator occurs in $\Omega_*^U(X)$.

We turn now to our example. We will need several preliminaries. We begin by recalling the relation between U-bordism and framed bordism (= stable homotopy), as developed in [9], [16], [22].

Let $\underset{\sim}{S}$ denote the sphere spectrum. The homology theory associated with the sphere spectrum is, by the famous Pontrjagin-Thom theorem ([3], [16], [22]), framed bordism theory, which we denote by $\Omega_*^{fr}(\ )$. The ring of coefficients $\Omega_*^{fr}(\text{point})$ is denoted simply by Ω_*^{fr}, and is isomorphic to the stable homotopy ring of spheres.

There is a natural inclusion $\underset{\sim}{S} \rightarrow \underset{\sim}{MU}$ and we have the cofibration sequence

$$\underset{\sim}{S} \xrightarrow{i} \underset{\sim}{MU} \xrightarrow{q} \underset{\sim}{MU}/\underset{\sim}{S}$$

defining the spectrum $\underset{\sim}{MU}/\underset{\sim}{S}$. The homology theory associated to $\underset{\sim}{MU}/\underset{\sim}{S}$ is called U-framed-bordism and is denoted by $\Omega_*^{U,fr}(\ )$. From the generalized Pontrjagin-Thom theorem we obtain the following bordism interpretation for $\Omega_*^{U,fr}(\ )$.

For any space X consider pairs (W, f) where W is a U-manifold together with a computable framing on ∂W, and $f: W \rightarrow X$ is a continuous map. We call such a pair a singular U-framed manifold on X. Two such pairs (W, f) and (V, g) are said to be U-framed cobordant over X iff there exists a framed manifold M, a map $\mu : M \rightarrow X$, a U-manifold N and a map $\eta : N \rightarrow X$ such that

1) $$\partial M = \partial W + (-\partial V);$$

2) $$\mu \mid \partial M = f \mid \partial W + g \mid \partial V;$$

3) $$\partial N = M \cup_\partial W + (-V);$$

4) $$\eta \mid \partial N = \mu \mid M \cup_\partial (f+g),$$

where $\cup_\partial$ denotes the boundary connected sum. The relation of U-framed-cobordism over X is an equivalence relation and the resulting equivalence classes form in the usual fashion a graded group that is naturally isomorphic to $\Omega_*^{U,fr}(X)$.

If (W, f) is a singular U-framed manifold on X and M is a framed manifold then $(W \times M, f \circ p_1)$ is also a singular U-framed manifold on X (here p_1 denotes projection on the first factor). This passes to the equivalence classes and provides $\Omega_*^{U,fr}(X)$ with the structure of an Ω_*^{fr}-module. The coefficients $\Omega_*^{U,fr}(\text{point})$ are as usual denoted simply by $\Omega_*^{U,fr}$.

Since a framed manifold of positive dimension always bounds a U-manifold, the cofibration sequence defining $\underset{\sim}{\mathrm{MU}}/\underset{\sim}{\mathrm{S}}$ yields the exact sequences

$$0 \to \Omega^{\mathrm{U}}_{n+1} \xrightarrow{i_*} \Omega^{\mathrm{U,fr}}_{n+1} \xrightarrow{\partial_*} \Omega^{\mathrm{fr}}_{n} \to 0, \qquad n>0$$

and
$$0 \to \Omega^{\mathrm{U}}_{1} \to \Omega^{\mathrm{U,fr}}_{1} \to \Omega^{\mathrm{fr}}_{0} \to \Omega^{\mathrm{U}}_{0} \to 0,$$

the latter sequence being somewhat trivial.

More generally for any finite complex X we obtain an exact triangle

$$\begin{tikzcd}
\Omega^{\mathrm{fr}}_*(\mathrm{X}) \arrow[rr, "i_*"] & & \Omega^{\mathrm{U}}_*(\mathrm{X}) \arrow[dl, "q_*"] \\
& \Omega^{\mathrm{U,fr}}_*(\mathrm{X}) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

of Ω^{fr}_*-modules and morphisms.

Proposition **(6.1)**. — *Let* X *be a finite complex,* $\alpha \in \Omega^{\mathrm{fr}}_n$, $\beta \in \Omega^{\mathrm{fr}}_m$ *and* $\gamma \in \Omega^{\mathrm{fr}}_k(\mathrm{X})$. *Suppose that* $\alpha\beta = 0 = \beta\gamma$. *Thus the Toda bracket* $\langle \alpha, \beta, \gamma \rangle \subset \Omega^{\mathrm{fr}}_{n+m+k+1}(\mathrm{X})$ *is defined.*

If $n>0$, *then* $i_*\langle \alpha, \beta, \gamma \rangle \subset \Omega^{\mathrm{U}}_{n+m+k+1}(\mathrm{X})$ *consists of a single element. Moreover, for any* $\lambda \in \Omega^{\mathrm{U,fr}}_{n+1}$ *with* $\partial_*(\lambda) = \alpha$, *there is a unique* $\mu \in \Omega^{\mathrm{U}}_{n+1}$ *such that* $i_*(\mu) = \lambda . \beta$; *and we then have* $i_*\langle \alpha, \beta, \gamma \rangle = (-1)^{k+m} \mu . i_*(\gamma) \in \Omega^{\mathrm{U}}_{n+m+k+1}(\mathrm{X})$.

Proof. — The proof is long and tedious, but straightforward. We proceed to the details.

The indeterminacy of the Toda bracket $\langle \alpha, \beta, \gamma \rangle$ is the homogenous component of degree $n+m+k+1$ of the graded group $\alpha . \Omega^{\mathrm{fr}}_*(\mathrm{X}) + \Omega^{\mathrm{fr}}_* . \gamma$. If $n>0$ then $i_*(\alpha) = 0$. Since any framed manifold of positive dimension bounds a U-manifold, we see that the indeterminacy of $\langle \alpha, \beta, \gamma \rangle$ is in the kernel of i_*. Thus $i_*\langle \alpha, \beta, \gamma \rangle \subset \Omega^{\mathrm{U}}_{n+m+k+1}(\mathrm{X})$ consists of a single element.

From the exact sequence

$$0 \to \Omega^{\mathrm{U}}_{n+1} \xrightarrow{i_*} \Omega^{\mathrm{U,fr}}_{n+1} \xrightarrow{\partial_*} \Omega^{\mathrm{fr}}_{n} \to 0$$

we see that we may choose a $\lambda \in \Omega^{\mathrm{U,fr}}_{n+1}$ with $\partial_*(\lambda) = \alpha$. Since ∂_* is a morphism of Ω^{fr}_*-modules we have

$$\partial_*(\lambda\beta) = \partial_*(\lambda) . \beta = \alpha\beta = 0.$$

Thus by exactness there is a unique $\mu \in \Omega^{\mathrm{U}}_{n+1}$ with $i_*(\mu) = \lambda\beta$. It remains to establish the equality $i_*\langle \alpha, \beta, \gamma \rangle = \mu . i_*(\gamma)$. We shall need several preliminaries.

Notations. — 1) If M′ and M″ are manifolds and $\partial\mathrm{M}' = \partial\mathrm{M}''$ we will denote by $\mathrm{M}' \cup_\partial \mathrm{M}''$ the manifold obtained by identifying $\partial\mathrm{M}'$ with $\partial\mathrm{M}''$. Note that $\mathrm{M}' \cup_\partial \mathrm{M}''$ is a manifold without boundary.

2) Let Y be a space and (M', f'), (M'', f'') be singular U-framed manifolds on Y. We will use the notation $(\mathrm{M}', f') \underset{\mathrm{U,fr}}{\sim} (\mathrm{M}'', f'')$ to indicate that (M', f') is U-framed

cobordant over Y to (M'', f''). We will use the similar notations $\underset{U}{\sim}$ and $\underset{fr}{\sim}$ to indicate U-cobordism over Y and framed cobordism over Y.

3) Let Y be a space and (M, f) a U-framed manifold on Y. We will denote by $[M, f]$ the U-framed equivalence class of (M, f). Note that $[M, f] \in \Omega_*^{U, fr}(Y)$.

4) If (M', f'), (M'', f'') are singular U-framed manifolds on Y and $\partial M' = \partial M''$, $f' \mid \partial M' = f'' \mid \partial M''$, then we obtain $f' \cup_\partial f'' : M' \cup_\partial M'' \to Y$ by requiring

$$f' \cup_\partial f'' \mid M' = f'$$
$$f' \cup_\partial f'' \mid M'' = f''.$$

The pair $(M' \cup_\partial M'', f' \cup_\partial f'')$ is then a closed singular U-manifold on Y.

Lemma. — Let R, S, T *be* U-*manifolds with* $\partial R = \partial S = \partial T$. *Suppose that* X *is a finite complex and*

$$\begin{tikzcd}
\partial R \arrow[r, hook] \arrow[d, equal] & R \arrow[dr, "\rho"] & \\
\partial S \arrow[r] \arrow[d, equal] & S \arrow[r, "\sigma"] & X \\
\partial T \arrow[r] & T \arrow[ur, "\tau"'] &
\end{tikzcd}$$

is a commutative diagram of continuous maps.

If

$$(R \cup_\partial R, \rho \cup_\partial \rho) \underset{U}{\sim} 0$$
$$(S \cup_\partial T, \sigma \cup_\partial \tau) \underset{U}{\sim} 0$$

then

$$(R \cup_\partial S, \rho \cup_\partial \sigma) \underset{U}{\sim} (R \cup_\partial T, \rho \cup_\partial \tau).$$

Proof. — We wish to construct a U-manifold H and a map $\varphi : H \to X$ such that:

1) $$\partial H = R \cup_\partial S + R \cup_\partial T$$

and 2) $$\varphi \mid \partial H = \rho \cup_\partial \sigma + \rho \cup_\partial \tau.$$

By hypothesis we may find U-manifolds P, Q and maps

$$\eta : P \to X \leftarrow Q : \zeta$$

such that

$$\partial P = R \cup_\partial R, \quad \eta \mid \partial P = \rho \cup_\partial \rho$$
$$\partial Q = S \cup_\partial T, \quad \zeta \mid \partial Q = \sigma \cup_\partial \tau.$$

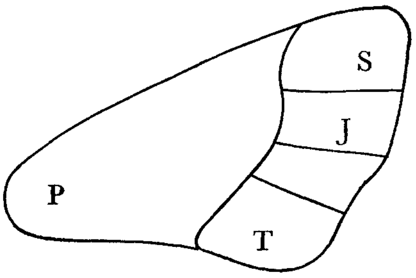
Let J denote the common boundary of R, S, and T. By the collaring theorem we may choose neighborhoods

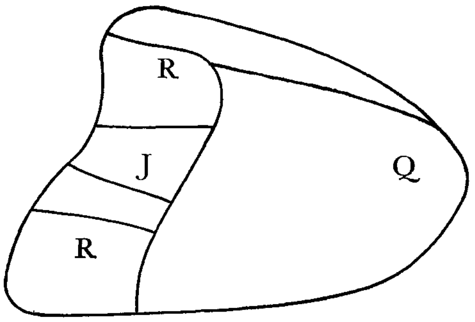
$$R \supset J \times [0, 1] \subset S$$
$$T \supset J \times [-1, 0] \subset R$$

and thus we obtain

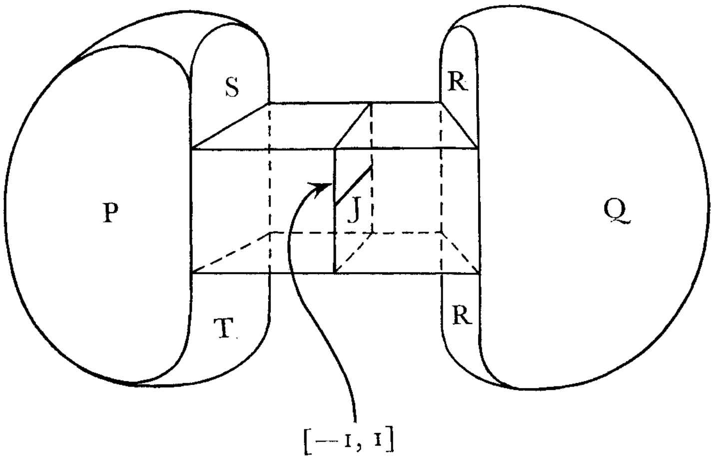
$$S \cup_\partial T \supset J \times [-1, 1] \subset R \cup_\partial R.$$

Pictorially we have





We then may form the smooth manifold H by identifying $J\times[-1, 1]\subset P$ with $J\times[-1, 1]\subset Q$. Pictorially we have



We then have $\partial H = RU_\partial S + RU_\partial T$. It is clear how to define φ with the required properties. □

We return now to the proof of Proposition (6.1).

Choose framed manifolds A, B to represent $\alpha, \beta\in\Omega_*^{fr}$ and a singular framed manifold (C, f) on X to represent $\gamma\in\Omega_*^{fr}(X)$. Choose a U-framed manifold L to represent $\lambda\in\Omega_*^{U,fr}$ and a U-manifold M to represent $\mu\in\Omega_*^{U}$.

Since $i_*(\mu)=\lambda\beta$ there exists a framed manifold D such that

$$\partial D = A\times B \quad (=\partial(L\times B))$$

and

$$M \underset{U}{\sim} ((L\times B)\cup_\partial D).$$

Choose a singular framed manifold (E, g) on X such that

$$\partial E = B\times C$$

and

$$g\,|\,\partial E = f.p_2 \quad (p_2 = \text{projection on the second factor}).$$

This is possible since $\beta\gamma = 0$.

Then $$\partial(D \times C) = A \times B \times C = \partial(A \times E)$$
and $$f \circ p_2 \,|\, \partial(D \times C) = g \circ p_2 \,|\, \partial(A \times E).$$
Thus $$(W, h) = ((A \times E) \cup_\partial (D \times C),\, g \circ p_2 \cup_\partial f \circ p_2)$$

is defined and represents an element of the coset $\langle \alpha, \beta, \gamma \rangle \in \Omega^{fr}_{n+m+k+1}(X)$.

What we wish to show is that
$$(W, h) \underset{U}{\sim} (M \times C, f.p_2).$$
We have $$(M \times C) \underset{U}{\sim} (((L \times B) \cup_\partial D) \times C) = (L \times B \times C) \cup_\partial (D \times C)$$
and we see $$(M \times C, fp_2) \underset{U}{\sim} ((L \times B \times C) \cup_\partial (D \times C),\, fp_3 \cup_\partial fp_2).$$

Now we observe that
$$(D \times C) \cup_\partial (D \times C) = (D \cup_\partial D) \times C.$$

Since D is a framed manifold, so is $D \cup_\partial D$ and thus as a U-manifold $D \cup_\partial D$ bounds. Therefore we have
$$((D \times C) \cup_\partial (D \times C), fp_2 \cup_\partial fp_2) \underset{U}{\sim} 0.$$

Consider the map $gp_2 : L \times E \rightarrow X$. We have
$$\partial(L \times E) = (\partial L \times E) \cup_\partial (L \times \partial E) = (A \times E) \cup_\partial (L \times B \times C)$$
and $$gp_2 \,|\, \partial(L \times E) = gp_2 \cup_\partial fp_3.$$
Thus if we set $$(R, \rho) = (D \times C, fp_2)$$
$$(S, \sigma) = (A \times E, gp_2)$$
$$(T, \tau) = (L \times B \times C, fp_3)$$

we find that the hypotheses of the Lemma are satisfied and hence we may conclude
$$((D \times C) \cup_\partial (A \times E), fp_2 \cup_\partial gp_2) \underset{U}{\sim} ((D \times C) \cup_\partial (L \times B \times C), fp_2 \cup_\partial fp_3)$$
which yields $$(W, h) \underset{U}{\sim} (M \times C, fp_2)$$

as required. □

We now obtain (compare [9], § 15):

Proposition **(6.2)**. — *Let* X *be a finite complex,* $\gamma \in \Omega^{fr}_*(X)$, $\mu \in \Omega^U_*$ *such that* $\deg \mu > 0$ *and*

1) $$2\gamma = 0 \ \in \Omega^{fr}_*(X)$$
2) $$2 \,|\, q_*(\mu) \in \Omega^{U,fr}_*.$$

Then there exists $\zeta \in \Omega^{fr}_*(X)$ *such that* $i_*(\zeta) = \mu . i_*(\gamma) \in \Omega^U_*(X)$.

Proof. — Let $\lambda \in \Omega^{U,fr}_*$ be such that $2\lambda = q_*(\mu)$. Let $\alpha = \partial_*(\lambda) \in \Omega^{fr}_*$. Since $\deg \mu > 0$ and $\Omega^U_1 = 0$ we may assume $\deg \mu > 1$ and hence $\deg \alpha > 0$. Thus the Toda bracket $\langle \alpha, 2, \mu \rangle \in \Omega^{fr}_*(X)$ is defined and by Proposition (6.1):
$$i_* \langle \alpha, 2, \mu \rangle = \pm \mu . \gamma$$

setting $\zeta = \mp \langle \alpha, 2, \mu \rangle$ the result follows. □

Corollary **(6.3)**. — *Let* X *be a finite complex and* $\gamma \in \Omega_*^{fr}(X)$ *with* $2\gamma = 0$. *Then for each positive integer n such that* $n \equiv 0$ *or* $n \equiv 1 \mod 4$, *there is a* $\zeta \in \Omega_*^{fr}(X)$ *such that*

$$i_*(\zeta) = [\mathbf{CP}(1)]^n i_*(\gamma).$$

Proof. — It is easy to see that $q_*([\mathbf{CP}(1)]) \in \Omega_*^{U, fr}$ is divisible by 2. Applying (6.2) now yields the result for $n = 1$. Standard properties of Toda brackets ([21], [23]) may now be applied to obtain the result inductively. The complete details for the general case, which is not needed here, will appear in a forthcoming publication of the authors in the *Proceedings of the 1969 Georgia conference on Topology.* □

Our example will be built from a suitable skeleton of $\mathbf{RP}(\infty) \times \mathbf{RP}(\infty)$. Recall that as an Ω_*^U-module, $\Omega_*^U(\mathbf{RP}(\infty))$ is generated by 1 and classes $\{\gamma_{2j+1}\}_{j=0}^{\infty}$ ([7], [9]). The class γ_{2i+1} may be taken to be the element of $\Omega_{2i+1}^U(\mathbf{RP}(\infty))$ represented by the standard inclusion $\mathbf{RP}(2i+1) \hookrightarrow \mathbf{RP}(\infty)$. The order of the class γ_{2n-1} is 2^{n-1} [7].

The manifolds $\mathbf{RP}(1)$ and $\mathbf{RP}(3)$ are framed and thus the elements γ_1 and γ_3 lie in the image of $\Omega_*^{fr}(\mathbf{RP}(\infty)) \to \Omega_*^U(\mathbf{RP}(\infty))$. It may be shown that $\tilde{\Omega}_1^{fr}(\mathbf{RP}(\infty)) \approx \mathbf{Z}/2\mathbf{Z}$ with generator σ_1 and $\tilde{\Omega}_3^{fr}(\mathbf{RP}(\infty)) \approx \mathbf{Z}/4\mathbf{Z}$ with generator σ_3. Moreover choices are possible such that $i_*\sigma_1 = \gamma_1$, and $i_*\sigma_3 = \gamma_3$.

The following are among the relations known to hold in $\Omega_*^U(\mathbf{RP}(\infty))$ [7]:

$$\begin{aligned} 2\gamma_1 &= 0 \\ 2\gamma_3 &= [\mathbf{CP}(1)]\gamma_1 \\ 2\gamma_5 &= [\mathbf{CP}(1)]\gamma_3 + [\mathbf{CP}(1)]^2\gamma_1 \\ 2\gamma_7 &= [\mathbf{CP}(1)]\gamma_5 + [\mathbf{CP}(1)]^2\gamma_3 + \lambda\gamma_1 \end{aligned}$$

where $\lambda \in \Omega_6^U$ and is not a zero divisor in $\Omega_*^U/(2, [\mathbf{CP}(1)])$.

Consider next the natural map

$$\nu : \Omega_*^U(\mathbf{RP}(\infty)) \otimes_{\Omega_*^U} \Omega_*^U(\mathbf{RP}(\infty)) \to \Omega_*^U(\mathbf{RP}(\infty) \times \mathbf{RP}(\infty))$$

arising from the exterior cross product. It may be shown (see for example section 8 in which we deal with the spectral Künneth Theorem for $\Omega_*^U(\cdot)$) that ν is a monomorphism. Thus we may write $\gamma_1 \otimes \gamma_1 \in \Omega_2^U(\mathbf{RP}(\infty) \times \mathbf{RP}(\infty))$ and other similar abuses of notation with the obvious meaning. Note that $\gamma_{2i+1} \otimes \gamma_{2j+1} \in \Omega_{2i+2j+2}^U(\mathbf{RP}(\infty) \times \mathbf{RP}(\infty))$ corresponds to the U-bordism class represented by the standard inclusion

$$\mathbf{RP}(2i+1) \times \mathbf{RP}(2j+1) \hookrightarrow \mathbf{RP}(\infty) \times \mathbf{RP}(\infty).$$

The classes $\gamma_1 \otimes \gamma_1$ and $\gamma_3 \otimes \gamma_3$ are in the image of the natural map $\Omega_*^{fr}(\mathbf{RP}(\infty) \times \mathbf{RP}(\infty)) \to \Omega_*^U(\mathbf{RP}(\infty) \times \mathbf{RP}(\infty))$, $\gamma_1 \otimes \gamma_1$ comming from an element σ_2 of order 2 and $\gamma_3 \otimes \gamma_3$ an element σ_6 of order 4.

Finally we recall ([7], [10]) that

$$A(\gamma_1 \otimes \gamma_1) = (2, [\mathbf{CP}(1)]) \subset \Omega_*^U.$$

Lemma **(6.4)**. — $2(\gamma_3 \otimes \gamma_3) = 0 \in \Omega_6^U(\mathbf{R}P(\infty) \times \mathbf{R}P(\infty))$.

Proof. — We have

$$\begin{aligned} 2(\gamma_3 \otimes \gamma_3) = 2\gamma_3 \otimes \gamma_3 &= [\mathbf{C}P(1)]\gamma_1 \otimes \gamma_3 \\ &= \gamma_1 \otimes [\mathbf{C}P(1)]\gamma_3 = \gamma_1 \otimes 2\gamma_5 - \gamma_1 \otimes [\mathbf{C}P(1)]^2\gamma_1 \\ &= 2\gamma_1 \otimes \gamma_5 - [\mathbf{C}P(1)]^2(\gamma_1 \otimes \gamma_1) = 0. \quad \square \end{aligned}$$

Lemma **(6.5)**. — $\lambda(\gamma_1 \otimes \gamma_1) = [\mathbf{C}P(1)](\gamma_3 \otimes \gamma_3) \in \Omega_8^U(\mathbf{R}P(\infty) \times \mathbf{R}P(\infty))$.

Proof. — We have

$$\begin{aligned} \lambda(\gamma_1 \otimes \gamma_1) &= \lambda\gamma_1 \otimes \gamma_1 \\ &= 2\gamma_1 \otimes \gamma_1 - [\mathbf{C}P(1)]\gamma_5 \otimes \gamma_1 - [\mathbf{C}P(1)]^2\gamma_1 \otimes \gamma_3 \\ &= \gamma_1 \otimes 2\gamma_1 - \gamma_5 \otimes [\mathbf{C}P(1)]\gamma_1 - [\mathbf{C}P(1)]([\mathbf{C}P(1)]\gamma_1 \otimes \gamma_3) \\ &= -\gamma_5 \otimes 2\gamma_3 - [\mathbf{C}P(1)]2(\gamma_3 \otimes \gamma_3) \\ &= -2\gamma_5 \otimes \gamma_3 = -[\mathbf{C}P(1)]\gamma_3 \otimes \gamma_3 - [\mathbf{C}P(1)]^2\gamma_1 \otimes \gamma_3 \\ &= [\mathbf{C}P(1)]\gamma_3 \otimes \gamma_3 - [\mathbf{C}P(1)]2(\gamma_3 \otimes \gamma_3) \\ &= [\mathbf{C}P(1)]\gamma_3 \otimes \gamma_3 \end{aligned}$$

as required. $\square$

Let $X \subset \mathbf{R}P(\infty) \times \mathbf{R}P(\infty)$ be the 9-skeleton. Then

$$\Omega_k^{fr}(X) \approx \Omega_k^{fr}(\mathbf{R}P(\infty) \times \mathbf{R}P(\infty)) \quad \text{for } 0 \leq k \leq 8$$

and

$$\Omega_k^U(X) \approx \Omega_k^U(\mathbf{R}P(\infty) \times \mathbf{R}P(\infty)) \quad \text{for } 0 \leq k \leq 8.$$

Thus we have classes $\gamma_2 \in \Omega_2^U(X), \gamma_6 \in \Omega_6^U(X)$ corresponding to $\gamma_1 \otimes \gamma_1$ and $\gamma_3 \otimes \gamma_3$. From the previous lemmas we have the relations

$$\begin{aligned} 2\gamma_2 = 0 = 2\gamma_6 \\ \lambda\gamma_2 = [\mathbf{C}P(1)]\gamma_6 \neq 0. \end{aligned}$$

There are also classes $\tau_2, \tau_6 \in \Omega_*^{fr}(X)$ with

$$\begin{aligned} 2\tau_2 &= 0 = 4\tau_6 \\ i_*\tau_2 &= \gamma_2 \\ i_*\tau_6 &= \gamma_6. \end{aligned}$$

Lemma **(6.6)**. — *Let* $f: S^{n-1} \rightarrow Y$ *be a map representing the* 0 *element of* $\Omega_*^U(Y)$. *Then*

$$\Omega_*^U(Y \cup_f e^n) \cong \Omega_*^U(Y) \oplus \widetilde{\Omega}_*^U(S^n).$$

Proof. — Form the cofibration sequence

$$Y \rightarrow Y \cup_f e^n \rightarrow S^n.$$

We then obtain an exact sequence

$$\begin{tikzcd} \Omega_*^U(Y) \arrow[rr] & & \Omega_*^U(Y \cup_f e^n) \arrow[dl, "j"] \\ & \widetilde{\Omega}_*^U(S^n) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

wherein $\partial_*=f_*=0$. Thus we have a short exact sequence of Ω_*^U-modules

$$0\to\Omega_*^U(Y)\to\Omega_*^U(Y\cup_f e^n)\to\tilde{\Omega}_*^U(S^n)\to 0.$$

Since $\tilde{\Omega}_*^U(S^n)$ is a free Ω_*^U-module this sequence splits and the result follows. $\square$

Now return to the 9-skeleton X of $\mathbf{RP}(\infty)\times\mathbf{RP}(\infty)$. Suspend X to $\Sigma^n X$ so that there is a map $g: S^{n+6}\to\Sigma^n X$ which represents $\Sigma^n\tau_6\in\Omega_{n+6}^{fr}(\Sigma^n X)$ as a spherical class. Let $f: S^{n+6}\to\Sigma^n X$ represent $2g\in\pi_{n+6}(\Sigma^n X)$. Let $Y=(\Sigma^n X)\cup_f e^{n+7}$. By Lemma (6.6) we have

$$\Omega_*^U(Y)\cong\Omega_*^U(\Sigma^n X)\oplus\tilde{\Omega}_*^U(S^{n+7})$$

since $2\Sigma^n\gamma_6=0\in\Omega_{n+6}(\Sigma^n X)$. Let $\tau_{n+6}\in\Omega_{n+6}^{fr}(Y)$ be the element represented by the composite

$$S^{n+6}\underset{g}{\to}X\to X\cup_f e^{n+7}.$$

Then we have
$$2\tau_{n+6}=0\in\Omega_{n+6}^{fr}(Y)$$
and
$$i_*\tau_{n+6}=\Sigma^n\gamma_6\in\Omega_{n+6}^U(Y).$$

Thus we may apply Corollary (6.3) to conclude that there is a $\zeta\in\Omega_{n+8}^{fr}(Y)$ with $i_*(\zeta)=[\mathbf{CP}(1)]\Sigma^n\gamma_6\in\Omega_{n+8}^U(Y)$. Recall that $[\mathbf{CP}(1)]\Sigma^n\gamma_6\neq 0$ and that

$$[\mathbf{CP}(1)]\Sigma^n\gamma_6=\lambda\Sigma^n\gamma_2\in\Omega_{n+8}^U(Y).$$

As noted previously $A(\gamma_1\otimes\gamma_1)=(2,[\mathbf{CP}(1)])\subset\Omega_*^U$. Thus $A(\gamma_2)=(2,[\mathbf{CP}(1)])$ and hence $[\mathbf{CP}(1)]\in A([\mathbf{CP}(1)]\Sigma^n\gamma_6)$.

