Psychological Review
2004, Vol. 111, No. 1, 236–241

0033-295X/04/$12.00 DOI: 10.1037/0033-295X.111.1.236

THEORETICAL NOTE

On the Composition of Risk Preference and Belief

Peter P. Wakker
Leiden University Medical Center

Prospect theory assumes nonadditive decision weights for preferences over risky gambles. Such decision weights generalize additive probabilities. This article proposes a decomposition of decision weights into a component reflecting risk attitude and a new component depending on belief. The decomposition is based on an observable preference condition and does not use other empirical primitives such as statements of judged probabilities. The preference condition is confirmed by most of the experimental findings in the literature. The implied properties of the belief component suggest that, besides the often-studied ambiguity aversion (a motivational factor reflecting a general aversion to unknown probabilities), perceptual and cognitive limitations play a role: It is harder to distinguish among various levels of likelihood, and to process them differently, when probabilities are unknown than when they are known.

Subjective expected utility theory posits a clear separation between value and belief. For example, consider an investment whose return is uncertain. The degrees of belief about the possible returns are quantified through subjective probabilities, and the subjective values of the possible returns are quantified through utilities. The investment is evaluated by its expected utility, that is, the probability-weighted average utility of its possible returns. In this approach, the assessments of values and of degrees of belief are taken as independent processes.

The situation changes when subjective expected utility is replaced by more general theories that use nonadditive decision weights instead of additive probabilities to calculate average utility. Such theories include prospect theory (Tversky & Kahneman, 1992) and rank-dependent utility theory (Quiggin, 1981). Under such theories, it is no longer obvious that decision weights inferred from preferences can be interpreted as pure measures of belief. It is reasonable to argue that they reflect additional considerations over and above pure belief. This is most clearly seen in the case of decision making under risk, that is, when for each uncertain event an objective probability is known. Then, it is natural to assume that beliefs are captured by the probabilities p and that the decision weights $w(p)$ reflect decision attitude, which gives rise to nonlinearity. For example, if the decision weight of a .001 probability is .01, then this need not mean that the relevant probability is overestimated by a factor of 10. It may mean only that the probability in question is overweighted when making decisions.

The interpretation of decision weights becomes more complicated when moving from risk to uncertainty. Uncertainty refers to situations in which probabilities of uncertain events need not be known. Two major complications enter the picture. Whereas the source that generates the risk (e.g., a roulette wheel, a die, or a standard urn) is not essential in the case of risk, provided that the probabilities are transparent, it is essential for uncertainty. People may have consistent preferences for one source over another (Ellsberg, 1961; Fellner, 1961, p. 672). The second complication when moving from risk to uncertainty is that belief, while equated with objective probability and, therefore, additive in the case of risk, seems to be subadditive in the case of uncertainty. That is, low likelihoods are overestimated and high likelihoods are underestimated.

This article decomposes decision weights for judgments under uncertainty into a component reflecting belief and a component reflecting decision attitude. Such a decomposition was suggested some decades ago by Fellner (1961, p. 672) and has been used in a two-stage model by Fox, Rogers, and Tversky (1996); Fox and Tversky (1998); Kilka and Weber (2001); Tversky and Fox (1995); and Wu and Gonzalez (1999), all of whom used judged probabilities as an additional empirical primitive to obtain the decomposition. For an alternative decomposition, see Epstein and Zhang (1999).

The essential feature of the decomposition in this article is that it is based solely on observable preference and does not use other empirical primitives. The decomposition, therefore, fits the behavioristically oriented revealed-preference approach that is common in the economics literature (Mas-Colell, Whinston, & Green, 1995, p. 5). The properties of the resulting index of belief agree well with

Peter P. Wakker, Medical Decision Making Department, Leiden University Medical Center, Leiden, the Netherlands.

This article started as a joint project with Amos Tversky. Due to his untimely death, I had to complete it on my own and am entirely responsible for any errors. In agreement with the judgment of people close to Amos (D. Kahneman, personal communication, October 4, 2000; B. Tversky, personal communication, September 27, 2000), I have become the sole author of this article.

Rich Gonzalez, Fabio Maccheroni, Mark Machina, Sujoy Mukerji, Daniel Read, and Horst Zank made helpful comments. Special thanks are due to Craig Fox for extensive discussions of the concepts analyzed in this article and for many helpful comments on the writing of the article.

