# ON THE COMPUTABILITY OF CONJUGATE POWERS IN FINITELY GENERATED FUCHSIAN GROUPS 

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## § 1. Introduction

We define the set of conjugate powers of elements $U$ and $W$ in any group $G$, denoted $C P_{G}(U, W)$, by

$$
C P_{G}(U, W)=\left\{(x, y) \in \mathbf{Z}^{2} ; U^{x} \sim_{G} W^{v}\right\}
$$

where " $\sim_{G}$ " denotes the conjugacy relation in $G$. In this paper we show how these sets $C P_{G}(U, W)$ can be effectively computed for most finitely generated (henceforth f.g.) Fuchsian groups. The Fuchsian groups are the discrete subgroups of the group of all $2 \times 2$ real matrices with determinant +1 .

By a result of Poincaré [11] (see also [8]), the class of f.g. Fuchsian groups consists of free products of cyclic groups, together with the groups

$$
G=\left\langle a_{1}, b_{1}, \ldots, a_{r}, b_{r}, c_{1}, \ldots, c_{s} ; c_{1}^{n_{1}}, \ldots, c_{s}^{n_{4}}, R\right\rangle
$$

where $R$ is the word $a_{1} b_{1} a_{1}^{-1} b_{1}^{-1} \ldots a_{r} b_{r} a_{r}^{-1} b_{r}^{-1} c_{1} \ldots c_{s}, r, s \geqslant 0, n_{t}>1$ for each $1 \leqslant i \leqslant s$, and

$$
2 r-2+\sum_{i=1}^{s}\left(1-n_{i}^{-1}\right)>0 .
$$

It therefore follows that the groups

$$
\begin{equation*}
G=\left\langle c_{1}, c_{2} ; c_{1}^{n_{1}}, c_{2}^{n_{2}},\left(c_{1} c_{2}\right)^{n_{0}}\right\rangle \tag{1}
\end{equation*}
$$

are Fuchsian when $n_{1}^{-1}+n_{2}^{-1}+n_{3}^{-1}<1$. The methods used in this paper do not apply to these groups. They do, however, apply to a more general class than the romaining f.g. Fuchsian groups. Let us denote this new class by $\Gamma_{1}$ and indicate how it is constructed.

To construct $\Gamma_{1}$ we start with a class $\Gamma_{0}$ consisting of free groups and certain treeproducts of one-relator groups with torsion. We then get $\Gamma_{1}$ by forming all free products of groups from $\Gamma_{0}$ with a cyclic subgroup amalgamated. Thus, $G \in \Gamma_{1}$ if and only if 18-772905 Acta mathematica 139. Imprimé le 30 Décembre 1977

$$
G=G_{1} *{ }_{C} G_{2}
$$

for some $G_{1}, G_{2} \in \Gamma_{0}$ and (possibly trivial) cyclic subgroup $C$ of $G_{1}$ and $G_{2}$. We include the trivial group in $\Gamma_{0}$, hence, $\Gamma_{0} \subseteq \Gamma_{1}$. The precise definition of $\Gamma_{0}$ is given in Section 2 , and from this it follows that all f.g. Fuchsian groups, except those given by (1), belong to $\Gamma_{1}$.

For the present and later use, let

$$
(r, s)+(a, b) \mathbf{Z}=\left\{(r+a x, s+b x) \in \mathbf{Z}^{2} ; x \in \mathbf{Z}\right\}
$$

for any integers $r, s, a$, and $b$. If $r, s=0$, then we write just $(a, b) Z$ for this set. Also, for any subsets $\Phi$ and $\Phi^{\prime}$ of $\mathbf{Z}^{2}$ we let $\Phi+\Phi^{\prime}$ be the obvious set of sums.

As our first result we have

Theorem A. We can effectively compute the order of elements in any $G$ from $\Gamma_{1}$.
For the next result, let $|U|$ denote the order of the element $U$ in $G$.

Theorem B. Given $U, W \in G \in \Gamma_{1}$, we can effectively compute integers $a, b$, and $c$ such that

$$
C P_{G}(U, W)=(a, b) \mathbf{Z} \cup(a c,-b c) \mathbf{Z}
$$

if $|U|,|W|=\infty$;

$$
C P_{G}(U, W)=(a, b) \mathbf{Z}+(|U| \mathbf{Z}) \times(|W| \mathbf{Z})
$$

if $|U|,|W|<\infty$.
Note that the sets $C P_{G}(U, W)$ are easily described if $|U|<|W|=\infty$, or vice versa.
We give the proof of Theorem B separately for $\Gamma_{0}$ and $\Gamma_{1}$. In the first of these we also show that $0 \leqslant c \leqslant 1$, while in the second $0 \leqslant c \leqslant 2$ or $0 \leqslant c \leqslant|C|$ according as $G=G_{1} *{ }_{C} G_{2}$ with $C$ infinite or finite.

The following is immediate from Theorems A and B.
Corollary C. All groups in $\Gamma_{1}$ have solvable conjugacy and power-conjugacy problems.
In Section 7 we indicate how these results can be generalized by iterating the process of forming free products with a cyclic subgroup amalgamated. The class $\Gamma$ obtained from $\Gamma_{0}$ through this process generalizes a class studied by the author in [5]. From the results in this paper, together with those in [5], it also follows that the HNN groups

$$
\left\langle G_{0}, p ; \text { rel } G_{0}, p S_{1} p^{-1}=S_{2}\right\rangle
$$

have solvable conjugacy and power-conjugacy problems, where $S_{1}, S_{2} \in G_{0} \in \Gamma,\left|S_{1}\right|=\left|S_{2}\right|$, and $p \notin G_{0}$.

This work is the result of considering a question of M. Anshel on the conjugacy problem for free products with cyclic amalgamations of one-relator groups with torsion. M. Anshel and P. Stebe [1] obtained a partial solution of this problem using different techniques.

## § 2. Some basic definitions

We state our definitions with respect to a fixed alphabet $\left\{a_{1}, a_{2}, \ldots\right\}$, but want them to carry over to any alphabet. Thus, let us call $U$ a word on $\left\{a_{1}, a_{2}, \ldots\right\}$ if $U$ is a word on $\left\{a_{1}, a_{2}, \ldots\right\} \cup\left\{a_{1}^{-1}, a_{2}^{-1}, \ldots\right\}$ in the usual sence. We use upper case Roman letters in the range $P, \ldots, W$, or variations of these such as $P^{\prime}, P_{i}$, etc., strictly to denote freely reduced words. If $U$ is such a word, then $l(U)$ denotes its length. As a special symbol, we also use $\Lambda$ for the empty word. If $a_{i}$ or $a_{i}^{-1}$ occurs in $U$, then we say that $U$ involves $a_{i}$; and to display the letters in $U$, we use the notation

$$
\text { gen }(U)=\left\{a_{i} ; U \text { involves } a_{i}\right\} .
$$

Let us call any non-trivial cyclically reduced word simple if it is not a proper power. Hence, we can define $\Gamma_{0}$ as the class of all groups

$$
\begin{equation*}
G=\left\langle a_{1}, a_{2}, \ldots ; R_{1}^{n_{1}}, \ldots, R_{k}^{n_{k}}\right\rangle \tag{2}
\end{equation*}
$$

with $k \geqslant 0$ ( $k=0$ means $G$ has no relators), where each $n_{i}>1$, each $R_{i}$ is simple, and

$$
\begin{equation*}
\text { gen }\left(R_{i}\right) \cap \operatorname{gen}\left(R_{i}\right) \nsubseteq \operatorname{gen}\left(R_{i}\right) \tag{3}
\end{equation*}
$$

for all $i \leqslant j \leqslant i^{\prime}$ with $i \neq i^{\prime}$. When $k>0$, we call $R_{1}, \ldots, R_{k}$ the roots of $G$, which is meaningful as long as we work with specific presentations for the groups in $\Gamma_{0}$. Recall that we also include the trivial group in $\Gamma_{0}$.

With this definition of $\Gamma_{0}$ it is easily verified that all f.g. Fuchsian groups, except those given by (1), must belong to $\Gamma_{1}$. Recall that $\Gamma_{1}$ consists of all free products of groups from $\Gamma_{0}$ with a cyclic subgroup amalgamated.

Requiring all groups $G$ to be given by specific presentations, we may denote the set of generators in the presentation of $G$ by gen $(G)$. Elements of $G$ can then be represented by freely reduced words on gen $(G)$. We also include 1 as a special symbol for the identity in any group. For any $U, W \in G, U=W$ means $U$ and $W$ define the same element in the abstract group) $G$, while $U \equiv W$ means they are identical as words.

Let us examine in some detail the groups (2). Those with just one relator form the class of one-relator groups with torsion, a subclass we denote by $\Gamma_{*}$. Most of the problems we need to consider for $\Gamma_{0}$ can be reduced to problems concerning $\Gamma_{*}$.

Suppose now that $G \in \Gamma_{0}$ is given by (2) with $k \geqslant 2$. For each $2 \leqslant i \leqslant k$ let $G_{i} \in \Gamma_{*}$ be the group

$$
G_{i}=\left\langle\operatorname{gen}\left(R_{i}\right) ; R_{i}^{n_{i}}\right\rangle ;
$$

and similarly, let $G_{1}^{\prime} \in \Gamma_{*}$ be the group

$$
G_{1}^{\prime}=\left\langle\operatorname{gen}\left(R_{1}\right) ; R_{1}^{n_{1}}\right\rangle .
$$

If we then let $G_{1} \in \Gamma_{*}$ be obtained from $G_{1}^{\prime}$ by adding the remaining generators of $G$, we can write $G$ as the tree-product (see A. Karrass and D. Solitar [4])

$$
\begin{equation*}
G=G_{1} *_{F_{1}} \ldots *_{F_{k-1}} G_{k} \tag{4}
\end{equation*}
$$

where each $F_{i}$ is given by

$$
F_{i}=G_{i} \cap G_{i+1}
$$

From the Freiheitssatz and the condition (3) imposed on the roots $R_{j}$, it follows that each

$$
F_{i}=\left\langle\operatorname{gen}\left(G_{i}\right) \cap \operatorname{gen}\left(G_{i+1}\right) ;\right\rangle
$$

as a free group.
Note that when $k \geqslant 2$ we can also write $G$ in the form

$$
\begin{equation*}
G=G^{\prime} *_{F} G_{k} \tag{5}
\end{equation*}
$$

where $F=F_{k-1}$ and $G^{\prime}$ is the subgroup generated by $\bigcup_{i=1}^{k-1}$ gen $\left(G_{i}\right)$. Of course, $G^{\prime}$ belongs to $\Gamma_{0}$ and has $k-1$ relators. Moreover, if $k \geqslant 3$, then $G^{\prime}$ can also be written as a tree-product (4) of length $k-1$. We utilize these facts to prove results about the class $\Gamma_{0}$ in Section 6.

Before pursuing the various problems in $\Gamma_{0}$, let us consider these for the subclass $\Gamma_{*}$.

## § 3. Conjugate powers in one-relator groups with torsion

Any group $G \in \Gamma_{*}$ with root $R$ of length $\geqslant 2$ can be presented in the form

$$
\begin{equation*}
G=\left\langle t, a_{0}, b_{0}, \ldots ; R^{n}\right\rangle \tag{6}
\end{equation*}
$$

where $t, a_{0} \in \operatorname{gen}(R)$ and $R$ begins with $a_{0}^{+1}$. If $R$ has exponent sum zero on $t$, then $G$ can also be realized as an HNN group

$$
\begin{equation*}
G=\left\langle H, t ; \widetilde{R}^{n}, t S t^{-1}=\theta(S)(\forall S \in X)\right\rangle \tag{7}
\end{equation*}
$$

where $H$ belongs to $\Gamma_{*}$ and has the root $\tilde{R}$ with $l(\tilde{R})<l(R)$.
We only sketch here how (7) is obtained from (6), referring the reader to the paper [7] by J. McCool and P. E. Schupp for the details. The first step is to set for each integer $i, a_{i}=t^{i} a_{0} t^{-i}, b_{i}=t^{i} b_{0} t^{-i}$, etc., and then rewrite $R$ as a word $\tilde{R}$ on these new generators.

If $\nu$ is minimal and $\mu$ maximal among the subscripts of $a$-symbols involved in $\tilde{R}$, then let $H$ have generators

$$
\operatorname{gen}(H)=\left\{a_{\nu}, \ldots, a_{\mu}\right\} \cup\left\{b_{i} ; i \in \mathbf{Z}\right\} \cup \ldots
$$

and relator $\tilde{R}^{n}$. By the Freiheitssatz, the subgroups $X$ and $Y$ generated by gen $(H)-\left\{a_{\mu}\right\}$ and gen $(H)-\left\{a_{\nu}\right\}$, respectively, are isomorphic under $\theta: X \rightarrow Y$ induced by $a_{i} \rightarrow a_{i+1}$, $b_{i} \rightarrow b_{i+1}$, etc.