Observe that $\Sigma^n X$ is the suspension of a 9-dimensional complex and Y was obtained by adjoining an $n+7$-cell. Thus $H_k(Y;\mathbf{Z})=0$ for $k>n+9$.

We next suspend Y to $\Sigma^m Y$ to obtain a map

$$h: S^{m+n+8}\to\Sigma^m Y$$

that represents $\Sigma^m\zeta\in\Omega_{m+n+8}(Y)$ as a spherical element. Adjoin a cell to obtain $W=(\Sigma^m Y)\cup_h e^{m+n+9}$.

Note that $H_k(W;\mathbf{Z})=0$ for $k>m+n+9$.

Recollection. — Let Ω be a graded connected algebra over a ring k. Let M be an Ω-module. The submodule $\tilde{\Omega}.M\subset M$ is called the module of decomposable elements. The graded k-module, $k\otimes_\Omega M\cong M/\tilde{\Omega}.M$ is called the k-module of indecomposable elements. By abuse of terminology an element $\eta\in M$ that projects to a non-zero element of $M/\tilde{\Omega}.M$ is referred to as an indecomposable element.

Consider the cofibration

$$S^{m+n+8}\xrightarrow{h}\Sigma^m Y\to W.$$

Applying the U-bordism functor yields the exact triangle

$$\begin{tikzcd}
\tilde{\Omega}_*^{\mathrm{U}}(\mathrm{S}^{m+n+8}) \arrow[rr, "h_*"] & & \tilde{\Omega}_*^{\mathrm{U}}(\Sigma^m \mathrm{Y}) \arrow[dl] \\
& \tilde{\Omega}_*^{\mathrm{U}}(\mathrm{W}) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

Note that we have $\operatorname{Im} \partial_* = \Sigma^{m+n}\mathrm{A}([\mathbf{C}\mathrm{P}(1)]\Sigma^{m+n}\mathrm{Y}_6) = \ker h_*$. Let

$$\theta_{m+n+8} \in \tilde{\Omega}^{\mathrm{U}}_{m+n+8}(\mathrm{S}^{m+n+8})$$

be the standard generator. Then

$$h_*([\mathbf{C}\mathrm{P}(1)]\theta_{m+n+8}) = [\mathbf{C}\mathrm{P}(1)]([\mathbf{C}\mathrm{P}(1)]\Sigma^{m+n}\mathrm{Y}_6) = 0.$$

Therefore there exists $\omega \in \Omega^{\mathrm{U}}_{m+n+11}(\mathrm{W})$ with $\partial_*(\omega) = [\mathbf{C}\mathrm{P}(1)]\theta_{m+n+8}$. Thus under the map

$$1 \otimes \partial_* : \ \mathbf{Z} \otimes_{\Omega_*^{\mathrm{U}}} \tilde{\Omega}_*^{\mathrm{U}}(\mathrm{W}) \to \mathbf{Z} \otimes_{\Omega_*^{\mathrm{U}}} \Sigma^{m+n}\mathrm{A}([\mathbf{C}\mathrm{P}(1)]\Sigma^{m+n}\mathrm{Y}_6)$$

we have
$$1 \otimes \partial_*(1 \otimes \omega) = 1 \otimes [\mathbf{C}\mathrm{P}(1)] \neq 0$$

since $[\mathbf{C}\mathrm{P}(1)]$ is indecomposable in Ω_*^{U}. Therefore $\omega \in \tilde{\Omega}_*^{\mathrm{U}}(\mathrm{W})$ is an indecomposable element, i.e., $1 \otimes \omega \neq 0 \in \mathbf{Z} \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{W})$.

Let us review what we have shown. We have constructed a finite complex W, and an element $\omega \in \Omega_*^{\mathrm{U}}(\mathrm{W})$, such that

1) $\dim \mathrm{W} = m+n+9$

2) $\dim \omega = m+n+11$

3) $\omega \neq 0 \in \mathbf{Z} \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{W})$.

Thus we have established:

Theorem **(6.7)**. — *There exists a finite complex* W *and an indecomposable element* $\omega \in \Omega_*^{\mathrm{U}}(\mathrm{W})$ *such that* $\dim \omega > \dim \mathrm{W}$. ☐

Corollary **(6.8)**. — *There exists a finite complex* W *such that the reduced Thom homomorphism*

$$\tilde{\mu} : \mathbf{Z} \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{W}) \to \mathrm{H}_*(\mathrm{W}; \mathbf{Z})$$

is not a monomorphism.

Proof. — Let W and $\omega \neq 0 \in \mathbf{Z} \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{W})$. Let $s = \dim \omega$. Then

$$\tilde{\mu}(\omega) \in \mathrm{H}_s(\mathrm{W}; \mathbf{Z}) = 0$$

since $s > \dim \mathrm{W}$, and the result follows. ☐

Corollary **(6.9)**. — *There exists a finite complex* W *such that the spectral sequence* $\{\mathrm{E}^r\langle \mathrm{W}\rangle, d^r\langle \mathrm{W}\rangle\}$ *is non-trivial.*

Proof. — Let W and $\omega \in \Omega_*^{\mathrm{U}}(\mathrm{W})$ be as in Theorem (6.7). Then

$$\omega \neq 0 \in \ker\{\tilde{\mu} : \mathbf{Z} \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{W}) \to \mathrm{H}_*(\mathrm{W}; \mathbf{Z})\}.$$

Since we may identify the edge map

$$E^2_{0,*}\langle W\rangle \to E^{\infty}_{0,*}\langle W\rangle$$

with $\tilde{\mu}$ it follows that there is a differential killing the class $\omega \in E^2_{0,*}\langle W\rangle$. ☐

Remark. — It is interesting to note that the example of non-triviality in the spectral sequence $\{E^r\langle\ \rangle, d^r\langle\ \rangle\}$ that we have constructed relies so heavily on Toda brackets in *framed* bordism.

Having now seen that $\Omega^U_*(X)$ need not be generated by classes of dimension not greater than the dimension of X we consider whether or not there is perhaps a uniform upper bound on the difference between the dimension of X and the largest dimension in which a generator occurs in $\Omega^U_*(X)$. We answer this question in the negative, by non-constructive methods, that are independent of our previous example, and thereby obtain alternate proofs of Theorem (6.7) and Corollaries (6.8), (6.9). Specifically we will prove:

Theorem **(6.10)**. — *Let n be a non-negative integer. Then there exists a finite complex* X *such that* $\Omega^U_*(X)$ *contains an indecomposable* U*-bordism class of dimension greater than* $n+\dim X$.

Proof. — The proof is by contradiction. We will therefore assume that there is an (even) integer 2N such that for any finite complex X and any $m > 2N+\dim X$ we have $(\mathbf{Z}\otimes_{\Omega^U_*}\Omega^U_*(X))_m = 0$. We are going to use this assumption to conclude that for every finite complex Y, $\text{hom.dim}_{\Omega^U_*}\Omega^U_*(Y) \leq N+3$. This will contradict Corollary (5.5) and establish the negation of our assumption, as desired. We proceed by induction on the number of cells, the case of one cell being trivial, and so we proceed to the inductive step. Let $f: S^n \to Y$ be a continuous map. Form the cofibration sequence

$$S^n \xrightarrow[f]{} Y \to Y\cup_f e^{n+1}.$$

Applying the U-bordism functor yields us an exact triangle

$$\begin{tikzcd}
\tilde{\Omega}^U_*(S^n) \arrow[rr] & & \tilde{\Omega}^U_*(Y) \arrow[dl] \\
& \Omega^U_*(Y\cup_f e^{n+1}) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

We thus obtain exact sequences

$$0 \to \Sigma^n A(\gamma) \to \tilde{\Omega}^U_*(S^n) \to M(\gamma) \to 0$$
$$0 \to M(\gamma) \to \tilde{\Omega}^U_*(Y) \to \Omega^U_*(Y)/M(\gamma) \to 0$$
$$0 \to \Omega^U_*(Y)/M(\gamma) \to \Omega^U_*(Y\cup_f e^{n+1}) \to \Sigma^n A(\gamma) \to 0$$

where $\gamma \in \tilde{\Omega}^U_*(Y)$ is the spherical bordism element represented by $f: S^n \to X$ and $M(\gamma)$ is the Ω^U_*-submodule of $\tilde{\Omega}^U_*(Y)$ generated by γ.

From the last exact sequence we learn that

$$(\mathbf{Z}\otimes_{\Omega^U_*}\Sigma^n A(\gamma))_m = 0$$

for $m>2N+\dim(Y\cup_f e^{n+1})>2N+n$. Hence the ideal $A(\gamma)\subset\Omega_*^U$ is generated by elements of dimension $\leq 2N$. If we now recall that $\Omega_*^U=\mathbf{Z}[x_2, x_4, x_6, \dots]$, $\deg x_i=i$, examination of the proof of Proposition (1.4) shows that $\mathrm{hom.dim}_{\Omega_*^U}A(\gamma)\leq N$. Hence the top exact sequence yields $\mathrm{hom.dim}_{\Omega_*^U}M(\gamma)\leq N+1$. By our inductive assumption $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(Y)\leq N+3$. Consideration of the middle exact sequence now yields by an easy argument (use for example the criterion given in ([5], VI, (2.1), *b)*) and exact sequences) that $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(Y)/M(\gamma)\leq N+3$. Thus by ([5], VI, (2.2)) we obtain from the last exact sequence that $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(Y\cup_f e^{n+1})\leq N+3$. This completes the inductive step and thus we have shown $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)\leq N+3$ for all finite complexes X. This contradicts Proposition (5.5) and as noted previously completes the proof. □

Our discussion of generators for U-bordism modules still leaves open the following question. Is there a function $\eta:\mathbf{Z}^+\to\mathbf{Z}^+$ such that for all finite complexes X of dimension n, $\Omega_*^U(X)$ is generated by elements of dimension no greater than $\eta(n)+\dim X$?

Note also that as a consequence of Theorem 4.4 a finite complex X with $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)\leq 2$ will always have the property that $\Omega_*^U(X)$ is generated by classes of dimension no greater than the dimension of X. This explains somewhat the complexity of examples such as those above (e.g. (6.7), etc.).

§ 7. Attaching Cells. Some Special Results.

In this section we will collect several special results concerning the change in $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)$ caused by attaching a cell to X. The results are very special in that they put severe restrictions on the " type " of cells attached. One pleasant consequence of these results is the characterization at the end of the section of the inequality

$$\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)\leq 2$$

in terms of the Thom homomorphism for X *and a family of subcomplexes of* X.

Before plunging into the technicalities we take this opportunity to construct some simple examples of spaces X with $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)=2$. These spaces will reappear in another context in the following section on the spectral Künneth Theorem.

Let k be a positive integer and M the Moore space obtained by attaching a $(k+1)$-cell to S^k by a map of degree 2. We find

$$\tilde{H}_*(M;\mathbf{Z})\cong\tilde{H}_k(M;\mathbf{Z})\cong\mathbf{Z}/2\mathbf{Z},$$

and

$$\tilde{\Omega}_*^U(M)\cong\Sigma^k[\Omega_*^U/(2)]$$

where $\tilde{\Omega}_*^U(\ )$ denotes the *reduced* U-bordism functor. Choose $\hat{\alpha}\neq 0\in\tilde{\Omega}_k^U(M)\cong\mathbf{Z}/2\mathbf{Z}$, and note that

$$\mu(\hat{\alpha})=\hat{a}\neq 0\in\tilde{H}_k(M;\mathbf{Z})\cong\mathbf{Z}/2\mathbf{Z}.$$

The class $\hat{\alpha}\in\tilde{\Omega}^{\mathrm{U}}_{k}(\mathrm{M})$ is in the image of $\tilde{\Omega}^{fr}_{k}(\mathrm{M})\to\tilde{\Omega}^{\mathrm{U}}_{k}(\mathrm{M})$ and hence by Corollary (6.3) so are the classes $[\mathbf{CP}(1)]^{n}\hat{\alpha}\in\Omega^{\mathrm{U}}_{k+2n}(\mathrm{M})$, for each $n\geq 0$ such that $n\equiv 0$ or $n\equiv 1$ mod 4. If we choose $k+2n$ to lie in the stable range (since M is $(k-1)$-connected this means $k+2n<2k-3$) then it follows from the suspension theorem ([21]) that we may find a map

$$f:\mathrm{S}^{2n+k}\to\mathrm{M}$$

representing an element of order 2 in $\pi_{2n+k}(\mathrm{M})$ such that

$$[\mathrm{S}^{2n+k},f]=[\mathrm{CP}(1)]^{n}\hat{\alpha}\in\Omega^{\mathrm{U}}_{2n+k}(\mathrm{M}).$$

We now let $\mathrm{X}=\mathrm{M}\cup_{f}e^{2n+k+1}$. We will compute $\tilde{\Omega}^{\mathrm{U}}_{*}(\mathrm{X})$ from the cofibration

$$\mathrm{M}\to\mathrm{X}\to\mathrm{S}^{2n+k+1}$$

and see that $\operatorname{hom.dim}_{\Omega^{\mathrm{U}}_{*}}\tilde{\Omega}^{\mathrm{U}}_{*}(\mathrm{X})=2$.

We have the exact triangle

$$\begin{tikzcd}
\tilde{\Omega}^{\mathrm{U}}_{*}(\mathrm{M}) \arrow[rr, "i_*"] & & \tilde{\Omega}^{\mathrm{U}}_{*}(\mathrm{X}) \arrow[dl, "j_*"] \\
& \tilde{\Omega}^{\mathrm{U}}_{*}(\mathrm{S}^{2n+k+1}) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

Let $\rho_{2n+k+1}\in\tilde{\Omega}^{\mathrm{U}}_{2n+k+1}(\mathrm{S}^{2n+k+1})$ be the usual generator. By construction

$$\partial_{*}(\rho_{2n+k+1})=[\mathbf{CP}(1)]^{n}\hat{\alpha},$$

and thus $$\tilde{\Omega}^{\mathrm{U}}_{*}(\mathrm{M})/\mathrm{Im}\ \partial_{*}\cong\Sigma^{k}\Omega^{\mathrm{U}}_{*}/(2,\ [\mathbf{CP}(1)]^{n}).$$

Next note that $$\mathrm{Im}\,j_{*}=\Sigma^{2n+k+1}\mathrm{A}(\hat{\alpha})$$

and $$\mathrm{A}(\hat{\alpha})=(2)\subset\Omega^{\mathrm{U}}_{*}.$$

Thus the exact triangle yields the exact sequence

$$0\to\Sigma^{k}[\Omega^{\mathrm{U}}_{*}/(2,\ [\mathbf{CP}(1)]^{n})]\to\tilde{\Omega}^{\mathrm{U}}_{*}(\mathrm{X})\to\Sigma^{2n+k+1}2\Omega^{\mathrm{U}}_{*}\to 0.$$

Since $\Sigma^{2n+k+1}2\Omega^{\mathrm{U}}_{*}$ is a free Ω^{U}_{*}-module this sequence must split and we obtain,

$$\tilde{\Omega}^{\mathrm{U}}_{*}(\mathrm{X})\cong(\Sigma^{k}[\Omega^{\mathrm{U}}_{*}/(2,\ [\mathbf{CP}(1)]^{n})])\oplus(\Sigma^{2n+k+1}2\Omega^{\mathrm{U}}_{*}).$$

Hence by explicit computation with the Koszul complex (see e.g. section 5 or [5], [18], etc.) we find $\operatorname{hom.dim}_{\Omega^{\mathrm{U}}_{*}}\tilde{\Omega}^{\mathrm{U}}_{*}(\mathrm{X})=2$. Since $\Omega^{\mathrm{U}}_{*}(\mathrm{X})=\Omega^{\mathrm{U}}_{*}\oplus\tilde{\Omega}^{\mathrm{U}}_{*}(\mathrm{X})$ we have shown:

Proposition (**7.1**). — *There exist finite complexes* X *with*

$$\tilde{\mathrm{H}}_{i}(\mathrm{X};\mathbf{Z})=\begin{cases}\mathbf{Z}_{2} & i=k\\ \mathbf{Z} & i=k+2n+1,\ n\equiv 0 \textit{ or } n\equiv 1 \bmod 4\\ 0 & \textit{otherwise}\end{cases}$$

such that $\operatorname{hom.dim}_{\Omega^{\mathrm{U}}_{*}}\Omega^{\mathrm{U}}_{*}(\mathrm{X})=2$.

Notice that in the explicit examples that we have constructed the manner in which the Thom homomorphism

$$\mu : \Omega_*^U(X) \to H_*(X; \mathbf{Z})$$

fails to be onto is quite simply that

$$\mu(\Omega^U_{k+2n+1}(X)) = 2\mathbf{Z} \subset \mathbf{Z} = H_{2n+k+1}(X; \mathbf{Z}).$$

These spaces X are the simpliest examples we know of where the Thom map fails to be onto.

We turn now to the study of attaching cells again. We begin with

Proposition **(7.2)**. — *Let* X *be a finite complex and* $f : S^n \to X$ *a continuous map. Let* $[S^n, f] = \gamma \in \Omega_n^U(X)$. *If*

$$\tilde{\mu} : \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(X \cup_f e^{n+1}) \to H_*(X \cup_f e^{n+1}; \mathbf{Z})$$

is a monomorphism then

$$n + 2\,\mathrm{hom.dim}_{\Omega_*^U} A(\gamma) < \max\{n+1, \dim X\} = \dim X \cup_f e^{n+1}.$$

As part of the proof is geometric and part is algebraic it will be convenient to separate out the main algebraic step.

Lemma. — *Let* $I \subset \Omega_*^U = \mathbf{Z}[x_2, x_4, x_6, \dots]$ *be an ideal. If there is a system of generators* $\{y_0, y_1, \dots, y_t\}$ *for* I *with* $y_i \in \mathbf{Z}[x_2, x_4, \dots, x_{2s}]$ *for* $i = 0, 1, \dots, t$, *then* $\mathrm{hom.dim}_{\Omega_*^U} I \leqslant s$.

Proof. — Let J be the ideal in $\mathbf{Z}[x_2, x_4, \dots, x_{2s}]$ generated by $y_0, y_1, \dots, y_t$. By Hilberts syzygy theorem ([5], VIII, 6; [15], VII) we may find a free resolution

$$0 \leftarrow J \xleftarrow{\varepsilon} P_0 \leftarrow P_1 \leftarrow \dots \leftarrow P_s \leftarrow 0$$

of J as a module over $\mathbf{Z}[x_2, x_4, \dots, x_{2s}]$. Since Ω_*^U is a free $\mathbf{Z}[x_2, x_4, \dots, x_{2s}]$-module, the sequence

$$\begin{tikzcd}
0 & J \otimes_{\mathbf{Z}[x_2, \dots, x_{2s}]} \Omega_*^U \arrow[l] \arrow[d, leftrightarrow, "\cong"] & P_0 \otimes_{\mathbf{Z}[x_2, \dots, x_{2s}]} \Omega_*^U \arrow[l] & \dots \arrow[l] & P_s \otimes_{\mathbf{Z}[x_2, \dots, x_{2s}]} \Omega_*^U \arrow[l] & 0 \arrow[l] \\
 & I & & & &
\end{tikzcd}$$

is then a free resolution of I as an Ω_*^U-module and the result follows.**

Proof of Proposition (7.2). — Actually we will prove a bit more than required, namely we will show $[\mathbf{Z} \otimes_{\Omega_*^U} A(\gamma)]_j = 0$ for all j with $n + j \geq \dim Y$. Thus $A(\gamma)$ is generated as an ideal by classes of dimension strictly less than $\dim Y - n + 1$. Since $\Omega_*^U = \mathbf{Z}[x_2, x_4, \dots]$ it follows that $A(\gamma) = (y_0, \dots, y_t)$ where $y_i \in \mathbf{Z}[x_2, x_4, \dots, x_{2s}]$ for some s with $n + 2s < \dim Y$. Hence by the lemma we will have

$$n + 2\,\mathrm{hom.dim}_{\Omega_*^U} A(\gamma) \leq n + 2s < \dim Y \quad \text{as required.}$$

So we have merely to establish our assertion about the placement of the generators of $A(\gamma)$.

Consider the cofibration

$$S^n \xrightarrow{f} X \to Y = X \cup_f e^{n+1}.$$

Applying $\widetilde{\Omega}^U_*(\ )$ yields an exact triangle

$$\begin{tikzcd}
\widetilde{\Omega}^U_*(S^n) \arrow[rr, "f_*"] & & \widetilde{\Omega}^U_*(X) \arrow[dl] \\
& \widetilde{\Omega}^U_*(Y) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

and we recall that $\quad \operatorname{Im} \partial_* = \Sigma^n A(\gamma) = \ker f_*.$

We thus have an epimorphism of degree -1

$$\partial_* : \ \widetilde{\Omega}^U_*(Y) \to \Sigma^n A(\gamma),$$

and hence we obtain an epimorphism of degree -1

$$1 \otimes \partial_* : \ \mathbf{Z} \otimes_{\Omega^U_*} \widetilde{\Omega}^U_*(Y) \to \mathbf{Z} \otimes_{\Omega^U_*} \Sigma^n A(\gamma).$$

Recall that by hypothesis $\quad \widetilde{\mu} : \ \mathbf{Z} \otimes_{\Omega^U_*} \widetilde{\Omega}^U_*(Y) \to \widetilde{H}_*(Y; \mathbf{Z})$

is a monomorphism. Since $\dim Y = \max\{n+1, \dim X\}$ we must have

$$[\mathbf{Z} \otimes_{\Omega^U_*} \widetilde{\Omega}^U_*(Y)]_j = 0 \quad \text{for } j > \max\{n+1, \dim X\}.$$

Therefore clearly

$$[\mathbf{Z} \otimes_{\Omega^U_*} \Sigma^n A(\gamma)]_i = 0 \quad \text{for } i \geq \max\{n+1, \dim X\}.$$

Hence $$[\mathbf{Z} \otimes_{\Omega^U_*} A(\gamma)]_j = 0 \quad \text{for } j \geq \max\{n+1, \dim X\} - n.$$

Which establishes our assertion about the placement of generators in $A(\gamma)$. □

Corollary **(7.3)**. — *Let* X *be a finite complex and* $f : S^n \to X$ *a continuous map. Let* $[S^n, f] = \gamma \in \Omega^U_n(X)$. *If*

$$n + 2\,\mathrm{hom.dim}_{\Omega^U_*} \Omega^U_*(X \cup_f e^{n+1}) \geq \max\{n+1, \dim X\} = \dim X \cup_f e^{n+1}$$

then $$\widetilde{\mu} : \ \mathbf{Z} \otimes_{\Omega^U_*} \Omega^U_*(X \cup_f e^{n+1}) \to H_*(X \cup_f e^{n+1}; \mathbf{Z})$$

has a non-trivial kernel. More precisely, $\Omega^U_*(X \cup_f e^{n+1})$ *is not generated as an* Ω^U_**-module by classes of dimension at most equal to* $\dim X \cup_f e^{n+1}$. □

Corollary **(7.4)**. — *Let* X *be a finite complex of dimension at most* $n+1$. *Suppose that* $f : S^n \to X$ *is a continuous map. Let* $\gamma = [S^n, f] \in \Omega^U_n(X)$. *If*

$$\widetilde{\mu} : \ \mathbf{Z} \otimes_{\Omega^U_*} \Omega^U_*(X \cup_f e^{n+1}) \to H_*(X \cup_f e^{n+1}; \mathbf{Z})$$

is a monomorphism then $\mathrm{hom.dim}_{\Omega^U_*} A(\gamma) = 0$.

Proof. — Note that $\dim(X \cup_f e^{n+1}) = n+1$ and apply Proposition (7.2). □

Proposition (7.5). — *Let* X *be a finite complex of dimension at most* $n+1$. *Suppose that* $f: \mathrm{S}^n \to \mathrm{X}$ *is a continuous map. If*

$$\tilde{\mu}: \quad \mathbf{Z} \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{X} \cup_f e^{n+1}) \to \mathrm{H}_*(\mathrm{X} \cup_f e^{n+1}; \mathbf{Z})$$

is a monomorphism and either

$$\text{hom.dim}_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{X}) > 2$$

or $$\text{hom.dim}_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{X} \cup_f e^{n+1}) > 2$$

then $$\text{hom.dim}_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{X}) = \text{hom.dim}_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{X} \cup_f e^{n+1}).$$

Proof. — Form the cofibration sequence

$$\mathrm{S}^n \xrightarrow{f} \mathrm{X} \to \mathrm{Y} = \mathrm{X} \cup_f e^{n+1}.$$

Applying $\tilde{\Omega}_*^{\mathrm{U}}(\ )$ yields the exact triangle

$$\begin{tikzcd}
\tilde{\Omega}_*^{\mathrm{U}}(\mathrm{S}^n) \arrow[rr, "f_*"] & & \tilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}) \arrow[dl] \\
& \tilde{\Omega}_*^{\mathrm{U}}(\mathrm{Y}) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

Let $\gamma = [\mathrm{S}^n, f] \in \Omega_n^{\mathrm{U}}(\mathrm{X})$. Let us denote by $\mathrm{M}(\gamma)$ the Ω_*^{U}-submodule of $\tilde{\Omega}_*^{\mathrm{U}}(\mathrm{X})$ generated by γ and by $\mathrm{N}(\gamma)$ the quotient module $\tilde{\Omega}_*^{\mathrm{U}}(\mathrm{X})/\mathrm{M}(\gamma)$.

We then have exact sequences

$$\begin{aligned}
&0 \to \Sigma^n \mathrm{A}(\gamma) \to \tilde{\Omega}_*^{\mathrm{U}}(\mathrm{S}^n) \to \mathrm{M}(\gamma) \to 0 \\
&0 \to \mathrm{M}(\gamma) \to \tilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}) \to \mathrm{N}(\gamma) \to 0 \\
&0 \to \mathrm{N}(\gamma) \to \tilde{\Omega}_*^{\mathrm{U}}(\mathrm{Y}) \to \Sigma^n \mathrm{A}(\gamma) \to 0.
\end{aligned}$$

By Corollary (7.4), $\Sigma^n \mathrm{A}(\gamma)$ is a projective Ω_*^{U}-module. Hence the last sequence splits and yields

$$(*) \qquad \text{hom.dim}_{\Omega_*^{\mathrm{U}}} \mathrm{N}(\gamma) = \text{hom.dim}_{\Omega_*^{\mathrm{U}}} \tilde{\Omega}_*^{\mathrm{U}}(\mathrm{Y}).$$

Moreover from the first sequence we obtain

$$(**) \qquad \text{hom.dim}_{\Omega_*^{\mathrm{U}}} \mathrm{M}(\gamma) = 1.$$

Consider now the middle sequence. From $(**)$ it follows that

$$(\overset{*}{*}) \qquad \mathrm{Ext}_{\Omega_*^{\mathrm{U}}}^{n,*}(\tilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}), \mathrm{C}) \cong \mathrm{Ext}_{\Omega_*^{\mathrm{U}}}^{n,*}(\mathrm{N}(\gamma), \mathrm{C})$$

for all $n > 2$ and any Ω_*^{U}-module C. From our hypotheses and equality $(*)$ we have either

$$\text{hom.dim}_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{X}) > 2$$

or $$\text{hom.dim}_{\Omega_*^{\mathrm{U}}} \mathrm{N}(\gamma) > 2.$$

A moment's reflection on the isomorphisms ($\overset{*}{*}$) and ([5], VI, (2.1)) show that

$$\mathrm{hom.dim}_{\Omega_*^U}\widetilde{\Omega}_*^U(X) = \mathrm{hom.dim}_{\Omega_*^U}N(\gamma),$$

and thus the result follows from equality ($*$). □

Corollary **(7.6)**. — *Let* X *be a finite complex of dimension at most* $n+1$ *and suppose that* $f : S^n \to X$ *is a continuous map. If* $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X \cup_f e^{n+1}) \leq 2$ *then* $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X) \leq 2$.

Proof. — Since $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X \cup_f e^{n+1}) \leq 2$ it follows from Theorem (4.4) that

$$\widetilde{\mu} : \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(X \cup_f e^{n+1}) \to H_*(X \cup_f e^{n+1}; \mathbf{Z})$$

is a monomorphism.