Correspondence concerning this article should be addressed to Peter P. Wakker, who is now at the Department of Economics, University of Amsterdam, Roetersstraat 11, Amsterdam 1018 WB, the Netherlands. E-mail: p.p.wakker@uva.nl

concepts that are not based on revealed preferences and that are studied in psychology, such as in support theory (Tversky & Koehler, 1994). Therefore, the result of this article may help to connect psychological concepts with economic concepts. Such connections give consistency bases to the psychological concepts considered and show the relevance of psychological concepts, such as judged probabilities, for economic decisions.

Experimental Findings on Nonlinear Decision Weights

This article considers only gain outcomes, that is, outcomes preferred to a given reference point. The reference point is called (receiving) nothing henceforth. For loss outcomes, similar theoretical results can be derived, but there is less evidence on what the prevailing empirical phenomena are. For risk, we consider only prospects (p,x) yielding a gain x with probability p and nothing otherwise and assume that their value is $w(p)v(x)$, with $v(x)$ the value or utility of outcome x and $w(p)$ the decision weight of p. The functions v and w can be nonlinear.

Figure 1 illustrates how two kinds of deviations from additive probabilities combine to create the probability weighting functions commonly found. Panel 1a depicts traditional expected utility with probabilities weighted linearly [i.e., $w(p) = p$]. Panel 1b depicts an aversion to risk because the good-outcome probability is underweighted at all levels. Panel 2a depicts another factor, sometimes called diminishing sensitivity. The weighting function is too shallow in the middle region, reflecting insufficient sensitivity to changes in likelihood. Correspondingly, there are jumps at the two bounds of the scale, reflecting too much sensitivity toward changes from impossible to possible and from possible to certain. The regressive shape in Panel 2a, with weights correlating imperfectly with probabilities and with as much overweighting of good as of bad outcomes, suggests that perceptual and cognitive limitations, prior to any consideration of value, underlie this effect. Panels 3a and 3b depict extreme cases of insufficient sensitivity. There are only three degrees of belief: certainly true, certainly not true, or possible.

The two factors in Panels 1b and 2a of Figure 1 have similar effects (underweighting) for high-probability-gain prospects but opposite effects for low-probability-gain prospects. The common experimental finding is a combination of the two factors, depicted in Panel 2b, with small probabilities overweighted. Psychological interpretations of the two factors have been given by Gonzalez and Wu (1999), Lopes (1987), Tversky and Fox (1995), and Weber (1994).

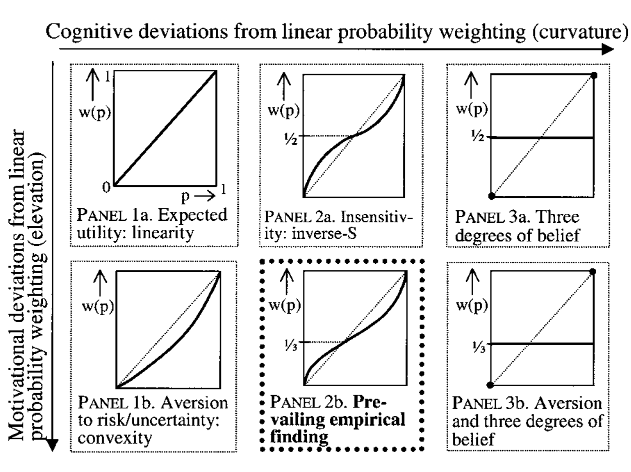

Figure 1. Two factors in probability weighting.

Since Keynes (1921) and Knight (1921), there has been interest in uncertain events with unknown probabilities. For such events, nonlinear decision weights are also needed. Suppose that a prospect, denoted (A,x), yields outcome x if an uncertain event A (e.g., rain tomorrow) occurs and nothing otherwise. The probability of event A is often unknown. We assume that the value of (A,x) is $W(A)v(x)$, with v the value function and W the decision weight. Like v and w, W may be nonadditive, that is, $W(A \cup B) \neq W(A) + W(B)$ is permitted for mutually exclusive events A and B. Here, $A \cup B$ denotes the disjunction, "A or B," of A and B.

