

ON THE COMPLEX OSCILLATION THEORY OF $f^{(k)} + Af = F$

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In this paper, we investigate the complex oscillation theory of

$$f^{(k)} + Af = F(z), \quad k \geq 1$$

where $A, F \not\equiv 0$ are entire functions, and obtain general estimates of the exponent of convergence of the zero-sequence and of the order of growth of solutions for the above equation.

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1. Introduction and results

For convenience in our statement, we first explain the notation used in this paper. We will use respectively the notation $\lambda(f)$ and $\bar{\lambda}(f)$ to denote the exponent of convergence of the zero-sequence and of the sequences of distinct zeros of $f(z)$, $\sigma(f)$ to denote the order of growth of $f(z)$, $\nu_f(r)$ to denote the central index of the entire function $f(z)$. By the Wiman–Valiron theory, we have

$$\sigma(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log \nu_f(r)}{\log r}.$$

In addition, other notation of the Nevanlinna theory is standard (e.g. see [4, 5]). Other notation will be shown when it appears.

In 1982, S. Bank and I. Laine investigated the complex oscillation theory of homogeneous linear differential equation, and proved in [1]:

Theorem A. *Let $A(z)$ be a nonconstant polynomial of degree n , and let $f(z) \not\equiv 0$ be a solution of the equation*

$$f'' + A(z)f = 0. \tag{1.1}$$

Then

- (a) *the order of growth of f is $(n+2)/2$,*
- (b) *if n is odd, the exponent of convergence of the zero-sequence of f is $(n+2)/2$,*
- (c) *if n is even, and if f_1 and f_2 are two linearly independent solutions of (1.1), then at*

least one of f_1, f_2 , has the property that the exponent of convergence of its zero-sequence is $(n+2)/2$.

Afterwards, I. Laine showed in [6].

Theorem B. Let $a_0, P_0, P_1 \not\equiv 0$ be polynomials such that $\deg a_0 = n, \deg p_0 < 1 + n/k$. Consider the equation

$$f^{(k)} + a_0 f = P_1 e^{P_0} \quad (k \geq 2). \tag{1.2}$$

(a) If $\deg P_1 < n$, then all solutions of (1.2) satisfy

$$\bar{\lambda}(f) = \lambda(f) = \sigma(f) = 1 + \frac{n}{k}. \tag{1.3}$$

(b) If $\deg P_1 \geq n$, then, apart from one possible exception, all solutions satisfy (1.3). The possible exceptional solution is of the form $f_0 = Qe^{P_0}$, where Q is a polynomial of degree $\deg Q = \deg P_1 - n$.

Gao Shi-an [3] had earlier addressed the case when $n=2$ in Theorem B.

In this paper, we consider the differential equation (DE)

$$f^{(k)} + Af = F(z), \tag{1.4}$$

where $A(z)$ and $F(z)$ are both entire functions of finite order, and where $A(z)$ is transcendental, or $A(z)$ is a polynomial, $F(z) \not\equiv 0$ is an entire function of finite order with infinitely many zeroes.

We will prove the following theorems in this paper.

Theorem 1. Let A and $F(z) \not\equiv 0$ be both entire functions of finite order, where A is transcendental. Then:

(a) All solutions f of (1.4) satisfy

$$\bar{\lambda}(f) = \lambda(f) = \sigma(f) = \infty, \tag{1.5}$$

except for at most one possible exceptional solution f_0 of finite order.

(b) The exceptional solution f_0 satisfies

$$\sigma(f_0) \leq \max \{ \sigma(A), \sigma(F), \bar{\lambda}(f_0) \}.$$

Furthermore, if $\sigma(A) \neq \sigma(F), \bar{\lambda}(f_0) < \sigma(f_0)$, then $\sigma(f_0) = \max \{ \sigma(A), \sigma(F) \}$.

Remark. If $\sigma(A) = \infty$, or $\sigma(F) = \infty$, then Theorem 1 does not hold.

Example 1. $f = e^{\sin z}$ solves

$$f'' + \sin z f = \cos^2 z e^{\sin z},$$

and

$$f'' + (\sin z - \cos^2 z + e^{-\sin z + z})f = e^z,$$

there $\lambda(f) = 0, \sigma(f) = \infty$.

