ON THE COMPOUND GENERALIZED POISSON DISTRIBUTIONS

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ABSTRACT

GOOVAERTS and KAAS (1991) present a recursive scheme, involving Panjer's recursion, to compute the compound generalized Poisson distribution (CGPD). In the present paper, we study the CGPD in detail. First, we express the generating functions in terms of Lambert's W function. An integral equation is derived for the pdf of CGPD, when the claim severities are absolutely continuous, from the basic principles. Also we derive the asymptotic formula for CGPD when the distribution of claim severity satisfies certain conditions. Then we present a recursive formula somewhat different and easier to implement than the recursive scheme of GOOVAERTS and KAAS (1991), when the distribution of claim severity follows an arithmetic distribution, which can be used to evaluate the CGPD. We illustrate the usage of this formula with a numerical example.

KEYWORDS

Compound generalized Poisson distributions; moments; integral equations; recursive equation; tail behaviour.

1. INTRODUCTION

Modelling the claim frequency data is one of the most important areas in risk theory. Traditionally, the Poisson distribution, when the mean number of claims is equal to its variance, and the negative binomial distribution, when the variance of the number of claims exceeds its mean, have been used because of their convenient mathematical properties. Several authors including GOSSIAUX and LEMAIRE (1981), SEAL (1982) and WILLMOT (1987) have considered alternatives to Poisson and negative binomial distributions for this purpose. CONSUL (1990) has compared the Generalized Poisson distribution (GPD) suggested by CONSUL and JAIN (1973) with several well known distributions and concluded that GPD is a plausible model for claim frequency data. GOOVAERTS and KAAS (1991) presented a recursive scheme to compute the total claim distribution under the assumptions that the claims are independently and identically distributed integer random variables with the GPD claim frequency.

In this paper, we discuss the compound generalized Poisson distribution (CGPD) in detail and derive a somewhat easy to programmable recursive relation than one given by GOOVAERTS and KAAS (1991). In Section 2, we present a brief summary

of the properties of the generalized Poisson distribution. In Section 3, we express the generating functions of CGPD in terms of Lambert's W function and illustrate the derivation of moments. In Section 4, we present an integral equation similar to Volterra's integral equation of second kind for the density function of CGPD when the distribution of claim severity is absolutely continuous. In addition, we discuss the tail behaviour of CGPD when the claim severity is non arithmetic. In Section 5, we present a recursive formula for the probability function of CGPD when the distribution of claim severity is arithmetic. We illustrate the usage of this formula through an example.

2. GENERALIZED POISSON DISTRIBUTION (GPD)

CONSUL and JAIN (1973) proposed a new generalization of the discrete Poisson distribution which was modified by CONSUL and SHOUKRI (1985) to: A discrete random variable N is said to have a generalized Poisson distribution (GPD) if its probability mass function is given by

$$(2.1) \quad \Pr(N=n) = p_n(\lambda, \theta) = \begin{cases} \lambda(\lambda + n\theta)^{n-1} \dfrac{\exp(-\lambda - n\theta)}{n!} & \text{for } n = 0, 1, 2 \dots \\ 0 & \text{for } n > m \text{ when } \theta < 0 \end{cases}$$

and zero otherwise, where $\lambda > 0$, $\max(-1, -\lambda/m) \le \theta < 1$ and $m(\ge 4)$ is the largest positive integer for which $\lambda + \theta m > 0$ when θ is negative. This generalization of the Poisson probability model in the sense that is probability generating function (pgf) is given by the Lagrange expansion of any pgf under a suitable transformation (CONSUL and SHENTON (1972)). The GPD reduces to the Poisson distribution when $\theta = 0$ and it possesses the twin properties of over-dispersion and under-dispersion according as $\theta > 0$ or $\theta < 0$. The GPD gets truncated for negative values of θ but the truncation error is always less than 0.07%. A recent book by CONSUL (1989) discusses various properties, inference and numerous applications of this model in biology, ecology, and other disciplines. For simplicity, from here on we assume the parameter $\theta > 0$. AMBAGASPITIYA and BALAKRISHNAN (1993) has recently expressed the moment generating function $M_N(t)$ and the probability generating function of the GPD in terms of Lambert's W function when $\theta > 0$ as follows:

$$(2.2) \qquad M_N(t) = \exp\left\{ -\frac{\lambda}{\theta} [W(-\theta \exp(-\theta + t)) + \theta] \right\}$$

$$(2.3) \qquad P_N(z) = \exp\left\{ -\frac{\lambda}{\theta} [W(-\theta z \exp(-\theta)) + \theta] \right\}$$

where W is the Lambert's W function defined as

$$W(x) \exp(W(x)) = x.$$

For more details about Lambert's W function see CORLESS et al. (1994).

2.1. Central moments of GPD

We can obtain the central moments of GPD by differentiating (2.2) with respect to t as illustrated by AMBAGASPITIYA and BALAKRISHNAN (1993), or from the basic principles as described by CONSUL (1989), or by using the method suggested by GOOVAERTS and KAAS (1991). The resulting expressions for first four central moments are as follows:

$$(2.4) \qquad \begin{aligned} \mu_1 &= \lambda M \\ \mu_2 &= \lambda M^3 \\ \mu_3 &= \lambda (3M - 2) M^4 \\ \mu_4 &= 3\lambda^2 M^6 + \lambda (15 M^2 - 20 M + 6) M^5 \end{aligned}$$

where $M = (1 - \theta)^{-1}$.

2.2. Maximum likelihood estimators of λ and θ

Let a random sample of n items be taken from the GPD model and let $x_1, x_2, \ldots, x_n$ be their corresponding values. If the sample values are classified into class frequencies and n_i denotes the frequency of the ith class ($n_i = \# \{x_j : 1 \leq j \leq n, x_j = i\}$), the ML estimate $\hat{\theta}$ as described in CONSUL and SHOUKRI (1984) is given by the unique root of θ given by the equation

$$(2.5) \qquad \sum_{i=0}^{k} n_i \frac{i(i-1)}{\bar{x} + (i - \bar{x})\theta} - n\bar{x} = 0$$

where k (≤ 2) is the number of classes, $n = \sum_{i=1}^{k} n_i$ and $\bar{x}$ is the sample mean. Note that (2.5) does not give a value for θ when $k = 0$ or 1. The ML estimate $\hat{\lambda}$ is then given by

$$(2.6) \qquad \hat{\lambda} = \bar{x}(1 - \hat{\theta})$$

2.3. Tail behaviour of GPD

Lemma 2.1: For fixed λ, θ and $n \to \infty$

$$(2.7) \qquad \Pr(N = n) \approx \frac{\lambda}{\theta \sqrt{2\pi}} \exp\left(-\lambda + \frac{\lambda}{\theta}\right) n^{-3/2} \cdot (\theta \exp(1 - \theta))^n$$

Proof:

For large n, using the Stirling approximation to $n!$ we can write the pmf in (2.1) as

$$(2.8) \qquad \Pr(N = n) \approx \frac{\lambda (\lambda + n\theta)^{n-1} \exp(-\lambda - n\theta)}{\sqrt{2\pi} n^{n+1/2} \exp\left(-n + \frac{\theta_1}{12n}\right)}$$

where $\theta_1 = \theta_1(n)$ satisfies $0 < \theta_1 < 1$. After some rearrangement, we have

$$(2.9) \quad \Pr(N = n) \approx \left\{ \frac{\lambda}{\theta\sqrt{2\pi}} \left[1 + \frac{\lambda}{\theta n} \right]^{n-1} \exp\left(-\lambda - \frac{\theta_1}{12n} \right) \right\} n^{-3/2} \left(\theta \exp(1-\theta) \right)^n;$$

Note that the term inside the { } tends to the required constant as $n \to \infty$ and hence the proof.