For uncertainty, effects similar to those in Figure 1 have been found. These effects can, however, not easily be illustrated in graphs. Fellner (1961, p. 684), Kahneman and Tversky (1979, p. 281, Paragraph −2), and Weber (1994, pp. 237–238) suggested that the effects are more pronounced for uncertainty than for risk. Experimental evidence supporting this suggestion, and a formalization, is given in the next section. The extreme case of insensitivity in Panel 3a is not uncommon for uncertain events. Then, all uncertainties have been lumped together in one category ("fifty-fifty": Fischhoff & Bruine de Bruin, 1999; see also Arrow & Hurwicz, 1972).

A Composition Theorem

In the formal analysis, we assume natural conditions such as, for risk, $w(0) = 0$, $w(1) = 1$, $w(p) > w(q)$ if $p > q$ (w is strictly increasing), and continuity (no jumps in the graph of w). Further, v is 0 at the reference point and $v(x) > 0$ for at least one outcome x. The conditions assumed for W, the weighting function for uncertainty, are similar to those assumed for w. Impossible events have decision weight 0, certain (universal) events have decision weight 1, and if an event contains (i.e., is implied by) another, then it has at least as high a decision weight.

The testable preference conditions for less sensitivity to uncertainty than to risk were introduced by Tversky and Fox (1995) and Tversky and Wakker (1995). They are explained next. We first consider disjoint events A and B with $W(A \cup B)$ bounded away from 1.[1] For uncertain events, the decision maker is less sensitive to changes in the middle of the region than she is for known probabilities. Therefore, in Equation 1, generating the same increase in weight [from $W(A)$ to $W(A \cup B)$, which is the same as from $w(p)$ to $w(p + q)$] requires adding a "heavier" event (B) than probability (q).

$$\text{If } W(A) = w(p) \text{ and } W(A \cup B) = w(p + q), \quad \text{then } W(B) \geq w(q). \quad (1)$$

Throughout this article, $\succ$ denotes strict preference, $\sim$ indifference, $\succcurlyeq$ weak preference (strict preference or indifference), and $\preccurlyeq$ and $\prec$ reversed preferences. Equation 1 can be reformulated di-

[1] Rigorous formulations of the boundary conditions required here and later are available in Tversky and Wakker (1995).

rectly in terms of preferences by relating its W values to the decision weights of the gain x in the following prospects:

$$\text{If } (A,x) \sim (p,x) \text{ and } (A \cup B,x) \sim (p+q,x), \quad \text{then } (B,x) \succcurlyeq (q,x). \quad (2)$$

Equations 1 and 2 concerned decision weights of receiving the gain x instead of nothing. Equations 3 and 4 are, in a way, dual to Equations 1 and 2. They describe exactly the same principle, but now for decision weights of receiving nothing instead of x (as can be seen). As before, A and B are disjoint events, and now $W((A \cup B)^c)$ is bounded away from 0, where a superscript c denotes the complementary event (negation). Generating the same decrease in weight [from $W(A^c)$ to $W((A \cup B)^c)$, which is the same as from $w(1-p)$ to $w(1-p-q)$] requires subtracting a heavier event (B) than probability (q):

$$\text{If } W(A^c) = w(1-p) \text{ and } W((A \cup B)^c) = w(1-p-q), \quad \text{then } W(B^c) \leq w(1-q). \quad (3)$$

$W(B^c) \leq w(1-q)$ means that B is heavier than q in the sense that subtracting B in isolation (reducing certainty by B) leads to a greater loss in decision weight than subtracting q in isolation. Equation 3 can be reformulated in terms of preferences by relating its W values to the decision weights of the gain x in the following prospects:

$$\text{If } (A^c,x) \sim (1-p,x) \text{ and } ((A \cup B)^c,x) \sim (1-p-q,x), \quad \text{then } (B^c,x) \preccurlyeq (1-q,x). \quad (4)$$

Equations 2 and 4 together define *less sensitivity to uncertainty than to risk*. The dual imposing of conditions for likelihood reflects the dual boundedness of the likelihood scale, with the impossible event at one end and the universal event at the other. Abdellaoui, Vossmann, and Weber (2003) and Tversky and Fox (1995) confirmed less sensitivity to uncertainty than to risk. Other articles did not specifically test the conditions, but their data were consistent (Fox et al., 1996; Fox & Tversky, 1998; Hogarth & Kunreuther, 1989; Kahn & Sarin, 1988, p. 271; MacCrimmon & Larsson, 1979, p. 390; Wu & Gonzalez, 1999). However, the three-color Ellsberg example, discussed in the following sections, provides counterevidence.