Suppose now that $G$ is given by (7). If $S$ is any word on gen $(H)$, let $S^{(x)}$ denote the word obtained from $S$ by shifting each subscript by $x$. If also gen $\left(S^{(x)}\right) \subseteq$ gen $(H)$, then $S^{(x)}=t^{x} S t^{-x}$.

Let any word $S$ in a group $G$ be called free in $G$ if gen $(S)$ generates the free group〈gen $(S)$; 〉 in $G$. By the Freiheitssatz, if $G \in \Gamma_{*}$ has the root $R$, then $S$ is free in $G$ if and only if gen $(R) \nsubseteq$ gen $(S)$. From a result of B. B. Newman [10] it follows that if $S$ is free in $G$, then gen $(S)$ generates a malnormal subgroup of $G$. Recall that $H$ is malnormal in $G$ if for any $U, W \in H, U=V W V^{-1} \neq 1$ implies $V \in H$.

From B. B. Newman's Lemma 2.1 in [10], we deduce
Lemma 3.1. Let $U \in G \in \Gamma_{*} b e \neq 1$. Then, up to cyclic permutation, there exists at most one cyclically reduced free word $S$ in $G$ with $U \sim_{G} S$. Moreover, if $S$ is such a word, then $U=W^{x}$ implies $S \equiv S_{0}^{x}$ with $W \sim_{G} S_{0}$.

Proof. Suppose that $S_{1}$ and $S_{2}$ both satisfy the lemma. If $R$ is the root of $G$, then $\operatorname{gen}(R) \notin \operatorname{gen}\left(S_{i}\right)$ for $i=1,2$. Thus, for some $a, b \in \operatorname{gen}(R), a \notin \operatorname{gen}\left(S_{1}\right)$ and $b \ddagger$ gen $\left(S_{2}\right)$. If gen $(R) \subseteq$ gen $\left(S_{1}\right) \cup$ gen $\left(S_{2}\right)$, then $b \in \operatorname{gen}\left(S_{1}\right)$ and $a \in \operatorname{gen}\left(S_{2}\right)$. Since $S_{1} \sim_{G} S_{2}$, this violates Lemma 2.1 in [10], hence, gen $(R) \nsubseteq$ gen $\left(S_{1}\right) \cup$ gen $\left(S_{2}\right)$. But then $S_{1}$ and $S_{2}$ belong to the malnormal subgroup $F$ of $G$ generated by gen $\left(S_{1}\right) \cup$ gen $\left(S_{2}\right)$, and therefore $S_{1} \sim_{F} S_{2}$. This proves the first half.

To complete the proof, suppose that $S$ is free in $G$ with $U=W^{x}=V S V^{-1}$. If we set $W_{0}=V^{-1} W V$, then $W_{0}^{x}=S$, and therefore $W_{0} S W_{0}^{-1}=S \neq 1$. If $F$ is the malnormal subgroup of $G$ generated by gen $(S)$, then $W_{0} \in F$ and the result follows.

The proof of the next lemma is given in Section 5 where the necessary techniques are developed.

Lemma 3.2. If $U \in G \in \Gamma_{*}$, then we can effectively decide if there exists any free word $S$ in $G$ conjugate to $U$. Moreover, we can effectively compute such an $S$, provided any exists.

Let us establish some terminology concerning elements of the HNN groups (7). Words on gen $(G)$ without any $t$ 's are called $t$-free. If $U$ involves $t$, then $U$ can be written in the form

$$
U \equiv U_{\mathbf{0}} t^{\varepsilon_{1}} U_{\mathbf{1}} \ldots t^{\varepsilon_{k}} U_{k}
$$

with each $U_{i} t$-free. (Lower case Greek letters denote $\pm 1$.) The number of $t$ 's occurring in $U$ is called the $t$-length of $U$, denoted $l_{t}(U)$. If $U$ contains no subword $t^{\varepsilon_{i}} U_{i} t^{-\varepsilon_{i}}$ with $\varepsilon_{i}=1$ and $U_{i} \in X$ or $\varepsilon_{i}=-1$ with $U_{i} \in Y$, then $U$ is called $t$-reduced. If all cyclic premutations of $U$ are $t$-reduced, and $U$ is either $t$-free or begins with $t^{ \pm 1}$, then $U$ is called cyclically $t$-reduced. Let us also call $U$ and $W$ t-parallel if $l_{t}(U)=l_{t}(W)=k$, and they contain identical $k$-tuples of $t^{ \pm 1}$.

To study the sets $C P_{G}(U, W)$ in $\Gamma_{*}$ we need
Lemma 3.3. Let $G$ be presented by (7). If $U, W \in H$, then $U \sim_{G} W$ implies

$$
U=V_{0} t^{x} V_{1} W V_{1}^{-1} t^{-x} V_{0}^{-1}
$$

for some $t$-free $V_{0}$ and $V_{1}$.
Proof. By Britton's Lemma [2], if $V$ is $t$-reduced and involves $t$, then $U=V W V^{-1}$ implies $U \sim_{H} S_{1}$ and $W \sim_{H} S_{2}$ for some cyclically reduced free words $S_{1}$ and $S_{2}$ in $H$. It thus suffices to prove the lemma for $U \equiv S_{1}$ and $W \equiv S_{2}$. Now, if $V \equiv V_{0} t^{\theta_{1}} V_{1} \ldots t^{\varepsilon_{k}} V_{k}$, then we may assume the words $t^{\varepsilon_{i}} V_{i} t^{-\varepsilon_{i}}$ to be $t$-reduced for each $1 \leqslant i \leqslant k$ with $V_{i} \neq \Lambda$. It remains to show that $V_{i} \equiv \Lambda$ for each $1 \leqslant i \leqslant k$. To this end, let $i$ be maximal with $V_{i} \neq \Lambda$. But then

$$
t^{\varepsilon_{i}} V_{i} S_{2}^{\left(e_{k}(k-t)\right)} V_{i}^{-1} t^{-\varepsilon_{i}}=t^{\varepsilon_{i}} T t^{-\varepsilon_{i}}=T^{\left(\varepsilon_{i}\right)}
$$

for some free word $T$. Lemma 3.1 now implies that $T \equiv P \bar{T} P^{-1}$ with $\bar{T}$ a cyclic premutation of $S_{2}^{\left(e_{k}(k-i)\right)}$. By malnormality of the subgroup generated by gen $(T)$, we easily see that $t^{\varepsilon_{i}} V_{i} t^{-\varepsilon_{i}}$ cannot be $t$-reduced, a contradiction.

Lemma 3.4. Let $U \in G \in \Gamma_{*} b e \neq 1$, and suppose that $(x, y) \in C P_{G}(U, U)$. Then
(i) $|U|=\infty$ implies $|x|=|y|$;
(ii) $|U|<\infty$ implies $U^{x}=U^{y}$.

Proof. We use induction on the length of the root $R$ of $G$. If this length is 1 , then $G$ is a free product of a finite cyclic group and a free group. Both (i) and (ii) are easily estab. lished in this case.

Suppose now that $l(R) \geqslant 2$ with the result established for all $G^{\prime} \in \Gamma_{*}$ having roots $R^{\prime}$ satisfying $l\left(R^{\prime}\right)<l(R)$. Assume first that $G$ can be realized as the HNN group (7). If $U$ is cyclically $t$-reduced and involves $t$, then $|U|=\infty$ and the result follows from Collins' Lemma (p. 123 in [3]). If $U \in H$, then Lemmas 3.1 and 3.3 imply $U^{x} \sim_{G} U^{y}$ only if $U^{x} \sim_{H} U^{y}$. Since $l(\tilde{R})<l(R)$, it remains to consider the case where no generator in gen $(R)$ has exponent sum zero in $R$. But J. McCool and P. E. Schupp showed in [7] that $G$ can then be
imbedded in an HNN group of the type (7) with the root $\tilde{R}$ of $H$ satisfying $l(\widetilde{R})<l(R)$. The case just considered can therefore be applied.

Note that $U \equiv a b a b^{-1}$ is of infinite order in $G=\left\langle a, b ; a^{2}\right\rangle \in \Gamma_{*}$ and satisfies $U \sim_{G} U^{-1}$.
B. B. Newman proved in [10] that the centralizer of any nontrivial $U \in G \in \Gamma_{*}$ is cyclic. Hence, if we denote the centralizer of $U$ in $G$ by $C_{G}(U)$, then for any $U \in G \in \Gamma_{*} \neq 1$, $C_{G}(U)=\langle T\rangle$ for some $T$. Here $\langle T\rangle$ denotes the subgroup generated by $T$.

Pending a proof of Lemma 3.2, we can now establish
Proposition 3.5. Let $U, W \in G \in \Gamma_{*}$ be of infinite order. Then we can effectively compute integers $a, b$, and $c$ with $0 \leqslant c \leqslant 1$ such that

$$
C P_{G}(U, W)=(a, b) \mathbf{Z} \cup(a c,-b c) \mathbf{Z} .
$$

Moreover, $a$ and $b$ are relatively prime if $a \neq 0$.
Proof. Note that we can determine the order of elements of $G$. Let us show first that $C P_{G}(U, W)$ has the asserted form. Because of Lemma 3.4 it suffices to show: If $(x z, y z) \in$ $C P_{G}(U, W)$ for some $x, y, z \in \mathbf{Z}$ with $z \neq 0$, then $(x, y) \in C P_{G}(U, W)$. This is trivial if $x=0$. If $x \neq 0$, then

$$
U^{x z}=V W^{y z} V^{-1}=\left(V W V^{-1}\right)^{y z}
$$

for some $V$. Let $C_{G}\left(U^{x z}\right)=\langle T\rangle$, and note that $U$ and $V W V^{-1}$ must therefore belong to $\langle T\rangle$. Now, if $U=T^{p}$ and $V W V^{-1}-T^{a}$, then $T^{p x z}=T^{a y z}$. Since $|T|=\infty$, we must have

$$
U^{x}=T^{p x}=T^{a y}=\left(V W V^{-1}\right)^{y}=V W^{y} V^{-1}
$$

By a result of B. B. Newman [9], $G$ has solvable conjugacy problem. Hence, it suffices to determine $a^{\prime}, b^{\prime}>0$ such that $a, b \neq 0$ implies $|a|=a^{\prime}$ and $|b|=b^{\prime}$. For this, let us proceed by induction on the length of the root $R$ of $G$. The case with $l(R)=1$ is trivial, so suppose that $l(R) \geqslant 2$ with the result established for all $G^{\prime} \in \Gamma_{*}$ having roots $R^{\prime}$ with $l\left(R^{\prime}\right)<l(R)$.

Suppose first that $G$ can be realized as the HNN group (7). J. McCool and P. E. Schupp [7] proved that $H$ has solvable generalized word problem with respect to $X$ and $Y$, hence, we can effectively cyclically $t$-reduce words in $G$. Suppose therefore that $U$ and $W$ are cyclically $t$-reduced. If only one of them is $t$-free, let $a^{\prime}, b^{\prime}=0$. If both $U$ and $W$ involve $t$, let $a^{\prime}, b^{\prime}>0$ be minimal such that $l_{t}\left(U^{a^{\prime}}\right)=l_{t}\left(W^{b^{\prime}}\right)$. If $U, W \in H$, we use the inductive hypothesis and compute $C P_{H}(U, W)$. From Lemmas 3.1 and 3.3 it follows that $C P_{H}(U, W) \neq$ $\{(0,0)\}$ implies $C P_{H}(U, W)=C P_{G}(U, W)$. Suppose therefore that $C P_{H}(U, W)=\{(0,0)\}$, and consider $U^{x} \sim_{G} W^{y}$. If $x, y \neq 0$, then $U \sim_{H} S_{1}$ and $W \sim_{H} S_{2}$ for some cyclically reduced free words $S_{1}$ and $S_{2}$ from $X$ or $Y$ (see Lemma 3.1). But then $S_{1}^{x} \sim_{G} S_{2}^{y}$, and so by Lemma 3.3, $S_{1}^{x}$ must be a cyclic permutation of $\left(S_{1}^{(z)}\right)^{y}$ for some $z$. By Lemma 3.2 we can compute $S_{1}$ and $S_{2}$ and therefore also determine $z$. It is now elementary to compute $a^{\prime}$ and $b^{\prime}$.

Finally, suppose that $G$ cannot be realized as the HNN group (7). By the remark at the end of the last proof, $G$ can be imbedded in such an HNN group with the inductive hypothesis applying to the base group. Since this imbedding is clearly effective, and we only need to compute the integers $a^{\prime}$ and $b^{\prime}$, the above case applies.