Suppose to the contrary that $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X) > 2$. Applying Proposition (7.5) we then find

$$2 \leq \mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X \cup_f e^{n+1}) = \mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X) < 2.$$

which is a contradiction. Therefore we must have $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X) \leq 2$. □

Theorem **(7.7)**. — *Let* Y *be a finite complex. Suppose that* $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(Y) > 2$. *Then* Y *contains a subcomplex* W *such that*

1) $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(W) = \mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(Y)$, *and*
2) $\widetilde{\mu} : \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(W) \to H_*(W; \mathbf{Z})$ *is* not *a monomorphism.*

Moreover $$Y = W \cup_{f_1} e^{n_1} \cup_{f_2} \ldots \cup_{f_t} e^{n_t}$$

where $n_t \geq n_{t-1} \geq \ldots \geq n_1 \geq n_1 \geq \dim W$, *and hence* Y/W *is* (dim W-1)*-connected.*

Proof. — Surprisingly enough this may be established by induction on the number of cells in Y.

If $\widetilde{\mu} : \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(Y) \to H_*(Y; \mathbf{Z})$ is not monic we take W=Y and there is nothing to prove. (This is what starts the induction.)

Suppose on the other hand that

$$\widetilde{\mu} : \quad \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(Y) \to H_*(Y, \mathbf{Z})$$

is monic. Then we may find a subcomplex $X \subset Y$ and a continuous map $f : S^n \to X$ such that

$$Y = X \cup_f e^{n+1}$$

and X has dimension at most n. By Proposition (7.5) we obtain

$$\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X) = \mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(Y) > 2.$$

Evidently X has one less cell than Y and the result follows from our inductive assumption. □

Corollary **(7.8)**. — *Let* Y *be a finite complex. Then* $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(Y) > 2$ *iff* Y *contains a subcomplex* W *such that*

1) $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(W) = \mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(Y)$, *and*
2) $\widetilde{\mu} : \mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(W) \to H_*(W; \mathbf{Z})$ *is* not *a monomorphism.*

Moreover we must have

$$Y = W \cup_{f_1} e^{n_1} \cup_{f_2} \dots \cup_{f_t} e^{n_t}$$

where $n_t \geq n_{t-1} \geq \dots \geq n_1 \geq \dim W$, *and hence* Y/W *is* $(\dim W - 1)$*-connected.*

Proof. — The « only if » part is just Theorem (7.7). To obtain the « if » part suppose that $W \subset Y$ is a subcomplex with the stated properties. Then $\text{hom.dim}_{\Omega^U_*}\Omega^U_*(W) > 2$ by Theorem (4.4) since

$$\tilde{\mu}: \quad \mathbf{Z} \otimes_{\Omega^U_*} \Omega^U_*(W) \to H_*(W; \mathbf{Z})$$

is *not* a monomorphism. From the cell decomposition

$$Y = W \cup_{f_1} e^{n_1} \cup_{f_2} \dots \cup_{f_t} e^{n_t}$$

it follows quite easily by iterated application of Proposition (7.5) that

$$\text{hom.dim}_{\Omega^U_*}\Omega^U_*(Y) = \text{hom.dim}_{\Omega^U_*}\Omega^U_*(W)$$

and hence $\text{hom.dim}_{\Omega^U_*}\Omega^U_*(Y) > 2$. □

Corollary **(7.9)**. — *Let* Y *be a finite complex. Then* $\text{hom.dim}_{\Omega^U_*}\Omega^U_*(Y) \leq 2$ *iff for every subcomplex* $W \subset Y$ *with*

$$Y = W \cup_{f_1} e^{n_1} \cup_{f_2} \dots \cup_{f_t} e^{n_t}$$

where $n_t \geq n_{t-1} \geq \dots \geq n_1 \geq \dim W$, *the reduced Thom homomorphism*

$$\tilde{\mu}: \quad \mathbf{Z} \otimes_{\Omega^U_*} \Omega^U_*(W) \to H_*(W; \mathbf{Z})$$

is a monomorphism.

Proof. — Suppose that $\text{hom.dim}_{\Omega^U_*}\Omega^U_*(Y) \leq 2$. Then by iterated application of Corollary (7.6) to the cell decomposition

$$Y = W \cup_{f_1} e^{n_1} \cup_{f_2} \dots \cup_{f_t} e^{n_t}$$

we obtain

$$\text{hom.dim}_{\Omega^U_*}\Omega^U_*(W) \leq 2$$

for every subcomplex $W \subset Y$ where $n_t \geq n_{t-1} \geq \dots \geq n_1 \geq \dim W$. Hence for every such W we obtain by Theorem (4.4) that the reduced Thom homomorphism

$$\tilde{\mu}: \quad \mathbf{Z} \otimes_{\Omega^U_*} \Omega^U_*(W) \to H_*(W; \mathbf{Z})$$

is a monomorphism.

To obtain the converse we proceed by induction on the number of cells of Y. If Y has only one cell there is nothing to prove. If Y has more than one cell we may find a subcomplex $X \subset Y$ and a map $f: S^n \to X$ such that

$$Y = X \cup_f e^{n+1}$$

and X has dimension at most n. Evidently X has one less cell than Y and a moment's reflection shows that our inductive assumption implies $\text{hom.dim}_{\Omega^U_*}\Omega^U_*(X) \leq 2$. Now if $\text{hom.dim}_{\Omega^U_*}\Omega^U_*(Y) > 2$ then by Proposition (7.5) we would have

$$\text{hom.dim}_{\Omega^U_*}\Omega^U_*(X) = \text{hom.dim}_{\Omega^U_*}\Omega^U_*(Y)$$

and hence $\hom.\dim_{\Omega_*^U}\Omega_*^U(X)>2$, which is a contradiction. Therefore we must have $\hom.\dim_{\Omega_*^U}\Omega_*^U(Y)\leq 2$. $\square$

Remark. — 1) Evidently Corollary (7.9) may be rephrased in the following manner.

Let Y be a finite complex. Then $\hom.\dim_{\Omega_*^U}\Omega_*^U(Y)\leq 2$ iff for every subcomplex $W\subset Y$ with

$$Y=W\cup_{f_1}e^{n_1}\cup_{f_2}\ldots\cup_{f_t}e^{n_t}$$

where $n_t\geq n_{t-1}\geq\ldots\geq n_1\geq\dim W$, we have $\hom.\dim_{\Omega_*^U}\Omega_*^U(W)\leq 2$.

2) It is also clear that the conditions on the cell structure that appear in Corollary (7.8) and Corollary (7.9) may be replaced by the condition that

$$W\rightarrow Y\rightarrow Y/W$$

is a cofibration and that

$$1+\text{connectivity}(Y/W)\leq\dim W.$$

We do not pursue this point further.

§ 8. The Spectral Künneth Theorem.

Our objective in this section is to establish the spectral Künneth Theorem for $\Omega_*^U(\cdot)$. The basic idea for the proof goes back to Atiyah [4] and Landweber [14]. A theorem of this type has also been obtained by D. S. Kahn (unpublished) by different methods, while J. F. Adams [2] has extended Atiyah's method to provide a Künneth type theorem in a very general setting.

The precise result we will establish is:

Theorem **(8.1)**. — *Let* X *and* Y *be finite complexes. Then there is a natural first quadrant homology spectral sequence* $\{E^r(X, Y), d^r(X, Y)\}$ *with*

$$E^r(X, Y)\Rightarrow\Omega_*^U(X\times Y)$$

and

$$E^2_{p,q}(X, Y)\cong\operatorname{Tor}^{\Omega_*^U}_{p,q}(\Omega_*^U(X), \Omega_*^U(Y)).$$

The convergence is in the naive sense. The edge homomorphism

$$\Omega_*^U(X)\otimes_{\Omega_*^U}\Omega_*^U(Y)=E^2_{0,*}(X, Y)\rightarrow E^\infty_{0,*}(X, Y)\rightarrow\Omega_*^U(X\times Y)$$

may be identified with the exterior cross-product

$$\Omega_*^U(X)\otimes_{\Omega_*^U}\Omega_*^U(Y)\rightarrow\Omega_*^U(X\times Y).$$

There are of course analogous results for smash products of pointed spaces. Actually it will be more convenient to begin with these and derive the unpointed case by simple manipulation from the pointed case. We will therefore require the pointed analog of the results of the first two sections and we begin with these.

Recollections. — A pointed space is a pair (X, x_0) where $x_0\in X$. If (X, x_0) and (Y, y_0) are pointed spaces, a map $f\colon(X, x_0)\rightarrow(Y, y_0)$ of pointed spaces is a continuous map $f\colon X\rightarrow Y$ such that $f(x_0)=y_0$.

Definition. — *If* (X, x_0) *is a pointed finite complex, a partial* U*-bordism resolution of* (X, x_0) *of length* k *and degree* l *consists of pointed complexes*

$$(*, *) = (A_{-1}, *) \subset \ldots \subset (A_{k-1}, *) \subset (A_k, *)$$

and a homotopy equivalence

$$\varphi : (S^l X, S^l x_0) \to (A_k, *)$$

($S^l X$ denotes the l-fold *reduced* suspension of (X, x_0), and $*$ denotes a fixed common base point for $A_{-1}, \ldots, A_k$.) *such that*

1) $\Omega_*^U(A_i, A_{i-1})$ *is a projective* Ω_*^U*-module for* $i = 0, 1, \ldots, k-1$ *and*
2) $\Omega_*^U(A_i, A_{i-1}) \to \Omega_*^U(A_k, A_{i-1})$ *is an epimorphism for* $i = 0, 1, \ldots, k$.

If in addition $\Omega_*^U(A_k, A_{k-1})$ is a projective Ω_*^U-module then we say that

$$* = (A_{-1}, *) \subset (A_0, *) \subset \ldots \subset (A_{k-1}, *) \subset (A_k, *) \sim (S^l X, S^l x_0)$$

is a U-*bordism resolution* of (X, x_0) of length k and degree l.

Quite clearly, with this definition, the results of section 2 are valid for pointed spaces and will be used without further comment.

Theorem **(8.2)**. — *Let* (X, x_0) *and* (Y, y_0) *be pointed finite complexes and* $(X \wedge Y, x_0 \wedge y_0)$ *their smash product. Then there exists a natural first quadrant homology spectral sequence* $\{\widetilde{E}^r((X, x_0), (Y, y_0)), d^r((X, x_0), (Y, y_0))\}$ *with*

$$\widetilde{E}^r((X, x_0), (Y, y_0)) \Rightarrow \Omega_*^U(X \wedge Y, x_0 \wedge y_0)$$

and

$$\widetilde{E}^2_{p,q}((X, x_0), (Y, y_0)) = \mathrm{Tor}^{\Omega_*^U}_{p,q}(\Omega_*^U(X, x_0), \Omega_*^U(Y, y_0)).$$

The convergence is in the naive sense. The edge homomorphism

$$\Omega_*^U(X, x_0) \otimes_{\Omega_*^U} \Omega_*^U(Y, y_0) = \widetilde{E}^2_{0,*}((X, x_0), (Y, y_0)) \to \widetilde{E}^\infty_{0,*}((X, x_0), (Y, y_0)) \to \Omega_*^U(X \wedge Y, x_0 \wedge y_0)$$

may be identified with the exterior product

$$\Omega_*^U(X, x_0) \otimes_{\Omega_*^U} \Omega_*^U(Y, y_0) \to \Omega_*^U(X \wedge Y, x_0 \wedge y_0).$$

Proof. — The spectral sequence will be obtained by taking the U-bordism exact couple of a suitable filtered pointed space. The procedure is similar to that of section 4. We proceed now to the details.

Construction. — Since (X, x_0) is a finite pointed complex we may, according to the pointed analog of Theorem (2.6), choose a U-bordism resolution of (X, x_0), say

$$* = (A_{-1}, *) \subset (A_0, *) \subset \ldots \subset (A_{k-1}, *) \subset (A_k, *) \sim (S^l X, S^l x_0).$$

We form the filtered pointed space

$$* \wedge y_0 = (A_{-1} \wedge Y, * \wedge y_0) \subset (A_0 \wedge Y, * \wedge y_0) \subset \ldots \subset (A_k \wedge Y, * \wedge y_0),$$

and note that $(A_k \wedge Y, * \wedge y_0)$ has the same homotopy type as $(S^l X \wedge Y, S^l x_0 \wedge y_0)$. Associated to this filtered space we have an exact couple

$$\begin{tikzcd} D \arrow[rr] & & D \arrow[dl] \\ & E \arrow[ul] & \end{tikzcd}$$

where

$$D_{p,q} = \Omega^U_{p+q}(A_p \wedge Y, * \wedge y_0)$$
$$E_{p,q} = \Omega^U_{p+q}(A_p \wedge Y, A_{p-1} \wedge Y)$$

the maps of the couple being the maps in the U-bordism exact triangles of the pointed pairs $(A_p \wedge Y, A_{p-1} \wedge Y, * \wedge y_0)$ for $p = 0, \ldots, k$.

Let $\{E^r, d^r\}$ be the spectral sequence of this exact couple. We define $\{\tilde{E}^r((X, x_0), (Y, y_0)), \tilde{d}^r((X, x_0), (Y, y_0))\}$ by

$$\tilde{E}^r_{p,q}((X, x_0), (Y, y_0)) = E^r_{p,q+l}$$
$$\tilde{d}^r_{p,q}((X, x_0), (Y, y_0)) = d^r_{p,q+l}.$$

It is evident that $\{\tilde{E}^r((X, x_0), (Y, y_0)), \tilde{d}^r((X, x_0), (Y, y_0))\}$ is a first quadrant homology spectral sequence.

Convergence. — From the construction of the spectral sequence $\{E^r, d^r\}$ as the spectral sequence of the *finitely* filtered pointed space

$$* \wedge y_0 = (A_{-1} \wedge Y, * \wedge y_0) \subset (A_0 \wedge Y, * \wedge y_0) \subset \ldots (A_k \wedge Y, * \wedge y_0)$$

it is immediately obvious that $\{E^r, d^r\}$ converges in the naive sense to $\Omega^U_*(A_k \wedge Y, * \wedge y_0)$. Taking into account the dimension shifts in the definition of

$$\{\tilde{E}^r((X, x_0), ((Y, y_0)), \tilde{d}^r((X, x_0), (Y, y_0))\}$$

and the suspension isomorphism (this is where $\wedge$-products are preferable to $\times$-products; namely $\wedge$ behaves well under suspension whereas $\times$ does not) we see that the spectral sequence $\{\tilde{E}^r((X, x_0), (Y, y_0)), \tilde{d}^r((X, x_0), (Y, y_0))\}$ converges in the naive sense to $\Omega^U_*(X \wedge Y, x_0 \wedge y_0)$.

Identification of $\tilde{E}^2((X, x_0), (Y, y_0))$. — We turn now to the identification of $\tilde{E}^2((X, x_0), (Y, y_0))$. We shall need the following elementary lemma. The proof may be constructed as in ([14]; (6.2)) or ([8]; 44).

Lemma **(8.3)**. — *Let* (A, a_0) *and* (B, b_0) *be finite complexes and assume that* $\Omega^U_*(A, a_0)$ *is a free* Ω^U_**-module. Then the bordism product*

$$\Omega^U_*(A, a_0) \otimes_{\Omega^U_*} \Omega^U_*(B, b_0) \to \Omega^U_*(A \wedge B, a_0 \wedge b_0)$$

is an isomorphism of Ω^U_**-modules.*

Proof. — Consider the functors $\Omega_*^U(A\wedge -, a_0\wedge -)$ and $\Omega_*^U(A, a_0)\otimes_{\Omega_*^U}\Omega_*^U(-, -)$. Since $\Omega_*^U(A, a_0)$ is a free Ω_*^U-module the second is a homology theory, while the first is a homology theory for elementary reasons. The bordism product

$$\Omega_*^U(A, a_0)\otimes_{\Omega_*^U}\Omega_*^U(-, -) \to \Omega_*^U(A\wedge -, a_0\wedge -)$$

is then a morphism of functors that induces an isomorphism of the coefficients. The result now follows from standard properties of generalized homology theories by induction over the number of cells in (B, b_0). □

We return now to the identification of $\widetilde{E}^2((X, x_0), (Y, y_0))$. From the definition of the spectral sequence $\{E^r, d^r\}$, it follows that E^2 is the homology of the complex

$$0\leftarrow\Omega_*^U(A_0\wedge Y, *\wedge y_0)\leftarrow\Omega_*^U(A_1\wedge Y, A_0\wedge Y)\leftarrow\dots\leftarrow\Omega_*^U((A_k\wedge Y, A_{k-1}\wedge Y))\leftarrow 0.$$

For each integer $p=0, 1, \dots, k$, $\Omega_*^U(A_p, A_{p-1})$ is a free Ω_*^U-module, and thus we have the isomorphism

$$\Omega_*^U(A_p\wedge Y, A_{p-1}\wedge Y)\cong\Omega_*^U(A_p, A_{p-1})\otimes_{\Omega_*^U}\Omega_*^U(Y, y_0)$$

by Lemma (8.3).

From the definition of a U-bordism resolution of (X, x_0) it follows that

$$0\leftarrow\Omega_*^U(A_k, *)\leftarrow\Omega_*^U(A_0, A_1)\leftarrow\dots\leftarrow\Omega_*^U(A_k, A_{k-1})\leftarrow 0$$

is a free resolution of $\Omega_*^U(A_k, *)$ as an Ω_*^U-module.

Since E^2 is the homology of the complex

$$0\leftarrow\Omega_*^U(A_0, *)\otimes_{\Omega_*^U}\Omega_*^U(Y, y_0)\leftarrow\dots\leftarrow\Omega_*^U(A_k, A_{k-1})\otimes_{\Omega_*^U}\Omega_*^U(Y, y_0)\leftarrow 0$$

it follows from the definition of derived functors that

$$E^2_{p,q}\cong\mathrm{Tor}^{\Omega_*^U}_{p,q}(\Omega_*^U(A_k, *), \Omega_*^U(Y, y_0)).$$

Taking into account the dimension shifts in the definition of

$$\{\widetilde{E}^r((X, x_0), (Y, y_0)),\ \widetilde{d}^r((X, x_0), (Y, y_0))\}$$

and the suspension isomorphism

$$\mathrm{Tor}^{\Omega_*^U}_{p,q}(\Omega_*^U(X, x_0), \Omega_*^U(Y, y_0))\cong\mathrm{Tor}^{\Omega_*^U}_{p,q+l}(\Omega_*^U(A_k, *), \Omega_*^U(Y, y_0))$$

we obtain

$$\widetilde{E}^2_{p,q}((X, x_0), (Y, y_0))=\mathrm{Tor}^{\Omega_*^U}_{p,q}(\Omega_*^U(X, x_0), \Omega_*^U(Y, y_0))$$

as claimed.

Naturality. — The proof of the naturality of the spectral sequence

$$\{\widetilde{E}^r((X, x_0), (Y, y_0)),\ \widetilde{d}^r((X, x_0), (Y, y_0))\}$$

is long and tedious, but follows the same procedure employed in Theorem (4.1). We leave the details to the reader. □

Remark. — Let $f : (\mathrm{X}', x_0') \to (\mathrm{X}'', x_0'')$, $g : (\mathrm{Y}', y_0') \to (\mathrm{Y}'', y_0'')$ be maps of finite complexes. Then f and g induce a map of spectral sequences

$$\{\widetilde{\mathrm{E}}^r(f, g)\} : \quad \{\widetilde{\mathrm{E}}^r((\mathrm{X}', x_0'), (\mathrm{Y}', y_0')),\ \widetilde{d}^r((\mathrm{X}', x_0'), (\mathrm{Y}', y_0'))\} \to$$
$$\to \{\widetilde{\mathrm{E}}^r((\mathrm{X}'', x_0''), (\mathrm{Y}'', y_0'')),\ \widetilde{d}((\mathrm{X}'', x_0''), (\mathrm{Y}'', y_0''))\}.$$

As in the proof of Theorem (4.1) it may be shown that $\widetilde{\mathrm{E}}^2(f, g)$ may be identified with the map

$$\begin{tikzcd}
\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}(f, g) : \arrow[d, equal] & \mathrm{Tor}^{\Omega^{\mathrm{U}}_*}(\Omega^{\mathrm{U}}_*(\mathrm{X}', x_0'),\ \Omega^{\mathrm{U}}_*(\mathrm{Y}', y_0')) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & \mathrm{Tor}^{\Omega^{\mathrm{U}}_*}(\Omega^{\mathrm{U}}_*(\mathrm{X}'', x_0''),\ \Omega^{\mathrm{U}}_*(\mathrm{Y}'', y_0'')) \arrow[d, leftrightarrow, "\cong"] \\
\widetilde{\mathrm{E}}^2(f, g) \ : & \widetilde{\mathrm{E}}^2((\mathrm{X}', x_0'), (\mathrm{Y}', y_0')) \arrow[r] & \widetilde{\mathrm{E}}^2((\mathrm{X}'', x_0''), (\mathrm{Y}'', y_0''))
\end{tikzcd}$$

Proof of Theorem (8.1). — For any space W, denote by $(\mathrm{W}^+, *)$ the pointed space obtained from W by adjoining a disjoint point $*$. Then

$$(\mathrm{X}^+ \wedge \mathrm{Y}^+, * \wedge *) = ((\mathrm{X} \times \mathrm{Y})^+, * \wedge *)$$

and the result follows by applying Theorem (8.2) to the pair of pointed spaces $(\mathrm{X}^+, *)$, $(\mathrm{Y}^+, *)$ together with the observation that $\Omega^{\mathrm{U}}_*(\mathrm{W}^+, *) = \Omega^{\mathrm{U}}_*(\mathrm{W})$ for any space W. ☐

The U-bordism Künneth Theorem of Landweber [14] follows from Theorem (8.1) and Corollary (3.11), viz:

Corollary **(8.4)**. — *Let* X *and* Y *be finite complexes and suppose that* $\mathrm{hom.dim}_{\Omega^{\mathrm{U}}_*} \Omega^{\mathrm{U}}_*(\mathrm{X}) \leq 1$. *Then there is a natural exact sequence*

$$0 \to \Omega^{\mathrm{U}}_*(\mathrm{X}) \otimes_{\Omega^{\mathrm{U}}_*} \Omega^{\mathrm{U}}_*(\mathrm{Y}) \to \Omega^{\mathrm{U}}_*(\mathrm{X} \times \mathrm{Y}) \to \mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{1,*}(\Omega^{\mathrm{U}}_*(\mathrm{X}), \Omega^{\mathrm{U}}_*(\mathrm{Y})) \to 0$$

Proof. — Instant from the definition of homological dimension and Theorem (8.1). ☐

Remark. — As for the spectral sequence $\{\mathrm{E}^r\langle\ \rangle, d^r\langle\ \rangle\}$ we may extend the validity of the spectral Künneth theorem to skeleton finite CW-complexes. The details are straightforward and left to the reader.

In view of our previous work on the Thom homomorphism a study of the kernel of the exterior product

$$\widetilde{\Omega}^{\mathrm{U}}_*(\mathrm{X}) \otimes_{\Omega^{\mathrm{U}}_*} \widetilde{\Omega}^{\mathrm{U}}_*(\mathrm{Y}) \to \widetilde{\Omega}^{\mathrm{U}}_*(\mathrm{X} \wedge \mathrm{Y})$$

seems in order. The fact that this kernel can be non-zero should not be too surprising in view of our previous examples. We will turn to an example of this phenomena now.

Let k be a positive integer and M the Moore space obtained by attaching a $(k+1)$-cell to S^k by a map of degree 2. Then

$$\widetilde{\mathrm{H}}_*(\mathrm{M}; \mathbf{Z}) = \mathrm{H}_k(\mathrm{M}; \mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z},$$

and one may easily show

$$\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}) \cong \Sigma^k[\Omega_*^{\mathrm{U}}/(2)].$$

Let $\hat{\alpha} \neq 0 \in \widetilde{\Omega}_k^{\mathrm{U}}(\mathrm{M}) \cong \mathbf{Z}/2\mathbf{Z}$. Note that $\mu(\hat{\alpha}) = \hat{a} \neq 0 \in \widetilde{\mathrm{H}}_k(\mathrm{M}; \mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$.

Let Y be a-pointed finite complex. Let us study the U-bordism of $\mathrm{M} \wedge \mathrm{Y}$.

Since $\mu : \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}) \to \widetilde{\mathrm{H}}_*(\mathrm{M}; \mathbf{Z})$ is onto we see that $\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}} \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}) = 1$. This is also easily seen from the isomorphism $\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}) = \Sigma^k[\Omega_*^{\mathrm{U}}/(2)]$. Thus by the reduced form of Corollary (8.3) we obtain an exact sequence

$$0 \to \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}) \otimes_{\Omega_*^{\mathrm{U}}} \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{Y}) \xrightarrow{\widetilde{\pi}} \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M} \wedge \mathrm{Y}) \xrightarrow{\widetilde{\eta}} \mathrm{Tor}_{1,*}^{\Omega_*^{\mathrm{U}}}(\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}), \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{Y})) \to 0$$

for any pointed complex Y.

Notation. — For any graded **Z**-module N, let $m(2) : \mathrm{N} \to \mathrm{N}$ denote multiplication by 2, i.e.:

$$m(2)(x) = 2x$$

for all $x \in \mathrm{N}$.

To compute the torsion product $\mathrm{Tor}_{1,*}^{\Omega_*^{\mathrm{U}}}(\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}), \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{Y}))$ we may choose as our resolution of $\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M})$ the exact sequence

$$0 \to \Sigma^k \Omega_*^{\mathrm{U}} \xrightarrow{m(2)} \Sigma^k \Omega_*^{\mathrm{U}} \to \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}) \to 0.$$

From the exact sequence

$$0 \to \mathrm{Tor}_{1,*}^{\Omega_*^{\mathrm{U}}}(\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}), \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{Y})) \to (\Sigma^k \Omega_*^{\mathrm{U}}) \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{Y}) \xrightarrow{m(2)\otimes 1} (\Sigma^k \Omega_*^{\mathrm{U}}) \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{Y})$$

we obtain:

Lemma **(8.5)**. — *For any pointed finite complex* Y

$$\mathrm{Tor}_{1,n}^{\Omega_*^{\mathrm{U}}}(\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}), \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{Y})) = \ker \{m(2) : \widetilde{\Omega}_{n+k}^{\mathrm{U}}(\mathrm{Y}) \circlearrowleft \}. \quad \square$$

Let us consider the special case $\mathrm{M} \wedge \mathrm{M}$.

Lemma **(8.6)**. — *The* U-*bordism spectral sequence for* $\mathrm{M} \wedge \mathrm{M}$ *collapses.*

Proof. — By the classical Künneth theorem we have

$$\widetilde{\mathrm{H}}_i(\mathrm{M} \wedge \mathrm{M}; \mathbf{Z}) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z} & \text{for} \quad i = 2k, \ 2k+1, \\ 0 \text{ otherwise.} \end{cases}$$

Consider the bordism spectral sequence

$$\widetilde{\mathrm{E}}^r \Rightarrow \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M} \wedge \mathrm{M})$$
$$\widetilde{\mathrm{E}}^2 = \widetilde{\mathrm{H}}_*(\mathrm{M} \wedge \mathrm{M}; \Omega_*^{\mathrm{U}}).$$

We find that

$$\widetilde{\mathrm{E}}_{p,*}^2 = \begin{cases} \Omega_*^{\mathrm{U}}/(2) & \text{for} \quad p = 2k, \ 2k+1, \\ 0 \text{ otherwise.} \end{cases}$$

Since $\deg(\widetilde{d}^r) = (-r, r-1)$ and $r > 1$ it follows that $\widetilde{d}^r = 0$ for all $r \geq 2$. $\square$

Lemma **(8.7).** — *There is a class* $\hat{\beta}\in\tilde{\Omega}^{U}_{2k+1}(M\wedge M)$ *such that:*

1) $2\hat{\beta}=0$ *and*

2) $\mu(\hat{\beta})=\hat{b}\neq 0\in\tilde{H}_{2k+1}(M\wedge M;\mathbf{Z})\cong\mathbf{Z}/2\mathbf{Z}.$

Proof. — Since the U-bordism spectral sequence for $M\wedge M$ collapses,

$$\tilde{\mu}:\ \mathbf{Z}\otimes_{\Omega^U_*}\tilde{\Omega}^U_*(M\wedge M)\rightarrow\tilde{H}_*(M\wedge M;\mathbf{Z})$$

is an isomorphism by the reduced version of Corollary (3.11). Choose $\hat{\beta}\in\tilde{\Omega}_{2k+1}(M\wedge M)$ such that $\mu(\hat{\beta})=\hat{b}$, where $\hat{b}\neq 0\in\tilde{H}_{2k+1}(M\wedge M;\ \mathbf{Z})\cong\mathbf{Z}/2\mathbf{Z}$. We claim that $\hat{\beta}$ has order 2. For suppose that $2\hat{\beta}\neq 0$. Since $\tilde{\mu}$ is an isomorphism we must have

$$1\otimes 2\hat{\beta}=0\in\mathbf{Z}\otimes_{\Omega^U_*}\tilde{\Omega}^U_*(M\wedge M).$$

Hence there exist $\gamma\in\tilde{\Omega}^U_*(M\wedge M)$ and $\lambda\in\Omega^U_*$ with $\deg\lambda=l>0$, $\deg\gamma=2k+1-l$ and

$$2\hat{\beta}=\lambda.\gamma\in\tilde{\Omega}^U_{2k+1}(M\wedge M).$$

Since $l>0$ and $\Omega^U_1=0$ we must have $l\geq 2$. Thus

$$\gamma\in\Omega^U_{2k+1-l}(M\wedge M)=0.$$

Therefore $2\hat{\beta}=\lambda.\gamma=0$. ☐

Lemma **(8.8).** — *The Künneth exact sequence*

$$0\rightarrow\tilde{\Omega}^U_*(M)\otimes_{\Omega^U_*}\tilde{\Omega}^U_*(M)\rightarrow\tilde{\Omega}^U_*(M\wedge M)\rightarrow\mathrm{Tor}^{\Omega^U_*}_{1,*}(\tilde{\Omega}^U_*(M),\ \tilde{\Omega}^U_*(M))\rightarrow 0$$

splits.