As we will see next, less sensitivity to uncertainty than to risk implies a decomposition of the weighting function W into two factors w and F, where w is the weighting function for risk and F depends on beliefs in uncertain events. The crucial property of F is *subadditivity*. For disjoint events A and B with $F(A \cup B)$ bounded away from 1, we have the following condition: (i) $F(A \cup B) - F(A) \leq F(B)$, meaning that B adds less to A than B's value in isolation. The same condition holds dually: (ii) $F(A^c) - F(A \cup B)^c \leq 1 - F(B^c)$, if $W((A \cup B)^c)$ is bounded away from 0. Subtracting B after having subtracted A leads to a smaller loss of weight than subtracting B alone. This condition is the analog for uncertainty of the condition illustrated in Panel 2a of Figure 1. Less sensitivity to uncertainty than to risk can be interpreted as greater subadditivity for uncertainty than for risk.

We next turn to the decomposition. Its proof is given in the Appendix. The theorem can be applied to virtually all nonlinear weighting theories (Birnbaum & Beeghley, 1997; Edwards, 1962; Einhorn & Hogarth, 1985; Ghirardato & Marinacci, 2002; Gilboa, 1987; Gilboa & Schmeidler, 1989; Gul, 1991; Kahneman & Tversky, 1979; Luce & Fishburn, 1991; Quiggin, 1981; Schmeidler, 1989; Tversky & Kahneman, 1992).

> Theorem 1: Assume that preferences over single-gain prospects are described by multiplicative representations $(A,x) \mapsto W(A)v(x)$ and $(p,x) \mapsto w(p)v(x)$, respectively. Then, the decision maker is less sensitive to uncertainty than to risk if and only if there exists a subadditive function F such that for all events A, $W(A) = w(F(A))$. F is uniquely determined by the equality $F(A) = p$, with p such that $(p,x) \sim (A,x)$.

A decomposition $W(A) = w(F(A))$ can always be obtained simply by defining $F = w^{\text{inv}}W$. The remarkable aspect of the theorem is that preference conditions, confirmed in experimental studies, exactly match the properties of F that are desirable for a belief interpretation and that have been assumed in two-stage models (e.g., Fox et al., 1996). Abdellaoui et al. (2003) studied the above decomposition experimentally. They elicited the belief component F through equivalences $(p,x) \sim (A,x)$, as above, for gains x. F exhibited subadditivity indeed. Assuming the same belief F for losses, they found that W and, therefore, w exhibited similar sensitivity but higher elevation for losses than for gains.

Source Preference and Subadditivity in the Ellsberg Examples

This section examines two examples that have greatly influenced the thinking of decision theorists about ambiguity: the two Ellsberg examples (Ellsberg, 1961). Prospect theory suggests different interpretations than are commonly adopted, as discussed further in the Discussion section, below.

Source preference means that people prefer to take their chances from one source rather than from another, even when the beliefs associated with each source are the same. Formally, if K and A are events related to two different sources of uncertainty, and $(K,100) \succcurlyeq (A,100)$ and $(K^c,100) \succcurlyeq (A^c,100)$, then Tversky and Fox (1995) and Tversky and Wakker (1995) say that *source preference* holds for K against A. The most famous case is the two-color Ellsberg example, in which K designates the event that a ball randomly drawn from a "known" urn with 50 red and 50 black balls will be red, and A the event that a ball randomly drawn from an "unknown," or "ambiguous," urn with 100 red and black balls in unknown proportion will be red. K has probability 1/2, and the probability of A is unknown. This example is commonly interpreted as evidence for ambiguity aversion, that is, a general aversion to unknown probabilities.