Theorem 2. Let $A(z)$ be a transcendental entire function with $\sigma(A) \neq 1, \sigma(A) < \infty$, let $F(z) \not\equiv 0$ be an entire function with $\sigma(F) < \infty$, and $\alpha (> 0)$, a constant, and let $f(z)$ be a solution of the DE

$$f^{(k)} + e^{-\alpha z} f' + Af = F(z). \tag{1.6}$$

Then:

(a) all solutions f of (1.6) satisfy (1.5), except for at most one possible exceptional solution f_0 of finite order,

(b) the exceptional solution f_0 satisfies

$$\sigma(f_0) \leq \max \{ \sigma(A), \sigma(F), \bar{\lambda}(f_0), 1 \},$$

Furthermore, if $\sigma(A) \neq \sigma(F), \bar{\lambda}(f_0) < \sigma(f_0)$, then $\sigma(f_0) = \max \{ \sigma(A), \sigma(F), 1 \}$.

Theorem 3. Let $A(z)$ be a polynomial with $\deg A = n \geq 1$, let $F(z) \not\equiv 0$ be an entire function with infinitely many zeros, and let $f(z)$ be a solution of the DE (1.4). If $(n+k)/k < \sigma(F) = \beta < \infty$, then

- (a) $\sigma(f) = \beta$,
- (b) if $\lambda(F) = \beta$, then every solution satisfies $\lambda(f) = \beta$,
- (c) if $\lambda(F) < \beta$, then all solutions f of (1.4) satisfy

$$\bar{\lambda}(f) = \lambda(f) = \sigma(f) = \beta,$$

except for at most one exceptional one f_0 with $\lambda(f_0) = \lambda(F)$.

Theorem 4. Let $A(z)$ be a nonconstant polynomial of degree n , and let $F(z) \not\equiv 0$ be an entire function with infinitely many zeros and $\sigma(F) = \beta$. Then:

- (a) if $\beta < (n+k)/k$, then all solutions f of (1.4) satisfy

$$\bar{\lambda}(f) = \lambda(f) = \sigma(f) = \frac{n+k}{k},$$

except for at most one exceptional one f_0 with $\sigma(f_0) = \beta$,

(b) if $\beta = (n+k)/k$, then all solutions f of (1.4) satisfy $\sigma(f) = (n+k)/k$, $\lambda(f) \geq \lambda(F)$.

2. Lemmas

Lemma 1. Let $A(z)$ be a transcendental entire function of finite order, Then every solution $g \not\equiv 0$ to the DE

$$g^{(k)} + Ag = 0 \tag{2.1}$$

satisfies $\sigma(g) = \infty$.

Proof. If $\sigma(g) < \infty$, then by $A = -g^{(k)}/g$, we have

$$m(r, A) = m\left(r, \frac{g^{(k)}}{g}\right) = O(\log r),$$

and this contradicts the hypothesis that $A(z)$ is transcendental.

Lemma 2. Let $A(z)$ be a transcendental entire function with $1 \not\equiv \sigma(A) < \infty$ and let $\alpha (> 0)$ be a constant. Then every solution $g \not\equiv 0$ to the DE

$$g^{(k)} + e^{-\alpha z}g' + A(z)g = 0 \tag{2.2}$$

satisfies $\sigma(g) = \infty$.

Proof. Using the same proof as in the proof of Theorem 1 in [2], we have $\sigma(g) = \infty$.

Lemma 3 (Wiman–Valiron). Let $g(z)$ be a transcendental entire function and let z be a point with $|z| = r$ at which $|g(z)| = M(r, g)$. Then for all $|z|$ outside a set E of r of finite logarithmic measure, we have

$$(a) \frac{g^{(k)}(z)}{g(z)} = \left(\frac{v_g(r)}{z}\right)^k (1 + o(1)) \quad (k \text{ is an integer, } r \notin E), \tag{2.3}$$

$$(b) \overline{\lim}_{\substack{r \rightarrow \infty \\ r \in [0, \infty)}} \frac{\log v_g(r)}{\log r} = \overline{\lim}_{\substack{r \rightarrow \infty \\ r \in [0, \infty) - E}} \frac{\log v_g(r)}{\log r},$$

where $v_g(r)$ is the central index of $g(z)$ (see [5, 7, 8]).

Proof. (a) This is the Wiman–Valiron theory (see [5, 7, 8]).