The sets $C P_{G}(U, W)$ with $|U|,|W|<\infty$ are considered in Section 6.

## § 4. Some more definitions

In this paper we use two constructions of generalized free products

$$
G=G_{1} *_{H} G_{2}
$$

of groups $G_{1}$ and $G_{2}$ from $\Gamma_{0}$. In the first of these we amalgamate a cyclic subgroup $H$ and get $G$ in $\Gamma_{1}$; in the second (see (5) in Section 2) the amalgamated subgroup $H$ is free on gen $\left(G_{1}\right) \cap$ gen $\left(G_{2}\right)$ and $G$ belongs to $\Gamma_{0}$.

For both of the above constructions we choose the natural presentation for $G$, hence,

$$
\text { gen }(G)=\operatorname{gen}\left(G_{1}\right) \cup \operatorname{gen}\left(G_{2}\right)
$$

Now, any word $U$ on gen $(G)$ can be written in the (not necessarily unique) form

$$
\begin{equation*}
U \equiv U_{1} \ldots U_{k} \tag{8}
\end{equation*}
$$

where gen $\left(U_{i}\right)$ is contained in gen $\left(G_{1}\right)$ or gen $\left(G_{2}\right)$ for each $i$. If $U \neq \Lambda$ and gen $\left(U_{1} U_{1+1}\right)$ is not contained in gen $\left(G_{1}\right)$ or gen $\left(G_{2}\right)$ for any $1 \leqslant i<k$, then we call each $U_{1}$ a syllable of $U$. The number of syllables in $U$ is called the $s$-length of $U$, denoted $l_{s}(U)$.

Words $U \in G$ are called $s$-reduced if either $l_{s}(U) \leqslant 1$ or no syllable of $U$ belongs to $H$; that is, for no decomposition (8) of $U$ into syllables $U_{1}^{\prime} \ldots U_{k}^{\prime}$ does $U_{i}^{\prime} \in H$ for any $1 \leqslant i \leqslant k$. If also all cyclic syllable-permutations (henceforth s-permutations) of $U$ are $s$-reduced, then we call $U$ cyclically s-reduced.

Both for generalized free products $G$ in $\Gamma_{0}$ and $\Gamma_{1}$ we need to determine for given cyclically $s$-reduced words whether or not these are conjugate in $G$. The main tool to deal with such problems is Solitar's Theorem (Thm. 4.6 in [6]). In part, this theorem asserts for cyclically $s$-reduced words $U$ and $W$ of the same s-length $\geqslant 2$, that $U \sim_{G} W$ if and only if $U=S W_{\pi} S^{-1}$ for some cyclic $s$-permutation $W_{\pi}$ of $W$ and $S \in H$. If we write $U \equiv U_{1} \ldots U_{k}$ and $W_{\pi} \equiv W_{1} \ldots W_{k}$ in terms of syllables, and examine the identity

$$
U_{1} \ldots U_{k} S W_{k}^{-1} \ldots W_{1}^{-1}=S
$$

then we note that $U_{k} S W_{k}^{-1}=S_{1} \in H, U_{k-1} S_{1} W_{k-1}^{-1}=S_{2} \in H$, etc. Thus, we are led to consider the sets

$$
\left\{\left(S, S^{\prime}\right) \in H \times H ; U_{t} S W_{i}^{-1}=S^{\prime}\right\}
$$

in groups from $\Gamma_{0}$. However, for some of our applications we need to consider a more general situation.

Let $G$ be a given group with subgroups $H$ and $K$. For any $U, W \in G$, consider the following subset of the direct product $H \times K$ :

$$
\operatorname{gph}(U, W ; H, K)=\{(S, T) \in H \times K ; U S W=T(\operatorname{in} G)\}
$$

This is the graph of the function $G \rightarrow G$, given by $V \rightarrow U V W$, restricted to the (possibly empty) subset of $H$ mapped into $K$.

For any subgroup $N$ of $H \times K$ and element $\left(S_{0}, T_{0}\right) \in H \times K$, let $N\left(S_{0}, T_{0}\right)$ and $\left(S_{0}, T_{0}\right) N$ denote the right and left translates of $N$ by ( $S_{0}, T_{0}$ ). For the next lemma, note that $\operatorname{gph}(U, W ; H, K)$ is a subgroup of $H \times K$ if and only if $W=U^{-1}$.

Lemma 4.1. If $\left(S_{0}, T_{0}\right) \in \operatorname{gph}(U, W ; H, K)$, then
$\operatorname{gph}(U, W ; H, K)=\left[\operatorname{gph}\left(U, U^{-1} ; H, K\right)\right]\left(S_{0}, T_{0}\right)=\left(S_{0}, T_{0}\right)\left[\operatorname{gph}\left(W^{-1}, W ; H, K\right)\right]$.
Proof. If $U S_{0} W=T_{0}$ then $U S W=T$ if and only if $U S W\left(U S_{0} W\right)^{-1}=U S S_{0}^{-1} U^{-1}=$ $T T_{0}^{-1}$. Hence $(S, T) \in \operatorname{gph}(U, W ; H, K)$ if and only if $\left(S S_{0}^{-1}, T T_{0}^{-1}\right) \in \operatorname{gph}\left(U, U^{-1} ; H, K\right)$. The other half of the proof is similar.

Much of the remaining work in this paper concerns the sets gph ( $U, W ; H, K$ ) for free and cyclic subgroups $H$ and $K$ of groups in the subclass $\Gamma_{*}$ of $\Gamma_{0}$. In the special case when $H=\langle S\rangle$ and $K=\langle T\rangle$ as infinite cyclic subgroups of $G$, then we identify $H$ and $K$ with $\mathbf{Z}$, and set

$$
\operatorname{gph}(U, W ; H, K)=\left\{(x, y) \in \mathbf{Z}^{2} ; U S^{x} W=T^{y}\right\}
$$

If this set is non-empty, then Lemma 4.1 implies that

$$
\operatorname{gph}(U, W ; H, K)=(r, s)+(a, b) \mathbf{Z}
$$

for any $r, s$ with $U S^{r} W=T^{s}$, provided $(a, b) \mathbf{Z}=\operatorname{gph}\left(U, U^{-1} ; H, K\right)$. In the next two sections we show how we can effectively compute such integers $r, s, a$, and $b$.

Let us also consider the graphs for free subgroups $H$ and $K$ of $G \in \Gamma_{*}$ generated by subsets of gen $(G)$.

Lemma 4.2. Let $H$ and $K$ be free subgroups of $G \in \Gamma_{*}$ generated by subsets of gen $(G)$, and set $N=H \cap K$. Then, if $\operatorname{gph}\left(U, U^{-1} ; H, K\right)$ is non-trivial, there exists at least one pair $(P, Q) \in H \times K$ with $U=Q P^{-1}$ and

$$
\operatorname{gph}\left(U, U^{-1} ; H, K\right)=(P, Q)[\operatorname{gph}(1,1 ; N, N)](P, Q)^{-1}
$$

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Proof. Suppose first that $U=Q P^{-1}$, and note that in this case, $(S, T) \in \operatorname{gph}\left(U, U^{-1} ; H, K\right)$ if and only if $P^{-1} S P=Q^{-1} T Q=S^{\prime} \in N=H \cap K$. Moreover, if $U S U^{-1}=T \neq 1$ for some $(S, T) \in$ $H \times K$, then Lemma 3.1 implies that $S \equiv S_{1} S^{\prime} S_{1}^{-1}$ and $T \equiv T_{1} T^{\prime} T_{1}^{-1}$ with $S^{\prime}, T^{\prime} \in N$. But then, since $N$ is malnormal in $G$ and $\left(T_{1}^{-1} U S_{1}\right) S^{\prime}\left(T_{1}^{-1} U S_{1}\right)^{-1}=T^{\prime} \neq 1$, we must have $T_{1}^{-1} U S_{1}=S_{0} \in N$. This in turn shows that $U=Q P^{-1}$ for $Q=T_{1} S_{0} \in K$ and $P=S_{1} \in H$.

In view of Lemmas 4.1 and 4.2, we may say that $\operatorname{gph}(U, W ; H, K)$ has been computed whenever elements $\left(S_{0}, T_{0}\right),(P, Q) \in H \times K$ have been effectively determined for which $\operatorname{gph}(U, W ; H, K) \neq \varnothing$ if and only if $U S_{0} W=T_{0}$, and $\operatorname{gph}\left(U, U^{-1} ; H, K\right)$ is nontrivial if and only if $U=Q P^{-1}$ and $H \cap K \neq(1)$. We assume here that $H, K$, and $G$ satisfy the hypotheses of Lemma 4.2. The problems involved in actually computing these sets are considered in Section 5.

Conjugacy between elements of $G_{1} *_{H} G_{2}$ belonging to the factors $G_{1}$ and $G_{2}$, will be considered in Section 6.

## § 5. Graphs in one-relator groups with torsion

Consider the graphs gph $\left(U, W ; C_{1}, C_{2}\right)$ for cyclic subgroups $C_{1}=\langle S\rangle$ and $C_{2}=\langle T\rangle$ of $G \in \Gamma_{*}$. To obtain results about such sets we proceed by induction on the length of the root of $G$. The HNN construction (7) allows us to apply the inductive hypothesis, but this construction also introduces new problems. To illustrate this, suppose $S$ and $T$ to be $t$-free while $U \equiv U_{0} t^{t_{1}} U_{1} \ldots t^{\ell_{k}} U_{k}$ and $W^{-1} \equiv W_{0}^{-1} t^{\varepsilon_{1}} W_{1}^{-1} \ldots t^{\ell_{k}} W_{k}^{-1}$ with $k \geqslant 2$. If now $U$ and $W$ are $t$-reduced and satisfy $U S^{x} W=T^{y}$, that is, if

$$
U_{0} t^{t_{1}} U_{1} \ldots t^{t_{k}} U_{k} S^{x} W_{k} t^{-\varepsilon_{k}} \ldots W_{1} t^{-\varepsilon_{1}} W_{0}=Y^{\nu}
$$

then $U_{k} S^{x} W_{k}=S_{1}$ in $X$ or $Y, U_{k-1} S_{1}^{\left(\varepsilon_{k}\right)} W_{k-1}=S_{2}$ in $X$ or $Y$, etc. Thus, we need to consider graphs gph ( $U, W ; H, K$ ) in groups $G \in \Gamma_{*}$ for the following combinations: $H$ and $K$ cyclic, $H$ cyclic and $K$ free on a subset of gen $(G)$ (or vice versa), and finally, both $H$ and $K$ free on subsets of gen $(G)$.

For the remainder of this section, let us use the following convention: All subgroups denoted by $F, F^{\prime}, F_{1}$, etc. of given groups $G \in \Gamma_{*}$ are assumed to be freely generated by recursive subsets of gen $(G)$. Cyclic subgroups are denoted by $C, C_{1}$, etc.

Before turning to the various problems involved in computing the relevant graphs, we need a definition and a lemma. But first, let us recall the "Spelling Theorem" of B. B. Newman [9] concerning groups $G \in \Gamma_{*}$ with relator $R^{n}$ ( $R$ simple). This theorem asserts for any $U, S \in G$ with $S$ free in $G$ and $U$ not free, that $U=S$ implies $U \equiv U_{1} U^{\prime} U_{2}$ with $U^{\prime} V$ a cyclic permutation of $R^{ \pm n}$ for some $V$ with $l(V)<l(R)$. Let us therefore call the process of replacing $U^{\prime}$ by $V^{-1}$ an $R$-reduction of $U$ where we assume $U^{\prime}$ to be maximal
so that $U_{1} V^{-1} U_{2}$ is freely reduced. Further $R$-reductions of $U_{1} V^{-1} U_{2}$ are also called $R$-reductions of $U$, etc.

Lemma 5.1. Let $S$ and $T$ be free in $G \in \Gamma_{*}$ where $G$ has the relator $R^{n}$. Then at most one $R$-reduction is possible in $S T$ (none if $n>2$ ).