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3. COMPOUND GENERALIZED POISSON DISTRIBUTION (CGPD)

Let N denote the number of claims produced by a portfolio of policies in a given time period. Let X_i denote the amount of the ith claim. Then

$$(3.1) \qquad\qquad\qquad S = X_1 + X_2 + \ldots + X_N$$

represents the aggregate claims generated by the portfolio for the period under study. In order to make the model tractable, two fundamental assumptions are made in risk theory and they are

1. $X_1, X_2, \ldots$ are identically distributed random variables with the distribution function $F(x)$.
2. The random variables $N, X_1, X_2, \ldots$ are mutually independent.

When a GPD is chosen for N, the distribution of S is called a compund generalized Poisson distribution. In terms of the convolution operation, we can write the distribution function of S as:

$$F_S(x) = \sum_{n=0}^{\infty} F^{*n}(x) \lambda (\lambda + n\theta)^{n-1} \frac{\exp(-\lambda - n\theta)}{n!}.$$

The moment generating function of S is given by

$$(3.2) \qquad\qquad M_S(t) = M_N(\log M_X(t)),$$

where $M_N(t)$ is the moment generating function (mgf) of the GPD and $M_X(t)$ is the mgf of the claim amount distribution. By using the expression given in (2.2), we can write the mgf of S as

$$(3.3) \qquad\qquad M_S(t) = \exp\left\{ -\frac{\lambda}{\theta} \left[W(-\theta \exp(-\theta) M_X(t)) + \theta \right] \right\}.$$

Similarly, the probability generating function (pgf) of S, when the distribution of claim severity is arithmetic, can be written as

$$(3.4) \qquad\qquad P_S(z) = \exp\left\{ -\frac{\lambda}{\theta} \left[W(-\theta \exp(-\theta) P_X(z)) + \theta \right] \right\},$$

where $P_X(z)$ is the pgf of claim amount distribution.

3.1. Central moments of S

The moments of S can be obtained by directly differentiating the mgf of S given in (3.3). For this differentiation, one may use the following identity, involving Lamberts W functions:

$$\frac{dW(x)}{dx} = \frac{W(x)}{x(1 + W(x))}.$$

After some lengthy algebra, we obtain the following expressions for the first three central moment of S:

$$E(S) = \lambda p_1 M$$
$$\text{Var}(S) = \lambda p_2 M^3 + \lambda (p_2 - p_1^2) M$$
$$E((S - E(S))^3) = \lambda (3M - 2) p_1^3 M^4 + 3\lambda p_1 (p_2 - p_1^2) M^3 + (p_3 - 3p_2 p_1 + 2p_1^3)\lambda M$$

where $M = (1 - \theta)^{-1}$ and p_i, $i = 1, 2, 3$ are the ith non-central moments of claim severity.

4. PROPERTIES OF CGPD: ABSOLUTELY CONTINUOUS SEVERITIES

Theorem 4.1: If the claim sizes are absolutely continuous with pdf $f(x)$ for $x > 0$, then the pdf $g(\lambda, \theta; x)$ of CGPD satisfy the integral equation

$$(4.1) \quad g(\lambda, \theta; x) = p_1(\lambda, \theta) f(x) + \frac{\lambda}{\lambda + \theta} \int_0^x \left(\theta + \lambda \frac{y}{x}\right) g(\lambda + \theta, \theta; x - y) f(y)\, dy$$

where $p_1(\lambda, \theta) = \Pr(N = 1)$ in the GPD with parameters λ and θ.