(b) We clearly have

$$\overline{\lim}_{\substack{r \rightarrow \infty \\ r \in (0, \infty)}} \frac{\log v_g(r)}{\log r} \geq \overline{\lim}_{\substack{r \rightarrow \infty \\ r \in (0, \infty) - E}} \frac{\log v_g(r)}{\log r}.$$

On the other hand, from $v_g(r)$ is the central index of $g(z)$, we have that $v_g(r) > 0$ and $v_g(r)$ is a nondecreasing function on $[0, +\infty)$. Setting $\int_E dr/r = \log \delta < \infty$ for a given $\{r'_n\}$, $r'_n \in [0, +\infty)$, $r'_n \rightarrow \infty$, there exists a point $r_n \in [r'_n, (\delta + 1)r'_n] - E$. From

$$\frac{\log v_g(r'_n)}{\log r'_n} \leq \frac{\log v_g(r_n)}{\log r'_n} \leq \frac{\log v_g(r)}{\log r_n + \log \frac{1}{\delta + 1}} = \frac{\log v_g(r)}{\log r_n \cdot (1 + o(1))},$$

it follows that

$$\overline{\lim}_{r'_n \rightarrow \infty} \frac{\log v_g(r'_n)}{\log r'_n} \leq \overline{\lim}_{r_n \rightarrow \infty} \frac{\log v_g(r_n)}{\log r_n} \leq \overline{\lim}_{\substack{r \rightarrow \infty \\ r \in (0, \infty) - E}} \frac{\log v_g(r)}{\log r}. \tag{2.4}$$

Since $\{r'_n\}$ is arbitrary, we have

$$\overline{\lim}_{\substack{r \rightarrow \infty \\ r \in (0, \infty)}} \frac{\log v_g(r)}{\log r} \leq \overline{\lim}_{\substack{r \rightarrow \infty \\ r \in (0, \infty) - E}} \frac{\log v_g(r)}{\log r}.$$

This proves Lemma 3(b).

From Lemma 3, we can deduce the following:

Lemma 4. *Let $A(z)$ be a nonconstant polynomial with $\deg A = n$. Then every solution $f \not\equiv 0$ to the DE*

$$f^{(k)} + A(z)f = 0 \tag{2.5}$$

satisfies $\sigma(f) = (n + k)/k$.

Lemma 5. *Let $A(z)$ be a polynomial with $\deg A = n \geq 1$, $F(z) \not\equiv 0$ be an entire function with $\sigma(F) = \beta < \infty$. Let f be a solution of the DE*

$$f^{(k)} + Af = F(z). \tag{2.6}$$

Then

(a) if $\beta \geq (n + k)/k$, then $\sigma(f) = \beta$,

(b) if $\beta < (n+k)/k$, then all solutions f of (2.6) satisfy $\sigma(f) = (n+k)/k$, except for at most one possible exceptional one f_0 with $\sigma(f_0) = \beta$.

Proof. It is easy to see that $\sigma(f) \geq \sigma(F) = \beta$ from (2.6). On the other hand we assume that $\{f_1, \dots, f_k\}$ is a fundamental solution set of (2.5) that is the corresponding homogeneous differential equation of (2.6). By Lemma 4, we have $\sigma(f_j) = (n+k)/k$ ($j=1, \dots, k$).

By variation of parameters, we can write

$$f = B_1(z)f_1 + \dots + B_k(z)f_k,$$

where $B_1(z), \dots, B_k(z)$ are determined by

$$\left\{ \begin{array}{l} B_1'f_1 + \dots + B_k'f_k = 0 \\ B_1'f_1' + \dots + B_k'f_k' = 0 \\ \dots\dots\dots \\ B_1'f_1^{(k-1)} + \dots + B_k'f_k^{(k-1)} = F. \end{array} \right.$$

Noting that the Wronskian $W(f_1, \dots, f_k)$ is a differential polynomial in f_1, \dots, f_k with constant coefficients, it is easy to deduce that $\sigma(W) \leq \sigma(f_j) = (n+k)/k$. Set

$$W_i = \begin{vmatrix} (i) \\ f_1, \dots, 0, \dots, f_k \\ \vdots & & \vdots \\ f_1^{(k-1)}, \dots, F, \dots, f_k^{(k-1)} \end{vmatrix} = F \cdot g_i \quad (i=1, \dots, k).$$

where g_i are differential polynomials in f_1, \dots, f_k with constant coefficients. So

$$\sigma(g_i) \leq \sigma(f_j) = \frac{n+k}{k}, B_i' = \frac{W_i}{W} = \frac{F \cdot g_i}{W},$$

and

$$\sigma(B_i) = \sigma(B'_i) \leq \max \left\{ \sigma(F), \frac{n+k}{k} \right\}.$$

$$\sigma(f) \leq \max \left\{ \beta, \frac{n+k}{k} \right\}.$$

Therefore,

(a) if $\beta \geq \frac{n+k}{k}$, then $\sigma(f) = \beta$,

(b) if $\beta < \frac{n+k}{k}$, then $\beta \leq \sigma(f) \leq \frac{n+k}{k}$.