The last line of Theorem 1 shows that in the above example $F(A) = F(A^c) < 1/2$ (the choice of w is immaterial). F violates *binary complementarity*, that is, the requirement $F(A) + F(A^c) = 1$. This implies that expected utility is also violated because expected utility requires, more restrictively, that F in Theorem 1 is a probability measure, in addition to the equality $w(p) = p$. If binary complementarity is taken as a reasonable requirement for belief, then the factor F of Theorem 1 cannot consist only of belief in the two-color Ellsberg example. In the terminology of Figure 1, F then captures motivational as well as cognitive factors. F may be a transform of belief, that is, $F = \varphi(\beta(A))$, with β reflecting belief and φ a catch-all factor comprising source preference. A three-stage decomposition, $W(A) = w(\varphi(\beta(A)))$, then results (Fox & Tversky, 1998, p. 893).[2]

The comparative ignorance hypothesis suggests an alternative interpretation of the two-color Ellsberg example: The source preference may have been generated by the fact that probabilistic information about the unknown urn is deliberately kept secret, rather than by the absence of this information itself (Chow & Sarin, 2001; Fox & Tversky, 1995; Frisch & Baron, 1988; Heath & Tversky, 1991; Keren & Gerritsen, 1999, Condition 1 in Table 2; Taylor, 1995; Viscusi & Magat, 1992, p. 380). The opposite of ambiguity aversion can result, that is, a preference for unknown probabilities, if people feel competent about them (Fox, 1999; Fox & Weber, 2002; Heath & Tversky, 1991; Tversky & Fox, 1995; Wakker, Timmermans, & Machielse, 2002). We next turn to an example in which a complementarity effect in the available information leads to a violation of subadditivity and, consequently, a violation of less sensitivity to uncertainty than to risk.

> Example (Ellsberg's three-color example): One ball is drawn at random from an urn with 30 red (R) balls and 60 black (B) and amber (A) balls in unknown proportion. Most people prefer to receive \$100 for R rather than \$100 for A or \$100 for B. This implies that W does not exhibit more sensitivity to risk than to uncertainty because it violates Equation 1: From the preferences, we derive $W(A) = W(B) < W(R) = w(1/3)$. Then, $W(A) = w(p)$ implies $p < 1/3$. $W(A \cup B) = w(2/3) = w(p + q)$ implies $q > 1/3$. We get $w(q) > w(1/3) > W(B)$, contradicting the conclusion of Equation 1. Similarly, under some plausible assumptions (e.g., that v is not very curved), we can derive $W(A) + W(B) < W(A \cup B)$, which violates subadditivity.

The lack of probabilistic information is again salient in the three-color example, as it was in the two-color example. In addition, there is an exceptional complementarity effect between the information about B and A, making each in isolation especially aversive. This complementarity effect induces the violation of subadditivity and, consequently, of less sensitivity to uncertainty than to risk.

That the three-color Ellsberg example violates subadditivity was pointed out to me by Martin Weber at Stanford University in March 1993. Amos Tversky's explanation was that the information structure in this example is exceptional and that uncertainty is a richer domain than risk, with exceptions existing for most laws. In a similar vein, utility for general outcomes is a richer domain than utility for money, and exceptions will exist for most laws regarding utility if general outcomes are considered.

Discussion

If source preference, as in the two-color Ellsberg example, is a genuine and important factor in individual preference, then a three-stage composition $W(A) = w(\varphi(\beta(A)))$, extending the model of Theorem 1, should be considered. Alternatively, source preference may mostly reflect social effects (suspicion about an opponent) and context effects (competence). Such effects, even if strong, are not intrinsic parts of transitive individual preference under uncertainty.

For likely events, less sensitivity to uncertainty reinforces the effect of ambiguity aversion: a relative dislike of uncertain events compared with risk. Many experimental studies have confirmed this effect. For unlikely events, less sensitivity generates an effect opposite to ambiguity aversion: People overweight uncertain events, which enhances ambiguity seeking. The latter is contrary to the universal ambiguity aversion that has been assumed in many theoretical studies. If both effects operate simultaneously, the ambiguity attitude for unlikely events may be close to neutral. Ellsberg (1962) himself predicted ambiguity seeking for gains contingent on unlikely events (pp. 268–270), and several experimental studies found it (Curley & Yates, 1989; Fox et al., 1996; Fox & Tversky, 1998; Kilka & Weber, 2001; Tversky & Fox, 1995; Wu & Gonzalez, 1999). However, there are mixed results in Einhorn and Hogarth (1985), Fox and Tversky (1995), Hogarth and Einhorn (1990), Kahn and Sarin (1988), and Sarin and Weber (1993). Gambling and insurance, whose coexistence is a classical paradox in economics, can both be explained by the overweighting of unlikely events (regarding loss outcomes in the case of insurance).