Proof. — First note that by Lemma (8.5)

$$\mathrm{Tor}^{\Omega^U_*}_{1,*}(\tilde{\Omega}^U_*(M))\cong\ker\{m(2):\Sigma^{2k}\Omega^U_*/(2)\ \circlearrowright\}.$$

But

$$m(2)=0:\Sigma^{2k}\Omega^U_*/(2)\rightarrow\Sigma^{2k}\Omega^U_*/(2).$$

Therefore

$$\mathrm{Tot}\ \mathrm{Tor}^{\Omega^U_*}_{1,*}(\tilde{\Omega}^U_*(M),\ \tilde{\Omega}^U_*(M))\cong\Sigma^{2k+1}\Omega^U_*/(2)$$

where Tot denotes totalization [15]. With $\hat{\beta}$ as in Lemma (8.7) the splitting is given by

$$\Sigma^{2k+1}\Omega^U_*/(2)\rightarrow\Omega^U_*(M\wedge M):\quad\Sigma^{2k+1}x=x\hat{\beta}$$

for all $x\in\Omega^U_*/(2)$. ☐

We thus find

$$\tilde{\Omega}^U_*(M\wedge M)\cong[[\Omega^U_*/(2)]\hat{\alpha}_k\otimes\hat{\alpha}_k]\oplus[[\Omega^U_*/(2)]\hat{\beta}_{2k+1}].$$

The class $\hat{\alpha}_k\in\tilde{\Omega}^U_k(M)$ is in the image of $\tilde{\Omega}^{fr}_*(M)\rightarrow\tilde{\Omega}^U_*(M)$. Thus by Corollary (6.3) (if $n\equiv 0$ or $n\equiv 1$ mod 4) so is $[\mathbf{CP}(1)]^n\hat{\alpha}_k\in\tilde{\Omega}^U_{k+2n}(M)$. Since M is $(k-1)$-connected it follows ([21], [22]) that if we choose $k+2n$ to lie in the stable range, i.e., if $k+2n\leq 2k-3$ then we may find a map

$$f: S^{2n+k}\to M$$

with

$$[S^{2n+k}, f]=[\mathbf{C}P(1)]^n\hat{\alpha}\in\tilde{\Omega}^U_{k+2n}(M).$$

We now set

$$X=M\cup_f e^{2n+k+1}.$$

We then have the cofibration

$$M\to X\to S^{2n+k+1}$$

and the exact triangle

$$\begin{tikzcd}
\tilde{\Omega}^U_*(M) \arrow[rr, "i_*"] & & \tilde{\Omega}^U_*(X) \arrow[dl, "j_*"] \\
& \tilde{\Omega}^U_*(S^{2n+k+1}) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

Let $\rho_{2n+k+1}\in\tilde{\Omega}^U_{2n+k+1}(S^{2n+k+1})$ be the usual generator. By construction

$$\partial_*(\rho_{2n+k+1})=[\mathbf{C}P(1)]^n\hat{\alpha}.$$

Let $i_*(\hat{\alpha})=\alpha\in\tilde{\Omega}^U_k(X)$. Since $2\hat{\alpha}=0$ we have

$$\partial_*(2\rho_{2n+k+1})=2[\mathbf{C}P(1)]^n\hat{\alpha}=[\mathbf{C}P(1)]^n2\hat{\alpha}=0.$$

Thus there exists $\beta\in\tilde{\Omega}^U_{2n+k+1}(X)$ with

$$j_*(\beta)=2\rho_{2n+k+1}.$$

Now note that

$$\deg\beta-\deg\alpha=2n+1.$$

Thus

$$\tilde{\Omega}^U_{2n+k+1}(M)\cong[\Sigma^k(\Omega^U_*/(2))]_{2n+k+1}=0,$$

since $\Omega^U_{2n+1}=0$. Therefore $\beta\in\tilde{\Omega}^U_{2n+k+1}(X)$ is unique.

Consider the exact sequence

$$\tilde{\Omega}^U_*(M)\to\tilde{\Omega}^U_*(X)\to\operatorname{Im} j_*\to 0.$$

We have

$$\operatorname{Im} j_*=\Sigma^{2n+k+1}A(\hat{\alpha})=\ker\partial_*$$

and

$$A(\hat{\alpha})=2\Omega^U_*.$$

Thus we see that α and β generate $\tilde{\Omega}^U_*(X)$.

Consider $\operatorname{Im}\partial_*\subset\tilde{\Omega}^U_*(M)$. One readily checks that

$$\tilde{\Omega}^U_*(M)/\operatorname{Im}\partial_*\cong\Sigma^k[\Omega^U_*/(2, [\mathbf{C}P(1)]^n)].$$

Thus we have an exact sequence

$$0\to\Sigma^k[\Omega^U_*/(2, [\mathbf{C}P(1)]^n)]\to\tilde{\Omega}^U_*(X)\to\Sigma^{2n+k+1}2\Omega^U_*\to 0.$$

Since $\Sigma^{2n+k+1}2\Omega^U_*$ is a free Ω^U_*-module this sequence splits and we obtain:

Lemma **(8.9)**. — $\widetilde{\Omega}^U_*(X) \cong \Sigma^k[\Omega^U_*/2, [\mathbf{C}P(1)]^n] \oplus \Sigma^{2n+k+1} 2\Omega^U_*$, *with generators*

$$\alpha \in \widetilde{\Omega}^U_k(X), \qquad \beta \in \widetilde{\Omega}^U_{2n+k+1}(X)$$

such that

$$A(\alpha) = (2, [\mathbf{C}P(1)]^n) \subset \Omega^U_*$$

$$A(\beta) = 0 \subset \Omega^U_*.$$

Proof. — Instant from the splitting of the above sequence. ☐

Our next objective is to show that the class

$$\alpha \otimes \beta - \beta \otimes \alpha \neq 0 \in \widetilde{\Omega}^U_*(X) \otimes_{\Omega^U_*} \widetilde{\Omega}^U_*(X)$$

lies in the kernel of the reduced bordism product,

$$\widetilde{\pi} : \quad \widetilde{\Omega}^U_*(X) \otimes_{\Omega^U_*} \widetilde{\Omega}^U_*(X) \to \widetilde{\Omega}^U_*(X \wedge X).$$

To this end consider the diagram

$$\begin{tikzcd}
 & X \wedge X & \\
M \wedge X \arrow[ur, "i'"] & & X \wedge M \arrow[ul, "j'"'] \\
 & M \wedge M \arrow[ul, "i"] \arrow[ur, "j"'] &
\end{tikzcd}$$

Lemma **(8.10)**. — $[\mathbf{C}P(1)]^n i_*(\beta) = \hat{\alpha} \otimes \beta \in \widetilde{\Omega}^U_*(M \wedge X)$

$$[\mathbf{C}P(1)]^n j_*(\beta) = \beta \otimes \hat{\alpha} \in \widetilde{\Omega}^U_*(X \wedge M).$$

Proof. — Let us begin by noting that

$$\mathrm{Tor}^{\Omega^U_*}_{1,*}(\widetilde{\Omega}^U_*(M), \widetilde{\Omega}^U_*(X)) = \ker\{m(2) : \Sigma^k \Omega^U_*(X)\}$$

by Lemma (8.5). From Lemma (8.9) we thus obtain

$$\mathrm{Tot}\ \mathrm{Tor}^{\Omega^U_*}_{1,*}(\widetilde{\Omega}^U_*(M), \widetilde{\Omega}^U_*(X)) \cong \Sigma^{2k+1}[\Omega^U_*/(2)\ [\mathbf{C}P(1)]^n],$$

and $i_*(\hat{\beta}) \in \widetilde{\Omega}^U_*(M \wedge X)$ maps onto the generator $\Sigma^{2k+1} 1$ under the map $\widetilde{\eta}$ of the Künneth exact sequence (Corollary (8.3) or the discussion preceeding Lemma (8.5)) for $M \wedge X$.

Consider now the cofibration

$$M \wedge M \to M \wedge X \to M \wedge S^{2n+k+1}.$$

We have the exact triangle

$$\begin{tikzcd}
\widetilde{\Omega}^U_*(M \wedge M) \arrow[rr] \arrow[dr] & & \widetilde{\Omega}^U_*(M \wedge X) \arrow[dl] \\
 & \widetilde{\Omega}^U_*(M \wedge S^{2n+k+1}) &
\end{tikzcd}$$

Now note $$\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}\wedge \mathrm{S}^{2n+k+1}) \cong \Sigma^{2n+k+1}\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M})$$
$$\cong \Sigma^{2n+2k+1}\Omega_*^{\mathrm{U}}/(2)$$

and thus $$\Omega_{2m}^{\mathrm{U}}(\mathrm{M}\wedge \mathrm{S}^{2n+k+1}) = 0$$

for all $m \geq 0$. Hence we obtain

$$0 \rightarrow \widetilde{\Omega}_{2m+1}^{\mathrm{U}}(\mathrm{M}\wedge \mathrm{M}) \rightarrow \widetilde{\Omega}_{2m+1}^{\mathrm{U}}(\mathrm{M}\wedge \mathrm{X})$$

is exact for all $m \geq 0$. Therefore

$$\mathrm{A}(i_*(\hat{\beta})) = \mathrm{A}(\hat{\beta}) = (2) \subset \Omega_*^{\mathrm{U}}$$

and hence $$[\mathbf{C}\mathrm{P}(1)]^n i_*(\hat{\beta}) \neq 0 \in \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}\wedge \mathrm{X})$$

since 2 does not divide $[\mathbf{C}\mathrm{P}(1)]^n \in \Omega_*^{\mathrm{U}}$.

Consider the exact sequence

$$0 \rightarrow \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}) \otimes_{\Omega_*^{\mathrm{U}}} \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}) \xrightarrow{\widetilde{\pi}} \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}\wedge \mathrm{X}) \xrightarrow{\widetilde{\eta}} \mathrm{Tor}_{1,*}^{\Omega_*^{\mathrm{U}}}(\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}), \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X})) \rightarrow 0.$$

We have 1) $[\mathbf{C}\mathrm{P}(1)]^n i_*(\hat{\beta}) \neq 0 \in \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}\wedge \mathrm{X})$.

2) $\widetilde{\eta}[\mathbf{C}\mathrm{P}(1)]^n i_*(\hat{\beta}) = 0$.

This second condition follows from our computation at the beginning of the proof of lemma (8.8).

From condition 2) and exactness of the Künneth sequence we obtain

$$[\mathbf{C}\mathrm{P}(1)]^n i_*(\hat{\beta}) = \widetilde{\pi}(\gamma)$$

for some $\gamma \in \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}) \otimes_{\Omega_*^{\mathrm{U}}} \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M})$.

Let us compute some degrees now. We have

$$\deg \gamma = \deg[\mathbf{C}\mathrm{P}(1)]^n i_*(\hat{\beta}) = 2n + 2k + 1.$$

Moreover $$[\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}) \otimes_{\Omega_*^{\mathrm{U}}} \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X})]_{2n+2k+1}$$
$$\cong [[[\Omega_*^{\mathrm{U}}/(2)]\hat{\alpha}_k] \otimes [[\Omega_*^{\mathrm{U}}/(2)[\mathbf{C}\mathrm{P}(1)]^n)]\alpha_k \oplus [2\Omega_*^{\mathrm{U}}\beta_{2k+1}]]]_{2n+2k+1}.$$

Thus $$\gamma = \hat{\alpha}_k \otimes \lambda\alpha_k + \hat{\alpha}_k \otimes \beta_{2k+1}$$

for some $\lambda \in \Omega_{2n+1}^{\mathrm{U}}$. But $\Omega_{2n+1}^{\mathrm{U}} = 0$ and behold

$$\gamma = \hat{\alpha}_k \otimes \beta_{2n+1}.$$

By a symmetric argument we obtain

$$[\mathbf{C}\mathrm{P}(1)]^n j_*(\hat{\beta}) = \beta \otimes \hat{\alpha} \in \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}\wedge \mathrm{M}),$$

and the result follows. □

Remark. — It is worthy of note that the formulas in Lemma (8.10) together with the computations in the early part of the proof show that the Künneth exact sequence

$$0 \rightarrow \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}) \otimes_{\Omega_*^{\mathrm{U}}} \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}) \rightarrow \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}\wedge \mathrm{X}) \rightarrow \mathrm{Tor}_{1,*}^{\Omega_*^{\mathrm{U}}}(\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{M}), \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X})) \rightarrow 0$$

is not split as a sequence of Ω_*^{U}-modules. Thus we see that the exact sequence of Corollary (8.3) need not split as a sequence of Ω_*^{U}-modules.

Proposition **(8.11)**. — *The element* $0 \neq \alpha\otimes\beta - \beta\otimes\alpha \in \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}) \otimes_{\Omega_*^{\mathrm{U}}} \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X})$ *lies in the kernel of the reduced* U-*bordism product*

$$\widetilde{\pi}: \quad \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}) \otimes_{\Omega_*^{\mathrm{U}}} \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}) \to \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}\wedge\mathrm{X}).$$

Proof. — According to Lemma (8.10) we have

$$[\mathbf{C}\mathrm{P}(1)]^n i_*(\hat{\beta}) = \hat{\alpha}\otimes\beta \in \Omega_*^{\mathrm{U}}(\mathrm{M}\wedge\mathrm{X})$$
$$[\mathbf{C}\mathrm{P}(1)]^n j_*(\hat{\beta}) = \beta\otimes\hat{\alpha} \in \Omega_*^{\mathrm{U}}(\mathrm{X}\wedge\mathrm{M}).$$

Therefore since the diagram prior to Lemma (8.10) is commutative we have in $\widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}\wedge\mathrm{X})$

$$\begin{aligned}\alpha\otimes\beta &= i'_*(\hat{\alpha}\otimes\beta) = i'_*([\mathbf{C}\mathrm{P}(1)]^n i_*(\hat{\beta}))\\ &= i'_* i_*([\mathbf{C}\mathrm{P}(1)]^n \hat{\beta}) = j'_* j_*([\mathbf{C}\mathrm{P}(1)]^n \hat{\beta})\\ &= j'_*([\mathbf{C}\mathrm{P}(1)]^n j_* \hat{\beta}) = j'_*(\beta\otimes\hat{\alpha}) = \beta\otimes\alpha.\end{aligned}$$

Therefore $$\alpha\otimes\beta = \beta\otimes\alpha \in \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}\wedge\mathrm{X})$$

and the result follows. ☐

Thus we have established:

Theorem **(8.12)**. — *There exists a finite complex* X *such that the reduced* U-*bordism product*

$$\widetilde{\pi}: \quad \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}) \otimes_{\Omega_*^{\mathrm{U}}} \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}) \to \widetilde{\Omega}_*^{\mathrm{U}}(\mathrm{X}\wedge\mathrm{X})$$

is not a monomorphism. ☐

Corollary **(8.13)**. — *There exists a finite complex* X *such that the reduced* U-*bordism Künneth spectral sequence* $\{\widetilde{\mathrm{E}}^r(\mathrm{X},\mathrm{X}), \widetilde{d}^r(\mathrm{X},\mathrm{X})\}$ *is non trivial.* ☐

Of course the unreduced versions of these results are valid, and for the same space X.

§ 9. The Relation of U-Bordism to K-theory.

Let $\mathrm{K}^*(\cdot)$ denote the $(\mathbf{Z}/2\mathbf{Z})$-graded cohomology theory associated to complex bundle theory ([9], [12]). Regard $\Omega_{\mathrm{U}}^*(\cdot)$ as a $(\mathbf{Z}/2\mathbf{Z})$-graded cohomology theory by its even and odd components. There is then a natural transformation of $(\mathbf{Z}/2\mathbf{Z})$-graded cohomology theories [9]

$$\mu_c: \quad \Omega_{\mathrm{U}}^*(\cdot) \to \mathrm{K}^*(\cdot).$$

given by the K^*-theory orientation of $\underset{\sim}{\mathrm{MU}}$.

We have then the following basic result [9]:

Theorem **(9.1)**. — *Let* X *be a finite complex. Then* μ_c *induces an isomorphism*

$$\widetilde{\mu}_c: \quad \mathbf{Z}\otimes_{\Omega_{\mathrm{U}}^*}\Omega_{\mathrm{U}}^*(\mathrm{X}) \to \mathrm{K}^*(\mathrm{X}).$$

Here we regard $\mathbf{Z}$ as a $(\mathbf{Z}/2\mathbf{Z})$-graded module over Ω_{U}^* via the map

$$\mu_c: \quad \Omega_{\mathrm{U}}^*(\text{point}) \to \mathrm{K}^*(\text{point}).$$

Our objective in this section is to obtain a proof of this result in the spirit of our study so far. A thorough discussion of the point at stake has been given by J. F. Adams [2] by related methods.

We begin with:

Proposition **(9.2)**. — *Let* X *be a finite complex, then the natural map*

$$\mu_c : \quad \Omega^*_U(X) \to K^*(X)$$

is onto.

Proof. — Since X is a finite complex we may find, according to Atiyah [4], a finite complex A and a map $f : X \to A$ such that

1) $H^*(A; \mathbf{Z})$ is a free abelian group, and
2) $f^* : K^*(A) \to K^*(X)$ is onto.

Since $H_*(A; \mathbf{Z})$ is free abelian it follows by routine arguments that

$$\mu_c : \quad \Omega^*_U(A) \to K^*(A)$$

is onto. We thus have a commutative diagram

$$\begin{tikzcd}
 & \Omega^*_U(X) \arrow[d, "\mu_c"] & \Omega^*_U(A) \arrow[l] \arrow[d, "\mu_c"] \\
0 & K^*(X) \arrow[l] & K^*(A) \arrow[l] \arrow[d] \\
 & & 0
\end{tikzcd}$$

from which it instantly follows that

$$\mu_c : \Omega^*_U(X) \to K^*(X)$$

is onto $\square$.

Proof of Theorem **(9.1)**. — We proceed by induction on $\text{hom.dim}_{\Omega^*_U} \Omega^*_U(X)$. Suppose first that $\text{hom.dim}_{\Omega^*_U} \Omega^*_U(X) = 0$. Then $\Omega^*_U(X)$ is a free Ω^*_U-module and $H^*(X; \mathbf{Z})$ is a free abelian group. The map

$$\tilde{\mu}_c : \quad \mathbf{Z} \otimes_{\Omega^*_U} \Omega^*_U(X) \to K^*(X)$$

is then seen to be an epimorphism between free abelian groups of the same rank, and hence is an isomorphism.

Proceeding inductively we will assume that we have established $\tilde{\mu}_c$ to be an isomorphism for all complexes Y for which $\text{hom.dim}_{\Omega^*_U} \Omega^*_U(Y) < n$, $n > 0$. Let X be a finite complex with

$$\text{hom.dim}_{\Omega^*_U} \Omega^*_U(X) = n.$$

We may find a finite complex A, with $\mathrm{H}^*(\mathrm{A};\mathbf{Z})$ free abelian, and a map (for some non-negative integer l)

$$f:\quad \Sigma^l \mathrm{X} \to \mathrm{A}$$

such that

$$f^*:\quad \Omega_{\mathrm{U}}^*(\mathrm{A}) \to \Omega_{\mathrm{U}}^*(\Sigma^l \mathrm{X})$$

is onto. From the cofibration

$$\Sigma^l \mathrm{X} \xrightarrow[f]{} \mathrm{A} \to \mathrm{A}/\Sigma^l \mathrm{X}$$

we obtain a commutative diagram

$$\begin{tikzcd}
0 & \mathbf{Z}\otimes_{\Omega_{\mathrm{U}}^*}\Omega_{\mathrm{U}}^*(\Sigma^l\mathrm{X}) \arrow[l] \arrow[d, "\widetilde{\mu}_c(\Sigma^l\mathrm{X})"] & \mathbf{Z}\otimes_{\Omega_{\mathrm{U}}^*}\Omega_{\mathrm{U}}^*(\mathrm{A}) \arrow[l] \arrow[d, "\widetilde{\mu}_c(\mathrm{A})", "\cong"'] & \mathbf{Z}\otimes_{\Omega_{\mathrm{U}}^*}\widetilde{\Omega}_{\mathrm{U}}^*(\mathrm{A}/\Sigma^l\mathrm{X}) \arrow[l] \arrow[d, "\widetilde{\mu}_c(\mathrm{A}/\Sigma^l\mathrm{X})", "\cong"'] \\
 & \mathrm{K}^*(\Sigma^l\mathrm{X}) & \mathrm{K}^*(\mathrm{A}) \arrow[l] & \widetilde{\mathrm{K}}^*(\mathrm{A}/\Sigma^l\mathrm{X}) \arrow[l]
\end{tikzcd}$$

where the isomorphic character of $\widetilde{\mu}_c(\mathrm{A})$ has already been noted and $\widetilde{\mu}_c(\mathrm{A}/\Sigma^l\mathrm{X})$ is an isomorphism by our inductive assumption (and stability) since

$$n = \mathrm{hom.dim}_{\Omega_{\mathrm{U}}^*}\Omega_{\mathrm{U}}^*(\Sigma^l\mathrm{X}) = 1 + \mathrm{hom.dim}_{\Omega_{\mathrm{U}}^*}\widetilde{\Omega}_{\mathrm{U}}^*(\mathrm{A}/\Sigma^l\mathrm{X}) > \mathrm{hom.dim}_{\Omega_{\mathrm{U}}^*}\widetilde{\Omega}_{\mathrm{U}}^*(\mathrm{A}/\Sigma^l\mathrm{X})$$

as may be seen from the exact sequence

$$0 \leftarrow \Omega_{\mathrm{U}}^*(\Sigma^l\mathrm{X}) \leftarrow \Omega_{\mathrm{U}}^*(\mathrm{A}) \leftarrow \widetilde{\Omega}_{\mathrm{U}}^*(\mathrm{A}/\Sigma^l\mathrm{X}) \leftarrow 0$$

of Ω_{U}^*-modules and ([5], VI, (2.8)).

It follows by stability and a routine chase of the above diagram that $\widetilde{\mu}_c(\mathrm{X})$ is a monomorphism. Since $\widetilde{\mu}_c$ is always an epimorphism by Proposition (9.1) we deduce that $\widetilde{\mu}_c(\mathrm{X})$ is an isomorphism. This completes the induction step.

Since $\mathrm{hom.dim}_{\Omega_{\mathrm{U}}^*}\Omega_{\mathrm{U}}^*(\mathrm{X})$ is finite for a finite complex X (by Theorem (1.6)), the result follows by induction □.

Having dealt with the relation between U-*co*bordism and K-theory we turn to an investigation of the relation between U-bordism and K-theory.

Convention. — Throughout this section we will regard $\Omega_*^{\mathrm{U}}(\cdot)$ as a $(\mathbf{Z}/2\mathbf{Z})$-graded homology theory, the $(\mathbf{Z}/2\mathbf{Z})$-grading being given by the even and odd part of the natural $\mathbf{Z}$-grading. The orientation $\mu_c : \underset{\sim}{\mathrm{MU}} \to \underset{\sim}{\mathrm{BU}}$ induces a natural transformation of homology functors $\Omega_*^{\mathrm{U}}(\cdot) \to \mathrm{K}_*(\cdot)$. The homomorphism

$$\mathrm{Td}:\quad \Omega_*^{\mathrm{U}} = \Omega_*^{\mathrm{U}}(\text{point}) \to \mathrm{K}_*(\text{point}) = \mathbf{Z}$$

may be identified (up to a sign fudge factor) with the classical Todd genus. The homomorphism Td provides $\mathbf{Z}$ with the correct trivial $(\mathbf{Z}/2\mathbf{Z})$-graded Ω_*^{U}-module structure of our further study. Throughout the remainder of this section all modules and algebras will be $(\mathbf{Z}/2\mathbf{Z})$-graded and the Ω_*^{U}-module structure of $\mathbf{Z}$ will be that given by Td.

We will begin by describing a natural transformation of functors

$$\zeta : \quad K^*(\cdot) \to \mathrm{Hom}_{\Omega_*^U}(\Omega_*^U(\cdot), \mathbf{Z}).$$

Let X be a finite complex. Suppose that $\alpha \in K^*(X)$ and $u \in \Omega_*^U(X)$. Represent u by a singular U-manifold $f: M \to X$. Recall ([9], [22]) that M is K^*-orientable. Let $[M] \in K_*(M)$ be the fundamental class of M. We now define ζ_X by

$$[\zeta_X(\alpha)](u) = \langle f^*(\alpha), [M] \rangle \in \mathbf{Z},$$

where

$$f^* : K^*(X) \to K^*(M)$$

is the map induced by f in K-theory.

The following formula is readily verified ([9], [22]): if M is a U-manifold and $\xi \in K^*(M)$ then

$$\langle \xi, [M] \rangle = \langle \mathrm{ch}(\xi) \mathrm{Td}(M), [M] \rangle$$

where [M] in the right hand formula denotes the fundamental homology class $[M] \in H_*(M; \mathbf{Z})$.

With the aid of this formula it is straightforward to verify that

$$\zeta_X(\alpha) : \quad \Omega_*^U(X) \to \mathbf{Z}$$

is Ω_*^U-linear. Thus we have

$$\zeta_X : \quad K^*(X) \to \mathrm{Hom}_{\Omega_*^U}(\Omega_*^U(X), \mathbf{Z})$$

which is easily seen to be a homomorphism that is natural in X.

Proposition **(9.3)**. — *Let* A *be a finite complex with* $\Omega_*^U(A)$ *a free* Ω_*^U*-module. Then*

$$\zeta_A : \quad K^*(A) \to \mathrm{Hom}_{\Omega_*^U}(\Omega_*^U(A), \mathbf{Z})$$

is an isomorphism.

Proof. — Since $\Omega_*^U(A)$ is a free Ω_*^U-module it follows that $H_*(A; \mathbf{Z})$ is a free abelian group and

$$\Omega_*^U(A) \cong \Omega_*^U \otimes_{\mathbf{Z}} H_*(A; \mathbf{Z})$$

as an Ω_*^U-module. Thus

$$\mathrm{Hom}_{\Omega_*^U}(\Omega_*^U(A); \mathbf{Z}) \cong \mathrm{Hom}_{\mathbf{Z}}(H_*(A; \mathbf{Z}), \mathbf{Z}).$$

Under this identification the homomorphism

$$\zeta_A : \quad K^*(A) \to \mathrm{Hom}_{\mathbf{Z}}(H_*(A; \mathbf{Z}), \mathbf{Z})$$

may be described as follows: for $\xi \in K^*(A)$ and $u \in H_*(A; \mathbf{Z})$ choose a U-manifold M_u and a map $f_u : M_u \to A$ such that

$$(f_u)_*[M_u] = u.$$

This is possible since $\mu : \Omega_*^U(A) \to H_*(A; \mathbf{Z})$ is onto. Thus for each $u \in H_*(A; \mathbf{Z})$ we obtain the polynomial $\mathrm{Td}(M_u) \in H^*(A; \mathbf{Z})$. Naturality of the Kronecker product then yields the formula

$$[\zeta_A(\xi)](u) = \langle \mathrm{ch}(\xi) \mathrm{Td}(M_u), u \rangle \in \mathbf{Z}.$$

We assert that ζ_A is onto. For by the universal coefficient theorem every $\varphi \in \mathrm{Hom}_{\mathbf{Z}}(H_*(A; \mathbf{Z}), \mathbf{Z})$ is given by

$$\varphi(u) = \langle v_\varphi, u \rangle$$

for some fixed $v_\varphi \in H^*(A; \mathbf{Z})$. Since $H^*(A; \mathbf{Z})$ is free abelian there exists $\xi \in K^*(A)$ such that

$$\mathrm{ch}(\xi) = v_\varphi + \text{higher terms}.$$

Since the constant term of the Todd polynomial is 1, we obtain

$$[\zeta_A(\xi)](u) = \langle \mathrm{ch}(\xi)\mathrm{Td}(M_u), u \rangle = \langle v, u \rangle = \varphi(u),$$

and thus ζ_A is onto.