We affirm that the DE (2.6) can only possess at most one exceptional solution f_0 with $\beta \leq \sigma(f_0) < (n+k)/k$. In fact, if f^* is a second solution with $\beta \leq \sigma(f^*) < (n+k)/k$, then $\sigma(f_0 - f^*) < (n+k)/k$. But $f_0 - f^*$ is a solution of the corresponding homogeneous equation (2.5) of (2.6). This contradicts Lemma 4.

Now we prove that the exceptional solution f_0 satisfies $\sigma(f_0) = \beta$. We assume $\beta < \sigma(f_0) < (n+k)/k$. Let z be a point with $|z| = r$ at which $|f_0(z)| = M(r, f_0)$. From Lemma 3(a)

$$\frac{f_0^{(k)}(z)}{f_0(z)} = \left(\frac{v_{f_0}(r)}{z} \right)^k (1 + o(1)) \quad r \notin E. \tag{2.7}$$

holds for all $|z|$ outside a set E of r of finite logarithmic measure. For sufficiently large $|z|$, we have $A = az^n(1 + o(1))$ ($a \neq 0$ is constant). Substituting (2.7) into (2.6), we have

$$\left(\frac{v_{f_0}(r)}{z} \right)^k (1 + o(1)) + az^n(1 + o(1)) = \frac{F(z)}{f_0(z)} \quad r \notin E. \tag{2.8}$$

Now for a given ε ($0 < 3\varepsilon < \sigma(f_0) - \beta$), there exists $\{\bar{r}_m\}$ ($\bar{r}_m \rightarrow \infty$) such that $M(\bar{r}_m, f_0) > \exp \{\bar{r}_m^{\sigma(f_0) - \varepsilon}\}$. Setting $\int_E dr/r = \log \delta < \infty$, there exists a point $r_m \in [\bar{r}_m, (\delta + 1)\bar{r}_m] - E$. At such points r_m , we have

$$M(r_m, f_0) \geq M(\bar{r}_m, f_0) > \exp \{\bar{r}_m^{\sigma(f_0) - \varepsilon}\}$$

$$\begin{aligned} &\geq \exp \left\{ \frac{r_m^{\sigma(f_0) - \varepsilon}}{(\delta + 1)^{\sigma(f_0)}} \right\} = \exp \left\{ r_m^{\sigma(f_0) - 2\varepsilon} \cdot \frac{r_m^\varepsilon}{(\delta + 1)^{\sigma(f_0)}} \right\} \\ &\geq \exp \{ r_m^{\sigma(f_0) - 2\varepsilon} \}. \end{aligned}$$

In addition for sufficiently large r_m , we have

$$|F(z)| \leq M(r_m, F) < \exp \{ r_m^{\beta + \varepsilon} \}.$$

So

$$\left| \frac{F(z)}{f_0(z)} \right| = \frac{|F(z)|}{M(r_m, f_0)} < \exp \{ r_m^{\beta + \varepsilon} - r_m^{\sigma(f_0) - 2\varepsilon} \} \rightarrow 0 (r_m \rightarrow \infty).$$

Therefore, at such points $|z_m| = r_m (r_m \notin E, |f_0(z_m)| = M(r_m, f_0))$, from (2.8) we have

$$\left(\frac{\nu_{f_0}(\gamma_m)}{z_m} \right)^k (1 + o(1)) + az_m^n (1 + o(1)) = o(1). \tag{2.9}$$

From the Wiman–Valiron theory, we obtain

$$\overline{\lim}_{r_m \rightarrow \infty} \frac{\log \nu_{f_0}(r_m)}{\log r_m} = \frac{n + k}{k}.$$

This contradicts that $\sigma(f_0) < (n + k)/k$. Hence $\sigma(f_0) = \beta$.

Lemma 6. Let $A_{k-j} (j = 1, \dots, k), B \not\equiv 0$ be entire functions. If f is a solution of the DE

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_0f = B, \tag{2.10}$$

and $\max \{ \sigma(B), \sigma(A_0), \dots, \sigma(A_{k-1}) \} = \beta < \sigma(f)$, then $\bar{\lambda}(f) = \lambda(f) = \sigma(f)$.