Proof. The remark about $n$ is obvious, so suppose that $n=2$. If an $R$-reduction is possible in $S T$, then $S \equiv S_{1} S^{\prime}, T \equiv T^{\prime} T_{1}$, and $S^{\prime} T^{\prime} V \equiv R_{*}^{2_{\varepsilon}}$ for some cyclic permutation $R_{*}$ of $R$ where $l(V)<l(R)$. But then gen $(R)=$ gen $\left(S^{\prime}\right) \cup$ gen $\left(T^{\prime}\right)$, and therefore $R_{s}^{*} \equiv U V$ with $V \equiv V_{1} V_{2}, S^{\prime} \equiv U V_{1}$, and $T^{\prime} \equiv V_{2} U$. Moreover, for some minimal subwords $W_{1}$ of $V_{1}$ and $W_{2}$ of $V_{2}$, we must have $\varnothing \neq \operatorname{gen}\left(V_{i}\right)$-gen $\left(V_{j}\right) \subseteq \operatorname{gen}\left(W_{i}\right)$ for $1 \leqslant i \neq j \leqslant 2$. Clearly, if $W_{i} W$ and $W_{i} W^{\prime}$ are cyclic permutations of $R^{\tau}$ and $R^{\tau^{\prime}}$ respectively, then $\tau=\tau^{\prime}=\varepsilon$ and $W \equiv W^{\prime}$. This follows since gen $\left(W_{i}\right) \subseteq \operatorname{gen}\left(W_{j}\right)$ only if $i=j$. Suppose now that $S_{1} V_{2}^{-1} V_{1}^{-1} T_{1}$ is the result of an $R$-reduction. Any new $R$-reduction of this word must involve all of $W_{1}^{-1}$ or $W_{2}^{-1}$, hence, must be with respect to a cyclic permutation of $R^{-2 \varepsilon}$. By the uniqueness of $W_{1}$ and $W_{2}$, we then get a contradiction to the necessary fact that $S_{1} U V_{1} V_{2} U T_{1}$ was freely reduced.

Most of the combinatorial difficulties involved in computing the sets gph ( $U, W ; H, K$ ) in $G \in \Gamma_{*}$ for $H=F_{1}$ or $C_{1}$ and $K=F_{2}$ or $C_{2}$, are handled by the next three lemmas.

Lemma 5.2. Let $G \in \Gamma_{*}$ have the root $R$ with exponent sum zero on $t$ where $R$ involves $t$. Then, if we can effectively compute the sets $\operatorname{gph}\left(U^{\prime}, W^{\prime} ; F_{1}^{\prime}, F_{2}^{\prime}\right)$ in any $G^{\prime} \in \Gamma_{*}$ having root $R^{\prime}$ with $l\left(R^{\prime}\right)<l(R)$, we can also effectively compute the sets $\operatorname{gph}\left(U, W ; F_{1}, F_{2}\right)$ in $G$.

Proof. We must show that given any $U, W \in G$, we can effectively determine at least one pair $(S, T) \in F_{1} \times F_{2}$ such that $\operatorname{gph}\left(U, W ; F_{1}, F_{2}\right) \neq \varnothing$ if and only if $U S W=T$.

Let $G$ be realized as the HNN group (7), and note that the hypothesis of the lemma applies to $H$.

Case 1. $t \notin F_{1} \cup F_{2}$. It follows that gen $\left(F_{i}\right) \subseteq$ gen $(H)$ for $i=1,2$. We may restrict ourselves to $t$-reduced words $U$ and $W$ with $U$ and $W^{-1} t$-parallel. Consider $U \equiv U_{0} t^{\varepsilon_{1}} U_{1} \ldots t^{\varepsilon_{k}} U_{k}$ and $W^{-1} \equiv W_{0}^{-1} t^{\varepsilon_{1}} W_{1}^{-1} \ldots t^{\varepsilon_{k}} W_{k}^{-1}$ with $k>0$, and suppose that $(S, T) \in F_{1} \times F_{2}$ satisfies $U S W=T$. It then follows from

$$
\begin{equation*}
U_{0} t^{t_{1}} U_{1} \ldots t^{\varepsilon_{k}} U_{k} S W_{k} t^{-\varepsilon_{k}} \ldots W_{1} t^{-s} W_{0}=T \tag{9}
\end{equation*}
$$

that $U_{k} S W_{k}=T^{\prime} \in F_{2}^{\prime}$, where $F_{2}^{\prime}=X$ if $\varepsilon_{k}=1$, and $F_{2}^{\prime}=Y$ if $\varepsilon_{k}=-1$. By assumption, we can effectively compute $\mathrm{gph}\left(U_{k}, W_{k} ; F_{1}, F_{2}^{\prime}\right)$ in $H$. If this set is finite, then $S$ is uniquely determined, so suppose that

$$
\operatorname{gph}\left(U_{k}, W_{k} ; F_{1} F_{2}^{\prime}\right)=(P, Q)[\operatorname{gph}(1,1 ; F, F)]\left(P^{-1} S_{0}, Q^{-1} T_{0}\right)
$$

as an infinite set, where $U_{k}=Q P^{-1}$ and $F=F_{1} \cap F_{2}^{\prime}$ as guaranteed by Lemma 4.2. Since we now have $U_{k} S W_{k}=T^{\prime}=Q \bar{T} Q^{-1} T_{0}$ for some $\bar{T} \in F$, let us replace $t^{\varepsilon_{k}} U_{k} S W_{k} t^{-\varepsilon_{k}}$ by $Q^{\left(\varepsilon_{k}\right)} \bar{T}^{\left(\epsilon_{k}\right)}\left(Q^{\left(\kappa_{k}\right)}\right)^{-1} T_{0}^{\left(\varepsilon_{k}\right)}$ in (9). This produces a new equation $U^{\prime} S^{\prime} W^{\prime}=T$ where $U^{\prime}$ and $\left(W^{\prime}\right)^{-1}$ are $t$-parallel, $S^{\prime} \equiv \bar{T}^{\left(e_{k}\right)} \in F_{1}^{\prime}=t^{\theta_{k}} F t^{-\epsilon_{k}}$ and $l_{t}\left(U^{\prime}\right)<l_{t}(U)$.

The above discussion shows how we can compute the sets gph ( $U, W ; F_{1}, F_{2}$ ) in $G$ for any free subgroups $F_{1}$ and $F_{2}$ of $H$ with gen $\left(F_{i}\right)$ recursive subsets of gen (H) for $i=1,2$, using induction of $l_{t}(U)$.

Case 2. $t \in F_{1}, t \notin F_{2}$. (The case with $t \notin F_{1}, t \in F_{2}$ is similar.) By relabelling the generators if necessary, we may assume that $a_{0} \ddagger F_{1}$. The elements of $F_{1}$ can then be written as $t$ reduced words $S t^{x}$ with $S \in F_{1}^{\prime}=F_{1} \cap H$. Since $U S t^{x} W=T \in F_{2}$ implies that $|x| \leqslant l_{t}(U)+$ $l_{t}(W)$, it suffices to consider $\operatorname{gph}\left(U, t^{x} W ; F_{1}^{\prime}, F_{2}\right)$ for each such $x$, using Case 1.

Case 3. $t \in F_{1} \cap F_{2}$. Let $F_{1}^{\prime}=F_{i} \cap H$ for $i=1,2$, and note that we can effectively compute $\operatorname{gph}\left(V, 1 ; F_{1}^{\prime}, F\right)$ and $\operatorname{gph}\left(1, V ; F_{2}^{\prime}, F\right)$ in $H$ for any $V \in H$ and $F=X, Y$. But then, if $U$ and $W$ are $t$-reduced, we may assume for any $(S, T) \in F_{1} \times F_{2}$ that $T^{-1} U S$ is $t$-reduced if $U \notin H$, and $S W T^{-1}$ is $t$-reduced if $W \notin H$. This in turn implies $T^{-1} U S W \neq 1$ or $W T^{-1} U S \neq 1$ if $T^{-1} U S$ is $t$-reduced and $S$ and $T$ are not both $t$-free. A similar statement holds when $S W T^{-1}$ is $t$-reduced. Since we can decide by Case 1 if $U S W=T$ is possible for any $(S, T) \in F_{1}^{\prime} \times F_{2}^{\prime}$, we need only consider $U, W \in H$.

Suppose now that $U, W \in H$. As in Case 2, we may assume that $a_{0} \notin F_{1}$ and write elements of $F_{1}$ as $t$-reduced words $S t^{x}$ with $S \in F_{1}^{\prime}$. Elements of $F_{2}$ can be written as $t$-reduced words $T_{0}{ }^{t^{\tau_{1}}} T_{1} \ldots t^{\tau_{r}} T_{r}$ with $T_{1} \in F_{2}^{\prime}$ for each $i$. If now

$$
U S t^{x} W=T_{0} t^{\tau_{1}} T_{1} \ldots t^{\tau_{r}} T_{r}
$$

then $|x|=r$ and $x=\tau_{i}|x|$ for each $i$. By symmetry, it suffices to treat the case with $x=r>0$ and thus each $\tau_{i}=1$. Note that we must have $W T^{\prime}=P \in X$ for some $T^{\prime} \in F_{2}^{\prime}$. Since $\operatorname{gph}\left(W, 1 ; F_{2}^{\prime}, X\right)$ can be effectively computed in $H$, we may as well assume that $W \equiv P \equiv \Lambda$. Moreover, let $P T^{\prime}$ be freely reduced for all $T^{\prime} \in F_{2}^{\prime}$. The words $T_{0} t T_{1} \ldots t T_{r}$ may be written such that each $T_{i}$ is either empty or begins with $a_{\mu}^{+1}$ for $1 \leqslant i \leqslant r$. Suppose now that $T_{q} \equiv \Lambda$ and $T_{i} \equiv \Lambda$ for each $i>q$ in the above. We then get

$$
\begin{equation*}
U S t^{q} P^{(r-q)}=T_{0} t T_{1} \ldots t T_{a} \tag{10}
\end{equation*}
$$

If $q \geqslant 1$, then we must have $P^{(r-q)} T_{q}^{-1}=P_{q} \in X$. From the restrictions above (on $P T^{\prime}$ ), it follows that $P_{q}$ must result after one $\tilde{R}$-reduction of $P^{(r-q)} T_{q}^{-1}$. By the proof of Lemma 5.1, $P^{(r-a)}$ must contain a certain unique subword $W_{1}$ which then uniquely determines $r-q$. Due to the shifting of subscripts in $P_{q}^{(1)}$, no $\tilde{R}$-reduction is possible in $P_{q}^{(1)} T_{q-1}^{-1}$,
therefore $T_{q-1} \equiv \Lambda$. Similarly, we must have $T_{i} \equiv \Lambda$ for each $l \leqslant i<q$. From the above equation (10) we now get

$$
U S P_{q}^{(\alpha)}=T_{0} .
$$

Since $P_{q}$ must involve $a$-symbols (from the $\tilde{R}$-reduction), we get $q \leqslant \mu-\nu$. For each such $q$ we can decide if $U S P_{q}^{(q)}=T_{0}$ is possible in $H$. It remains to consider the case with $q=0$ in (10). Then $U S P^{(r)}=T_{0}$, and if $P$ involves $a$-symbols, then the remark about $P_{q}$ applies. If $P$ involves no $a$-symbols, then the words $S P^{(r)}$ and $T_{0}$ are both free in $H$. Since $U^{-1}$ must result after free reductions and possibly one $\tilde{R}$-reduction of the word $S P^{(r)} T_{0}^{-1}$, it follows that $r$ is uniquely determined, hence, we can solve $U S P^{(r)}=T_{0}$ in $H$.

Lemma 5.3. Exactly like Lemma 5.2 with $F_{1}^{\prime}$ and $F_{1}$ replaced by $C_{1}^{\prime}$ and $C_{1}$. Also include the assertion from Lemma 5.2 about $\operatorname{gph}\left(U^{\prime}, W^{\prime} ; F_{1}^{\prime}, F_{2}^{\prime}\right)$.

Proof. The case with $C_{1}$ finite is trivial, so suppose that $C_{1}=\langle S\rangle$ with $|S|=\infty$. Lemma 3.1 together with Lemma 4.1 show that if $\operatorname{gph}\left(U, W ; C_{1}, F_{2}\right)$ is infinite, then $U S U^{-1} \in F_{2}$ and therefore $U W \in F_{2}$. It suffices therefore to find just one $x$ such that $U S^{x} W \in F_{2}$ if and only if $\operatorname{gph}\left(U, W ; C_{1}, F_{2}\right) \neq \varnothing$.

Let $G$ be realized as the HNN group (7), and note that the hypotheses of the lemma apply to $H$.

By changing $U$ and $W$ if necessary, assume that $S$ is cyclically $t$-reduced.
Case 1. $t \notin F_{2}$. If $S \notin H$, then $U S^{x} W \in F_{2}$ forces a bound on $|x|$, so we need only consider the case with $S \in H$. It then suffices to consider $t$-reduced $U$ and $W$ with $U$ and $W^{-1} t$ parallel. The result is now easily obtained by induction on $l_{t}(U)$ (see Case 1 in the proof of Lemma 5.2 ), using the remark above concerning $\operatorname{gph}\left(U^{\prime}, W^{\prime} ; C_{1}, F_{2}^{\prime}\right)$ when this is infinite.