Since A is a finite complex and $H_*(A; \mathbf{Z})$ is free abelian it follows that $K^*(A)$ and $\mathrm{Hom}_{\mathbf{Z}}(H_*(A; \mathbf{Z}), \mathbf{Z})$ are free abelian groups with the same rank as $H^*(A; \mathbf{Z})$. Thus since

$$\zeta_A : \quad K^*(A) \to \mathrm{Hom}_{\Omega_*^U}(\Omega_*^U(A), \mathbf{Z})$$

is epic it must be an isomorphism. $\square$

Theorem **(9.4)**. — *Let* X *be a finite complex. Then there exists a natural spectral sequence* $\{E_r[X], d_r[X]\}$ *with*

$$E_r[X] \Rightarrow K^*(X)$$
$$E_2^{p,q}[X] \cong \mathrm{Ext}_{\Omega_*^U}^{p,q}(\Omega_*^U(X); \mathbf{Z}).$$

Here $\mathrm{Ext}_{\Omega_*^U}^{*,*}(\Omega_*^U(X), \mathbf{Z})$ *and* $E_r^{*,*}[X]$ *are* $(\mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z}))$*-graded and the differential has bidegree* $(r, 1-r)$. *The convergence is in the naive sense.*

Proof. — Since X is a finite complex we may choose a U-bordism resolution of X, say

$$\emptyset = A_{-1} \subset A_0 \subset \dots \subset A_k \sim \Sigma^l X.$$

Associated with the filtered space

$$\emptyset = A_{-1} \subset A_0 \subset \dots \subset A_k$$

we have the exact couple

$$\begin{tikzcd} D \arrow[rr] & & D \arrow[dl] \\ & E \arrow[ul] & \end{tikzcd}$$

where

$$D_{p,q} = K^{p+q}(A_p)$$
$$E^{p,q} = K^{p+q}(A_p, A_{p-1})$$

the maps of the couple being induced by the K-theory exact triangles of the pairs (A_p, A_{p-1}) for $p = 0, \dots, k$. Denote the resulting spectral sequence by $\{E_r, d_r\}$. Note that E_2 is the homology of the complex

$$0 \to K^*(A_0) \to K^*(A_1, A_0) \to \dots \to K^*(A_k, A_{k-1}) \to 0.$$

By Proposition (9.3) we have for $i=0, \ldots, k$.

$$K^*(A_i, A_{i-1}) \cong \operatorname{Hom}_{\Omega_*^U}(\Omega_*^U(A_i, A_{i-1}), \mathbf{Z})$$

since $\Omega_*^U(A_i, A_{i-1})$ is a free Ω_*^U-module. Thus E_2 is the homology of the complex

$$0 \rightarrow \operatorname{Hom}_{\Omega_*^U}(\Omega_*^U(A_0), \mathbf{Z}) \rightarrow \ldots \rightarrow \operatorname{Hom}_{\Omega_*^U}(\Omega_*^U(A_k, A_{k-1}), \mathbf{Z}) \rightarrow 0.$$

Since
$$0 \leftarrow \Omega_*^U(A_k) \xleftarrow{\varepsilon} \Omega_*^U(A_0) \leftarrow \ldots \leftarrow \Omega_*^U(A_k, A_{k-1}) \leftarrow 0$$

is a free resolution of $\Omega_*^U(A_k)$ as an Ω_*^U-module it follows from the definition of derived functors that

$$E_2 \cong \operatorname{Ext}_{\Omega_*^U}(\Omega_*^U(X), \mathbf{Z}).$$

We now define $\{E_r[X], d_r[X]\}$ by the usual indexing trick, viz

$$E_r^{p,q}[X] = E_r^{p,q+l}$$
$$d_r^{p,q}[X] = d_r^{p,q+l}.$$

The usual suspension argument shows

$$E_2^{p,q}[X] \cong \operatorname{Ext}_{\Omega_*^U}^{p,q}(\Omega_*^U(X), \mathbf{Z}).$$

Since the filtration used is finite there is no trouble with convergence. Naturality follows by the usual nasty argument (compare Theorem (4.1)). □

Note that the edge homomorphism

$$K^*(X) \rightarrow E_\infty^{0,*}[X] \rightarrow E_2^{0,*}[X] \cong \operatorname{Hom}_{\Omega_*^U}(\Omega_*^U(X), \mathbf{Z})$$

coincides with the natural transformation ζ_X.

Proposition **(9.5)**. — *Let* X *be a finite complex*; *then*

$$\operatorname{Tor}_{p,*}^{\Omega_*^U}(\mathbf{Z}, \Omega_*^U(X)) = 0$$

for all $p > 0$.

Warning. — Remember everything is $(\mathbf{Z}/2\mathbf{Z})$-graded and the Ω_*^U-module structure on $\mathbf{Z}$ comes from $\operatorname{Td} : \Omega_*^U \rightarrow \mathbf{Z}$.

Proof. — By either applying Spanier-Whitehead duality to Theorem (9.2) or proving the duals of Proposition (9.1) and Theorem (9.2) directly we obtain a natural isomorphism of functors

$$\mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(\ ) \rightarrow K_*(\ )$$

on the category of finite complexes. Thus the functor $\mathbf{Z} \otimes_{\Omega_*^U} \Omega_*^U(\ )$ is a homology theory.

Let X be a finite complex. Choose a bordism resolution

$$\emptyset = A_{-1} \subset A_0 \subset \ldots \subset A_k \sim \Sigma^l X.$$

From the commutative diagram

$$\begin{tikzcd}
\Omega^{\mathrm{U}}_*(\mathrm{A}_i, \mathrm{A}_{i-1}) \arrow[r] \arrow[d] & \Omega^{\mathrm{U}}_*(\mathrm{A}_k, \mathrm{A}_{i-1}) \arrow[r] \arrow[d] & 0 \\
\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_i, \mathrm{A}_{i-1}) \arrow[r] \arrow[d] & \mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_k, \mathrm{A}_{i-1}) \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

it follows that

$$\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_i, \mathrm{A}_{i-1}) \rightarrow \mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_k, \mathrm{A}_{i-1})$$

is onto.

Thus since $\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\ )$ is a homology theory we obtain the exact sequences

$$0\leftarrow\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_k, \mathrm{A}_{i-1})\leftarrow\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_i, \mathrm{A}_{i-1})\leftarrow\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_k, \mathrm{A}_i)\leftarrow 0.$$

Thus as in the proof of Proposition (2.1) we obtain the *exact* sequence

$$0\leftarrow\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_k)\overset{\varepsilon}{\leftarrow}\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_0)\leftarrow\ldots\leftarrow\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_k, \mathrm{A}_{k-1})\leftarrow 0.$$

Since $$0\leftarrow\Omega^{\mathrm{U}}_*(\mathrm{A}_k)\overset{\varepsilon}{\leftarrow}\Omega^{\mathrm{U}}_*(\mathrm{A}_0)\leftarrow\ldots\leftarrow\Omega^{\mathrm{U}}_*(\mathrm{A}_k, \mathrm{A}_{k-1})\leftarrow 0$$

is a free resolution of $\Omega^{\mathrm{U}}_*(\mathrm{A}_k)$ we obtain from the definition of derived functors that

$$\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{p,*}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{A}_k))=0$$

for all $p>0$. The result now follows by stability. ☐

Corollary **(9.6)**. — *If* X *is a finite complex, then there is a natural isomorphism*

$$\mathrm{Ext}^{n,*}_{\Omega^{\mathrm{U}}_*}(\Omega^{\mathrm{U}}_*(\mathrm{X}), \mathbf{Z}) \cong \mathrm{Ext}^{n,*}_{\mathbf{Z}}(\mathrm{K}_*(\mathrm{X}), \mathbf{Z}).$$

Proof. — Consider the Todd homomorphism $\mathrm{Td} : \Omega^{\mathrm{U}}_*\rightarrow\mathbf{Z}$ as a change of rings. By Proposition (9.5).

$$\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{n,*}(\Omega^{\mathrm{U}}_*(\mathrm{X}), \mathbf{Z})=0$$

for all $n>0$. Hence by ([5], VI, (4.1.3)) the natural map

$$\mathrm{Ext}^{n,*}_{\mathbf{Z}}(\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{X}), \mathbf{Z}) \rightarrow \mathrm{Ext}^{n,*}_{\Omega^{\mathrm{U}}_*}(\Omega^{\mathrm{U}}_*(\mathrm{X}), \mathbf{Z})$$

is an isomorphism for all $n\geq 0$. As noted in Proposition (9.5)

$$\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{X}) \cong \mathrm{K}_*(\mathrm{X})$$

and the result follows. ☐

Corollary **(9.7)**. — *If* X *is a finite complex, then*

$$\mathrm{Ext}^{n,*}_{\Omega^U_*}(\Omega^U_*(X), \mathbf{Z}) = 0$$

for all $n > 1$.

Proof. — Immediate from Corollary (9.6) and the known properties of $\mathrm{Ext}^{n,*}_{\mathbf{Z}}(\ ,\ )$. ☐

Theorem **(9.8)**. — *Let* X *be a finite complex. Then there exists a natural exact sequence*

$$0 \rightarrow \mathrm{Ext}^{1,*}_{\Omega^U_*}(\Omega^U_*(X), \mathbf{Z}) \rightarrow K^*(X) \xrightarrow{\zeta} \mathrm{Hom}_{\Omega^U_*}(\Omega^U_*(X), \mathbf{Z}) \rightarrow 0.$$

The sequence splits, although not naturally, and $\mathrm{Ext}^{1,*}_{\Omega^U_*}(\Omega^U_*(X), \mathbf{Z})$ *is naturally isomorphic to the torsion subgroup of* $K^*(X)$.

Proof. — Consider the spectral sequence $\{E_r[X], d_r[X]\}$ of Theorem (9.4). By Corollary (9.7).

$$E_2^{p,*}[X] = 0 \quad \text{for } p \neq 0, 1.$$

The result now follows from elementary nonsense about spectral sequences and the fact that $\mathrm{Hom}_{\Omega^U_*}(\Omega^U_*(X), \mathbf{Z})$ is free abelian and $\zeta \otimes_{\mathbf{Z}} \mathbf{Q}$ is an isomorphism. ☐

Remark. — It follows from Corollary (9.6) that the exact sequence of Theorem (9.8) may be viewed as the Pontrjagin duality rule

$$0 \rightarrow \mathrm{Ext}_{\mathbf{Z}}(K_*(X), \mathbf{Z}) \rightarrow K^*(X) \rightarrow \mathrm{Hom}_{\mathbf{Z}}(K_*(X), \mathbf{Z}) \rightarrow 0$$

for K-theory.

Remark. — It is interesting to note that T.tom-Dieck has shown (private communication) that under the isomorphism

$$\widetilde{\mu}_c : \quad \mathbf{Z} \otimes_{\Omega^*_U} \Omega^*_U(X) \xrightarrow{\cong} K^*(X)$$

the γ-filtration on $K^*(X)$ coincides with the filtration of $\Omega^*_U(X) \otimes_{\Omega^*_U} \mathbf{Z}$ by the degree in $\Omega^*_U(X)$.

§ 10. On the Relation of U-Bordism to Connective K-Theory.

In this section we will study the connection between U-bordism and the homology theory associated to the connective BU-spectrum $\underset{\sim}{bu}$. We find that many of the results relating U-bordism to integral homology may be extended to analogous results relating U-bordism to connective K-theory. In this way we obtain a further understanding of the condition $\mathrm{hom.dim}_{\Omega^U_*} \Omega^U_*(X) = 2$. These results are indicative of a more extensive theory which we hope to discuss on another occassion.

Recollections. — For any space X let us denote by

$$X(n, \ldots, \infty) \rightarrow X$$

the $(n-1)$-connective fibering over X. We will denote the loop space of X by ΩX (not to be confused with the oriented bordism $\Omega^{SO}_*(X)$).

Consider the collection of spaces $\{bu_n\}$ defined by

$$bu_{2n} = \mathrm{BU}(2n, \ldots, \infty)$$
$$bu_{2n+1} = \mathrm{BU}(2n+1, \ldots, \infty).$$

Note that Bott periodicity implies

$$\Omega\mathrm{U}(2n+1, \ldots, \infty) = \mathrm{BU}(2n, \ldots, \infty)$$
$$\Omega\mathrm{BU}(2n, \ldots, \infty) = \mathrm{U}(2n-1, \ldots, \infty).$$

Thus
$$\Omega bu_m = bu_{m-1}$$

for all $m>0$. With the aid of these identifications we may form the Ω-spectrum $\underset{\sim}{bu} = \{bu_n;\ \beta_n\}$ in the standard fashion. This spectrum is referred to as the stable or connective BU-spectrum. We denote the homology and cohomology theories associated to $\underset{\sim}{bu}$ by

$$k_*(\cdot) = \mathrm{H}_*(\cdot\,;\underset{\sim}{bu})$$
$$k^*(\cdot) = \mathrm{H}^*(\cdot\,;\underset{\sim}{bu}).$$

It may be shown that $\underset{\sim}{bu}$ is a ring spectrum and thus $k_*(\cdot)$ and $k^*(\cdot)$ are multiplicative homology and cohomology theories. As a consequence of Bott periodicity we find

$$k_*(\text{point}) = \mathbf{Z}[t]\ :\quad \deg t = 2.$$

The K-theory orientation morphisms

$$\mathrm{MU}(n) \to \mathrm{BU}$$

may be lifted to morphisms

$$\mathrm{MU}(n) \to bu_{2n} = \mathrm{BU}(2n, \ldots, \infty)$$

by standard obstruction theory (recall that $\mathrm{MU}(n)$ is $(2n-1)$-connected). These in turn fit together to yield a morphism of ring spectra

$$\underset{\sim}{\mathrm{MU}} \to \underset{\sim}{bu}.$$

We thus have morphisms of ring spectra

$$\begin{tikzcd}
 & \underset{\sim}{bu} \arrow[rd, "\eta"] \arrow[dd, bend left=90, "\lambda"] & \\
\underset{\sim}{\mathrm{MU}} \arrow[ru, "\zeta"] \arrow[rr, "\mu"'] \arrow[rd, "\mu_c"'] & & \underset{\sim}{\mathrm{K}(\mathbf{Z})} \\
 & \underset{\sim}{\mathrm{BU}} &
\end{tikzcd}$$

that commutes (up to homotopy). We thus obtain morphisms of multiplicative homology theories

$$\begin{tikzcd}
\Omega^{\mathrm{U}}_*(\cdot) \arrow[r, "\zeta"] \arrow[rr, bend right, "\mu"'] & k_*(\cdot) \arrow[r, "\eta"] & \mathrm{H}_*(\cdot\,;\mathbf{Z})
\end{tikzcd}$$

where μ is the Thom homomorphism. The morphisms of coefficients

$$\zeta: \quad \Omega_*^{\mathrm{U}} \to k_* \equiv k_*(\text{point}) \cong \mathbf{Z}[t]$$
$$\eta: \quad k_* \to \mathrm{H}_*(\text{point}); \mathbf{Z}) \cong \mathbf{Z}$$

are given by

$$\zeta([\mathrm{M}^{2n}]) = \mathrm{Td}[\mathrm{M}^{2n}]t^n$$
$$\eta(a_n t^n) = \begin{cases} a_0 & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}$$

where $t \in k_2$ is an appropriatly chosen generator e.g. $t = \zeta[\mathbf{C}\mathrm{P}(1)]$.

Notice that U-manifolds are k^*-orientable and hence satisfy Poincaré duality for connective K-theory.

Our objective in this section is to study the relation between $\Omega_*^{\mathrm{U}}(\cdot)$ and $k_*(\cdot)$ via the homomorphism $\zeta: \Omega_*^{\mathrm{U}}(\cdot) \to k_*(\cdot)$ of homology theories. To this end we shall need:

Lemma (**10.1**). — *Let* A *be a finite complex and suppose that* $\Omega_*^{\mathrm{U}}(\mathrm{A})$ *is a projective* Ω_*^{U}*-module. Then the morphism*

$$\tilde{\zeta}: \quad \mathbf{Z}[t] \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{A}) \to k_*(\mathrm{A})$$

induced by ζ *is an isomorphism.*

Proof. — By Corollary (3.10) $\mathrm{H}_*(\mathrm{A}; \mathbf{Z})$ is free abelian. Hence there are filtrations

$$\{\mathrm{F}_p \Omega_*^{\mathrm{U}}(\mathrm{A})\}, \quad \{\mathrm{F}_p k_*(\mathrm{A})\}$$

with

$$\mathrm{E}^0 \Omega_*^{\mathrm{U}}(\mathrm{A}) \cong \mathrm{H}_*(\mathrm{A}; \mathbf{Z}) \otimes_{\mathbf{Z}} \Omega_*$$
$$\mathrm{E}^0 k_*(\mathrm{A}) \cong \mathrm{H}_*(\mathrm{A}; \mathbf{Z}) \otimes_{\mathbf{Z}} k_*.$$

The morphisms ζ induces a filtration preserving morphism

$$\zeta: \quad \mathbf{Z}[t] \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{A}) \to k_*(\mathrm{A}).$$

Passing to associated gradeds we find that

$$\mathrm{E}^0\zeta: \quad \mathbf{Z}[t] \otimes_{\Omega_*^{\mathrm{U}}} (\Omega_*^{\mathrm{U}} \otimes_{\mathbf{Z}} \mathrm{H}_*(\mathrm{A}; \mathbf{Z})) \to \mathbf{Z}[t] \otimes_{\mathbf{Z}} \mathrm{H}_*(\mathrm{A}; \mathbf{Z})$$

is the standard associativity isomorphism. Thus it follows that

$$\zeta: \quad \mathbf{Z}[t] \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{A}) \to k_*(\mathrm{A})$$

is an isomorphism by the usual induction over the filtration argument. □

Theorem (**10.2**). — *Let* X *be a finite complex. Then there exists a natural spectral sequence* $\{\mathrm{E}^r[\mathrm{X}], d^r[\mathrm{X}]\}$ *with*

$$\mathrm{E}^r[\mathrm{X}] \Rightarrow k_*(\mathrm{X})$$

and

$$\mathrm{E}^2_{p,q}[\mathrm{X}] = \mathrm{Tor}_{p,q}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}[t], \Omega_*^{\mathrm{U}}(\mathrm{X})).$$

Proof. — This follows by forming (and reindexing) the $k_*(\ )$ exact couple of a U-bordism resolution of X. The procedure is analagous to that employed in Theorem (4.1) with Lemma (10.1) playing the role of Lemma (3.1) in the identification of $\mathrm{E}^2[\mathrm{X}]$. □

As in the case of Theorem (4.1) the edge map

$$\mathbf{Z}[t]\otimes_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{X})=\mathrm{E}^2_{0,*}[\mathrm{X}]\xrightarrow{e}\mathrm{E}^{\infty}_{0,*}[\mathrm{X}]\hookrightarrow k_*(\mathrm{X})$$

may be identified with the reduced morphism $\tilde{\zeta}$. The morphism $\eta : k_*(\ )\to\mathrm{H}_*(\,;\mathbf{Z})$ induces a morphism of spectral sequences

$$\{\eta^r\}:\quad\{\mathrm{E}^r[\mathrm{X}],\,d^r[\mathrm{X}]\}\to\{\mathrm{E}^r\langle\mathrm{X}\rangle,\,d^r\langle\mathrm{X}\rangle\}$$

where $\{\mathrm{E}^r\langle\mathrm{X}\rangle,\,d^r\langle\mathrm{X}\rangle\}$ denotes the spectral sequence of section 4.

Our study of the relation between $\Omega_*^{\mathrm{U}}(\cdot)$ and $k_*(\cdot)$ depends quite crucially on the following technical result. Because of its pivotal role we will present two slightly different proofs. While these proofs are (clearly) logically equivalent, we feel that they emphasize different aspects of our previous results, and offer different insights into the situation. The result in question is:

Proposition **(10.4)**. — *Let* X *be a finite complex,* $p, q\in\mathbf{Z}$ *with* $p>0$. *Then for any element*

$$\alpha\in\mathrm{Tor}^{\Omega_*^{\mathrm{U}}}_{p,q}(\mathbf{Z}[t],\,\Omega_*^{\mathrm{U}}(\mathrm{X}))$$

there exists an integer N, *depending on* α, *such that*

$$t^{\mathrm{N}}\alpha=0\in\mathrm{Tor}^{\Omega_*^{\mathrm{U}}}_{p,q+2\mathrm{N}}(\mathbf{Z}[t],\,\Omega_*^{\mathrm{U}}(\mathrm{X})).$$

The two proofs of Proposition (10.4) that we will present depend on the $\mathbf{Z}_2$-graded and $\mathbf{Z}$-graded versions of Theorem (9.1). We will begin with the proof dependent on the $\mathbf{Z}_2$-graded version of Theorem (9.1), i.e., the version presented in section 9.

Consider the exact sequence of $\mathbf{Z}_2$-graded Ω_*^{U}-modules

$$0\to\mathbf{Z}[t]\xrightarrow[m_{1-t}]{}\mathbf{Z}[t]\to\mathbf{Z}\to 0$$

where m_{1-t} denotes multiplication by $1-t$. The Ω_*^{U}-module structure on $\mathbf{Z}$ is just that given by the homomorphism $\mathrm{Td}:\Omega_*^{\mathrm{U}}\to\mathbf{Z}$ employed in the previous section. As we shall also be dealing with the trivial Ω_*^{U}-module structure on $\mathbf{Z}$ we will denote by $\mathbf{Z}_{\mathrm{Td}}$ the Ω_*^{U}-module structure on $\mathbf{Z}$ given by Td.

Proposition **(10.5)**. — *Let* X *be a finite complex and p a (strictly) positive integer. Then*

$$m_{1-t}:\quad\mathrm{Tor}^{\Omega_*^{\mathrm{U}}}_{p,*}(\mathbf{Z}[t],\,\Omega_*^{\mathrm{U}}(\mathrm{X}))\to\mathrm{Tor}^{\Omega_*^{\mathrm{U}}}_{p,*}(\mathbf{Z}[t],\,\Omega_*^{\mathrm{U}}(\mathrm{X}))$$

is an isomorphism.

Proof. — From the exact sequence

$$0\to\mathbf{Z}[t]\xrightarrow[m_{1-t}]{}\mathbf{Z}[t]\to\mathbf{Z}_{\mathrm{Td}}\to 0$$

of $\mathbf{Z}_2$-graded Ω_*^{U}-modules we obtain a long exact sequence

$$\begin{aligned}\ldots\to\mathrm{Tor}^{\Omega_*^{\mathrm{U}}}_{p+1,*}(\mathbf{Z}_{\mathrm{Td}},\,\Omega_*^{\mathrm{U}}(\mathrm{X}))\to\mathrm{Tor}^{\Omega_*^{\mathrm{U}}}_{p,*}(\mathbf{Z}[t],\,\Omega_*^{\mathrm{U}}(\mathrm{X}))&\\ \xrightarrow{m_{1-t}}\mathrm{Tor}^{\Omega_*^{\mathrm{U}}}_{p,*}(\mathbf{Z}[t],\,\Omega_*^{\mathrm{U}}(\mathrm{X}))\to\mathrm{Tor}^{\Omega_*^{\mathrm{U}}}_{p,*}(\mathbf{Z}_{\mathrm{Td}},\,\Omega_*^{\mathrm{U}}(\mathrm{X}))\to\ldots&\end{aligned}$$

By Proposition (9.5)

$$\mathrm{Tor}^{\Omega^U_*}_{p,*}(\mathbf{Z}_{\mathrm{Td}}, \Omega^U_*(X)) = 0$$

for all $p > 0$, and the result follows from exactness. □

First proof of (10.4). — Suppose that

$$\nu \in \mathrm{Tor}^{\Omega^U_*}_{p,*}(\mathbf{Z}[t], \Omega^U_*(X))$$

where we are regarding $\mathrm{Tor}^{\Omega^U_*}_{p,*}(\mathbf{Z}[t], \Omega^U_*(X))$ as being $\mathbf{Z}_2$-graded by its even and odd dimensional components. Then the equation

$$(*) \qquad (1-t)\xi = \nu$$

has a unique solution in $\mathrm{Tor}^{\Omega^U_*}_{p,*}(\mathbf{Z}[t], \Omega^U_*(X))$. Recalling that $\mathrm{Tor}^{\Omega^U_*}_{p,*}(\mathbf{Z}[t], \Omega^U_*(X))$ is actually $\mathbf{Z}$-graded we may write ν and ξ in terms of components

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots)$$
$$\xi = (\xi_0, \xi_1, \xi_2, \ldots)$$

and the equation

$$(1-t)\xi = \nu$$

takes the component form

$$\xi_n - t\xi_{n-2} = \nu_n.$$

This leads to the recursion formula

$$\xi = \sum_{j=0}^{i} t^j \nu_{j-2i}.$$

Now suppose that p is a (strictly) positive integer and

$$\alpha \in \mathrm{Tor}^{\Omega^U_*}_{p,q}(\mathbf{Z}[t], \Omega^U_*(X)).$$

Then there exist classes

$$\xi_{q+2i} \in \mathrm{Tor}^{\Omega^U_*}_{p,q+2i}(\mathbf{Z}[t], \Omega^U_*(X))$$

for $-\infty < i < \infty$, $\xi_{q+2i} = 0$ for large i, and satisfying, for non-negative i, the equation

$$\xi_{q+2i} = t^i \alpha.$$

As $\xi_{q+2i} = 0$ for suitably large non-negative i we must have

$$t^i \alpha = 0$$

for i large and positive as required. □

We turn now to a second proof of (10.4) that uses a $\mathbf{Z}$-graded formulation of the results of section 9. We will begin by describing the precise form these results take.

Notation and convention. — Until noted to the contrary we will denote by $K_*(\cdot)$ the homology theory determined by the spectrum $\underset{\sim}{BU}$. Recall that $K_*(\cdot)$ is thus $\mathbf{Z}$-graded and that

$$K_*(\text{point}) \cong \mathbf{Z}[t, t^{-1}],$$

where $\mathbf{Z}[t, t^{-1}]$ denotes the $\mathbf{Z}$-graded ring of finite Laurent series in a degree 2 variable t.

There is a natural transformation of **Z**-graded homology theories

$$\mu: \quad \Omega_*^U(\cdot) \to K_*(\cdot)$$

given as before by the K-theory orientation of $\underset{\sim}{MU}$. On the coefficients, μ is given by

$$\mu([M^{2n}]) = (-1)^n \operatorname{Todd}[M^{2n}]t^n,$$

and more generally there is a commutative diagram of **Z**-graded homology theories

$$\begin{tikzcd}
\Omega_*^U(\cdot) \arrow[rr, "\mu"] \arrow[dr, "\zeta"'] & & K_*(\cdot) \\
& k_*(\cdot) \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

The **Z**-graded formulation of ([9], (10.1)) (see also Theorem (9.1)) of the present paper) is then

(*) *For any finite complex* X *the natural transformation*

$$\tilde{\mu}: \quad \mathbf{Z}[t, t^{-1}] \otimes_{\Omega_*^U} \Omega_*^U(X) \to K_*(X)$$

is an isomorphism.

Next we recall that there is the natural transformation of multiplicative homology theories

$$\lambda: \quad k_*(\cdot) \to K_*(\cdot)$$

that on coefficients is simply the inclusion

$$\mathbf{Z}[t] \hookrightarrow \mathbf{Z}[t, t^{-1}].$$

As $\mathbf{Z}[t, t^{-1}]$ is a flat $\mathbf{Z}[t]$-module we find that

$$\mathbf{Z}[t, t^{-1}] \otimes_{\mathbf{Z}[t]} k_*(\cdot)$$

is again a homology theory. Hence a simple induction argument over the number of cells shows that

$$\tilde{\lambda}: \quad \mathbf{Z}[t, t^{-1}] \otimes_{\mathbf{Z}[t]} k_*(\cdot) \to K_*(\cdot)$$

is an isomorphism of functors on the category of finite CW-complexes, and hence by continuity on the category of CW-complexes.