Proof. We can write from (2.10)

$$\frac{1}{f} = \frac{1}{B} \left(\frac{f^{(k)}}{f} + A_{k-1} \frac{f^{(k-1)}}{f} + \dots + A_0 \right). \tag{2.11}$$

If f has a zero at z_0 of order $\alpha (> k)$, then B must have a zero at z_0 of order $\alpha - k$. Hence,

$$n \left(r, \frac{1}{f} \right) \leq k\bar{n} \left(r, \frac{1}{f} \right) + n \left(r, \frac{1}{B} \right)$$

and

$$N\left(r, \frac{1}{f}\right) \leq k\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{B}\right). \tag{2.12}$$

From (2.11), we have

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{1}{B}\right) + \sum_{j=1}^k m(r, A_{k-j}) + O(\log T(r, f) + \log r) \quad r \notin E \tag{2.13}$$

holds for all r outside a set E of r of finite linear measure (if $\sigma(f) < \infty$, then $E = \emptyset$). (2.12) and (2.13) give

$$\begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &\leq k\bar{N}\left(r, \frac{1}{f}\right) + T\left(r, \frac{1}{B}\right) + \sum_{j=1}^k T(r, A_{k-j}) + O(\log T(r, f) + \log r) \\ &= k\bar{N}\left(r, \frac{1}{f}\right) + T(r, B) + \sum_{j=1}^k T(r, A_{k-j}) + O(\log T(r, f) + \log r) \quad (r \notin E) \end{aligned} \tag{2.14}$$

and there exists $\{r'_n\}$ ($r'_n \rightarrow \infty$) such that

$$\lim_{r'_n \rightarrow \infty} \frac{\log T(r'_n, f)}{\log r'_n} = \sigma(f).$$

Setting $mE = \delta < \infty$, then there exists a point $r_n \in [r'_n, r'_n + \delta + 1] - E$. For such r_n , we have

$$\frac{\log T(r_n, f)}{\log r_n} \geq \frac{\log T(r'_n, f)}{\log(r'_n + \delta + 1)} = \frac{\log T(r'_n, f)}{\log r'_n + \log\left(1 + \frac{\delta + 1}{r'_n}\right)}.$$

So

$$\lim_{r_n \rightarrow \infty} \frac{\log T(r_n, f)}{\log r_n} \geq \lim_{r'_n \rightarrow \infty} \frac{\log T(r'_n, f)}{\log r'_n + \log\left(1 + \frac{\delta + 1}{r'_n}\right)} = \sigma(f) \quad (r_n \notin E)$$

and

$$\lim_{r_n \rightarrow \infty} \frac{\log T(r_n, f)}{\log r_n} = \sigma(f).$$

For a given c such that $\beta < c < \sigma(f)$, we have

$$T(r_n, f) \geq r_n^c$$

for sufficiently large r_n . On the other hand, for a given ε ($0 < \varepsilon < c - \beta$), we have

$$T(r_n, B) < r_n^{\beta + \varepsilon}, \quad T(r_n, A_{k-j}) < r_n^{\beta + \varepsilon} \quad (j = 1, \dots, k).$$

Therefore

$$T(r_n, B) \leq \frac{1}{k+3} T(r_n, f), \tag{2.15}$$

$$T(r_n, A_{k-j}) \leq \frac{1}{k+3} T(r_n, f) \quad (j = 1, \dots, k) \tag{2.16}$$

hold for sufficiently large r_n . Since

$$O\{\log T(r_n, f) + \log r_n\} = o\{T(r_n, f)\},$$

for sufficiently large r_n

$$O\{\log T(r_n, f) + \log r_n\} \leq \frac{1}{k+3} T(r_n, f) \tag{2.17}$$

holds. (2.14) and (2.15), (2.16), (2.17) give

$$T(r_n, f) \leq (k+3)K\bar{N}\left(r_n, \frac{1}{f}\right).$$

So

$$\sigma(f) = \lim_{r_n \rightarrow \infty} \frac{\log T(r_n, f)}{\log r_n} \leq \overline{\lim}_{r_n \rightarrow \infty} \frac{\log \bar{N}\left(r_n, \frac{1}{f}\right)}{\log r_n} \leq \bar{\lambda}(f).$$

Therefore $\bar{\lambda}(f) = \lambda(f) = \sigma(f)$.