Case 2. $t \in F_{2}$. By relabelling the generators if necessary, we may assume that $a_{0} \notin F_{2}$. Elements of $F_{2}$ may therefore be written in the form $T t^{y}$ with $T \in F_{2}^{\prime}=F_{2} \cap H$.

If $S \in H$, then $U S^{x} W=T t^{y}$ implies $|y| \leqslant l_{t}(U)+l_{t}(W)$. For each such $y$ we can use induction on $l_{t}(U)$, just like we indicated for Case 1 , and consider $U S^{x}\left(W t^{-y}\right)=T$.

For the remainder of the proof, assume that $S \equiv t^{\varepsilon_{1}} S_{1} \ldots t^{\varepsilon_{r}} S_{r}$ with $r \geqslant 1$. It suffices to obtain a bound on $|x|$, so by symmetry, we need only treat the case with $x>0$. We may assume $x$ to be large enough for us to $t$-reduce $U S^{x} W$ and obtain a $t$-reduced word $U^{\prime} S^{x^{\prime}} W^{\prime}$ with $x^{\prime}>0$. But then we may also assume all $\varepsilon_{i}$ 's in $S$ to be equal. By symmetry, let us only consider the case with each $\varepsilon_{i}=1$. If now $U^{\prime} S^{x^{\prime}} W^{\prime}=T t^{y}=t^{y} T^{(-y)}$ with $x^{\prime}, y>0$, then we must have $S W^{\prime} t^{-z}=Q \in Y$ and $t^{-z^{\prime}} U^{\prime}=Q_{0} \in Y$ for some $z$ and $z^{\prime}$. All the above reductions are effective, so we may as well assume that $U \equiv Q_{0} \in Y, W \equiv Q \in Y$, and $x^{\prime}=x$. We now have

$$
Q_{0}\left(t S_{1} \ldots t S_{r}\right)^{x} Q=T t^{x r}
$$

If $x>0$, then we must have $S_{r} Q=P_{r} \in X, S_{r-1} P_{r}^{(1)}=P_{r-1} \in X$, etc. By changing $Q_{0}$ and $Q$ if necessary, we may therefore assume $S_{i}$ to be a word on gen $(Y)$ for each $i$. Note that the $P_{i}$ 's are unique.

Let us now write

$$
t S_{1} \ldots t S_{r}=S_{0} t S_{1}^{\prime} \ldots t S_{r}^{\prime} \equiv S^{\prime}
$$

where each $S_{i}^{\prime}$ is either empty or begins with $a_{\mu}^{ \pm 1}$ for $i>0$. Next we write

$$
t S_{1}^{\prime} \ldots t S_{r}^{\prime} S_{0}=S_{0}^{\prime} t S_{1}^{\prime \prime} \ldots t S_{r}^{\prime \prime} \equiv S^{\prime \prime}
$$

with $S^{\prime \prime}$ satisfying the same conditions as $S^{\prime}$. Note that if $S_{i}^{\prime} \neq \Lambda$ for some $i>0$, then we either get $S_{0}^{\prime} \equiv \Lambda$ or $l\left(S^{\prime \prime}\right)<l\left(S^{\prime}\right)$. Thus, after a finite number of steps, that we can keep track of, we must arrive at either $\bar{S}_{0} t^{r}$ with $\bar{S}_{0}$ involving no $a_{i}$ 's, or $t \bar{S}_{1} \ldots t \bar{S}_{r}$ with each $S_{1}$ empty or beginning with $a_{\mu}^{ \pm 1}$. By changing $Q_{0}$ and $Q$ if necessary, we may assume that $S \equiv S_{0} t^{r} \equiv \bar{S}_{0} t^{r}$ or $S \equiv t S_{1} \ldots t S_{r} \equiv t \bar{S}_{1} \ldots t \bar{S}_{r}$ with the conditions just mentioned satisfied. We may also assume that $S_{0} S_{0}^{(r)}$ is freely reduced, otherwise we can replace $S_{0}$ with a shorter word $S_{0}^{\prime}$.

Suppose now that $Q_{0}\left(S_{0} t^{r}\right)^{x} Q=T t^{x r}$ for some $x>0$. We then arrive at

$$
Q_{0} S_{0} S_{0}^{(r)} \ldots S_{0}^{(x r-r)} Q^{(x r)}=T \in F_{2}^{\prime}
$$

If $Q$ involves some $a_{i}$, then $x r \leqslant \mu-\nu$. Freely reducing both sides gives an equation in $Y \cap F_{2}^{\prime}$ that can be solved by inspection. Finally assume that $S \equiv t S_{1} \ldots t S_{r}$ with at least one $S_{i} \equiv \Lambda$. Let $q$ be maximal with $S_{q} \equiv \Lambda$, and note that $S_{q} Q^{(r-q)}=P \in X \cap Y$. Since $S_{q}$ begins with $a_{\mu}^{+1}$, it must be completely absorbed in $Q^{(r-Q)}$, hence $l(P)<l(Q)$. Repeating this argument, note that if $S^{k} Q t^{-k r}=P_{k} \in X$ for each $k>0$, then $l\left(P_{k+1}\right)<l\left(P_{k}\right)$ for each such $k$. From this it is easy to bound $x$.

Lemma 5.4. Exactly like Lemma 5.2 with $F_{i}^{\prime}$ and $F_{i}$ replaced by $C_{i}^{\prime}$ and $C_{1}$ for $i=1,2$. Also include the assertions from the previous two lemmas about $\operatorname{gph}\left(U^{\prime}, W^{\prime} ; F_{1}^{\prime}, F_{2}^{\prime}\right)$ and $\operatorname{gph}\left(U^{\prime}, W^{\prime} ; C_{1}^{\prime}, F_{2}^{\prime}\right)$.

Proof. Let $C_{1}=\langle S\rangle$ and $S_{2}=\langle T\rangle$. If $S$ or $T$ is of finite order, then we can easily list all pairs $\left(S^{x}, T^{y}\right)$ with $U S^{x} W=T^{y}$. Suppose therefore that $|S|=|T|=\infty$.

Let $G$ be realized as the HNN group (7), and assume $S$ and $T$ to be cyclically $t$-reduced. If now $T \in H$ with $S \notin H$, then $U S^{x} W=T^{y}$ forces a bound on $|x|$. For each such $x$ we can decide if $U S^{x} W=V_{x} \in H$, and then if $V_{x} \in\langle T\rangle$. Just let $C_{0}=\langle\mathbf{l}\rangle$ in $H$, and consider $\operatorname{gph}\left(V_{x}, 1 ; C_{0}, C_{2}\right)$. The case with $S \in H$ and $T \notin H$ is similar. Two cases remain.

Case 1. $S, T \in H$. It suffices to consider $t$-reduced $U$ and $W$ with $U$ and $W^{-1} t$-parallel. Since we can compute gph ( $U^{\prime}, W^{\prime} ; C_{1}, F_{2}^{\prime}$ ) in $H$, the result is easily obtained by induction on $l_{t}(U)$.

Case 2. $S, T \notin H$. Note that $|y| l_{t}(T) \leqslant l_{t}(U)+l_{t}(W)+|x| l_{t}(S)$ whenever $U S^{x} W=T^{y}$, hence, it suffices to bound $|x|$. By symmetry we need only treat the case with $x, y>0$. Let $a, b>0$ be minimal such that $S^{a}$ and $T^{b}$ have the same $t$-length. Then $U S^{x} W=T^{y}$ if and only if $U_{r} S^{a x^{\prime}} W_{s}=T^{b y^{\prime}}$ where $x=a x^{\prime}+r, y=b y^{\prime}+s, 0 \leqslant r<a, 0 \leqslant s<b, U_{r}=U S^{r}$, and $W_{s}=W T^{-s}$. Since there are only finitely many such pairs $(r, s)$ to consider, it suffices to treat the case with $a=b=1$.

Following a remark in Case 2 of the last proof, we may assume that $U S^{x} W$ is $t$-reduced for all $x>0$. Suppose now that $S \equiv t^{\varepsilon_{1}} S_{1} \ldots t^{\ell_{r}} S_{r}$ and $T \equiv t^{\tau_{1}} T_{1} \ldots t^{\tau_{r}} T_{r}$ satisfy $U S^{x} W=T^{y}$ for some $x>0$. This implies

$$
S W T^{-z}\left(T^{\prime}\right)^{-1}=Q \in F_{2}^{\prime}
$$

for some $z \geqslant 0$ and terminal segment $T^{\prime}$ of $T$ beginning with $t \pm$; and

$$
\left(T^{\prime \prime}\right)^{-1} T^{-z^{\prime}} U=Q_{0} \in F_{2}^{\prime}
$$

for some $z^{\prime} \geqslant 0$ and initial segment $T^{\prime \prime}$ of $T$ for which $T \equiv T^{\prime \prime} \bar{T}$ and $\bar{T}$ begins with $t^{ \pm 1}$. Here $F_{2}^{\prime}=Y$ if $\varepsilon_{1}=1, F_{2}^{\prime}=X$ if $\varepsilon_{1}=-1$. Without loss of generality, we may assume that $U \equiv Q_{0}$, $W \equiv Q$, and then consider

$$
Q_{0}\left(t^{\varepsilon_{1}} S_{1} \ldots t^{e_{r}} S_{r}\right)^{x} Q=\left(t^{\varepsilon_{1}} T_{1} \ldots t^{\varepsilon_{r}} T_{r}\right)^{x} .
$$

Note that we must have $y=x$ and $S$ and $T$-parallel. Suppose now that $\varepsilon_{i+1}=-\varepsilon_{i}$ for some $1 \leqslant i<r$, and let $S^{\prime}$ and $T^{\prime}$ be the terminal segments of $S$ and $T$ beginning with $t^{\varepsilon+1}$. If we set

$$
S^{\prime} S^{z} Q T^{-z}\left(T^{\prime}\right)^{-1}=P_{z} \in F^{\prime}
$$

for $z=0$, 1 where $F^{\prime}$ depends on $\varepsilon_{i+1}$, then we must have $S_{1} P_{1} P_{0}^{-1} S_{i}^{-1} \in F^{\prime}$ as well. Hence, by malnormality of $F^{\prime}$ we get $S_{i} \in F^{\prime}$ if $P_{1} \neq P_{0}$. Since $S_{i} \in F^{\prime}$ would violate the assumption that $t^{\varepsilon_{i}} S_{i} t^{-\varepsilon_{i}}$ is t-reduced, we must have either all $\varepsilon_{i}$ 's equal or $P_{1}=P_{0}$. The latter implies $S^{x} Q T^{-x}=Q$ for all $x$, thus bounding $x$ by 0 .

By symmetry, it suffices now to treat the case with $S \equiv t S_{1} \ldots t S_{r}$ and $T \equiv t T_{1} \ldots t T_{r}$. If $S Q T^{-1} \neq Q$, then for each $i$ we must have $\operatorname{gph}\left(S_{i}, T_{1}^{-1} ; Y, X\right)$ infinite; otherwise no $x>1$ can satisfy $Q_{0} S^{x} Q=T^{x}$. But then, by Lemma 4.2 we must have each $S_{i}=P_{i} Q_{i}^{-1}$ for some $\left(P_{i}, Q_{i}\right) \in X \times Y$. By assumption we can effectively compute such pairs, and by changing $Q_{0}$ and $Q$ if necessary, we may assume each $S_{i}$ to be a word on gen ( $Y$ ). By a similar argument we may also assume each $T_{i}$ to be a word on gen ( $Y$ ).

We can now assume that we have applied the cyclic reduction process from the last proof to $S$ and $T$, and consider the following subcases:

Subcase 1. $S \equiv S_{0} t^{r}$ and $T \equiv T_{0} t^{r}$, with $S_{0}$ and $T_{0}$ involving no $a_{i}$ 's, and $S_{0} S_{0}^{(r)}$ and $T_{0} T_{0}^{(r)}$ freely reduced. From $Q_{0} S^{x} Q=T^{x}$ we then get the equation

$$
Q_{0} S_{0} S_{0}^{(r)} \ldots S_{0}^{(x r-r)} Q^{(r r)}=T_{0} T_{0}^{(r)} \ldots T_{0}^{(x r-r)}
$$

in $Y$. It is easy to bound $x$ if $l\left(S_{0}\right) \neq l\left(T_{0}\right)$. Also, if $l\left(S_{0}\right)=l\left(T_{0}\right)$, then we can decide by inspection if the equation can hold for any $x$ with $(x-2) l\left(S_{0}\right)>l\left(Q_{0}\right)+l(Q)$.