Proof:

Consider

$$(4.2) \qquad g(\lambda, \theta; x) = \sum_{i=1}^{\infty} p_i(\lambda, \theta) f^{*i}(x)$$

$$(4.3) \qquad = p_1(\lambda, \theta) f(x) + \sum_{i=2}^{\infty} p_i(\lambda, \theta) f^{*i}(x)$$

By using the following identity of GPD,

$$(4.4) \quad p_i(\lambda, \theta) = \frac{\lambda}{\lambda + \theta} \left(\theta p_{i-1}(\lambda + \theta, \theta) + \frac{\lambda}{i} p_{i-1}(\lambda + \theta, \theta)\right) \qquad i = 1, 2, \ldots$$

we have

$$(4.5) \qquad \sum_{i=2}^{\infty} p_i(\lambda, \theta) f^{*i}(x) = \frac{\lambda}{\lambda + \theta} \left(\theta \sum_{i=2}^{\infty} p_{i-1}(\lambda + \theta, \theta) f^{*i}(x) + \right.$$

$$\left. + \lambda \sum_{i=2}^{\infty} \frac{p_{i-1}(\lambda + \theta, \theta)}{i} f^{*i}(x)\right)$$

Using the identities

$$(4.6) \qquad f^{*i}(x) = \int_0^x f^{*(i-1)}(x-y)f(y)\,dy$$

and

$$(4.7) \qquad \frac{f^{*i}(x)}{i} = \int_0^x \frac{y}{x} f^{*(i-1)}(x-y)f(y)\,dy$$

we have

$$(4.8) \quad \sum_{i=2}^{\infty} p_i(\lambda,\theta)f^{*i}(x) = \frac{\lambda}{\lambda+\theta}\left(\theta \sum_{i=2}^{\infty} p_{i-1}(\lambda+\theta,\theta)\int_0^x f^{*(i-1)}(x-y)f(y)\,dy \right.$$

$$\left. + \lambda \sum_{i=2}^{\infty} p_{i-1}(\lambda+\theta,\theta)\int_0^x \frac{y}{x}f^{*(i-1)}(x-y)f(y)\,dy\right)$$

By interchanging the order of summation and the integration and realizing the fact

$$(4.9) \qquad \sum_{i=2}^{\infty} p_{i-1}(\lambda+\theta,\theta)f^{*(i-1)}(x-y) = \sum_{i=1}^{\infty} p_i(\lambda+\theta,\theta)f^{*i}(x-y)$$

$$(4.10) \qquad\qquad\qquad\qquad = g(\lambda+\theta,\theta;x-y)$$

we have

$$(4.11) \qquad \sum_{i=2}^{\infty} p_i(\lambda,\theta)f^{*i}(x) = \frac{\lambda}{\lambda+\theta}\int_0^x \left(\theta+\lambda\frac{y}{x}\right)g(\lambda+\theta,x-y)f(y)\,dy$$

Substitution of (4.11) in (4.3) yields the required result and hence the theorem.

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One has to solve the integral equation (4.1) numerically. Although, there are many algorithms and implementations available to solve Volterra integral equations of the second kind, one has to modify them to solve (4.1). We are currently investigating the problem of finding the best algorithm and we hope to report this finding in a future article.

4.1. Tail behaviour of CGPD

Theorem 4.2: If there exists a number $\kappa > 0$ satisfying

$$(4.12) \qquad\qquad \frac{\exp(\theta)}{e\theta} = \mathcal{L}_X(-\kappa)$$

for X non-arithmetic and if $- \mathcal{L}'_X(-\kappa) < \infty$, then

$$(4.13) \qquad\qquad 1 - F_S(x) \sim C x^{-3/2} \exp(-\kappa x),$$

where C is given by

$$C = \frac{\lambda}{\theta\sqrt{2\pi}} \exp\left(-\lambda + \frac{\lambda}{\theta}\right) \sqrt{\kappa\left[-\frac{\exp(\theta-1)}{\theta} \mathcal{L}'_x(-\kappa)\right]}$$

Proof:

The Proof of this theorem directly follows from the Lemma 2.1 and from the theorem of EMBRECHTS, MAEJIMA, and TEUGELS (1982).