A number of theoretical studies in economics have avoided the use of objective probabilities. Probabilistic beliefs are then defined in the case in which W (and consequently F) is any ordinal transformation of an additive measure, which is endogenously derived from preference (Machina & Schmeidler, 1992, "probabilistic sophistication"). Ambiguity aversion is likewise taken as a purely ordinal property of likelihood weighting, and some subjective neutrality benchmarks have been proposed (Epstein & Zhang, 2001; Ghirardato & Marinacci, 2002). I have adopted an absolute scale F, and thus, more restrictive concepts have resulted. The aforementioned studies are normatively oriented, whereas this article is descriptively oriented. Assuming objective probabilities, one can take these probabilities as a convenient neutrality benchmark for aversion or attraction to ambiguity.

Primarily because of the Ellsberg examples, most of the current literature has focused on ambiguity aversion and the move from Panel 1a to Panel 1b in Figure 1. This article has suggested that the implications of the Ellsberg examples may be less clear than is usually thought and that cognitive and perceptual processes and the move from Panel 1a to Panel 2a also deserve attention.

[2] Representation theorems in work in progress (Wakker, 2003) imply this decomposition rather than $\varphi(w(\beta(A)))$.

References

Abdellaoui, M., Vossmann, F., & Weber, M. (2003). *Choice-based elicitation and decomposition of decision weights for gains and losses under uncertainty.* GRID-CNRS, Ecole Normale Supérieure, Cachan, France.

Arrow, K. J., & Hurwicz, L. (1972). An optimality criterion for decision making under ignorance. In C. F. Carter & J. L. Ford (Eds.), *Uncertainty and expectations in economics: Essays in honour of G. L. S. Shackle* (pp. 1–11). Oxford, England: Basil Blackwell.

Birnbaum, M. H., & Beeghley, D. (1997). Violations of branch independence in judgments of the value of gambles. *Psychological Science, 8,* 87–94.

Chow, C. C., & Sarin, R. K. (2001). Comparative ignorance and the Ellsberg paradox. *Journal of Risk and Uncertainty, 22,* 129−139.

Curley, S. P., & Yates, J. F. (1989). An empirical evaluation of descriptive models of ambiguity reactions in choice situations. *Journal of Mathematical Psychology, 33,* 397−427.

Edwards, W. (1962). Subjective probabilities inferred from decisions. *Psychological Review, 69,* 109−135.

Einhorn, H. J., & Hogarth, R. M. (1985). Ambiguity and uncertainty in probabilistic inference. *Psychological Review, 93,* 433−461.

Ellsberg, D. (1961). Risk, ambiguity and the savage axioms. *Quarterly Journal of Economics, 75,* 643−669.

Ellsberg, D. (1962). *Risk, ambiguity and decision.* Unpublished manuscript, Harvard University.

Epstein, L. G., & Zhang, J. (1999). Least convex capacities. *Economic Theory, 13,* 263−286.

Epstein, L. G., & Zhang, J. (2001). Subjective probabilities on subjectively unambiguous events. *Econometrica, 69,* 265–306.

Fellner, W. (1961). Distortion of subjective probabilities as a reaction to uncertainty. *Quarterly Journal of Economics, 75,* 670−689.

Fischhoff, B., & Bruine de Bruin, W. (1999). Fifty-fifty = 50%? *Journal of Behavioral Decision Making, 12,* 149−163.

Fox, C. R. (1999). Strength of evidence, judged probability, and choice under uncertainty. *Cognitive Psychology, 38,* 167–189.

Fox, C. R., Rogers, B. A., & Tversky, A. (1996). Options traders exhibit subadditive decision weights. *Journal of Risk and Uncertainty, 13,* 5−17.

Fox, C. R., & Tversky, A. (1995). Ambiguity aversion and comparative ignorance. *Quarterly Journal of Economics, 110,* 585−603.

Fox, C. R., & Tversky, A. (1998). A belief-based account of decision under uncertainty. *Management Science, 44,* 879−895.