Finally we shall need some elementary facts about localizations. We shall need to know the kernel of the natural map

$$M \to \mathbf{Z}[t, t^{-1}] \otimes_{\mathbf{Z}[t]} M$$

for a $\mathbf{Z}[t]$-module M. To this end we introduce for any $\mathbf{Z}[t]$-module M, the submodule

$N(M) \subseteq M$ defined by $\qquad N(M) = \bigcup_{i=0}^{\infty} N(M, i)$

where $\qquad N(M, i) = \{x \in M \mid t^i x = 0\}$.

Lemma **(10.6)**. — *For any* $\mathbf{Z}[t]$*-module* M *there is a natural isomorphism*

$$\mathrm{Tor}_{1,*}^{\mathbf{Z}[t]}(\mathbf{Z}[t,t^{-1}]/\mathbf{Z}[t],\mathrm{M})\cong \mathrm{N}(\mathrm{M}).$$

Proof. — For each non-negative integer i introduce the exact sequence of $\mathbf{Z}[t]$-modules

$$0\to\mathbf{Z}[t]\xrightarrow[m_i]{}s^{-2i}\mathbf{Z}[t]\to s^{-2i}\mathbf{Z}[t]/\mathbf{Z}[t]\to 0$$

where $s^{-2i}\mathbf{Z}[t]$ denotes the $\mathbf{Z}[t]$-module of finite Laurent series

$$\sum_{n\geq -i} a_n t^n$$

and m_i is the standard inclusion.

We then have for any $\mathbf{Z}[t]$-module M the exact sequence

$$\begin{aligned}&0\to\mathrm{Tor}_{1,*}^{\mathbf{Z}[t]}(s^{-2i}\mathbf{Z}[t]/\mathbf{Z}[t],\mathrm{M})\to\mathbf{Z}[t]\otimes_{\mathbf{Z}[t]}\mathrm{M}\\ &\to s^{-2i}\mathbf{Z}[t]\otimes_{\mathbf{Z}[t]}\mathrm{M}\to(s^{-2i}\mathbf{Z}[t]/\mathbf{Z}[t])\otimes_{\mathbf{Z}[t]}\mathrm{M}\to 0.\end{aligned}$$

Now we may identify

$$\mathbf{Z}[t]\otimes_{\mathbf{Z}[t]}\mathrm{M}\to s^{-2i}\mathbf{Z}[t]\otimes_{\mathbf{Z}[t]}\mathrm{M}$$

with

$$\mu_i:\quad \mathrm{M}\to s^{-2i}\mathrm{M}$$

where μ_i denotes multiplication by t^i, and $s^{-2i}\mathrm{M}$ denotes the $(-2i)$-fold suspension of M. Thus we see

$$\mathrm{Tor}_{1,*}^{\Omega_*^{\mathrm{U}}}(s^{-2i}\mathbf{Z}[t]/\mathbf{Z}[t],\mathrm{M})\cong\mathrm{N}(\mathrm{M},i).$$

Passing to the limit as $i\to\infty$ we obtain by the definitions

$$\mathrm{Tor}_{1,*}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}[t,t^{-1}]/\mathbf{Z}[t],\mathrm{M})\cong\mathrm{N}(\mathrm{M})$$

as desired. □

Proposition **(10.7)**. — *If* M *is a* $\mathbf{Z}[t]$*-module then the kernel of the natural map*

$$\mathrm{M}\to\mathbf{Z}[t,t^{-1}]\otimes_{\mathbf{Z}[t]}\mathrm{M}$$

is exactly $\mathrm{N}(\mathrm{M})\subseteq\mathrm{M}$.

Proof. — This results from Lemma (10.6) and the natural exact sequence

$$\begin{aligned}&0\to\mathrm{Tor}_{1,*}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}[t,t^{-1}]/\mathbf{Z}[t],\mathrm{M})\to\mathbf{Z}[t]\otimes_{\mathbf{Z}[t]}\mathrm{M}\\ &\to\mathbf{Z}[t,t^{-1}]\otimes_{\mathbf{Z}[t]}\mathrm{M}\to\mathbf{Z}[t,t^{-1}]\otimes_{\mathbf{Z}[t]}\mathrm{M}\to 0\end{aligned}$$

quite quickly. □

Second proof of **(10.4)**. — The natural inclusion of rings

$$\mathbf{Z}[t]\hookrightarrow\mathbf{Z}[t,t^{-1}]$$

induces a morphism of functors

$$\tilde{\lambda}:\quad\mathbf{Z}[t,t^{-1}]\otimes_{\mathbf{Z}[t]}\mathrm{Tor}_{p,q}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}[t],-)\to\mathrm{Tor}_{p,q}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}[t,t^{-1}],-)$$

for all $p,q\in\mathbf{Z}$, which in view of the exactness of the functor $\mathbf{Z}[t,t^{-1}]\otimes_{\mathbf{Z}[t]}$ is readily seen to be a natural equivalence.

In view of our discussion above of the **Z**-graded formulation of (9.1) we find that the **Z**-graded formulation of (9.5) yields

$$\mathrm{Tor}_{p,q}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}[t, t^{-1}], \Omega_*^{\mathrm{U}}(\mathrm{X}))=0$$

for all $p>0$.

Thus we find

$$\mathbf{Z}[t, t^{-1}]\otimes_{\mathbf{Z}[t]}\mathrm{Tor}_{p,*}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}[t], \Omega_*^{\mathrm{U}}(\mathrm{X}))=0$$

for all $p>0$. In view of Proposition (10.7) this yields

$$t^i\alpha=0$$

for any $\alpha\in\mathrm{Tor}_{p,*}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}[t], \Omega_*^{\mathrm{U}}(\mathrm{X}))$, $p>0$, and i sufficiently large (depending on α). □

Theorem **(10.8)**. — *Let* X *be a finite complex and* $u\in k_*(\mathrm{X})$. *Then there exists an integer* s, *depending on* u, *such that*

$$t^s u\in\mathrm{Im}\,\{\Omega_*^{\mathrm{U}}(\mathrm{X})\xrightarrow{\zeta}k_*(\mathrm{X})\}.$$

Proof. — Consider the spectral sequence of (10.2)

$$\mathrm{E}^r[\mathrm{X}]\Rightarrow k_*(\mathrm{X})$$

where

$$\mathrm{E}_{p,q}^2[\mathrm{X}]=\mathrm{Tor}_{p,q}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}[t], \Omega_*^{\mathrm{U}}(\mathrm{X})).$$

Let us denote by $\{\mathrm{F}_p k_*(\mathrm{X})\}$ the filtration on $k_*(\mathrm{X})$ determined by the above spectral sequence. We will say that the element $u\in k_*(\mathrm{X})$ has filtration p iff

$$u\in\mathrm{F}_p k_*(\mathrm{X}) \quad\text{and}\quad u\notin\mathrm{F}_{p-1}k_*(\mathrm{X}).$$

We then have:

Lemma. — *Let* X *be a finite complex and* $v\in k_*(\mathrm{X})$ *a class of filtration* $p>0$. *Then there exists an integer* $n=n(v)$ *such that* $t^n v\in k_*(\mathrm{X})$ *has filtration* $p-1$.

Proof. — Let us first note that the spectral sequence $\{\mathrm{E}^r[\mathrm{X}], d^r[\mathrm{X}]\}$ is a spectral sequence of **Z**[t]-modules and that for each $p\geq 0$

$$\mathrm{E}_{p,*}^0 k_*(\mathrm{X})\cong\mathrm{E}_{p,*}^\infty[\mathrm{X}]$$

as graded **Z**[t]-modules. Note that of course

$$\ldots\subseteq\mathrm{F}_i k_*(\mathrm{X})\subseteq\mathrm{F}_{i-1}k_*(\mathrm{X})\subseteq\ldots\subseteq k_*(\mathrm{X})$$

are **Z**[t]-submodules.

Now let us write

$$[v]=\text{class }\{v\}\in\mathrm{F}_p k_*(\mathrm{X})/\mathrm{F}_{p-1}k_*(\mathrm{X}).$$

Notice that as $p>0$,

$$t^n[v]=0\in\mathrm{E}_{p,*}^\infty[\mathrm{X}]$$

for suitably large n by (10.4). But this means

$$t^n v\in\mathrm{F}_{p-1}k_*(\mathrm{X})$$

as desired. □

Returning to the proof of (10.8) we suppose that $u \in k_*(X)$ has filtration p. By repeated application of the above lemma we may find an integer $s = s(u)$ such that $t^s u$ has filtration 0. But

$$F_0 k_*(X) \equiv \mathrm{Im}\,\{\Omega_*^U(X) \xrightarrow{\zeta} k_*(X)\}$$

which yields the desired result. □

It is perhaps of interest to note that (10.8) may be proved directly from the **Z**-graded version of the result of section 9 and thereby avoiding (10.4). The details are very similar to (10.4) and may be organized as follows.

Alternate proof of **(10.8)**. — Introduce the commutative diagram

$$\begin{tikzcd}
\Omega_*^U(X) \arrow[rr, "\zeta"] \arrow[d, "\nu"'] & & k_*(X) \arrow[d, "\lambda"] \\
\mathbf{Z}[t, t^{-1}] \otimes_{\Omega_*^U} \Omega_*^U(X) \arrow[rr, dash, "\hat{\zeta}"] \arrow[dr, "\cong"'] & & \mathbf{Z}[t, t^{-1}] \otimes_{\mathbf{Z}[t]} k_*(X) \arrow[dl, "\cong"] \\
& K_*(X) &
\end{tikzcd}$$

from which we find that $\hat{\zeta}$ is an isomorphism.

We shall require the following elementary lemma:

Lemma. — *Let* X *be a finite complex and* $\alpha \in \mathbf{Z}[t, t^{-1}] \otimes_{\Omega_*^U} \Omega_*^U(X)$. *Then there exists an integer* $m = m(a)$ *such that*

$$t^m \alpha \in \mathrm{Im}\,\{\Omega_*^U(X) \to \mathbf{Z}[t, t^{-1}] \otimes_{\Omega_*^U} \Omega_*^U(X)\}$$

Proof. — A typical element α has the form

$$\sum_{i \geq n(\alpha)} t^i \otimes \alpha_i : \quad \alpha_i \in \Omega_*^U(X).$$

Thus there exists an integer m (say $m > |n(\alpha)|$) so that

$$t^m \alpha = \sum_{m+i \geq 0} t^{i+m} \otimes \alpha_i = \sum_{j \geq 0} t^j \otimes \alpha_{j-m}.$$

As t has been chosen so that

$$\zeta([\mathbf{CP}(1)]) = t$$

we find that

$$t^m \alpha = \sum_{j \geq 0} t^j \otimes \alpha_{j-m} = \sum_{j \geq 0} 1 \otimes [\mathbf{CP}(1)]^j \alpha_{j-m}$$
$$= \zeta(\sum_{j \geq 0} [\mathbf{CP}(1)]^j \alpha_{j-m})$$

as desired. □

Our second proof of (10.8) is now readily completed as follows. Let $v \in k_*(\mathrm{X})$. Then

$$\lambda(v) = \hat{\zeta}(\alpha)$$

for some $\alpha \in \mathbf{Z}[t, t^{-1}] \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{X})$. By the preceeding lemma there exists an integer $m = m(\alpha)$ such that

$$t^m \alpha \in \operatorname{Im} \{\Omega_*^{\mathrm{U}}(\mathrm{X}) \overset{\nu}{\to} \mathbf{Z}[t, t^{-1}] \otimes_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(\mathrm{X})\}.$$

Let $\gamma \in \Omega_*^{\mathrm{U}}(\mathrm{X})$ be such that $\nu(\gamma) = t^m \alpha.$

Then we find

$$\begin{aligned} \lambda(t^m v - \zeta(\gamma)) &= t^m \lambda(v) - \lambda\zeta(\gamma) \\ &= t^m \hat{\zeta}(\alpha) - \hat{\zeta}\nu(\gamma) \\ &= t^m \hat{\zeta}(\alpha) - \hat{\zeta}(t^m \alpha) = 0. \end{aligned}$$

Thus

$$t^m v - \zeta(\gamma) \in \ker \lambda = \mathrm{N}(k_*(\mathrm{X}))$$

by Proposition (10.7). Thus there exists an integer n such that

$$t^n(t^m v - \zeta(\gamma)) = 0 \in k_*(\mathrm{X}).$$

Thus we find

$$\zeta([\mathbf{C}\mathrm{P}(1)]\gamma) = t^{m+n} v$$

as required. □

We note that standard mod $\mathscr{C}$ theory [21] may be applied to deduce that some integral multiple of every class $v \in k_*(\mathrm{X})$ is in the image of ζ, i.e., is represented by a closed singular U-manifold on X. Theorem (10.8) is a sort of complement to this. Both results imply that some localization functor is an isomorphism.

Taken together these two results suggest that for each class $v \in k_*(\mathrm{X})$, X a finite complex, we introduce the ideal $\mathrm{U}(v) \subseteq \mathbf{Z}[t]$ by

$$\rho \in \mathrm{U}(\nu) \Leftrightarrow \rho . \nu \in \operatorname{Im} \{\Omega_*^{\mathrm{U}}(\mathrm{X}) \overset{\zeta}{\to} k_*(\mathrm{X})\}.$$

It follows from (10.7) that there exist non-negative integers m, n such that

$$m,\ t^n \in \mathrm{U}(v).$$

It would be interesting to have a more complete structure theory for these ideals $\mathrm{U}(v)$. For example, when does $\mathrm{U}(v) = (m, t^n)$? The answer is certainly affirmative if $m = p$, a prime. On the other hand if m is not a prime, when does $\mathrm{U}(v) \in qt^s$ for a proper divisor q of m and a proper divisor s of n? Note that $\mathrm{A}(v) \subseteq \mathrm{U}(v)$, when does equality hold?, etc.

§ 11. More on the Relation of U-Bordism to Connective K-Theory.

We turn now to the relation between the numerical invariant $\operatorname{hom.dim}_{\Omega_*^{\mathrm{U}}} \Omega_*^{\mathrm{U}}(-)$ and the homology theory $k_*(-)$. Our goal in this section is an analog of Corollary (3.11). We shall need the following preliminary result.

Proposition (**11.1**). — *Let* X *be a finite complex and* k *a strictly positive integer. Then*

1) *if* $k>1$, $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X)\leq k$ *iff* $\mathrm{Tor}^{\Omega^U_*}_{j,*}(\mathbf{Z}[t], \Omega^U_*(X))=0$ *for all* $j\geq k-1$;

2) *if* $k=1$, $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X)\leq 1$ *iff* $\mathrm{Tor}^{\Omega^U_*}_{j,*}(\mathbf{Z}[t], \Omega^U_*(X))=0$ *for all* $j\geq 1$ *and*

$$m_t : \mathbf{Z}[t]\otimes_{\Omega^U_*}\Omega^U_*(X)\to \mathbf{Z}[t]\otimes_{\Omega^U_*}\Omega^U_*(X)$$

is a monomorphism.

Here as usual m_t denotes multiplication by t.

Proof. — Consider the short exact sequence of $\mathbf{Z}$-graded Ω^U_*-modules

$$0\to \mathbf{Z}[t]\xrightarrow[m_t]{}\mathbf{Z}[t]\to\mathbf{Z}\to 0$$

where $\mathbf{Z}$ has its usual trivial Ω^U_*-module structure. This yields an exact sequence

$$(*)\qquad \begin{aligned}\ldots\to \mathrm{Tor}^{\Omega^U_*}_{j,*}(\mathbf{Z}[t], \Omega^U_*(X))\xrightarrow[m_t]{}\mathrm{Tor}^{\Omega^U_*}_{j,*}(\mathbf{Z}[t], \Omega^U_*(X))\to\mathrm{Tor}^{\Omega^U_*}_{j,*}(\mathbf{Z}, \Omega^U_*(X))\to\\ \to \mathrm{Tor}^{\Omega^U_*}_{j-1,*}(\mathbf{Z}[t], \Omega^U_*(X))\xrightarrow[m_t]{}\ldots\end{aligned}$$

Let us suppose that $k>1$ and that $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X)\leq k$. By Corollary (4.3) we thus obtain

$$\mathrm{Tor}^{\Omega^U_*}_{j,*}(\mathbf{Z}, \Omega^U_*(X))=0$$

for all $j\geq k$. Exactness of the above sequence then yields

$$\mathrm{Tor}^{\Omega^U_*}_{j,*}(\mathbf{Z}[t], \Omega^U_*(X))=0$$

for all $j\geq k$ and

$$m_t : \mathrm{Tor}^{\Omega^U_*}_{k,*}(\mathbf{Z}[t], \Omega^U_*(X))\to\mathrm{Tor}^{\Omega^U_*}_{k,*+2}(\mathbf{Z}[t], \Omega^U_*(X))$$

is a monomorphism. Examination of Proposition (10.4) shows that this is possible iff

$$\mathrm{Tor}^{\Omega^U_*}_{k,*}(\mathbf{Z}[t], \Omega^U_*(X))=0.$$

Conversely, if $k>1$ and

$$\mathrm{Tor}^{\Omega^U_*}_{j,*}(\mathbf{Z}[t], \Omega^U_*(X))=0$$

for all $j\geq k-1$, then exactness of the sequence $(*)$ yields

$$\mathrm{Tor}^{\Omega^U_*}_{j,*}(\mathbf{Z}, \Omega^U_*(X))=0$$

for all $j\geq k$ and an application of Corollary (4.3) thus establishes 1).

The proof of 2) is similar and left to the reader. □

Theorem (**11.2**). — *Let* X *be a finite complex. Then* $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X)\leq 2$ *iff the natural morphism*

$$\zeta : \quad \mathbf{Z}[t]\otimes_{\Omega^U_*}\Omega^U_*(X)\to k_*(X)$$

is an isomorphism.

Proof. — Suppose that $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X)\leq 2$. By Proposition (11.1) it follows that

$$\mathrm{Tor}^{\Omega^U_*}_{p,q}(\mathbf{Z}[t], \Omega^U_*(X))=0$$

for all $p>0$. Thus the spectral sequence of Theorem (10.2) collapses to the edge isomorphism

$$\tilde{\zeta}: \quad \mathbf{Z}[t]\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{X}) \to k_*(\mathrm{X})$$

as required.

The converse implication is somewhat complicated, owing to the absence of a spectral sequence of Dold-Atiyah-Hirzebruch type relating $\Omega^{\mathrm{U}}_*(\cdot)$ and $k_*(\cdot)$.

We will proceed as follows.

Let us suppose that X is a finite complex and

$$\tilde{\zeta}: \quad \mathbf{Z}[t]\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{X}) \to k_*(\mathrm{X})$$

is an isomorphism. Apply Proposition (2.5) to choose a *partial* U-bordism resolution of X

$$\emptyset=\mathrm{A}_{-1}\subset\mathrm{A}_0\subset\mathrm{A}_1\subset\mathrm{A}_2\sim\Sigma^l\mathrm{X}$$

of length 2. By stability

$$\tilde{\zeta}: \quad \mathbf{Z}[t]\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_2) \to k_*(\mathrm{A}_2)$$

is also an isomorphism. By construction we also have the exact sequences

$$0\leftarrow\Omega^{\mathrm{U}}_*(\mathrm{A}_2, \mathrm{A}_{-1})\leftarrow\Omega^{\mathrm{U}}_*(\mathrm{A}_0, \mathrm{A}_{-1})\leftarrow\Omega^{\mathrm{U}}_*(\mathrm{A}_2, \mathrm{A}_0)\leftarrow 0$$
$$0\leftarrow\Omega^{\mathrm{U}}_*(\mathrm{A}_2, \mathrm{A}_0)\leftarrow\Omega^{\mathrm{U}}_*(\mathrm{A}_1, \mathrm{A}_0)\leftarrow\Omega^{\mathrm{U}}_*(\mathrm{A}_2, \mathrm{A}_1)\leftarrow 0$$

where $\Omega^{\mathrm{U}}_*(\mathrm{A}_0, \mathrm{A}_{-1})$ and $\Omega^{\mathrm{U}}_*(\mathrm{A}_1, \mathrm{A}_0)$ are free Ω^{U}_*-modules.

From the first of these exact sequences we obtain a diagram

$$\begin{tikzcd}
0 & \mathbf{Z}[t]\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_2) \arrow[l] \arrow[d, "\tilde{\zeta}'"', "\cong"] & \mathbf{Z}[t]\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_0) \arrow[l] \arrow[d, "\tilde{\zeta}"', "\cong"] & \mathbf{Z}[t]\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_2, \mathrm{A}_0) \arrow[l] \arrow[d, "\tilde{\zeta}''"'] \\
 & k_*(\mathrm{A}_2) \arrow[rr, bend right=30, "j_*"'] & k_*(\mathrm{A}_0) \arrow[l, "i_*"'] & k_*(\mathrm{A}_2, \mathrm{A}_0) \arrow[l]
\end{tikzcd}$$

From the left hand square it follows that i_* is onto. Hence, by exactness, $j_*=0$. Thus we obtain a diagram

$$\begin{tikzcd}
0 & \mathbf{Z}[t]\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_2) \arrow[l] \arrow[d, "\tilde{\zeta}'"', "\cong"] & \mathbf{Z}[t]\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_0) \arrow[l] \arrow[d, "\tilde{\zeta}"', "\cong"] & \mathbf{Z}[t]\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{A}_2, \mathrm{A}_0) \arrow[l] \arrow[d, "\tilde{\zeta}''"'] & \\
0 & k_*(\mathrm{A}_2) \arrow[l] & k_*(\mathrm{A}_0) \arrow[l] & k_*(\mathrm{A}_2, \mathrm{A}_0) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

from which it readily follows that

$$\mathbf{Z}[t]\otimes_{\Omega^U_*}\Omega^U_*(A_2, A_0) \to k_*(A_2, A_0)$$

is onto. Thus we obtain a commutative diagram

$$\begin{tikzcd}
0 & \mathbf{Z}[t]\otimes_{\Omega^U_*}\Omega^U_*(A_2, A_0) \arrow[l] \arrow[d, "\text{onto}"] & \mathbf{Z}[t]\otimes_{\Omega^U_*}\Omega^U_*(A_1, A_0) \arrow[l] \arrow[d, "\cong"] & \mathbf{Z}[t]\otimes_{\Omega^U_*}\Omega^U_*(A_2, A_1) \arrow[l] \arrow[d] \\
 & k_*(A_2, A_0) \arrow[rr, bend right, "j'_*"'] & k_*(A_1, A_0) \arrow[l, "i'_*"'] & k_*(A_2, A_1) \arrow[l]
\end{tikzcd}$$

From the left hand square it evidently follows that i'_* is onto and hence j'_* is 0 by exactness. Thus we obtain the exact sequence

$$0 \leftarrow k_*(A_2, A_0) \leftarrow k_*(A_1, A_0) \leftarrow k_*(A_2, A_1) \leftarrow 0.$$

Thus we find that the sequence

$$0 \leftarrow k_*(A_2) \leftarrow k_*(A_0) \leftarrow k_*(A_1, A_0) \leftarrow k_*(A_2, A_1) \leftarrow 0$$

is an exact sequence of $\mathbf{Z}[t]$-modules. Now recall that by construction $H_*(A_0; \mathbf{Z})$ and $H_*(A_1, A_0; \mathbf{Z})$ are free $\mathbf{Z}$-modules. It quickly follows that $k_*(A_0)$ and $k_*(A_1, A_0)$ are free $\mathbf{Z}[t]$-modules. Since the global dimension of $\mathbf{Z}[t]$ is 2 it follows by ([5], IV, (2.1)) that $k_*(A_2, A_1)$ is a projective $\mathbf{Z}[t]$-module and hence by Proposition (3.2) is a free $\mathbf{Z}[t]$-module.

We shall now need the following:

Lemma. — *Let* W *be a finite complex with* $k_*(W)$ *a free* $k_*(=\mathbf{Z}[t])$*-module. Then* $H_*(W; \mathbf{Z})$ *is a free* $\mathbf{Z}$*-module.*

Proof. — Suppose that $H_*(W; \mathbf{Z})$ is not a free $\mathbf{Z}$-module. Let $u \in H_*(W; \mathbf{Z})$ be a torsion element of lowest dimension. Consider the spectral sequence

$$E^r \Rightarrow k_*(W)$$
$$E^2 = H_*(W; \mathbf{Z}[t]) \cong H_*(W; \mathbf{Z}) \otimes_Z \mathbf{Z}[t].$$

Since the differentials of this spectral sequence are torsion valued a simple degree check shows that $u \in E^2_{*,0}$ is an infinite cycle and hence u is in the image of the edge map

$$\eta : k_*(W) \to H_*(W; \mathbf{Z}).$$

Thus there exists a class $\alpha \in \mathbf{Z} \otimes_{\mathbf{Z}[t]} k_*(W)$

with $\tilde{\eta}(\alpha) = u$.

Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{Z}\otimes_{\mathbf{Z}[t]}k_*(W) \arrow[r, "\tilde{\eta}"] \arrow[d, "\xi"] & H_*(W;\mathbf{Z}) \arrow[d, "\theta"] \\
\mathbf{Q}\otimes_{\mathbf{Z}}\otimes\mathbf{Z}\otimes_{\mathbf{Z}[t]}k_*(W) \arrow[r, "\hat{\eta}", "\cong"'] & H_*(W;\mathbf{Q})
\end{tikzcd}$$

Since $k_*(W)$ is a free $\mathbf{Z}[t]$-module it follows that $\mathbf{Z}\otimes_{\mathbf{Z}[t]}k_*(W)$ is a free $\mathbf{Z}$-module. Hence ξ is a monomorphism. We also have

$$\hat{\eta}\xi(\alpha)=\theta\tilde{\eta}(\alpha)=\theta(u)=0.$$

But since $\hat{\eta}$ and ξ are monomorphisms this implies that $\alpha=0$ and hence $u=0$, which is a contradiction. Therefore $H_*(W;\mathbf{Z})$ is torsion free; W being a finite complex, this implies that $H_*(W;\mathbf{Z})$ is a free $\mathbf{Z}$-module. □

Returning to the proof of Theorem (11.2) we find that

$$H_*(A_2, A_1;\mathbf{Z})$$

is a free $\mathbf{Z}$-module. Therefore

$$\Omega_*^U(A_2, A_1)$$

is a free Ω_*^U-module and hence by Proposition (2.1)

$$0\leftarrow\Omega_*^U(A_2)\leftarrow\Omega_*^U(A_0)\leftarrow\Omega_*^U(A_1, A_0)\leftarrow\Omega_*^U(A_2, A_1)\leftarrow 0$$

is a free Ω_*^U-resolution of $\Omega_*^U(A)$. Hence

$$\text{hom.dim}_{\Omega_*^U}\Omega_*^U(A_2)\leq 2.$$

The result now follows by stability. □

As an example of how Theorem (11.2) might be applied consider the complex

$$X=M(\mathbf{Z}_2, k)\cup_{[CP(1)]^n i_k}e^{2n+k+1}$$

constructed in section 7. It is not possible to distinguish the spaces

$$X \quad\text{and}\quad M(\mathbf{Z}_2, k)\vee S^{2n+k+1}$$

by examining their ordinary homology or cohomology; or examining primary operations. However from our discussion in section 7 we find

$$k_i(X)=\begin{cases}\mathbf{Z}_2 & \text{generated by } a \text{ for } i=k\\ \mathbf{Z} & \text{generated by } b \text{ for } i=2n+k+1\\ 0 & \text{otherwise}\end{cases}$$

and since $$\mathrm{Td}([\mathbf{CP}(1)])=t$$
we find $$t^n a=0$$

Thus $k_*(\cdot)$ distinguishes X from $M(\mathbf{Z}_2, k)\vee S^{2n+k+1}$.

Theorem **(11.3)**. — *Let* X *be a finite complex and suppose that* $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)\leq 3$. *Then there is a natural exact sequence*

$$0\to\mathbf{Z}[t]\otimes_{\Omega_*^U}\Omega_*^U(X)\to k_*(X)\to\mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}[t],\ \Omega_*^U(X))\to 0.$$

Proof. — From Proposition (11.1) we find

$$\mathrm{Tor}_{j,*}^{\Omega_*^U}(\mathbf{Z}[t],\ \Omega_*^U(X))=0 \quad \text{for } j>1.$$

Thus the spectral sequence of Theorem (10.2) collapses to the required exact sequence. □

Finally we may combine Theorem (4.4) with Proposition (11.1) and Theorem (11.2) to obtain:

Theorem **(11.4)**. — *Let* X *be a finite complex and suppose that* $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)\leq 2$. *Then there is a natural exact sequence*

$$0\to\mathbf{Z}\otimes_{\Omega_*^U}\Omega_*^U(X)\xrightarrow{\tilde{\mu}}H_*(X;\mathbf{Z})\xrightarrow{\triangle}k_*(X)\xrightarrow{m_t}k_*(X)\xrightarrow{\square}\mathbf{Z}\otimes_{\Omega_*^U}\Omega_*^U(X)\to 0$$

where $\tilde{\mu}$ *is the reduced Thom map,* Δ *has degree* -3, m_t *is multiplication by* $t\in k_2$ *and* □ *has degree* 0.