Lemma 7. *Let F be the same as in Theorem 3, let Q be the canonical product formed with the nonzero zeros of F , with $\sigma(Q) < \beta = \sigma(F)$, let $m (\geq 0)$ be an integer, and let b_{k-r}, \dots, b_0 be polynomials with $\deg b_{k-i} = i(\beta - 1)$. Then the DE*

$$g^{(k)} + b_{k-1}g^{(k-1)} + \dots + b_0g = z^m Q \tag{2.18}$$

may have at most one exceptional solution g_0 with $\lambda(g_0) = \sigma(g_0) = \sigma(Q) = \lambda(Q)$, and all the other solutions g of (2.18) satisfy

$$\bar{\lambda}(g) = \lambda(g) = \sigma(g) = \beta.$$

Proof. It is not difficult to see that all solutions of (2.18) and its corresponding homogeneous equation

$$g^{(k)} + b_{k-1}g^{(k-1)} + \dots + b_0g = 0 \tag{2.19}$$

are entire functions. For the DE (2.19), from Lemma 3(a), we have basic formulas

$$\frac{g^{(j)}(z)}{g(z)} = \left(\frac{v_g(r)}{z}\right)^j (1 + o(1)) \quad r \notin E, (j = 1, \dots, k) \tag{2.20}$$

where $|z| = r$, $|g(z)| = M(r, g)$, $\int_E dr/r < \infty$, $v_g(r)$ denotes the central index of g . As $r \rightarrow \infty$ set $b_{k-i} = d_{k-i}z^{i(\beta-1)}(1 + o(1))$ d_{k-1}, \dots, d_0 are nonzero constants). Substituting them and (2.20) into (2.19), we have

$$\left(\frac{v_g(r)}{z}\right)^k (1 + o(1)) + d_{k-1}z^{\beta-1} \left(\frac{v_g(r)}{z}\right)^{k-1} (1 + o(1)) + \dots + d_0z^{k(\beta-1)}(1 + o(1)) = 0 \quad (r \notin E). \tag{2.21}$$

By the reasoning in [7, pp. 106–108] for sufficiently large r , we have $v_g(r) \sim cz^\alpha$ ($|z| = r \notin E$, $c \neq 0$ a constant), substituting it into (2.21), it is easy to see that the degrees of all terms of (2.21) are respectively

$$k(\alpha - 1), j(\beta - 1) + (k - j)(\alpha - 1) \quad (j = 1, \dots, k - 1), k(\beta - 1).$$

From the Wiman–Valiron theory (see [5, pp. 227–229], [7, 8]), we see that $\alpha = \beta$ is the only possible value. Therefore, all solutions of (2.19) satisfy $\sigma(g) = \beta$.

Using the same proof (variation of parameters) as in the proof of Lemma 5, we have that all solution g of the DE (2.18) satisfy $\sigma(g) \leq \beta$.

Using the same proof as in the proof of Lemma 5, it is easy to see that the DE (2.18) may have at most one exceptional solution g_0 with $\sigma(g_0) < \beta$.

Next we are going to work out the order of the exceptional solution g_0 . By the above proof and (2.18), we have $\sigma(Q) \leq \sigma(g_0) < \beta$. We will now prove that $\sigma(Q) < \sigma(g_0) < \beta$ fails.

Suppose that $\sigma(Q) < \sigma(g_0) < \beta$. From the Wiman–Valiron theory, we have basic formulas

$$\frac{g_0^{(j)}(z)}{g_0(z)} = \left(\frac{v_{g_0}(r)}{z}\right)^j (1 + o(1)) \quad r \notin E, j = 1, \dots, k \tag{2.22}$$

where $|z|=r$, $|g_0(z)|=M(r, g_0)$, $\int_E dr/r < \infty$. As $r \rightarrow \infty$, we set $b_{k-i} = d_{k-i}z^{i(\beta-1)}(1+o(1))$ ($d_{k-i} \neq 0$ are constants). Substituting them and (2.22) into (2.18), and using the same proof as in the proof of Lemma 5(b), we can obtain a sequence $\{z_n\}$. The sequence $\{z_n\}$ satisfies $|z_n|=r_n \notin E$, $|g_0(z_n)|=M(r_n, g_0)$ and for sufficiently large r_n

$$\left(\frac{v_{g_0}(r_n)}{z_n}\right)^k (1+o(1)) + d_{k-1}z_n^{\beta-1} \left(\frac{v_{g_0}(r_n)}{z_n}\right)^{k-1} (1+o(1)) + \dots + d_0z_n^{k(\beta-1)}(1+o(1)) = o(1). \tag{2.23}$$

Setting $\sigma(g_0) = \delta < \beta$, by the reasoning in [7, pp. 106–108], for sufficiently larger r_n , we have $v_{g_0}(r_n) \sim c_1z_n^\delta$ ($|z_n|=r_n \notin E$, $c_1 \neq 0$ a constant). It is easy to see that the degrees of all terms of (2.23) are respectively

$$k(\delta-1), j(\beta-1) + (k-j)(\delta-1) (j=1, \dots, k-1), k(\beta-1).$$

Then there is only one term $d_0z_n^{k(\beta-1)}$ ($d_0 \neq 0$) with the degree $k(\beta-1)$ being the highest one in (2.23). This is impossible. Therefore, the order of g_0 can only be $\sigma(g_0) = \sigma(Q)$.