Fox, C. R., & Weber, M. (2002). Ambiguity aversion, comparative ignorance, and decision context. *Organizational Behavior and Human Decision Processes, 88,* 476−498.

Frisch, D., & Baron, J. (1988). Ambiguity and rationality. *Journal of Behavioral Decision Making, 1,* 149−157.

Ghirardato, P., & Marinacci, M. (2002). Ambiguity made precise: A comparative foundation. *Journal of Economic Theory, 102,* 251–289.

Gilboa, I. (1987). Expected utility with purely subjective non-additive probabilities. *Journal of Mathematical Economics, 16,* 65–88.

Gilboa, I., & Schmeidler, D. (1989). Maxmin expected utility with a non-unique prior. *Journal of Mathematical Economics, 18,* 141–153.

Gonzalez, R., & Wu, G. (1999). On the shape of the probability weighting function. *Cognitive Psychology, 38,* 129−166.

Gul, F. (1991). A theory of disappointment aversion. *Econometrica, 59,* 667–686.

Heath, C., & Tversky, A. (1991). Preference and belief: Ambiguity and competence in choice under uncertainty. *Journal of Risk and Uncertainty, 4,* 5−28.

Hogarth, R. M., & Einhorn, H. J. (1990). Venture theory: A model of decision weights. *Management Science, 36,* 780−803.

Hogarth, R. M., & Kunreuther, H. C. (1989). Risk ambiguity and insurance. *Journal of Risk and Uncertainty, 2,* 5−35.

Kahn, B. E., & Sarin, R. K. (1988). Modeling ambiguity in decisions under uncertainty. *Journal of Consumer Research, 15,* 265−272.

Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica, 47,* 263−291.

Keren, G. B., & Gerritsen, L. E. M. (1999). On the robustness and possible accounts of ambiguity aversion. *Acta Psychologica, 103,* 149−172.

Keynes, J. M. (1921). *A treatise on probability.* London: McMillan.

Kilka, M., & Weber, M. (2001). What determines the shape of the probability weighting function under uncertainty. *Management Science, 47,* 1712–1726.

Knight, F. H. (1921). *Risk, uncertainty, and profit.* New York: Houghton Mifflin.

Krantz, D. H., & Tversky, A. (1975). Similarity of rectangles: An analysis of subjective dimensions. *Journal of Mathematical Psychology, 12,* 4–34.

Lopes, L. L. (1987). Between hope and fear: The psychology of risk. *Advances in Experimental Psychology, 20,* 255−295.

Luce, R. D., & Fishburn, P. C. (1991). Rank- and sign-dependent linear utility models for finite first-order gambles. *Journal of Risk and Uncertainty, 4,* 29−59.

MacCrimmon, K. R., & Larsson, S. (1979). Utility theory: Axioms versus "paradoxes." In M. Allais & O. Hagen (Eds.), *Expected utility hypotheses and the Allais paradox* (pp. 333–409). Dordrecht, the Netherlands: Reidel.

Machina, M. J., & Schmeidler, D. (1992). A more robust definition of subjective probability. *Econometrica, 60,* 745−780.

Mas-Colell, A., Whinston, M. D., & Green, J. R. (1995). *Microeconomic theory.* New York: Oxford University Press.

Pratt, J. W. (1964). Risk aversion in the small and in the large. *Econometrica, 32,* 122−136.

Quiggin, J. (1981). Risk perception and risk aversion among Australian farmers. *Australian Journal of Agricultural Economics, 25,* 160−169.

Sarin, R. K., & Weber, M. (1993). Effects of ambiguity in market experiments. *Management Science, 39,* 602−615.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica, 57,* 571−587.

Taylor, K. A. (1995). Testing credit and blame attributions as explanation for choices under ambiguity. *Organizational Behavior and Human Decision Processes, 54,* 128−137.

Tversky, A., & Fox, C. R. (1995). Weighing risk and uncertainty. *Psychological Review, 102,* 269−283.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty, 5,* 297−323.

Tversky, A., & Koehler, D. J. (1994). Support theory: A nonextensional representation of subjective probability. *Psychological Review, 101,* 547−567.

Tversky, A., & Wakker, P. P. (1995). Risk attitudes and decision weights. *Econometrica, 63,* 1255−1280.

