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ON THE COMPUTATION OF FIXED POINTS IN THE PRODUCT SPACE OF UNIT SIMPLICES AND AN APPLICATION TO NONCOOPERATIVE *N* PERSON GAMES*

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In this paper an algorithm based on the principle of simplicial approximation is introduced to compute fixed points of upper semicontinuous point to set mappings from the product space S of unit simplices into itself. The algorithm is a modification of an algorithm, introduced in an earlier paper. The main feature is that it starts with an arbitrary chosen point in S and that the triangulation of S depends on the starting point. Moreover, the algorithm can terminate with a non-full-dimensional subsimplex, yielding a good approximation. An application is given for non cooperative n person games, where S is the strategy space. Some computational experiences are given.

1. Introduction. In their paper [9] Lemke and Howson showed that a bimatrix game can be formulated as a linear complementarity problem. The generalization to n person games was found independently by Rosenmüller [12] and Wilson [17]. Using features of this nonconstructive method Garcia, Lemke and Lüthi [3] developed an algorithm to compute an equilibrium point for noncooperative n person games. This algorithm is based on the ideas of the algorithm introduced by Scarf [13] and [14] for computing fixed points of a continuous function on the unit simplex. Scarf's algorithm is characterized by a particular subdivision of the unit simplex and by a start in a corner. More efficient methods on the unit simplex have been given by Eaves [1], Kuhn and MacKinnon [5] and van der Laan and Talman [6]. The generalization to mappings on R^n can be found in Eaves and Saigal [2], Merrill [10], Todd [16] and van der Laan and Talman [7].

In this paper we will modify the algorithm of van der Laan and Talman to compute a fixed point of a continuous function on the product space of unit simplices. In particular the equilibrium strategies of n person games will be computed. §2 gives a short description of the algorithm developed in [6]. The triangulation of the product space of unit simplices is given in §3. In §4 an integer labelling rule is given and it is proved that the algorithm's terminal subsimplex, provides a good approximation of a fixed point. §5 presents the modification of the algorithm. The generalization for vector labelling is given in §6. In §7 we consider the problem of the n person game. §8 shows some computational experiences and conclusions are drawn.

2. Description of van der Laan and Talman's algorithm for the unit simplex. In this section we give a concise description of the algorithm of van der Laan and Talman [6] for a continuous function f from the unit simplex S^m into itself. We restrict ourselves to integer labelling.

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Assume that we use the standard triangulation of the $(m-1)$-dimensional unit simplex, i.e., this triangulation is the analogon of the triangulation of R^n, introduced by Kuhn [4] (see also Scarf [14] and Todd [15]). Any point x of the simplex receives an integer label $l(x)$ defined by $l(x) = k$ if $f_k(x) - x_k \leqslant f_i(x) - x_i$, $i = 1, \ldots, m$, and $x_k > 0$ where k is the lowest index which satisfies this condition. Note that this is a proper labelling. A completely labelled subsimplex, i.e., a simplex having all its n vertices labelled differently, is a good approximation of a fixed point.

The algorithm starts with one point of the grid, say v^0, which can be chosen arbitrarily, e.g., by prior information. From that point, a zero-dimensional face, a path of adjacent faces is generated by unique replacement steps, until a completely labelled subsimplex is found. To do so the unit simplex is subdivided into regions $A(T)$, with T a proper subset of $I_m = \{1, \ldots, m\}$, defined by

$$A(T) = \{x \in S^m \mid x = v^0 + \textstyle\sum_{j \in T} \lambda_j (e(j+1) - e(j)), \lambda_j \geqslant 0, j \in T\}$$

where $e(j)$ is the jth unit vector and $j + 1 = 1$ if $j = m$.