Subcase 2. $S \equiv S_{0} t^{r}$ as in Subcase 1 and $T \equiv t T_{1} \ldots t T_{r} \neq t^{r}$ with each $T_{i}$ empty or beginning with $a_{\mu}^{ \pm 1}$. (The case with $S$ and $T$ interchanged is similar.) From $Q_{0} S^{x} Q=T^{x}$ with $x>0$ we then get

$$
Q_{0} S_{0} S_{0}^{(r)} \ldots S_{0}^{(2 r-r)} t^{z r} Q=\left(t T_{1} \ldots t T_{r}\right)^{x}
$$

But then $Q T_{r}^{-1}=P \in X \cap Y$, and therefore all of $T_{r}^{-1}$ must be absorbed in $Q$. If $T_{r} \equiv \Lambda$, then $l(P)<l(Q)$. Since $T_{j} \neq \Lambda$ for at least one $j$, we must have $t^{r} Q T^{-1}=Q_{1} \in Y$ with $l\left(Q_{1}\right)<l(Q)$. This forces a bound on $x$.

Subcase 3. $S \equiv t S_{1} \ldots t S_{r} \equiv t^{r}$ and $T \equiv t T_{1} \ldots t T_{r} \neq t^{r}$ where each $S_{i}$ and $T_{j}$ is either empty or begins with $a_{\mu}^{ \pm 1}$. If now

$$
Q_{0}\left(t S_{1} \ldots t S_{r}\right)^{x} Q=\left(t T_{1} \ldots t T_{r}\right)^{x}
$$

for some $x>0$, then we must have $S_{r} Q T_{r}^{-1}=P_{r} \in X \cap Y, S_{r-1} P_{r}^{(1)} T_{r-1}^{-1}=P_{r-1} \in X \cap Y$, etc. All of $S_{r}$ and $T_{r}^{-1}$ must be absorbed in the free reductions of $S_{r} Q T_{r}^{-1}$. Hence, unless $S_{r}$ and $T_{r}$ are empty, we must have $l\left(P_{r}\right)<l(Q)$. Since at least one $S_{1}$ is non-empty, it follows that $S Q T^{-1}=Q_{1} \in Y$ with $l\left(Q_{1}\right)<l(Q)$. This forces a bound on $x$, and completes the proof of the lemma.

We can now establish all the needed results about computability of graphs in any $G \in \Gamma_{*}$. Recall our assertions about free groups $F_{1}$ and $F_{2}$ in $G$.

Proposition 5.5. We can effectively compute the sets $\operatorname{gph}(U, W ; H, K)$ for any $U, W \in G \in \Gamma_{*}$ where $H=F_{1}$ or $C_{1}$ and $K=F_{2}$ or $C_{2}$ as subgroups of $G$.

Proof. We use induction on the length of the root $R$ of $G$. If $l(R)=1$, then $G=C * F$ for some finite cyclic group $C$ and free group $F . F_{1}$ and $F_{2}$ must be subgroups of $F$, which we may assume to be nonempty. All three types of graphs can be effectively computed in $C$ and $F$. But then, by modifying the techniques in the three last lemmas, we can also compute these sets in $G$. It is of course considerably easier to work with free products than with HNN groups.

Suppose now that $l(R) \geqslant 2$, and that the proposition holds for all $G^{\prime} \in \Gamma_{*}$ having roots $R^{\prime}$ with $l\left(R^{\prime}\right)<l(R)$. If $R$ has exponent sum zero on one of its generators, then the above lemmas apply. Finally assume that the exponent sum is non-zero on all generators in $R$, and in particular, assume that $x \neq 0$ and $y \neq 0$ are the exponent sums of $t$ and $a_{0}$ respectively in $R$. Then let $\hat{G} \in \Gamma_{*}$ be obtained from $G$ by replacing the generators $t$ and $a_{0}$ by $\hat{t}$ and $\hat{a}_{0}$, and
then replacing the root $R$ by the cyclic reduction of $\hat{R}$, where $\hat{R}$ is obtained from $R$ by replacing each $t$ by $\hat{a}_{0} \hat{t}^{-y}$ and each $a_{0}$ by $\hat{t}^{x}$. This construction also defines an imbedding

$$
\Psi: G \rightarrow G .
$$

Now, $\hat{G}$ can be realized as an HNN group with stable letter $\hat{t}$ and base $\hat{H} \in \Gamma_{*}$ having root of length less that $l(R)$ (see [7]). For any $V \in G$, let $\hat{V}$ be the $\hat{t}$-reduced form of $\Psi^{\prime}(V) \in \hat{G}$. Similarly, if $H$ is a subgroup of $G$, let $\hat{H}$ be the image in $\hat{G}$ of $H$ under $\Psi$.

The case with $C_{1}$ or $C_{2}$ finite is trivial, so suppose that both are infinite. Then

$$
\operatorname{gph}\left(U, W ; C_{1}, C_{2}\right)=\operatorname{gph}\left(\hat{U}, \hat{W} ; \hat{C}_{1}, \hat{C}_{2}\right)
$$

in $Z^{2}$, so this set can be effectively computed since $\Psi$ is clearly effective. Next, consider gph ( $U, W ; C_{1}, F_{2}$ ) with $C_{1}=\langle S\rangle$. By relabelling the generators of $G$ if necessary, we may assume that $t \notin F_{2}$. If also $a_{0} \notin F_{2}$, let $\bar{F}_{2}=\hat{F}_{2}$ in $\hat{G}$; otherwise let $\bar{F}_{2}$ be generated by gen $\left(F_{2}\right)-\left\{a_{0}\right\}$ together with $\hat{t}$. Now compute $\operatorname{gph}\left(\hat{O}, \hat{W} ; \hat{C}_{1}, \bar{F}_{2}\right)$ in $\hat{G}$. The case with this set finite is easy, so suppose it to be infinite. By an earlier remark, we must have $O S O^{-1}=$ $\bar{S}_{0} \in \bar{F}_{2}$ and $\hat{O} \hat{W}=\bar{T}_{0} \in \bar{F}_{2}$. But then, $U S^{x} W=T \in F_{2}$ if and only if $\left(\hat{O} \bar{S} \hat{O}^{-1}\right)^{x} \hat{O} \hat{W}=\bar{S}_{0}^{x} \bar{T}_{0}=\hat{T}$. We can now decide if any $x \in \mathbf{Z}$ and $\hat{T} \in \hat{F}_{2}$ can satisfy this equation in $\bar{F}_{2}$.

It remains to consider $\operatorname{gph}\left(U, W ; F_{1}, F_{2}\right)$. Up to relabelling of the generators, the following three cases exhaust all possibilities.

Case 1. $t, a_{0} \in F_{3}, t \notin F_{2}$. Let $\bar{F}_{1}$ be generated by (gen $\left.\left(F_{1}\right)-\left\{t, a_{0}\right\}\right) \cup\left\{\hat{t}, \hat{a}_{0}\right\}$ in $\hat{G}$, while $\bar{F}_{2}$ is the group just considered above. Now compute $\operatorname{gph}\left(\hat{O}, \hat{W} ; \bar{F}_{1}, \bar{F}_{2}\right)$ in $\hat{G}$. The case with this set finite is easy, so suppose that

$$
\operatorname{gph}\left(\hat{O}, \hat{W} ; \bar{F}_{1}, \bar{F}_{2}\right)=(P, Q)[\operatorname{gph}(1,1 ; \bar{F}, \bar{F})]\left(P^{-1} S_{0}, Q^{-1} T_{0}\right)
$$

as an infinite set where $\hat{O}=Q P^{-1},(P, Q) \in \bar{F}_{1} \times \bar{F}_{2}$, and $\bar{F}=\bar{F}_{1} \cap \bar{F}_{2}$ (see Lemma 4.2). If now $U S W=T$ for some $(S, T) \in F_{1} \times F_{2}$, then

$$
S=P \bar{S} P^{-1} S_{0} \quad \text { and } \quad \hat{T}=Q \bar{S} Q^{-1} T_{0}
$$

for some $\bar{S} \in \bar{F}$. Since $\hat{a}_{0} \notin \bar{F}$, it is easy to decide if these equations have a solution in $\bar{F}_{1}$ and $\bar{F}_{2}$.

Case 2. $t \ddagger F_{1} \cup F_{2}$. This is essentially like Case 1, only easier.
Case 3. gen $(R)=\left\{t, a_{0}\right\}, t \in F_{1}, a_{0} \in F_{2}$. The cases with $F_{1}$ or $F_{2}$ cyclic have been considered, so suppose that $F_{1}=\langle t ;\rangle * F_{1}^{\prime}$ and $F_{2}=\left\langle a_{0} ;\right\rangle * F_{2}^{\prime}$ with $F_{1}^{\prime}$ and $F_{2}^{\prime}$ non-trivial. Hence, we must also have $G=G_{0} * F$ for some non-trivial free group $F$ where $G_{0}=$ $\left\langle t, a_{0} ; R^{n}\right\rangle$. Let us assume that no terminal segment of $U$ or initial segment of $W$ belongs to $F_{1}$, and similarly for $U^{-1}$ and $W^{-1}$ with respect to $F_{2}$. It is now easy to decide if $U S W=T$ is possible for any $(S, T) \in F_{1} \times F_{2}$ with $U, W, S$, and $T$ s-reduced.

Note that we have also shown: All groups in $\Gamma_{*}$ have solvable generalized word problem with respect to cyclic subgroups. This because $U \in C_{2}$ if and only if $\mathrm{gph}(U, \mathbf{1}$; $\left.F_{1}, C_{2}\right) \neq \varnothing$ where $F_{1}$ is the trivial group.

We can now prove Lemma 3.2 as well.
Proof of Lemma 3.2. Let $U \in G \in \Gamma_{*}$ where $G$ has the root $R$. We must decide if $U \sim_{G} S$ is possible for any free word $S$ in $G$; then we must show how such an $S$ can be effectively constructed whenever solutions exist. We proceed by induction on $l(R)$, observing that the problem is trivial when $l(R)=1$. Suppose now that the result has been established for all $G^{\prime} \in \Gamma_{*}$ having roots $R^{\prime}$ of length less than $l(R)$, where $l(R) \geqslant 2$.

First we consider the case when $G$ can be realized as the HNN group (7). Let $U$ be cyclically $t$-reduced. Note that if $S$ is free in $G \in \Gamma_{*}$ and the $t$-reduced form $S^{\prime}$ of $S$ belongs to $H$, then $S^{\prime}$ is also free in $H \in \Gamma_{*}$. Suppose now first that $U$ is $t$-free. By the inductive hypothesis, suppose that we have determined a free word $S$ in $H$ with $U \sim_{H} S$. Moreover, assume this $S$ to be cyclically reduced. Suppose also that there exists a free word $T$ in $G \in \Gamma_{*}$ with $U \sim_{G} T$. The cyclically $t$-reduced form $T^{\prime}$ of $T$ must belong to $H$, moreover, for some $V \equiv V_{0} t^{x} V_{1}$ we must have $T^{\prime}=V S V^{-1}$ (see Lemma 3.3). It now suffices to set $V_{0}, V_{1} \equiv$ $\Lambda$, and check if $T^{\prime} \equiv S^{(x)}$ for any $x \in \mathbf{Z}$ and free word $T$ in $G \in \Gamma_{*}$. Finally, suppose that $U \equiv t^{\epsilon_{1}} U_{1} \ldots t^{\varepsilon_{k}} U_{k}$ with $k \geqslant 1$. By Collins' Lemma (p. 123 in [3]), if $U \sim_{G} S$ with $S$ cyclically $t$-reduced, then we must also have $l_{t}(S)=k$. Replacing $a_{0}$ successively by the elements from gen $(R)-\{t\}$ in the HNN construction of $G$, we may assume that $S$ involves no $a$ symbols. Hence, $S$ can be written $S \equiv S_{0} t^{x}$ with $|x|=k$. It is now enough to consider the case with $U \equiv t U_{1} \ldots t U_{\kappa}$ and $S \equiv S_{0} t^{\kappa}$. By Collins' Lemma, if $U \sim_{G} S$, then we may assume that

$$
\begin{equation*}
t U_{1} \ldots t U_{k}=Q S_{0} t^{k} Q^{-1} \tag{11}
\end{equation*}
$$

for some $Q \in Y$. But then $U_{k} Q=P \in X, U_{k-1} P^{(1)}=P^{\prime} \in X$, etc. By Proposition 5.5 we can effectively compute pairs $\left(P_{i}, Q_{i}\right) \in X \times Y$ with $U_{i}=P_{i} Q_{i}^{-1}$ for each $1 \leqslant i \leqslant k$ (if (ll) can be satisfied), hence, we may as well assume each $U_{i}$ to be a word on gen ( $Y$ ). We can now use the cyclic reduction process on $U$ that we applied to the generator of $C_{1}$ in the proof of Lemma 5.3. If we arrive at a word without $a$-symbols, then we are done. Suppose therefore that we arrive at the word $V \equiv t V_{1} \ldots t V_{k}$ with each $V_{i}$ empty or beginning with $a_{\mu 1}^{ \pm 1}$ and $V_{i} \neq \Lambda$ for at least one $i$. If now