Viscusi, W. K., & Magat, W. A. (1992). Bayesian decisions with ambiguous belief aversion. *Journal of Risk and Uncertainty, 5,* 371–387.

Wakker, P. P. (1994). Separating marginal utility and probabilistic risk aversion. *Theory and Decision, 36,* 1−44.

Wakker, P. P. (2003). *A three-factor decomposition of chance attitude.* Manuscript in preparation, University of Amsterdam, the Netherlands.

Wakker, P. P., Timmermans, D. R. M., & Machielse, I. A. (2002). *The effects of information about probabilities on insurance decisions.* Unpublished manuscript, University of Amsterdam, the Netherlands.

Weber, E. U. (1994). From subjective probabilities to decision weights: The effects of asymmetric loss functions on the evaluation of uncertain outcomes and events. *Psychological Bulletin, 115,* 228−242.

Wu, G. (1999). Anxiety and decision making with delayed resolution of uncertainty. *Theory and Decision, 46,* 159−198.

Wu, G., & Gonzalez, R. (1999). Nonlinear decision weights in choice under uncertainty. *Management Science, 45,* 74−85.

Yaari, M. E. (1969). Some remarks on measures of risk aversion and on their uses. *Journal of Economic Theory, 1,* 315−329.

Appendix

Proof of Theorem 1

Theorem 1 is of a different nature than previous transformation results in the literature, including those by Pratt (1964) and Yaari (1969) for utility and Tversky and Wakker (1995), Wakker (1994), and Wu (1999) for weighting functions. The existing results invariably applied the transformation to the images (output) of the function, that is, outside the brackets. In Theorem 1, the transformation F applies to the arguments (input) of the function, that is, inside the brackets. The theorem of this article is based on Krantz and Tversky's (1975) more-curved-than technique. We now turn to the proof.

Because w is continuous and strictly increasing, we can apply its inverse, denoted by w^{inv}, to Equation 1, to obtain the following:

$$\text{If } w^{\text{inv}}(W(A)) = p \text{ and } w^{\text{inv}}(W(A \cup B)) = p + q, \text{ then } w^{\text{inv}}(W(B)) \geq q.$$

For A and B as described, this implication is equivalent to

$$w^{\text{inv}}(W(B)) \geq w^{\text{inv}}(W(A \cup B)) - w^{\text{inv}}(W(A))$$

because p and q can simply be defined as above. The latter inequality is exactly condition (i) of the subadditivity definition for $w^{\text{inv}}(W(.))$.

We next apply w^{inv} to Equation 3 to obtain the following:

$$\text{If } w^{\text{inv}}(W(A^c)) = 1 - p \text{ and } w^{\text{inv}}(W(A \cup B)^c) = 1 - p - q,$$
$$\text{then } w^{\text{inv}}(W(B^c)) \leq 1 - q.$$

With the last inequality rewritten as $1 - w^{\text{inv}}(W(B)^c) \geq q$ for the events A and B as described, we get, equivalently,

$$1 - w^{\text{inv}}(W(B)^c) \geq w^{\text{inv}}(W(A^c)) - w^{\text{inv}}(W(A \cup B)^c).$$

This inequality is precisely condition (ii) of subadditivity for $w^{\text{inv}}(W(.))$. We conclude that $F(.) = w^{\text{inv}}(W(.))$ satisfies subadditivity if and only if Equations 1 and 3 hold. These equations imply Equations 2 and 4. The reversed implication holds because there exists, besides the reference point, an outcome x with $v(x) > 0$. Therefore, $F(.) = w^{\text{inv}}(W(.))$ satisfies subadditivity if and only if Equations 2 and 4 hold. The uniqueness of F follows from w being strictly increasing, so that the p at the end of Theorem 1 is unique for $v(x) > 0$.

On the domain considered in this article, that is, the single-gain prospects, the multiplicative representation $W \times v$ determines W (and w and v) only up to a power. All formal results in this article are invariant with respect to power transformations, including the uniqueness of F and the boundary conditions alluded to in Footnote 1. For determining absolute levels of w and W and, for example, for verifying their additivity, powers should be determined. Under prospect theory, powers can be uniquely identified through prospects with two (or more) nonzero outcomes.

Received March 29, 2000
Revision received February 10, 2003
Accepted February 10, 2003 ■