As $\sigma(g_0) = \sigma(Q)$, by [3], we have $\lambda(g_0) \geq \lambda(z^m Q) = \sigma(Q)$. Hence

$$\lambda(g_0) = \sigma(g_0) = \sigma(Q) = \lambda(Q).$$

As $\sigma(g) = \beta > \sigma(Q)$, by Lemma 6, we have

$$\bar{\lambda}(g) = \lambda(g) = \sigma(g) = \beta.$$

3. Proof of theorems

Proof of Theorem 1. (a) Now assume f_0 is a solution of (1.4) with $\sigma(f_0) < \infty$. If f^* is a second solution with $\sigma(f^*) < \infty$, then $\sigma(f^* - f_0) < \infty$. And $f^* - f_0$ is a solution of (2.1), that is the corresponding homogeneous differential equation of (1.4). But by Lemma 1, we have $\sigma(f^* - f_0) = \infty$.

Now assume f is a solution of (1.4) with $\sigma(f) = \infty$. Then $\max\{\sigma(A), \sigma(F)\} < \sigma(f)$. By Lemma 6, we have $\bar{\lambda}(f) = \lambda(f) = \sigma(f) = \infty$.

(b) Assume f is an exceptional solution of (1.4) with $\sigma(f_0) < \infty$. Using the same proof as in the proof of Lemma 6, we have

$$T(r, f_0) \leq k\bar{N}\left(r, \frac{1}{f_0}\right) + T(r, F) + T(r, A) + O(\log r). \tag{3.1}$$

Now set $\max\{\sigma(A), \sigma(F)\} = \bar{\alpha}$. Then for sufficiently large r , we have

$$T(r, F) < r^{\bar{\alpha}+\epsilon}, T(r, A) < r^{\bar{\alpha}+\epsilon}.$$

By (3.1) we have

$$T(r, f_0) < k\bar{N}\left(r, \frac{1}{f_0}\right) + 2r^{\epsilon+\epsilon} + O(\log r).$$

Therefore

$$\sigma(f_0) \leq \max \{ \bar{\lambda}(f_0), \bar{\alpha} \} = \max \{ \bar{\lambda}(f_0), \sigma(A), \sigma(F) \}. \tag{3.2}$$

If $\sigma(A) \neq \sigma(F)$, $\bar{\lambda}(f_0) < \sigma(f_0)$, then from (3.2), we get

$$\sigma(f_0) \leq \max \{ \sigma(A), \sigma(F) \},$$

and by (1.4), we have $\sigma(f_0) \geq \max \{ \sigma(A), \sigma(F) \}$. Therefore,

$$\sigma(f_0) = \max \{ \sigma(A), \sigma(F) \}.$$

Proof of Theorem 2. By Lemma 2, every solution $g \neq 0$ of (2.2), that is the corresponding homogeneous equation of (1.6), satisfies $\sigma(g) = \infty$. Using the analogous proof to that in Theorem 1, we can prove that Theorem 2 holds.

Proof of Theorem 3. (a) By Lemma 5, we have $\sigma(f) = \beta$.

(b) If $\lambda(F) = \beta$, then by [3] we have $\lambda(f) \geq \lambda(F)$. Hence $\lambda(f) = \sigma(f) = \beta$.

(c) If $\lambda(F) < \beta$, set $F = Z^m Q e^P$, (Q is the canonical product formed with the nonzero zeros of F , P is a polynomial with $\deg P = \beta$). Set $f = g e^P$, then $\lambda(g) = \lambda(f)$, $\bar{\lambda}(g) = \bar{\lambda}(f)$. Substituting $f = g e^P$ into (1.4), we have

$$g^{(k)} + d_{k-1}g^{(k-1)} + \dots + d_0g = z^m Q. \tag{3.3}$$

To work out the degrees of d_{k-j} for $j = 1, \dots, k$, we need d_{k-j} ($j = 1, \dots, k$) in more detailed form. It is easy to check by induction that we have for $k \geq 2$ (see [6])