Proof. — By Theorem (4.4) we have an exact sequence

$$0\to\mathbf{Z}\otimes_{\Omega_*^U}\Omega_*^U(X)\to H_*(X;\mathbf{Z})\to\mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z},\ \Omega_*^U(X)\to 0.$$

From the exact sequence of graded Ω_*^U-modules

$$0\to\mathbf{Z}[t]\xrightarrow[m_t]{}\mathbf{Z}[t]\to\mathbf{Z}\to 0$$

we obtain from Proposition (11.1) the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z},\ \Omega_*^U(X)) \arrow[r, "\triangle"] & \mathbf{Z}[t]\otimes_{\Omega_*^U}\Omega_*^U(X) \arrow[dll, "m_t"'] & \\
\mathbf{Z}[t]\otimes_{\Omega_*^U}\Omega_*^U(X) \arrow[r] & \mathbf{Z}\otimes_{\Omega_*^U}\Omega_*^U(X) \arrow[r] & 0.
\end{tikzcd}$$

From Theorem (11.2) we obtain the natural isomorphism

$$\mathbf{Z}[t]\otimes_{\Omega_*^U}\Omega_*^U(X)\to k_*(X).$$

Splicing the above exact sequences now yields the result. □

We hope that the results of this section provide further understanding of the numerical invariant $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(X)$.

In view of our previous work it should not be suprising that the spectral sequence of Theorem (10.2) is in general non-trivial. We postpone a discussion of this point to another occassion.

§ 12. More on Attaching Cells.

The results of the previous section may be applied to the study of the variation in the numerical invariant $\operatorname{hom.dim}_{\Omega^U_*}\Omega^U_*(\cdot)$ under cell attachment.

It will be recalled (5.11) that if X is a finite complex with $\operatorname{hom.dim}_{\Omega^U_*}\Omega^U_*(X)=2$ and $\gamma\in\Omega^U_*(X)$ has prime order p, then $A(\gamma)=(p)$ or (p, σ), where $p, \sigma\in\Omega^U_*$ is an ESP-sequence of length 2. If γ is also a spherical bordism class represented by $f: S^{n-1}\to X$ then it follows (from (5.12)) that $\operatorname{hom.dim}_{\Omega^U_*}\Omega^U_*(X\cup_f e^n)\leq 3$. We shall present a necessary and sufficient condition that this dimension equal 3.

Definition. — Let Ω *be a graded algebra and* M *an* Ω*-module. If* $\gamma\in M$ *and* $I\subset\Omega$ *is an ideal denote by* $A(\gamma; I)$ *the ideal of* Ω *given by*

$$A(\gamma, I)=\{\alpha\in\Omega \mid \alpha.\gamma\in I.M\}.$$

Note that $A(\gamma, (0))=A(\gamma)$. Note that $A(\gamma; I)\supset A(\gamma), I$ and hence the ideal generated by both.

Notation. — Let $\zeta:\Omega^U_*\to k_*=\mathbf{Z}[t]$ be the natural transformation of the previous section. Denote by $I(t)\subset\Omega^U_*$ the kernel of ζ.

Note that for any Ω^U_*-module M

$$\mathbf{Z}[t]\otimes_{\Omega^U_*}M=M/I(t).M$$

as may be readily checked.

Theorem (**12.1**). — *Let* X *be a finite complex with* $\operatorname{hom.dim}_{\Omega^U_*}\Omega^U_*(X)=2$. *Suppose that* $\gamma\in\Omega^U_*(X)$ *is a spherical bordism element of prime order represented by* $f: S^{n-1}\to X$. *Then*

$$\operatorname{hom.dim}_{\Omega^U_*}\Omega^U_*(X\cup_f e^n)=3 \quad \textit{iff} \quad A(\gamma, I(t))/(A(\gamma), I(t))\neq 0.$$

Proof. — As usual we form the cofibration sequence

$$S^{n-1}\xrightarrow{f}X\to Y=X\cup_f e^n$$

from which we obtain the exact triangle

$$\begin{tikzcd}
\tilde{\Omega}^U_*(S^{n-1}) \arrow[rr, "f_*"'] & & \tilde{\Omega}^U_*(X) \arrow[dl] \\
& \tilde{\Omega}^U_*(Y) \arrow[ul] &
\end{tikzcd}$$

We thus obtain exact sequences

$$0\to\Sigma^{n-1}A(\gamma)\to\tilde{\Omega}^U_*(S^{n-1})\to M(\gamma)\to 0$$

$$0\to M(\gamma)\to\tilde{\Omega}^U_*(X)\to C(\gamma)\to 0$$

$$0\to C(\gamma)\to\tilde{\Omega}^U_*(Y)\to\Sigma^{n-1}A(\gamma)\to 0$$

where

$$M(\gamma)=\Omega^U_*.\gamma\subset\tilde{\Omega}^U_*(X)$$

$$C(\gamma)=\tilde{\Omega}^U_*(X)/M(\gamma).$$

Now we contend that

$$\mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}[t], \Sigma^{n-1}\mathrm{A}(\gamma)) = 0.$$

For suppose the contrary. Then the first exact sequence yields

$$\mathrm{Tor}_{2,*}^{\Omega_*^U}(\mathbf{Z}[t], \mathrm{M}(\gamma)) \cong \mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}[t], \Sigma^{n-1}\mathrm{A}(\gamma)) \neq 0.$$

By (11.1)

$$\mathrm{Tor}_{j,*}^{\Omega_*^U}(\mathbf{Z}[t], \tilde{\Omega}_*^U(\mathrm{X})) = 0$$

for $j \geq 1$.

Thus the middle exact sequence yields

$$\mathrm{Tor}_{3,*}^{\Omega_*^U}(\mathbf{Z}[t], \mathrm{C}(\gamma)) \neq 0.$$

By Corollary (5.12) $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(\mathrm{Y}) \leq 3$. Hence by Proposition (10.5)

$$\mathrm{Tor}_{j,*}^{\Omega_*^U}(\mathbf{Z}[t], \tilde{\Omega}_*^U(\mathrm{Y})) = 0$$

for $j \geq 2$. Thus the last exact sequence yields

$$\mathrm{Tor}_{4,*}^{\Omega_*^U}(\mathbf{Z}[t], \Sigma^{n-1}\mathrm{A}(\gamma)) \cong \mathrm{Tor}_{3,*}^{\Omega_*^U}(\mathbf{Z}[t], \mathrm{C}(\gamma)) \neq 0.$$

However this is contrary to the fact that (5.11) $\mathrm{hom.dim}_{\Omega_*^U}\mathrm{A}(\gamma) \leq 1$. Therefore our original supposition must be false and hence

$$\mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}[t], \Sigma^{n-1}\mathrm{A}(\gamma)) = 0$$

Since $\mathrm{hom.dim}_{\Omega_*^U}\Sigma^{n-1}\mathrm{A}(\gamma) \leq 1$ it follows that

$$\mathrm{Tor}_{j,*}^{\Omega_*^U}(\mathbf{Z}[t], \Sigma^{n-1}\mathrm{A}(\gamma)) = 0$$

for $j \geq 2$. Combining these two facts with the last exact sequence above we find an isomorphism

$$\mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}[t], \mathrm{C}(\gamma)) = \mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}[t], \tilde{\Omega}_*^U(\mathrm{Y})).$$

By Proposition (11.1) $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(\mathrm{Y}) = 3$ iff

$$\mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}[t], \Omega_*^U(\mathrm{Y})) \neq 0.$$

Combined with the previous equality this yields; $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(\mathrm{Y}) = 3$ iff

$$\mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}[t], \mathrm{C}(\gamma)) \neq 0.$$

Consider now the middle exact sequence above. It yields, in view of Proposition (11.1), the exact sequence

$$0 \to \mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}[t], \mathrm{C}(\gamma)) \to \mathbf{Z}[t] \otimes_{\Omega_*^U} \mathrm{M}(\gamma) \to \mathbf{Z}[t] \otimes_{\Omega_*^U} \tilde{\Omega}_*^U(\mathrm{X}) \to \mathbf{Z}[t] \otimes_{\Omega_*^U} \mathrm{C}(\gamma) \to 0.$$

Thus we find that $\mathrm{hom.dim}_{\Omega_*^U}\Omega_*^U(\mathrm{Y}) = 3$ iff the map

$$\mathbf{Z}[t] \otimes_{\Omega_*^U} \mathrm{M}(\gamma) \to \mathbf{Z}[t] \otimes_{\Omega_*^U} \tilde{\Omega}_*^U(\mathrm{X})$$

has a non-trivial kernel. Note that

$$\mathbf{Z}[t] \otimes_{\Omega_*^U} \mathrm{M}(\gamma) = \Sigma^{n-1}[\Omega_*^U/(\mathrm{A}(\gamma), \mathrm{I}(t))].$$

An element $\alpha\in\Sigma^{n-1}[\Omega_*^{\mathrm{U}}/(\mathrm{A}(\gamma), \mathrm{I}(t))]$ with $\alpha=0\in\mathbf{Z}[t]\otimes_{\Omega_*^{\mathrm{U}}}\tilde{\Omega}_*^{\mathrm{U}}(\mathrm{X})$ means

$$\alpha.\gamma=0\in\tilde{\Omega}_*^{\mathrm{U}}(\mathrm{X})/\mathrm{I}(t).\tilde{\Omega}_*^{\mathrm{U}}(\mathrm{X})$$

i.e. $\alpha\in\mathrm{A}(\gamma, \mathrm{I}(t))$. Thus we find

$$\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{Y})=3 \quad \text{iff} \quad \mathrm{A}(\gamma, \mathrm{I}(t))/(\mathrm{A}(\gamma), \mathrm{I}(t))\neq 0$$

as required. □

This result may be rephrased in terms of $k_*(\ )$ theory in the following manner.

Corollary (**12.2**). — *Let* X *be a finite complex with* $\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{X})=2$. *Suppose that* $\gamma\in\Omega_*^{\mathrm{U}}(\mathrm{X})$ *is a spherical bordism element of prime order represented by* $f: \mathrm{S}^{n-1}\to\mathrm{X}$. *Let* $\mathrm{A}(\zeta\gamma)\subset\mathbf{Z}[t]$ *denote the annihilator ideal of* $\zeta\gamma\in k_*(\mathrm{X})$. *Then* $\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{X}\cup_f e^n)=3$ *iff* $\mathrm{A}(\zeta\gamma)/\zeta\mathrm{A}(\gamma)\neq 0$.

Proof. — Clearly $\zeta: \Omega_*^{\mathrm{U}}\to k_*=\mathbf{Z}[t]$ maps $\mathrm{A}(\gamma)$ into $\mathrm{A}(\zeta\gamma)$. By Theorem (12.1) we obtain $\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{X}\cup_f e^n)=3$ iff $\mathrm{A}(\gamma, \mathrm{I}(t))/(\mathrm{A}(\gamma), \mathrm{I}(t))\neq 0$. But clearly ζ induces an isomorphism

$$\zeta: \ \frac{\mathrm{A}(\gamma, \mathrm{I}(t))}{(\mathrm{A}(\gamma), \mathrm{I}(t))}\xrightarrow{\cong}\frac{\mathrm{A}(\zeta\gamma)}{\zeta\mathrm{A}(\gamma)}$$

and the result follows. □

Of course if $\zeta\gamma=0$ then $\mathrm{A}(\gamma; \mathrm{I}(t))=\Omega_*^{\mathrm{U}}$ and hence $1\neq 0\in\mathrm{A}(\gamma, \mathrm{I}(t))/(\mathrm{A}(\gamma), \mathrm{I}(t))$, and thus $\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{Y})=3$. A somewhat suprising result.

As an example of these phenomena let X be a large skeleton of $\mathbf{R}\mathrm{P}(\infty)\times\mathbf{R}\mathrm{P}(\infty)$. Employing the notation of section 6, we have

$$\gamma_3\otimes\gamma_3\in\Omega_6^{\mathrm{U}}(\mathrm{X})$$

and

$$\mathrm{A}(\gamma_3\otimes\gamma_3)=(2, [\mathbf{C}\mathrm{P}(1)]^2).$$

Next note that by Lemma (6.5)

$$[\mathbf{C}\mathrm{P}(1)](\gamma_3\otimes\gamma_3)=\lambda(\gamma_1\otimes\gamma_1).$$

Since

$$[\mathbf{C}\mathrm{P}(1)](\gamma_1\otimes\gamma_1)=0$$

we may replace λ by $\lambda-\mathrm{Td}(\lambda)[\mathbf{C}\mathrm{P}(1)]^2$. Thus we may assume $\mathrm{Td}(\lambda)=0$, i.e., $\lambda\in\mathrm{I}(t)$. Hence

$$[\mathbf{C}\mathrm{P}(1)]\neq 0\in\mathrm{A}(\gamma_3\otimes\gamma_3; \mathrm{I}(t))/(\mathrm{A}(\gamma), \mathrm{I}(t))$$

and therefore suspending X to $\Sigma^n\mathrm{X}$ and attaching a cell (recall $\gamma_3\otimes\gamma_3$ is a framed class) by $\Sigma^n\gamma_3\otimes\gamma_3$ we obtain a complex $\mathrm{Y}=\Sigma^n\mathrm{X}\cup_{\Sigma^n\gamma_3\otimes\gamma_3}e^{n+7}$ with $\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{Y})=3$. This would seem to be the simplest example of a complex for which $\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\ )=3$.

Before continuing with our study of the transition from homological dimension 2 to homological dimension 3 we introduce an exact sequence associated with cell attachment. This exact sequence collects many of the things that we have be concerned with so far. Taken in conjunction with some further results we will find that

$$\tilde{\mu} : \quad \mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\Sigma^n \mathrm{X}\cup_{\Sigma^n\gamma_3\otimes\gamma_3} e^{n+7}) \to \mathrm{H}_*(\Sigma^n \mathrm{X}\cup_{\Sigma^n\gamma_3\otimes\gamma_3} e^{n+7};\mathbf{Z})$$

has a non-trivial kernel, thus simplifying our previous example of this phenomena.

Definition. — Let W *be a finite complex. Define* $\mathscr{K}_*(\mathrm{W})$ *and* $\mathscr{C}_*(\mathrm{W})$ *by*

$$\mathscr{K}_j(\mathrm{W})=\ker\{\tilde{\mu} : [\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{W})]_j \to \mathrm{H}_j(\mathrm{W};\mathbf{Z})\}$$
$$\mathscr{C}_j(\mathrm{W})=\operatorname{coker}\{\tilde{\mu} : [\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\Omega^{\mathrm{U}}_*(\mathrm{W})]_j \to \mathrm{H}_j(\mathrm{W};\mathbf{Z})\}.$$

Theorem **(12.3)**. — *Let* X *be a finite complex and* $\gamma\in\Omega^{\mathrm{U}}_*(\mathrm{X})$ *a spherical bordism class represented by* $f : \mathrm{S}^{n-1}\to\mathrm{X}$. *Then there is a natural exact sequence*

$$\mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{1,m+n}(\mathbf{Z}, \Omega^{\mathrm{U}}_*(\mathrm{X}\cup_f e^n)) \to \mathrm{Tor}^{\Omega^{\mathrm{U}}_*}_{1,m}(\mathbf{Z}, \mathrm{A}(\gamma)) \xrightarrow{\tilde{\Delta}} \mathscr{K}_{m+n}(\mathrm{X})$$
$$\xrightarrow{\varphi} \mathscr{K}_{m+n}(\mathrm{X}\cup_f e^n) \xrightarrow{\psi} [\mathbf{Z}\otimes_{\Omega^{\mathrm{U}}_*}\mathrm{A}(\gamma)]_m \xrightarrow{\theta} \mathscr{C}_{m+n}(\mathrm{X}) \xrightarrow{\hat{i}_*} \mathscr{C}_{m+n}(\mathrm{X}\cup_f e^n) \to 0$$

for each positive integer m.

Proof. — The proof is a straightforward, although tedious procedure. As usual introduce the cofibration

$$\mathrm{S}^{n-1}\xrightarrow{f}\mathrm{X}\to\mathrm{Y}=\mathrm{X}\cup_f e^n$$

We thus obtain the exact triangle

$$\begin{tikzcd}
\tilde{\Omega}^{\mathrm{U}}_*(\mathrm{S}^{n-1}) \arrow[rr, "f_*"'] & & \tilde{\Omega}^{\mathrm{U}}_*(\mathrm{X}) \arrow[dl] \\
& \tilde{\Omega}^{\mathrm{U}}_*(\mathrm{Y}) \arrow[ul] &
\end{tikzcd}$$

and the associated exact sequences

$$0\to\Sigma^{n-1}\mathrm{A}(\gamma)\to\tilde{\Omega}^{\mathrm{U}}_*(\mathrm{S}^{n-1})\to\mathrm{M}(\gamma)\to 0$$
$$0\to\mathrm{M}(\gamma)\to\tilde{\Omega}^{\mathrm{U}}_*(\mathrm{X})\to\mathrm{C}(\gamma)\to 0$$
$$0\to\mathrm{C}(\gamma)\to\tilde{\Omega}^{\mathrm{U}}_*(\mathrm{Y})\to\Sigma^{n-1}\mathrm{A}(\gamma)\to 0$$

From the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_{m+n}(\mathrm{S}^{n-1};\mathbf{Z}) \arrow[r] \arrow[d, equal] & \mathrm{H}_{m+n}(\mathrm{X};\mathbf{Z}) \arrow[r] \arrow[dd] & \mathrm{H}_{m+n}(\mathrm{Y};\mathbf{Z}) \arrow[r] \arrow[dd] & \mathrm{H}_{m+n-1}(\mathrm{S}^{n-1};\mathbf{Z}) \arrow[d, equal] \\
0 & & & 0 \\
& \mathscr{C}_{m+n}(\mathrm{X}) \arrow[r, "i_*"] \arrow[d] & \mathscr{C}_{m+n}(\mathrm{Y}) \arrow[d] & \\
& 0 & 0 &
\end{tikzcd}$$

for $m>0$, we find that $\mathscr{C}_{m+1}(\mathrm{X}) \to \mathscr{C}_{m+1}(\mathrm{Y})$ is onto.

Let us next define

$$[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{A}(\gamma)]_m \to \mathscr{C}_{m+n}(\mathrm{X}).$$

As before supposing m a positive integer we may introduce a diagram with exact rows

$$\begin{tikzcd}
 & \mathscr{K}_{m+n}(\mathrm{X}) \arrow[r, dashed, "\varphi"] \arrow[d, "k_{\mathrm{X}} = \text{kernel } \tilde{\mu}_{\mathrm{X}}"'] & \mathscr{K}_{m+n}(\mathrm{Y}) \arrow[d, "k_{\mathrm{Y}} = \text{kernel } \tilde{\mu}_{\mathrm{Y}}"] \arrow[dr, "\psi = \tilde{c}_* k_{\mathrm{Y}}"] & & \\
 & {[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{X})]_{m+n}} \arrow[r, "\tilde{j}_*"] \arrow[d, "\tilde{\mu}_{\mathrm{X}}"'] & {[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{Y})]_{m+n}} \arrow[r, "\tilde{c}_*"] \arrow[d, "\tilde{\mu}_{\mathrm{Y}}"] & {[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{A}(\gamma)]_m} \arrow[r] \arrow[ddll, dashed, bend left=60, "\theta = c_{\mathrm{X}} i_*^{-1} \tilde{\mu}_{\mathrm{Y}} \tilde{c}_*^{-1}"] & 0 \\
0 = \mathrm{H}_{m+n}(\mathrm{S}^{n-1};\mathbf{Z}) \arrow[r] & \mathrm{H}_{m+n}(\mathrm{X};\mathbf{Z}) \arrow[r, "i_*"] \arrow[d, "c_{\mathrm{X}} = \text{cokernel } \tilde{\mu}_{\mathrm{X}}"'] & \mathrm{H}_{m+n}(\mathrm{Y};\mathbf{Z}) \arrow[r] \arrow[d, "c_{\mathrm{Y}} = \text{cokernel } \tilde{\mu}_{\mathrm{Y}}"] & \mathrm{H}_{m+n-1}(\mathrm{S}^{n-1};\mathbf{Z}) = 0 & \\
 & \mathscr{C}_{m+n}(\mathrm{X}) \arrow[r, "\hat{i}_*"] & \mathscr{C}_{m+n}(\mathrm{Y}) \arrow[r] & 0 &
\end{tikzcd}$$

defining the morphisms φ, ψ and θ and yielding an exact sequence

$$\mathscr{K}_{m+n}(\mathrm{X}) \xrightarrow{\varphi} \mathscr{K}_{m+n}(\mathrm{Y}) \xrightarrow{\psi} [\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{A}(\gamma)]_m \xrightarrow{\theta} \mathscr{C}_{m+n}(\mathrm{X}) \xrightarrow{\hat{i}_*} \mathscr{C}_{m+n}(\mathrm{Y}) \to 0$$

Next we note that

$$[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{M}(\gamma)]_j = 0$$

for $j \geq n$, as $\mathrm{M}(\gamma)$ is a cyclic Ω_*^{U}-module with generator γ of dimension $n-1$. Thus we find

$$[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{X})]_{m+n} \xrightarrow{\cong} [\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{C}(\gamma)]_{m+n}.$$

Thus we obtain an exact sequence

$$\begin{tikzcd}
\mathrm{Tor}_{1,m+n}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}, \Omega_*^{\mathrm{U}}(\mathrm{Y})) \arrow[r] & \mathrm{Tor}_{1,m}^{\Omega_*^{\mathrm{U}}}(\mathbf{Z}, \mathrm{A}(\gamma)) \arrow[dl] & & \\
{[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{C}(\gamma)]_{m+n}} \arrow[r] \arrow[d, phantom, "\wr\|"] & {[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{Y})]_{m+n}} \arrow[r] & {[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{A}(\gamma)]_m} \arrow[r] & 0 \\
{[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{X})]_{m+n}.} & & &
\end{tikzcd}$$

Consider now the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{K}_{m+n}(\mathrm{X}) \arrow[r, "\varphi"] \arrow[d, "k_\mathrm{X}"] & \mathcal{K}_{m+n}(\mathrm{Y}) \arrow[r, "\psi"] \arrow[d, "k_\mathrm{Y}"] & {[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{A}(\gamma)]_m} \arrow[d, dotted, equal] & \\
\mathrm{Tor}^{\Omega_*^{\mathrm{U}}}_{1,m+n}(\mathbf{Z}, \mathrm{A}(\gamma)) \arrow[ur, dashed, "\tilde{\Delta}"] \arrow[r, "\Delta"'] & {[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{X})]_{m+n}} \arrow[r, "\tilde{j}_*"] \arrow[d, "\tilde{\mu}_\mathrm{X}"] & {[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{Y})]_{m+n}} \arrow[r, "\tilde{\partial}_*"] \arrow[d, "\tilde{\mu}_\mathrm{Y}"] & {[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{A}(\gamma)]_m} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{H}_{m+n}(\mathrm{X}; \mathbf{Z}) \arrow[r, "i_*"] & \mathrm{H}_{m+n}(\mathrm{Y}; \mathbf{Z}) \arrow[r] & 0 &
\end{tikzcd}$$

Observe that

$$i_*\mu_\mathrm{X}\Delta = \mu_\mathrm{Y}\tilde{j}_*\Delta = 0$$

and since i_* is monic

$$\mu_\mathrm{X}\Delta = 0.$$

Thus $\tilde{\Delta}$ exists and splicing leads to the exact sequence

$$\begin{tikzcd}[column sep=small]
\mathrm{Tor}^{\Omega_*^{\mathrm{U}}}_{1,m+n}(\mathbf{Z}, \Omega_*^{\mathrm{U}}(\mathrm{Y})) \arrow[r] & \mathrm{Tor}^{\Omega_*^{\mathrm{U}}}_{1,m}(\mathbf{Z}, \mathrm{A}(\gamma)) \arrow[r, "\tilde{\Delta}"] & \mathcal{K}_{m+n}(\mathrm{X}) \arrow[dll, "\varphi"] & & \\
\mathcal{K}_{m+n}(\mathrm{Y}) \arrow[r, "\psi"] & {[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{A}(\gamma)]_m} \arrow[r, "\theta"] & \mathcal{C}_{m+n}(\mathrm{X}) \arrow[r, "\hat{i}_*"] & \mathcal{C}_{m+n}(\mathrm{Y}) \arrow[r] & 0,
\end{tikzcd}$$

as required. ☐

The maps $\tilde{\Delta}$ and θ may be described in a more geometric fashion as follows.

Suppose that $[\mathrm{M}^{2t}] \in \mathrm{A}(\gamma)$. By the cellular approximation theorem we may assume that γ is represented by

$$f : \mathrm{S}^{n-1} \to \mathrm{X}^{n-1}$$

where $\mathrm{X}^{n-1} \subset \mathrm{X}$ denotes the $(n-1)$-skeleton. Then since $[\mathrm{M}^{2t}] \in \mathrm{A}(\gamma)$ we may find

$$\mathrm{F} : \mathrm{W} \to \mathrm{X}$$

such that

$$\mathrm{F} \mid \partial\mathrm{W} = f.p : \quad \mathrm{M}^{2t} \times \mathrm{S}^{n-1} \to \mathrm{S}^{n-1} \to \mathrm{X}^{n-1} \hookrightarrow \mathrm{X}$$

where p is the projection. Thus we have a map of pairs

$$\mathrm{F} : (\mathrm{W}, \partial\mathrm{W}) \to (\mathrm{X}, \mathrm{X}^{n-1})$$

and hence the homology class

$$\mathrm{F}_*[\mathrm{W}, \partial\mathrm{W}] \in \mathrm{H}_{2t+n}(\mathrm{X}, \mathrm{X}^{n-1}; \mathbf{Z}).$$

By assumption $t>0$ and hence

$$H_{2t+n}(X; \mathbf{Z}) \to H_{2t+n}(X, X^{n-1}; \mathbf{Z})$$

is an isomorphism, yielding the class

$$F_*[W, \partial W] \in H_{2t+n}(X; \mathbf{Z}).$$

One readily checks that

$$F_*[W, \partial W] \in \mathscr{C}_{2t+n}(X)$$

is independent of the choice of W and the map f, and we have thus defined

$$A(\gamma)_m \to \mathscr{C}_{m+n}(X).$$

One readily checks that there is induced a map

$$[\mathbf{Z} \otimes_{\Omega_*^U} A(\gamma)]_m \to \mathscr{C}_{m+n}(X)$$

that coincides with θ.

To describe $\widetilde{\Delta}$, consider the exact sequence

$$0 \to \mathrm{Tor}_{1,*}^{\Omega_*^U}(\mathbf{Z}, A(\gamma)) \to \widetilde{\Omega}_*^U \otimes_{\Omega_*^U} A(\gamma) \to A(\gamma) \to \mathbf{Z} \otimes_{\Omega_*^U} A(\gamma) \to 0$$

obtained from $\quad 0 \to \widetilde{\Omega}_*^U \to \Omega_*^U \to \mathbf{Z} \to 0.$

Thus an element $\quad \alpha \in \mathrm{Tor}_{1,m}^{\Omega_*^U}(\mathbf{Z}, A(\gamma))$

may be written as $\quad \alpha = \sum_i \lambda_i \otimes \alpha_i$

where $\quad \alpha_i \in A(\gamma), \ \lambda_i \in \widetilde{\Omega}_*^U \quad$ and $\quad \sum_i \lambda_i \alpha_i = 0 \in A(\gamma)$

and of course $\deg \lambda_i + \deg \alpha_i = m$. Denote by $\sigma_{n-1} \in \Omega_{n-1}^U(S^{n-1})$ the generator. Choose classes $x_i \in \Omega_*^U(Y)$ such that

$$\partial_*: \quad x_i \mapsto \alpha_i \sigma_{n-1}.$$

Then we find

$$\partial_*(\sum_i \lambda_i x_i) = \sum_i \lambda_i \alpha_i \sigma_{n-1} = 0$$

since $\sum_i \lambda_i \alpha_i = 0$. Therefore there exists $y \in \Omega_*^U(X)$ with

$$j_*: \quad y \mapsto \sum_i \lambda_i x_i$$

From the commutative diagram

$$\begin{tikzcd}
\Omega_{m+n}^U(X) \arrow[r, "j_*"] \arrow[d, "\mu_X"] & \Omega_{m+n}^U(Y) \arrow[d, "\mu_Y"] \\
H_{m+n}(X; \mathbf{Z}) \arrow[r, "i_*", "\cong"'] & H_{m+n}(Y; \mathbf{Z})
\end{tikzcd}$$

we find $\mu_X(y)=0$, for $j_*(y)$ being decomposable, $\mu_Y j_*(y)=0$ and hence $i_*\mu_X(y)=0$, but i_* is isomorphic. Thus we have defined

$$\mathrm{Tor}^{\Omega^U_*}_{1,m}(\mathbf{Z}, A(\gamma)) \to \mathscr{K}_{m+n}(X)$$

and one readily checks that this describes $\widetilde{\Delta}$ directly.