$$
t V_{1} \ldots t V_{k}=Q S_{0} t^{k} Q^{-1}
$$

then $V_{k} Q=P_{k} \in X, \quad V_{k-1} P_{k}^{(1)}=P_{k-1} \in X$, etc. Thus, $Q \equiv V_{k}^{-1} P_{k}, P_{k}^{(1)} \equiv V_{k-1}^{-1} P_{k-1}, \ldots, P_{2}^{(1)} \equiv$ $V_{1}^{-1} P_{1}$. After these $t$-reductions we arrive at

$$
P_{1}^{(1)}=Q S_{0} .
$$

But this equation cannot be satisfied for any $S_{0}$ without $a$-symbols. To see this, note that $P_{1}^{(1)}=Q S_{0}$ is an equation in $Y$, and

$$
Q \equiv V_{k}^{-1}\left(V_{k-1}^{-1}\right)^{(-1)} \ldots\left(V_{1}^{-1}\right)^{(-k+1)} P_{1}^{(-k+1)}
$$

contains strictly more $a_{i}$ 's than $P_{1}^{(1)}$.
The case remains where $G$ cannot be realized directly as the HNN group (7). We then use the imbedding

$$
\Psi: G \rightarrow \hat{G}
$$

from the proof of Proposition 5.5. Suppose now that there exists a free word $S \in G$ and a free word $T \in G$, such that

$$
U \sim_{G} S \quad \text { and } \quad \hat{O} \sim_{\hat{G}} T
$$

If $t \notin \operatorname{gen}(S)$, where $\Psi(t)=\hat{a}_{0} \hat{t}^{-y}$, then $S$ is also a free word in $G$, hence, by Lemma 3.1 $S$ must be a cyclic permutation of $T$. We assume here that $S$ and $T$ are cyclically reduced. Since the imbedding $\Psi$ depends on the particular choice of $t, a_{0} \in \operatorname{gen}(R)$, we must repeat this imbedding for each such choice, and then check if $T \in \Psi^{\prime \prime}(G)$ for some such $\Psi^{\prime \prime}$.

## § 6. The main results for the class $\boldsymbol{\Gamma}_{\mathbf{0}}$

In this section we complete our study of the class $\Gamma_{0}$. To this end, let $G \in \Gamma_{0}$ be given by

$$
\begin{equation*}
G=\left\langle a_{1}, a_{2}, \ldots ; R_{1}^{n_{1}}, \ldots, R_{k}^{n_{k}}\right\rangle \tag{12}
\end{equation*}
$$

subject to the conditions on (2) in Section 2. In that section we also showed that if $k \geqslant 2$, then $G$ can be realized as a tree-product

$$
\begin{equation*}
G=G_{1} *_{F_{1}} \ldots * F_{k}{ }_{1} G_{k} \tag{13}
\end{equation*}
$$

where each $G_{i} \in \Gamma_{*}$ and each $F_{t}$ is free on gen $\left(G_{i}\right) \cap$ gen $\left(G_{t+1}\right)$. Moreover, if $G^{\prime} \in \Gamma_{0}$ is the subgroup of $G$ generated by $\bigcup_{i=1}^{k-1}$ gen $\left(G_{i}\right)$, then

$$
\begin{equation*}
G=G^{\prime} *_{F} G_{k} \tag{14}
\end{equation*}
$$

where $F=F_{k-1}$. Note that $G^{\prime}$ has $k-1$ relators, hence, this gives us a means of proving results about $\Gamma_{0}$ by induction on $k$ in (12).
B. B. Newman proved in [10] that if $J$ is malnormal in $H_{1}$ and $H_{2}$, then $H_{1}$ and $H_{2}$ are malnormal in $H_{1} *_{J} H_{2}$. Using this result together with transitivity of malnormality, it is easy to prove by induction on $k$ :

Lemma 6.1. Let $G \in \Gamma_{0}$ be given by (12) with $k \geqslant 2$. Then the subgroups $G_{i}$ in (13) and $G^{\prime}$ in (14) are malnormal in $G$.

Suppose now that $G$ is given by (12) with $k \geqslant 1$. If $k=1$, let $G_{1}=G$; otherwise let the
subgroups $G_{i}$ be defined by (13). Now, if $U$ is a word on gen $(G)$ with gen $\left(U^{\prime}\right) \subseteq \operatorname{gen}\left(G_{i}\right)$ for some subword $U^{\prime}$ of $U$ and $l \leqslant i \leqslant k$, then we call any $R_{i}$-reduction (see Section 5) of $U^{\prime}$ in $G_{i}$ an $R$-reduction of $U$. If no $R$-reductions are possible in $U$, then we call $U R$ reduced. If also all cyclic permutations of $U$ are $R$-reduced, then we call $U$ cyclically $R$. reduced. Note-that these reductions are effective.

Let $k \geqslant 2$ and consider $G$ as the generalized free product (14). Then, if $U$ is an $R$ reduced word on gen $(G)$, we claim that $U$ is also $s$-reduced (see Section 4) as an element of $G^{\prime} *_{F} G_{k}$. This is so because any $s$-reduction of $U$ must involve an $R$-reduction. Similarly, if $U$ is cyclically $R$-reduced, then it must also be cyclically $s$-reduced.

Using the above ideas we can prove Theorem A for the class $\Gamma_{0}$. We state this as a lemma.

## Lemma 6.2. We can effectively compute the order of elements in any $G$ from $\Gamma_{0}$.

Proof. We use induction on $k$ where $G$ is given by (12). The result is well-known for $k \leqslant 1$, so suppose that $k \geqslant 2$ with the lemma established for all $G^{\prime} \in \Gamma_{0}$ having less that $k$ relators. Write $G=G^{\prime} *_{F} G_{k}$ as in (14), and consider $U \in G$. By the remarks above, we may assume that $U$ is cyclically $R$-reduced. Now, if $U$ belongs to $G^{\prime}$ or $G_{k}$, then the inductive hypothesis applies, while otherwise, $|U|=\infty$.

Our next result generalizes Proposition 3.5 and establishes Theorem B for the class $\Gamma_{0}$.

Theorem 6.3. Given $U, W \in G \in \Gamma_{0}$, we can effectively compute integers $a$, $b$, and $c$ with $0 \leqslant c \leqslant 1$ such that

$$
C P_{G}(U, W)=(a, b) \mathbf{Z} \cup(a c,-b c) \mathbf{Z}
$$

if $|U|,|W|=\infty$;

$$
G P_{G}(U, W)=(a, b) \mathbf{Z}+(|U| \mathbf{Z}) \times(|W| \mathbf{Z})
$$

if $|U|,|W|<\infty$.
Proof. We use induction on the number $k$ of roots in the presentation (12) of $G$. The case with $k=0$ is trivial. Also, $k=1$ with $|U|,|W|=\infty$ is covered by Proposition 3.5.

To complete the case with $k=1$, let $U$ and $W$ be of finite order in $G \in \Gamma_{*}$ where $G$ has the relator $R^{n}$. Now, if $U$ or $W$ equals 1 , then we can clearly take $a, b=0$. Suppose therefore that $U, W \neq 1$. Hence, $U \sim_{G} R^{p}$ and $W \sim_{G} R^{\alpha}$ for some $0<p, q<n$. From Lemma 3.4 it follows that $U^{x} \sim_{G} W^{y}$ if and only if $R^{p x}=R^{o y}$. The integers $p$ and $q$ can be effectively computed, hence, we can decide if $U$ and $W$ are power-conjugate, that is, if $U^{x} \sim_{G} W^{y} \neq 1$ for some $x, y \in \mathbf{Z}$. If $U$ and $W$ are not power-conjugate, let $a, b=0$; otherwise determine the mi-
nimal $a>0$ in $\mathbf{Z}$ with $R^{p a}=R^{a b^{\prime}} \neq 1$ for some $b^{\prime} \in \mathbf{Z}$. Then determine the minimal $b>0$ for which $R^{p a}=R^{a b}$. It remains to show that $U^{x} \sim_{G} W^{\nu}$ implies $U^{x}=U^{a z}$ and $W^{y}=W^{b z}$ for some $z \in \mathbf{Z}$. But this is easy enough, just use the Euclidean algorithm and write $x=a z+r$ with $0 \leqslant r<a$. Then observe that

$$
R^{p r}=R^{p(x-a z)}=R^{p x} R^{-p a z}=R^{q y} R^{-q b z}=R^{q(y-b z)}
$$

By minimality of $a$, it follows that $R^{p r}=1$, and hence,

$$
U^{x} \sim_{G} R^{p x}=R^{p a z} \sim_{G} U^{a z}
$$

and

$$
W^{y} \sim_{G} R^{a y}=R^{a b z} \sim_{G} W^{b z} .
$$

The result now follows from Lemma 3.4.
Suppose next that $G$ has $k \geqslant 2$ roots, and that the theorem is valid for all $G^{\prime} \in \Gamma_{0}$ with less than $k$ roots. We can then write $G=G^{\prime} *_{F} G_{k}$ as in (14) and apply the inductive hypothesis to both $G^{\prime}$ and $G_{k}$. Let $U$ and $W$ be cyclically $R$-reduced. By Lemma 6.1 it is clear that

$$
C P_{G}(U, W)=C P_{G^{*}}(U, W)
$$

if $U, W \in G^{\prime \prime}$ for $G^{\prime \prime}=G^{\prime}$ or $G_{k}$. Moreover, if $U$ and $W$ belong to distinct factors, then $U^{x} \sim_{G} W^{y}$ implies $U^{x}$ and $W^{y}$ must both be conjugate in their factor to some $T \in F$. By symmetry we may assume that $W \in G_{k}$. If $W^{y} \neq 1$, then $W$ must be conjugate in $G_{k}$ to some $T_{0} \in F$ with $T=T_{0}^{\nu}$. By Lemma 3.2 we can effectively determine such a $T_{0}$ if it exists, so because we then get

$$
C P_{G}(U, W)=C P_{G}\left(U, T_{0}\right)
$$

it remains to consider $W^{y}=1$. But in this case (i.e. $W^{\nu} \sim_{G} T \in F$ implies $T=1$ ) we may set $a, b, c=0$. The case with $l_{s}(U), l_{s}(W) \geqslant 2$ remains. Since we can easily determine minimal integers $a^{\prime}, b^{\prime}>0$ with $l_{s}\left(U^{a^{\prime}}\right)=l_{s}\left(W^{b^{\prime}}\right)$, we may as well assume that $l_{s}(U)=l_{s}(W)$. Now, by Solitar's Theorem we know that $U^{x} \sim_{G} W^{x}$ for some $x \neq 0$ if and only if $U^{x}=S\left(W^{x}\right)_{\pi} S^{-1}$ for some $s$-permutation $\left(W^{x}\right)_{\pi}$ of $W^{x}$ and $S \in F$. Since $\left(W^{x}\right)_{\pi}=W_{\pi}^{x}$ for some $s$-permutation $W_{\pi}$ of $W$, note that if $x>1$, then

$$
U^{i} S W_{\pi}^{-i}=S_{i} \in F
$$

for each $1 \leqslant i \leqslant x$. But then we must have $S=S_{1}$; otherwise $U S S_{1}^{-1} U^{-1}=S_{1} S_{2}^{-1} \neq 1$, and by malnormality of $F$ in $G$, we then get $U \in F$. The case with $x<-1$ is similar, so we can conclude that $U^{x} \sim_{G} W^{x}$ for some $x \neq 0$ if and only if $U \sim_{G} W$. Thus, it suffices to determine whether or not $U \sim_{G} W^{\varepsilon}$ for $\varepsilon= \pm 1$. Let us just consider $\varepsilon=1$. Now, if $U \equiv U_{1} \ldots U_{r}$ and $W \equiv W_{1} \ldots W_{r}$ in terms of syllables and $U \sim_{G} W$, then

$$
U_{1} \ldots U_{r} S\left(W_{1} \ldots W_{r}\right)_{\pi}^{-1}=S
$$

for some $s$-permutation $\left(W_{1} \ldots W_{r}\right)_{\pi}$ of $W$. There are only finitely many such $s$-permutations, so let us just consider the trivial one. By considering $S=U^{-1} S W$ if necessary, we may assume that $U_{r}, W_{r} \in G_{k}$. In this factor we can effectively compute $g \mathrm{ph}\left(U_{r}, W_{r}^{-1} ; F, F\right)$ by Proposition 5.5. Moreover, this set can contain at most one pair ( $S, T$ ), otherwise $U_{r}, W_{r} \in F$. Now all we need to do is to check if $U S W^{-1} S^{-1}=1$ for this $S$.