$$f^{(k)} = \left\{ g^{(k)} + k p' g^{(k-1)} + \sum_{j=2}^k [C_k^j (p')^j + H_{j-1}(p')] g^{(k-j)} \right\} e^P \tag{3.4}$$

where $H_{j-1}(p')$ are differential polynomials in p' and its derivatives of total degree $j-1$ with constant coefficients. It is easy to see that the derivatives of $H_{j-1}(p')$ as to z are of the same form $H_{j-1}(p')$. C_k^j is the usual notation for the binomial coefficients. (1.4) and (3.4) give

$$d_{k-1} = k p', \quad d_{k-j} = C_k^j (p')^j + H_{j-1}(p') \quad j = 2, \dots, k-1,$$

$$d_0 = (p')^k + H_{k-1}(p') + A.$$

So $\deg d_{k-j} = j(\beta-1)$ ($j = 1, \dots, k-1$). Since $\beta > (n+k)/k$, we have $\deg d_0 = k(\beta-1)$. By

Lemma 7, the DE (3.3) may have at most one exceptional solution g_0 with $\lambda(g_0) = \sigma(g_0) = \sigma(Q) = \lambda(Q)$, and all the other solutions g of (3.3) satisfy $\bar{\lambda}(g) = \lambda(g) = \sigma(g) = \beta$.

Therefore, the DE (1.4) may have at most one exceptional solution $f_0 = g_0 e^p$ with $\lambda(f_0) = \lambda(F)$, and all the other solutions $f = g e^p$ of (1.4) satisfy

$$\bar{\lambda}(f) = \lambda(f) = \sigma(f) = \beta.$$

Proof of Theorem 4. (a) If $\beta < (n+k)/k$, then by Lemma 5, the DE(1.4) may have at most one exceptional solution f_0 with $\sigma(f_0) = \beta$, and all the other solutions f of (1.4) satisfy $\sigma(f) = (n+k)/k$. By Lemma 6, all the other solutions f of (1.4) satisfy

$$\bar{\lambda}(f) = \lambda(f) = \sigma(f) = \frac{n+k}{k}.$$

(b) If $\beta = (n+k)/k$, then by Lemma 5, all solutions f of (1.4) satisfy $\sigma(f) = (n+k)/k$.

From [3], we have $\lambda(f) \geq \lambda(F)$.

4. Examples of the exceptional solution

Example 2 (concerning the exceptional solution in Theorem 1). $f_0 = e^{z^2}$ solves

$$f'' + (\sin z - 4z^2 - 2)f = e^{z^2} \sin z,$$

there $\sigma(A) < \sigma(F)$, $\sigma(f_0) = \sigma(F)$, $\lambda(f_0) = 0 < \sigma(f_0)$.

Example 3 (concerning the exceptional solution in Theorem 2). Let G be a given transcendental entire function with $\sigma(G) \neq 1$, $\sigma(G) < \infty$, Then $f_0 = e^{3z}$ solves

$$f'' + e^{-z} f' - Gf = e^{3z}(9 + 3e^{-z} - G),$$

there $\sigma(f_0) = 1$, $\bar{\lambda}(f_0) = 0$.

Example 4 (concerning the exceptional solution in Theorem 3). The DE

$$f'' + (1 - 6z)f = 3z^2(2 \cos z + 3z^2 \sin z)e^{z^3}$$

has exceptional solution $f_0 = \sin z \cdot e^{z^3}$, there $\sigma(F) = 3 > (n+k)/k$, and $\sigma(f_0) = 3$, $\lambda(f_0) = 1$. Now we prove that $\lambda(F) = 1$. For the real function $2 \cos x + 3x^2 \sin x$, $\sin m\pi = 0$ and $2 \cos m\pi + 3(m\pi)^2 \cdot \sin m\pi \neq 0$ ($m = \pm 1, \pm 2, \dots$), the zeros of $2 \cos x + 3x^2 \sin x$ are zeros of $\text{ctgx} + \frac{3}{2}x^2$. $\text{ctgx} + \frac{3}{2}x^2$ has zeros $x_m \in (m\pi, (m+1)\pi)$ ($m = \pm 1, \pm 2, \dots$). Hence $\lambda(2 \cos x + 3x^2 \sin x) = 1$. Therefore $\lambda(F) = 1$.

Example 5 (concerning the exceptional solution in Theorem 4). $f_0 = \sin z$ solves

$$f'' + (z^2 + 1)f = z^2 \sin z,$$

there $(n+k)/k = 2 > \sigma(F) = 1$, $\sigma(f_0) = \sigma(F)$.

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