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5. PROPERTIES OF CGPD: ARITHMETIC SEVERITIES

Theorem 5.1: If the claim sizes are random variables on the positive integers with probability mass function $f(x) = \Pr(X = x)$, $x = 0, 1, 2, \ldots$, then the probability mass function $g(\lambda, \theta; x)$ of CGPD satisfies the recurrence equation

$$(5.1) \qquad g(\lambda, \theta; x) = \frac{\lambda}{\lambda + \theta} \sum_{y=1}^{x} \left(\theta + \lambda \frac{y}{x}\right) g(\lambda + \theta, \theta; x - y) f(y).$$

Proof:

This theorem can be proved following the same line of reasoning as Theorem 4.1 or the standard proof of Panjer's recursion (see Theorem 6.6.1 and Corollary 6.6.1 in PANJER and WILLMOT (1992)).

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A result analogue to Theorem 4.2 can be established for discrete severity case using Lemma 2.1 and the theorem given in WILMOT (1989).

5.1. Recursive evaluation

The recursive formula given in (5.1) is easily programmable and also simple to use for manual calculations. For the latter, one may use the following schematic approach:

$g(\lambda, \theta, 0) \quad g(\lambda + \theta, \theta, 0) \quad g(\lambda + 2\theta, \theta, 0) \quad g(\lambda + 3\theta, \theta, 0) \quad g(\lambda + 4\theta, \theta, 0)$

$g(\lambda, \theta, 1) \quad g(\lambda + \theta, \theta, 1) \quad g(\lambda + 2\theta, \theta, 1) \quad g(\lambda + 3\theta, \theta, 1)$

$g(\lambda, \theta, 2) \quad g(\lambda + \theta, \theta, 2) \quad g(\lambda + 2\theta, \theta, 2)$

$g(\lambda, \theta, 3) \quad g(\lambda + \theta, \theta, 3)$

$g(\lambda, \theta, 4)$

The first row of the above scheme is obtained by using the fact that $g(\lambda + i\theta, 0) = p_0(\lambda + i\theta, \theta) = \exp(-\lambda - i\theta)$ for $i = 0, 1 \ldots$. To calculate the probability mass function given in the (i, j)th location, one has to use the elements in $(l, j + 1)$ where $l = 0, 1, \ldots, i - 1$. Since the scheme is of an upper diagonal form, we can carry out the computations for each row starting from right to left. For example, if one wishes to compute $g(\lambda, \theta, 4)$ one may start from $g(\lambda + 4\theta, \theta, 0)$ and move along the diagonal from right to left, i.e. calculate $g(\lambda + (4 - i)\theta, \theta, i)$, $i = 0, 1, 2, 3, 4$ in that order.

Example:

Suppose that S has a CGPD with $\lambda = 0.8$, $\theta = 0.5$ and the distribution of individual claim amounts is as follows:

x	$Pr(X = x)$
1	0.25
2	0.45
3	0.30

Then, by using the recursive method described above, the pmf of S has been tabulated for $s = 0(1)59$ and these values are presented in Table 1.

TABLE 1

THE PROBABILITY MASS FUNCTION OF S

s	$Pr(S = s)$	s	$Pr(S = s)$	s	$Pr(S = s)$
0	.44933	20	.00269	40	.00017
1	.05451	21	.00231	41	.00015
2	.10555	22	.00198	42	.00013
3	.09329	23	.00171	43	.00012
4	.04809	24	.00148	44	.00010
5	.04813	25	.00128	45	.00009
6	.03595	26	.00111	46	.00008
7	.02737	27	.00096	47	.00007
8	.02320	28	.00083	48	.00006
9	.01835	29	.00073	49	.00006
10	.01505	30	.00063	50	.00005
11	.01248	31	.00055	51	.00004
12	.01029	32	.00048	52	.00004
13	.00860	33	.00042	53	.00003
14	.00720	34	.00037	54	.00003
15	.00605	35	.00032	55	.00003
16	.00512	36	.00028	56	.00002
17	.00434	37	.00025	57	.00002
18	.00369	38	.00022	58	.00002
19	.00315	39	.00019	59	.00002

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