We may of course apply a similar analysis to the natural transformation

$$k_*(\ ) \to H_*(\ ;\mathbf{Z})$$

and obtain an exact sequence of a similar sort. More precisely let $f: S^{n-1} \to X$ be a continuous map and $Y = X \cup_f e^n$. Denote by $A(\zeta\gamma) \subset \mathbf{Z}[t]$ the annihilator ideal of the spherical k_*-element represented by $f: S^{n-1} \to X$.

For any complex W introduce

$$\overline{\mathscr{C}}_j(W) = \mathrm{coker}\,\{\widetilde{\eta}: \quad \mathbf{Z}\otimes_{\mathbf{Z}[t]} k_*(W) \to H_*(W;\mathbf{Z})\}$$
$$\overline{\mathscr{K}}_j(W) = \ker\,\{\widetilde{\eta}: \quad \mathbf{Z}\otimes_{\mathbf{Z}[t]} k_*(W) \to H_*(W;\mathbf{Z})\}.$$

We may then construct an exact sequence

$$\mathrm{Tor}^{\mathbf{Z}[t]}_{1,m+n}(\mathbf{Z}, k_*(Y)) \to \mathrm{Tor}^{\mathbf{Z}[t]}_{1,m}(\mathbf{Z}, A(\zeta(\gamma))) \to \overline{\mathscr{K}}_{m+n}(X) \to$$
$$\to \mathscr{K}_{m+n}(Y) \to [\mathbf{Z}\otimes_{\mathbf{Z}[t]} A(\zeta(\gamma))]_m \to \overline{\mathscr{C}}_{m+n}(X) \to \overline{\mathscr{C}}_{m+n}(Y) \to 0$$

and a morphism of the exact sequence for Ω^U_* into that for k_*.

Clearly the same analysis may be applied to

$$\widetilde{\Omega}^U_* \to \widetilde{H}_*(\ )$$

however we shall not need this here.

We are now ready to return to the study of attaching cells.

Lemma (**12.4**). — *Let* X *be a finite complex with* $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X) = 2$. *Suppose that* $\gamma \in \Omega^U_*(X)$ *is a spherical bordism element of prime order represented by* $f: S^{n-1} \to X$. *Then if* $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X \cup_f e^n) = 3$ *then* $A(\gamma) = (p, \lambda)$ *where* $p, \lambda \in \Omega^U_*$ *is an* ESP-*sequence of length* 2.

Proof. — By (5.11) $A(\gamma) = (p)$ or (p, λ). Let us suppose that $A(\gamma) = (p)$. As usual we have the exact sequences (where $Y = X \cup_f e^n$)

$$0 \to \Sigma^{n-1}A(\gamma) \to \widetilde{\Omega}^U_*(S^{n-1}) \to M(\gamma) \to 0$$
$$0 \to M(\gamma) \to \widetilde{\Omega}^U_*(X) \to C(\gamma) \to 0$$
$$0 \to C(\gamma) \to \widetilde{\Omega}^U_*(Y) \to \Sigma^{n-1}A(\gamma) \to 0.$$

By our assumption $\Sigma^{n-1}A(\gamma)$ is a free Ω^U_*-module and hence the last sequence splits, yielding

$$\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(Y) = \mathrm{hom.dim}_{\Omega^U_*}C(\gamma).$$

From the freeness of $A(\gamma)$ and the first exact sequence we find

$$\mathrm{hom.dim}_{\Omega^U_*}M(\gamma) = 1.$$

Let N be any Ω^U_*-module. From the middle exact sequence we obtain

$$\mathrm{Ext}^{3,*}_{\Omega^U_*}(\Omega^U_*(X), N) \leftarrow \mathrm{Ext}^{3,*}_{\Omega^U_*}(C(\gamma), N) \leftarrow \mathrm{Ext}^{2,*}_{\Omega^U_*}(M(\gamma), N).$$

The two end groups vanish by ([5], VI, (2.16)) and hence

$$\mathrm{hom.dim}_{\Omega^U_*} C(\gamma) = 2.$$

Therefore if $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(Y)=3$ we must have $A(\gamma)=(p, \lambda)$ where $p, \lambda \in \Omega^U_*$ is an ESP-sequence of length 2. □

We are now ready to combine (1.2) and (12.3) to study the transition from homological dimension 2 to homological dimension 3.

Theorem (**12.5**). — *Let* X *be a finite complex with* $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X)=2$. *Suppose that* $\gamma \in \Omega^U_*(X)$ *is a spherical bordism element of prime order represented by* $f: S^{n-1} \rightarrow X$. *Then* $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X \cup_f e^n)=3$ *iff the reduced Thom homomorphism*

$$\tilde{\mu}: \quad \mathbf{Z} \otimes_{\Omega^U_*} \Omega^U_*(X \cup_f e^n) \rightarrow H_*(X \cup_f e^n; \mathbf{Z})$$

has a non-trivial kernel.

Proof. — By Corollary (5.12) $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X \cup_f e^n) \leq 3$. Clearly if

$$\tilde{\mu}: \quad \mathbf{Z} \otimes_{\Omega^U_*} \Omega^U_*(X \cup_f e^n) \rightarrow H_*(X \cup_f e^n; \mathbf{Z})$$

has a non-trivial kernel then by (4.4) we have $\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(X \cup_f e^n)=3$.

Conversely, let $Y = X \cup_f e^n$ and suppose that

$$\mathrm{hom.dim}_{\Omega^U_*}\Omega^U_*(Y)=3.$$

By Lemma (11.4) we have $A(\gamma)=(p, \lambda)$, where $p, \lambda \in \Omega^U_*$ is an ESP-sequence of length 2. Since the global dimension of $\mathbf{Z}[t]$ is 2 one finds that $A(\zeta(\gamma))=(p, at^n) \subset \mathbf{Z}[t]$. By Corollary (12.2) we have

$$\zeta A(\gamma) \subsetneq A(\zeta(\gamma)).$$

Therefore we must have $\zeta(\lambda)=t\alpha$ for some $\alpha \in A(\zeta\gamma)$. Hence the homomorphism

$$\bar{\zeta}: \quad [\mathbf{Z} \otimes_{\Omega^U_*} A(\gamma)]_m \rightarrow [\mathbf{Z} \otimes_{\mathbf{Z}[t]} A(\zeta\gamma)]_m$$

is the trivial (i.e., zero) homomorphism for $m>0$.

By Theorem (12.3) we obtain an exact sequence

$$0 = \mathcal{H}_{m+n}(X) \rightarrow \mathcal{H}_{m+n}(Y) \rightarrow [\mathbf{Z} \otimes_{\Omega^U_*} A(\gamma)]_m \xrightarrow{\theta} \mathcal{C}_{m+n}(X)$$

where $\mathcal{H}_{m+n}(X)=0$ by (4.4).

As noted above we have a commutative diagram

$$\begin{tikzcd}
{[\mathbf{Z} \otimes_{\Omega^U_*} A(\gamma)]_m} \arrow[r, "\bar{\zeta}"] \arrow[d, "\theta"] & {[\mathbf{Z} \otimes_{\mathbf{Z}[t]} A(\zeta(\gamma))]_m} \arrow[d, "\bar{\theta}"] \\
\mathcal{C}_{m+n}(X) \arrow[r, "\bar{\zeta}"] & \bar{\mathcal{C}}_{m+n}(X)
\end{tikzcd}$$

It follows from Theorem (11.2) that the bottom $\overline{\zeta}$ is isomorphic. Since the top $\overline{\zeta}$ is the zero map for positive m we must have that

$$[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{A}(\gamma)]_m \xrightarrow{0} \mathscr{C}_{m+n}(\mathrm{X})$$

is the zero map for $m>0$. The exactness of the sequence

$$0=\mathscr{K}_{m+n}(\mathrm{X})\to\mathscr{K}_{m+n}(\mathrm{Y})\to[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{A}(\gamma)]_m\to 0$$

together with $$[1\otimes\lambda]\neq 0\in\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{A}(\gamma)$$

($\deg\lambda>0$) now completes the proof. □

Actually we have obtained a more precise knowledge of $\mathscr{K}_{m+n}(\mathrm{Y})$ then the statement of Theorem (12.5) gives. Our proof actually shows:

Corollary **(12.6).** — *Let* X *be a finite complex with* $\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{X})=2$. *Suppose that* $\gamma\in\Omega_*^{\mathrm{U}}(\mathrm{X})$ *is a spherical bordism element of prime order represented by* $f:\mathrm{S}^{n-1}\to\mathrm{X}$. *Suppose that* $\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\mathrm{X}\cup_f e^n)=3$.

Then $\mathrm{A}(\gamma)=(p,\lambda)$, *where* $p,\lambda\in\Omega_*^{\mathrm{U}}$ *is an* ESP-*sequence of length* 2, $p\in\mathbf{Z}$ *a prime, and* $\mathscr{K}_{m+n}(\mathrm{X}\cup_f e^n)=\mathbf{Z}_p$ *where* $m=\deg\lambda$, *while* $\mathscr{K}_j(\mathrm{X}\cup_f e^n)=0$ *otherwise.*

Proof. — In the notation of (12.5) this follows by observing

$$[\mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\mathrm{A}(\gamma)]_j=\begin{cases}\mathbf{Z}_p & j=0\\ \lambda\mathbf{Z}_p & j=m\\ 0 & \text{otherwise}\end{cases}$$

and reexamining the proof of (12.5). □

As an example let X be a large skeleton of $\mathbf{RP}(\infty)\times\mathbf{RP}(\infty)$ and

$$f:\mathrm{S}^{n+6}\to\Sigma^n\mathrm{X}$$

a map representing $\Sigma^n\gamma_3\otimes\gamma_3\in\Omega_{n+6}^{\mathrm{U}}(\Sigma^n\mathrm{X})$. As noted prior to Theorem (11.3) $\mathrm{hom.dim}_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\Sigma^n\mathrm{X}\cup_f e^{n+7})=3$. Thus from Corollary (11.6) we find

$$\tilde{\mu}:\quad \mathbf{Z}\otimes_{\Omega_*^{\mathrm{U}}}\Omega_*^{\mathrm{U}}(\Sigma^n\mathrm{X}\cup_f e^{n+7})\to\mathrm{H}_*(\Sigma^n\mathrm{X}\cup_f e^{n+7};\mathbf{Z})$$

has a kernel that is cyclic of order 2 in degree $n+11$. This is a considerable simplification of our previous explicit example (in § 6) of a complex for which the reduced Thom homomorphism has a non-trivial kernel.

§ 13. An Application to U-fr cobordism.

Recall from [9] or section 5 that there is a cofibration sequence

$$\underset{\sim}{\mathrm{S}}\to\underset{\sim}{\mathrm{MU}}\to\underset{\sim}{\mathrm{MU}}/\underset{\sim}{\mathrm{S}}$$

of spectra defining the spectrum $\underset{\sim}{\mathrm{MU}}/\underset{\sim}{\mathrm{S}}$. The homology theory associated to the spectrum $\underset{\sim}{\mathrm{MU}}/\underset{\sim}{\mathrm{S}}$ is denoted by $\Omega_*^{\mathrm{U,fr}}(\cdot)$ and referred to as U-framed cobordism. As noted in section 5 (see [16] for a thorough treatement; also [9], § 15) the coefficients $\Omega_*^{U, fr}$ may be described as cobordism classes of compact U-manifolds together with a compatable framing along their boundary. Thus a U-fr manifold (M, ∂M) has Chern classes

$$C_i(M, \partial M) \in H^{2i}(M, \partial M; \mathbf{Z})$$

and thus we many define Chern numbers for U-fr manifolds by

$$C_\omega[M, \partial M] = \langle C_{i_1}(M, \partial M) \ldots C_{i_n}(M, \partial M), [M, \partial M] \rangle$$

where $\omega = (i_1, \ldots, i_n)$. The usual argument ([22], III) may be applied to show that the Chern numbers of U-framed manifolds

$$C_\omega[M, \partial M] \in \mathbf{Z}$$

are invariants of the U-framed cobordism class of $[M, \partial M] \in \Omega_*^{U, fr}$.

The main purpose of this section is to solve the following:

Problem. — Given a compact U-framed manifold (M, ∂M), when does there exist a *closed* U-manifold N having the same Chern numbers as (M, ∂M)?

This problem was originally considered in ([9], § 15) where it was solved with the aid of the Hattori-Stong theorem ([22], VII). In this section we will present a solution based on the results of section 10. The fundamental idea is to exploit the special role played by $[\mathbf{CP}(1)] \in \Omega_2^U$ vis-a-vis $k_*(\ )$ theory resulting from the fact that $\mathrm{Td}[\mathbf{CP}(1)] = 1$.

Let us first recall how the rational number $\mathrm{Td}[M, \partial M] \in \mathbf{Q}$ may be defined for U-framed manifolds (M, ∂M).

Let n be a positive integer and introduce the *non-commutative* diagram

$$\begin{tikzcd}
K^*(MU(n)) \arrow[r, "ch"] & H^*(MU(n); \mathbf{Q}) \\
K^*(BU(n)) \arrow[r, "ch"'] \arrow[u, "\varphi_K"', "\cong"] & H^*(BU(n); \mathbf{Q}) \arrow[u, "\varphi_H"', "\cong"]
\end{tikzcd}$$

where φ_K and φ_H are the Thom isomorphisms in K-theory and rational cohomology respectively. We define

$$\mathrm{Td}_n \in H^*(BU(n); \mathbf{Q})$$

by

$$\mathrm{Td}_n = \varphi_H^{-1}\, ch\, \varphi_K(1) \in H^*(BU(n); \mathbf{Q}).$$

Naturality of the Thom isomorphism then yields

$$\theta_n^* \mathrm{Td}_n = \mathrm{Td}_{n-1} : \quad (n > 0)$$

where

$$\theta_n : \quad BU(n) \hookrightarrow BU(n+1)$$

is the usual mapping. Thus we may define a class

$$\mathrm{Td} \in H^{**}(BU; \mathbf{Q})$$

that restricts to Td_n on $BU(n)$. As Td is a power series in the rational Chern classes the rational number

$$\mathrm{Td}[M, \partial M] = \langle \mathrm{Td}(M, \partial M), [M, \partial M] \rangle \in \mathbf{Q}$$

is defined for any compact U-framed manifold $(M, \partial M)$ and depends only on the U-framed cobordism class of $(M, \partial M)$. It may be shown that for closed U-manifolds N, Td[N] as defined above coincides with the classical Todd genus of N ([22], VII); hence our notation is consistent with previous usage. Note further that Td[N] is an integer whenever N is a closed U-manifold. The converse of this observation is the key that was first noted in [9] to the solution of the problem of this section.

We turn now to this. The precise result that we shall establish ([9], (16.1)) is:

Theorem **(13.1)**. — *Let* $(M, \partial M)$ *be a compact* U*-framed manifold. Then there exists a closed* U*-manifold* N *with the same Chern numbers as* $(M, \partial M)$ *iff* $\mathrm{Td}[M, \partial M] \in \mathbf{Q}$ *is an integer.*

For the application of this result to the study of Ω_*^{fr} and the Adams e_C invariant the reader is referred to ([9], § 15-18). We will here concern ourselves only with the proof of (13.1). We will of course need several preliminary results.

Proposition **(3.2)**. — *Let* X *be a finite complex with* $\mathrm{hom.dim}_{\Omega_*^U} \Omega_*^U(X) = 1$. *Suppose that* $\gamma \in \Omega_*^U(X)$ *is a stably framed class;* i.e., *in the image of the natural map*

$$\Omega_*^{fr}(X) \to \Omega_*^U(X).$$

Then the natural map $\mathbf{Z}[t] \otimes_{\Omega_*^U} A(\gamma) \to A(\zeta\gamma)$

is an isomorphism, where $\zeta: \Omega_*^U(\cdot) \to k_*(\cdot)$

is the connective K*-theory orientation of* $\underset{\sim}{MU}$.

Proof. — There is no loss of generality in assuming that γ is represented by a map

$$f: S^n \to X;$$

as is easily seen by stability. Form the cofibration

$$S^n \xrightarrow{f} X \to Y \equiv X \cup_f e^{n+1}.$$

By Proposition (5.10) we have

1) $A(\gamma) \subset \Omega_*^U$ is a free Ω_*^U-module;
2) $\mathrm{hom.dim}_{\Omega_*^U} \Omega_*^U(Y) \leq 2$.

As usual we set

$$M(\gamma) = \Omega_*^U . \gamma \subset \Omega_*^U(X)$$
$$C(\gamma) = \Omega_*^U(X) / M(\gamma).$$

Similarly we introduce

$$M(\zeta\gamma) = \mathbf{Z}[t] . \gamma \subset k_*(X)$$
$$C(\zeta\gamma) = k_*(X) / M(\zeta\gamma).$$

We thus obtain two commutative diagrams with exact rows:

I

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}[t]\otimes_{\Omega^U_*} C(\gamma) \arrow[r] \arrow[d, "\zeta"] & \mathbf{Z}[t]\otimes_{\Omega^U_*} \Omega^U_*(Y) \arrow[r] \arrow[d, "\zeta", "\cong"'] & \mathbf{Z}[t]\otimes_{\Omega^U_*} \Sigma^n A(\gamma) \arrow[r] \arrow[d, "\zeta"] & 0 \\
0 \arrow[r] & C(\zeta\gamma) \arrow[r] & k_*(Y) \arrow[r] & \Sigma^n A(\zeta\gamma) \arrow[r] & 0
\end{tikzcd}$$

where the zero on the left of the top now results because $A(\gamma)$ is a free Ω^U_*-module and the indicated isomorphism follows from (10.6):

II

$$\begin{tikzcd}
 & \mathbf{Z}[t]\otimes_{\Omega^U_*} M(\gamma) \arrow[r] \arrow[d, "\zeta"] & \mathbf{Z}[t]\otimes_{\Omega^U_*} \Omega^U_*(X) \arrow[r] \arrow[d, "\zeta", "\cong"'] & \mathbf{Z}[t]\otimes_{\Omega^U_*} C(\gamma) \arrow[r] \arrow[d, "\zeta"] & 0 \\
0 \arrow[r] & M(\zeta\gamma) \arrow[r] & k_*(X) \arrow[r] & C(\zeta\gamma) \arrow[r] & 0
\end{tikzcd}$$

where the indicated isomorphism arises by (11.2).

Consider $\zeta: \mathbf{Z}[t]\otimes_{\Omega^U_*} C(\gamma) \rightarrow C(\zeta\gamma)$.

Evidently the second diagram implies it is epic while the first implies that it is monic. Hence it must be an isomorphism. Thus applying the five lemma to diagram I yields

$$\mathbf{Z}[t]\otimes_{\Omega^U_*} \Sigma^n A(\gamma) \rightarrow \Sigma^n A(\zeta\gamma)$$

is an isomorphism, and the result follows by desuspending. ☐

Corollary **(13.3).** — *Let* X *be a finite complex with* $\hom.\dim_{\Omega^U_*}\Omega^U_*(X)=1$. *Suppose that* $\gamma\in\Omega^U_*(X)$ *is a stably framed class. Then the natural map*

$$\mathbf{Z}[t]\otimes_{\Omega^U_*} M(\gamma) \rightarrow M(\zeta\gamma)$$

is an isomorphism, where

$$M(\gamma)=\Omega^U_*.\gamma\subset\Omega^U_*(X)$$
$$M(\zeta\gamma)=\mathbf{Z}[t].\zeta\gamma\subset k_*(X).$$

Proof. — Let us continue to employ the notation of the proof of Proposition (13.2). We then obtain a commutative diagram

III

$$\begin{tikzcd}
 & \mathbf{Z}[t]\otimes_{\Omega^U_*} \Sigma^n A(\gamma) \arrow[r, "i_*"] \arrow[d, "\cong"] & \mathbf{Z}[t]\otimes_{\Omega^U_*} \Sigma^n \Omega^U_* \arrow[r] \arrow[d, "\cong"] & \mathbf{Z}[t]\otimes_{\Omega^U_*} M(\gamma) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Sigma^n A(\zeta\gamma) \arrow[r] & \Sigma^n \mathbf{Z}[t] \arrow[r] & M(\zeta\gamma) \arrow[r] & 0
\end{tikzcd}$$

where the indicated isomorphisms arise from (13.2) and trivia. The left hand square clearly implies that i_* is monic and the result follows from the five-lemma. ☐

Corollary **(13.4)**. — *Let* X *be a finite complex and* $\gamma \in \Omega_*^U(X)$ *a non-zero stably framed class. Suppose that* $\text{hom.dim}_{\Omega_*^U} \Omega_*^U(X) = 1$. *Then*

$$\gamma \notin I(t)\Omega_*^U(X)$$

where $I(t)$ *is the kernel of the natural map*

$$\zeta : \Omega_*^U \to \mathbf{Z}[t].$$

Proof. — Suppose to the contrary that $\gamma \in I(t)\Omega_*^U(X)$. Then

$$\gamma = \sum_i \omega_i \lambda_i$$

where $\omega_i \in I(t), \quad \lambda_i \in \Omega_*^U(X).$

Hence $\zeta\gamma = 0 \in k_*(X).$

Therefore $M(\zeta\gamma) = 0$

and hence in view of (13.3) we must have

$$\mathbf{Z}[t] \otimes_{\Omega_*^U} M(\gamma) = 0.$$

From the natural epimorphism $\eta : \mathbf{Z}[t] \to \mathbf{Z}$

we therefore conclude that $\mathbf{Z} \otimes_{\Omega_*^U} M(\gamma) = 0.$

But this is impossible, for by assumption $\gamma \neq 0$ and hence

$$1 \otimes \gamma \neq 0 \in \mathbf{Z} \otimes_{\Omega_*^U} M(\gamma).$$

Therefore our original assumption must be incorrect and hence $\gamma \notin I(t)\Omega_*^U(X)$ as required. □

Proposition **(13.5)**. — *Let* X *be a finite complex with* $\text{hom.dim}_{\Omega_*^U} \Omega_*^U(X) = 1$. *Suppose that we are given a class* $\gamma \in \Omega_*^U(X)$ *of finite order m and a closed* U*-manifold* M^{2k} *such that*:

1) $\text{Td}[M^{2k}] \equiv 0 \mod m$, *and*
2) $[M^{2k}]\gamma \in \Omega_*^U(X)$ *is stably spherical.*

Then

$$[M^{2k}] \in m.\Omega_*^U$$

and hence $[M^{2k}]\gamma = 0 \in \Omega_*^U(X).$

Proof. — Let $\text{Td}[M^{2k}] = m.r.$ Let

$$[N^{2k}] = [M^{2k}] - m.r[\mathbf{C}P(1)]^k.$$

Note that
$$\begin{aligned}[N^{2k}]\gamma &= [M^{2k}]\gamma - m.r[\mathbf{C}P(1)]^k\gamma \\ &= [M^{2k}]\gamma - r[\mathbf{C}P(1)]^k m.\gamma \\ &= [M^{2k}]\gamma.\end{aligned}$$

Moreover $\text{Td}[N^{2k}] = \text{Td}[M^{2k}] - mr\,\text{Td}[\mathbf{C}P(1)]^k = 0.$

Thus $[N^{2k}] \in I(t) \subset \Omega_*^U$ and therefore

$$[M^{2k}]\gamma = [N^{2k}]\gamma \in I(t)\Omega_*^U(X).$$

Therefore by Corollary (13.4) $[M^{2k}]\gamma = 0$

and hence $[M^{2k}] \in A(\gamma)$.

By Proposition (5.9) $A(\gamma) = (m) \subset \Omega_*^U$

and the result follows. □

Theorem **(13.6)**. — *Let* (M, ∂M) *be a compact* U*-framed manifold. Then there exists a closed* U*-manifold* N *such that*

$$[M, \partial M] - [N] \in \Omega_*^{U, fr}$$

has finite order iff Td[M, ∂M] *is an integer.*

Proof. — Recall ([22], VII) that we have an exact sequence

$$0 \to \Omega_{2k}^U \underset{\Phi}{\to} \Omega_{2k}^{U, fr} \overset{\partial}{\to} \Omega_{2k-1}^{fr} \to 0 : k > 0$$

and that $\Omega_{2k+1}^{U, fr} = 0$ for all $k \geq 0$. Next recall that by the fundamental result of Serre (see e.g. [21]) Ω_{2k-1}^{fr} has finite order for all positive integers k. Thus there exists an integer m such that

$$m . \partial_*[M, \partial M] = [\partial M] = 0 \in \Omega_{2k-1}^{fr}$$

where $\dim M = 2k$. Hence there exists a unique $[V^{2k}] \in \Omega_{2k}^U$ such that

$$\Phi[V] = m[M, \partial M] \in \Omega_{2k}^{U, fr}$$

By hypothesis $Td[M, \partial M] \in \mathbf{Q}$ is an integer and hence we evidently have

$$Td[V] = m \, Td[M, \partial M] \equiv 0 \quad \mod m.$$

Let C be the space obtained from S^1 by attaching a 2-cell by a map of degree m. Let

$$\rho \in \Omega_1^U(C)$$

be the standard generator. We then have that ρ has order m. Next note that by construction m divides $\Phi[V] \in \Phi_{2k}^{U, fr}$. Hence by our study of Toda brackets ((6.1) and (6.2)) we obtain that $[V]\rho \in \Omega_{2k+1}^U(C)$ is a stably spherical class.

Let us review our progress. We have the finite complex C and the class $\rho \in \Omega_1^U(C)$ of finite order m. Moreover $\text{hom.dim}_{\Omega_*^U} \Omega_*^U(C) = 1$ and we have $[V] \in \Omega_{2k}^U$ such that

a) $Td[V] \equiv 0 \quad \mod m$, and

b) $[V] . \rho \in \Omega_{2k+1}^U(C)$ is a stably spherical class.

Therefore by Proposition (13.5)

$$[V] \in (m) \subset \Omega_*^U.$$

Write $[V] = m[N] \in \Omega_*^U$

and note that

$$m(\Phi[\mathrm{N}]-[\mathrm{M}, \partial\mathrm{M}])=\Phi[\mathrm{V}]-m[\mathrm{M}, \partial\mathrm{M}]=0$$

which is the desired result.

To prove the converse implication note that for any class $\alpha\in\Omega_*^{\mathrm{U, fr}}$ of finite order

$$\mathrm{Td}(\alpha)\in\mathbf{Q}$$

is of necessity zero. Thus if

$$m(\Phi[\mathrm{N}]-[\mathrm{M}, \partial\mathrm{M}])=0$$

for some closed U-manifold N we must have

$$\mathrm{Td}[\mathrm{M}, \partial\mathrm{M}]=\mathrm{Td}[\mathrm{N}]\in\mathbf{Z}\subset\mathbf{Q}$$

which completes the proof. □

Proof of Theorem (13.1). — Suppose that (M, ∂M) is a compact U-framed manifold with Td[M, ∂M]$\in\mathbf{Z}$. Then by (13.6) there exists a closed U-manifold N such that

$$\alpha=[\mathrm{M}, \partial\mathrm{M}]-[\mathrm{N}]\in\Omega_*^{\mathrm{U, fr}}$$

has finite order. Thus $\mathrm{C}_\omega(\alpha)\in\mathbf{Z}$

must vanish for all ω and hence

$$\mathrm{C}_\omega[\mathrm{M}, \partial\mathrm{M}]=\mathrm{C}_\omega[\mathrm{N}]$$

as required.

As the converse implication is clear the result follows. □

It is interesting to note that Stong has applied the ideas of ([9], § 15) to the study of the forgetful functor

$$\Omega_*^{\mathrm{fr}}(\ )\to\mathfrak{N}_*(\ )$$

and has obtained results similar to ([9], (15.1)) in this case that place the classical Hopf invariant into a cobordism setting. (For a very readable account see ([22], p. 102-106).)

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Louisiana State University,
I.H.E.S., Bures-sur-Yvette
and The University of Virginia.

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