The final result we need for the class $\Gamma_{0}$ is the following generalization of part of Proposition 5.5.

Lemma 6.4. For any cyclic subgroups $C_{1}$ and $C_{2}$ of $G \in \Gamma_{0}$, and elements $U, W \in G$, we can effectively compute $\operatorname{gph}\left(U, W ; C_{1}, C_{2}\right)$.

Proof. The case with $C_{1}$ or $C_{2}$ finite is trivial, so suppose that $C_{1}=\langle S\rangle$ and $C_{2}=\langle T\rangle$ with $|S|,|T|=\infty$. In this case we identify $C_{1}$ and $C_{2}$ with $\mathbf{Z}$ and set

$$
\operatorname{gph}\left(U, W ; C_{1}, C_{2}\right)=\left\{(x, y) \in \mathbf{Z}^{2} ; U S^{x} W=T^{y}\right\}
$$

As a consequence of Lemma 4.1, we know that this set is either empty or takes the form

$$
\operatorname{gph}\left(U, W ; C_{1}, C_{2}\right)=(r, s)+(a, b) \mathbf{Z}
$$

Let us proceed by induction on the number $k$ of roots in $G$. The case with $k=0$ was treated in [5], while $k=1$ is covered by Proposition 5.5. Suppose now that $k \geqslant 2$ with the lemma established for all $G^{\prime} \in \Gamma_{0}$ having less than $k$ roots. Then write $G=G^{\prime} *_{F} G_{k}$ as in (14), noting that the inductive hypothesis applies to both factors. By standard arguments, we may assume that $S$ and $T$ are cyclically $R$-reduced, $U$ and $W R$-reduced. The case with $l_{s}(S)=$ $l<l_{s}(T)$, or vice versa, is trivial since we can then bound $|x|$ or $|y|$. Let us now consider

Case 1. $l_{s}(S), l_{s}(T) \geqslant 2$. By considering a finite number of cases, we may assume that $S$ and $T$ have the same $s$-length, and that $\operatorname{gph}\left(U, W ; C_{1}, C_{2}\right) \neq \varnothing$ if and only if $U S^{x} W=T^{y}$ for some $x, y \geqslant 0$. It suffices to bound $x$ since this also yields a bound on $y$. Assume therefore that $x$ is large enough so that we can $s$-reduce $U S^{x} W$ and obtain an $s$-reduced word $U^{\prime} \mathcal{S}^{x^{\prime}} W^{\prime}$ with $x^{\prime} \geqslant 2$. (We accomplish this by $R$-reductions.) If now $U^{\prime} S^{x^{\prime}} W^{\prime}=T^{y}$, then

$$
S W^{\prime} T^{-y_{2}} T_{1}^{-1}=P \in F
$$

and

$$
T_{2}^{-1} T^{-y_{s}} U^{\prime}=Q \in F
$$

for some syllable-segments $T_{1}$ and $T_{2}$ of $T$ for which $T^{\prime} T_{1} \equiv T \equiv T_{2} T^{\prime \prime}$, where $y_{1}, y_{2} \geqslant 0$. Since we now have

$$
Q S^{x^{\prime}-1} P=T^{\prime \prime} T^{y^{\prime}} T^{\prime}=T_{0}^{x^{\prime}-1}
$$

where $T_{0} \equiv T^{\prime \prime} T_{2} \equiv T_{1} T^{\prime \prime}$ (comparing $s$-lengths), it suffices to show that we can take $x^{\prime}-1=1$. But clearly, if $x^{\prime}-1>1$, then

$$
S^{4} P T_{0}^{-i}=P_{1} \in F
$$

for each $1 \leqslant i \leqslant x^{\prime}-1$, hence, malnormality of $F$ in $G$ implies $P=P_{1}$.
Case 2. $l_{s}(S), l_{s}(T)=1$. Suppose that $U \equiv U_{1} \ldots U_{p}$ and $W \equiv W_{1} \ldots W_{q}$ as s-reduced decompositions into syllables where we allow $U \equiv U_{1} \equiv \Lambda$ and $W \equiv W_{1} \equiv \Lambda$. Suppose further that

$$
U_{1} \ldots U_{p} S^{x} W_{1} \ldots W_{q}=T^{y}
$$

for some $x$ and $y$. If $U, S, W$, and $T$ belong to one and the same factor, then the inductive hypothesis applies, and otherwise we must have $U_{p} S^{x}, S^{x} W_{1}$, or $U_{p} S^{x} W_{1}$ in $F$. If this is an equation in $G_{k}$, then by Proposition 5.5 we can compute the corresponding graph. Moreover, unless the third possibility occurs with $U_{p} S U_{p}^{-1}=S_{0} \in F$ and $U_{p} W_{1} \in F$, the $x$ is unique. Also, in this case with $x$ not unique we can shorten $U$ and $W$ by a syllable, and repeat the argument with $S$ replaced by $S_{0} \in F$. If $S \notin G_{k}$ and $U_{p}$ or $W_{1}$, as the case may be, belongs to $G^{\prime}$, then let $U_{*}$ and $W_{*}$ be the remaining segments of $U$ and $W$. We now get $U_{*}^{-1} T^{y} W_{*}^{-1} \in F$, and hence, if $U_{*}, W_{*} \ddagger G_{k}$, then we arrive at the graph gph ( $U_{i}, W_{j}$; $F, F)$ in $G_{k}$, where $U_{1} \equiv U_{p}$ or $U_{p-1}(\Lambda$ if $p=1)$ and $W_{j} \equiv W_{1}$ or $W_{2}(\Lambda$ if $q=1)$. By Lemma 4.l, at most one pair $S_{0}, T_{0} \in F$ can satisfy $U_{i} S_{0} W_{j}=T_{0}$, otherwise $U_{i}, W_{,} \in F$. By Proposition 5.5 we can compute this pair ( $S_{0}, T_{0}$ ), and hence, also determine $x$ and $y$. It remains to consider the case with $U_{*}, W_{*} \in G_{k}$. But this is just like the first part.

The following corollary is immediate.

Corollary 6.5. The groups in $\Gamma_{0}$ have solvable generalized word problem with respect to cyclic subgroups.

With these results for $\Gamma_{0}$ we can turn to the main theorems for $\Gamma_{1}$.

## § 7. Proofs of the main theorems

With the results thus far established in this paper and techniques used in [5], the following is easy to prove, hence we omit the proof here.

Lemma 7.1. Let $G=G_{1} *_{C} G_{2} \in \Gamma_{1}$. Then for any $U, W \in G$ we can effectively compute $\operatorname{gph}(U, W ; C, C)$ in $G$.

Note that by Corollary 6.5 we can effectively $s$-reduce and cyclically $s$-reduce elements in any $G=G_{1} *_{C} G_{2} \in \Gamma_{1}$.

Proof of Theorem A. Let $U \in G=G_{1} *_{C} G_{2} \in \Gamma_{1}$. To compute $|U|$, let us first cyclically $s$-reduce $U$ and obtain $U^{\prime}$. If now $l_{s}\left(U^{\prime}\right)>1$, then $|U|=\infty$. If instead $U^{\prime}$ belongs to $G_{1}$ or $G_{2}$, then we can apply Lemma 6.2.

We can also give the
Proof of Theorem B. We have already proved this for all groups in $\Gamma_{0}$, so assume that $G=G_{1} *{ }_{C} G_{2} \in \Gamma_{1}$ with $C$ non-trivial. Let $U, W \in G$ be cyclically $s$-reduced, and consider first

Case 1. $l_{s}(U), l_{3}(W)>1$. As usual, we only treat the case with $U$ and $W$ of the same $s$-length. Moreover, by Lemma 7.1 and Solitar's Theorem, it suffices to obtain a bound on $x>0$ for which $U^{x} \sim_{G} W^{e x}$ is possible for $\varepsilon= \pm 1$. By a result in [5], we may take $x \leqslant 2$ if $C$ is infinite. Also, if $C=\langle S\rangle$ in $G$ with $|S|<\infty$, then $U^{x} \sim_{G} W^{\varepsilon x}$ implies

$$
U^{i} S^{z} W_{\pi}^{-\varepsilon t}=S^{z_{i}}
$$

for each $1 \leqslant i \leqslant x$, where

$$
U^{x}=S^{z} W_{\pi}^{\epsilon x} S^{-z}
$$

Here $W_{\pi}$ is some cyclic $s$-permutation of $W$. But then, if $x>|S|$, we get $S^{z_{i}}=S^{z_{j}}$ for some $j<i$, hence,
implies

$$
U^{i} S^{z} W_{\pi}^{-\varepsilon i}=U^{j} S^{z} W_{\pi}^{-\varepsilon t}
$$

with $i-j<x$.

$$
U^{i-f}=S^{2} W_{\pi}^{(i-j) \varepsilon} S^{-z}
$$

Case 2. $l_{s}(U), l_{s}(W)=1$. If $U, W \in G_{i}(i=1$ or 2$)$ and some $(x, y) \in C P_{G}(U, W)$ does not belong to $C P_{G_{i}}(U, W)$, then

$$
U^{x} \sim_{G_{i}} S^{z}=T^{1 z} \sim_{G_{y}} T^{z^{\prime}}=S^{z^{\prime}} \sim_{G_{1}} W^{y}
$$

where $C=\langle S\rangle$ in $G_{i}$ and $C=\langle T\rangle$ in $G_{j}(j \neq i)$. If $|T|<\infty$, then $T^{z}=T^{z^{\prime}}$ and therefore $S^{z}=S^{z^{\prime}}$. Hence, we must have $|T|=\infty$. But then $z^{\prime}=-z$ and thus $(x,-y) \in C P_{G_{i}}(U, W)$. Since by Theorem 6.3,

$$
C P_{i}(U, W)=\left(a^{\prime}, b^{\prime}\right) \mathbf{Z} \cup\left(a^{\prime} c^{\prime},-b^{\prime} c^{\prime}\right) \mathbf{Z}
$$

with $0 \leqslant c^{\prime} \leqslant 1$, we must have $c^{\prime}=0$. Therefore, if $T \sim_{G_{j}} T^{-1}$, then we get

$$
C P_{G}(U, W)=(a, b) \mathbf{Z} \cup(a,-b) \mathbf{Z}
$$

with $a^{\prime}=a, b=b^{\prime}$; if $T r_{G_{,}} T^{-1}$, then $C P_{G}(U, W)=C P_{G_{i}}(U, W)$.
Finally consider $U \in G_{i}$ and $W \in G_{j}$ with $i \neq j$. Since $U^{x} \sim_{G} W^{y}$ if and only if

$$
U^{x} \sim_{G_{i}} S^{z}=T^{z} \sim_{G_{g}} W^{y}
$$

we can construct $C P_{G}(U, W)$ from $C P_{G_{i}}(U, S)$ and $C P_{G_{j}}(T, W)$.

As for the generalizations alluded to in the introduction, let $\Gamma$ be the smallest class containing $\Gamma_{0}$ which is closed under the formation of free products with cyclic subgroups amalgamated.

Using techniques from this paper and from [5], we can generalize Lemmas 6.4 and 7.1 to groups in $\Gamma$. Also, if $G=G_{1}{ }_{C} G_{2} \in \Gamma$, and $|U|<\infty$ in $G$, then we must have $U$ conjugate to an element of a factor. Continuing, we note that $U$ must be conjugate to an element of a subgroup $G^{\prime}$ of $G$ with $G^{\prime} \in \Gamma_{0}$.

It follows from the above that we must get
Theorem $\mathbf{B}^{\prime}$. Given $U, W \in G \in \Gamma$, we can effectively compute integers $a, b, c_{1}, \ldots, c_{n}$ such that

$$
C P_{G}(U, W)=(a, b) \mathbf{Z} \cup\left[\bigcup_{i=1}^{n}\left(a c_{i},-b c_{i}\right) \mathbf{Z}\right]
$$

if $|U|,|W|=\infty$;

$$
C P_{G}(U, W)=(a, b) \mathbf{Z}+(|U| \mathbf{Z}) \times(|W| \mathbf{Z})
$$

if $|U|,|W|<\infty$.
Note that the generalized version of Lemma 6.4 implies we can cyclically $s$-reduce elements of any $G \in \Gamma$, hence we can compute $|U|$ and $|W|$ in Theorem $\mathrm{B}^{\prime}$.

The corresponding generalization of the results in [5] about HNN groups are straightforward.

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Received July 16, 1976
19†-772905 Acta mathematica 139. Imprimé le 30 Décembre 1977