A t-dimensional face σ will be called a simplex of region $A(T)$, if all points of the relative interior of σ are also interior points of $A(T)$. Then the algorithm generates simplices of $A(T)$ if and only if T is the current set of labels found. As soon as a new label is found, say label j, the algorithm continues with simplices of $A(T \cup \{j\})$ by extending the current simplex τ of $A(T)$ to a simplex σ of $A(T \cup \{j\})$, such that τ is a facet of σ. If, however, a replacement step would imply a change from a simplex of $A(T)$ to an adjacent simplex of $A(T \cup \{k\} \setminus \{j\})$ for some $j \in T, k \notin T$, then the vertex, which is to be removed, is the only one in the interior of $A(T)$ and the other are points of $A(T \setminus \{j\})$. Instead of replacing this vertex by a new one in the interior of $A(T \cup \{k\} \setminus \{j\})$, it is deleted, whereas the vertex with label j (there will be exactly one, since $j \in T$) is now removed. Then the algorithm continues in a unique way with simplices of $A(T \setminus \{j\})$. In van der Laan and Talman [6] it is proved that a completely labelled subsimplex will always be found within a finite number of iterations.

3. The triangulation of the product space of unit simplices. Let S^m be the $(m-1)$-dimensional unit simplex, i.e., $S^m = \{x \in R^m_+ \mid \sum_{k=1}^m x_k = 1\}$. Let $S = \prod_{j=1}^n S^{m_j}$ be the product space of n unit simplices S^{m_j}. We denote $\sum_{j=1}^n m_j$ by M and an element x of S by $x = (x^1, x^2, \ldots, x^n)$ where x^j is an element of S^{m_j}. Let y be a continuous function from S into itself, i.e., $y(x) = (y^1(x), \ldots y^n(x))$ is an element of S. To compute a fixed point of y we will modify the algorithm described in the previous section. To do so we need a triangulation of S. For some t let $w^0, \ldots, w^t$ be $t+1$ affinely independent points in R^n. Then the convex hull $\sigma(w^0, \ldots, w^t) = \{\sum_{i=0}^t \lambda_i w^i \mid \lambda_i \geqslant 0$ for all i and $\sum_{i=0}^t \lambda_i = 1\}$ is called a t-dimensional simplex or t-simplex with vertices $w^0, \ldots, w^t$. A simplex τ is a face of a simplex σ if all the vertices of τ are vertices of σ. If the number of vertices of the face τ of σ is one less than the number of vertices of σ, then τ is called a facet of σ.

DEFINITION 3.1. A collection G of m-simplices is a triangulation of C if the relative interiors of the faces of all simplices partition C.

To triangulate S we define the block diagonal matrix Q by

$$Q = \begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & Q_n \end{bmatrix},$$

where Q_j is the $m_j \times m_j$ matrix defined by

$$Q_j = \begin{bmatrix} -1 & 0 & \cdot & \cdot & & 1 \\ 1 & -1 & 0 & \cdot & \cdot & 0 \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & & \cdot & \cdot & 0 \;\; 1 & -1 \end{bmatrix}.$$

The ith column of Q will be denoted by $q(i) i = 1, \ldots, M$.

It can easily be seen that the matrix Q_j induces the standard triangulation of the unit simplex S^{m_j}. Define $S^{m_j}(d_j)$ as the set of points of S^{m_j} induced by the regular grid of size d_j, i.e., the elements of $S^{m_j}(d_j)$ are the points $(x_1^j, \ldots, x_{m_j}^j)$ such that

$$x_k^j = z_k^j / d_j \text{ where } z_k^j \text{ is a nonnegative integer and } \sum_{k=1}^{m_j} z_k^j = d_j.$$

Let D be the diagonal matrix defined by

$$D = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & & \vdots \\ \vdots & \cdot & \cdot & 0 \\ 0 & \cdots & 0 & D_n \end{bmatrix}$$

where D_j is the $m_j \times m_j$ diagonal matrix with elements $1/d_j$ on the diagonal. Let $S(d)$ be the set of points $x = (x^1, \ldots, x^n)$ such that x^j is an element of $S^{m_j}(d_j)$ and let v^0 be an arbitrarily chosen point of $S(d)$.

Moreover, let $I(j, m_j)$ be the set of elements (j, k) such that $k \in I_{m_j}$. Let $\mathfrak{T}^1$ be the collection of all subsets T of $\bigcup_j I(j, m_j)$ such that for all j there is at least one element $(j, k) \in I(j, m_j)$ not in T, and let $\mathfrak{T}^2$ be the subset of $\mathfrak{T}^1$ such that for all j there is exactly one element $(j, k) \in I(j, m_j)$ not in T. For all $T \in \mathfrak{T}^1$ the subset $A(T)$ of S is defined by

$$A(T) = \left\{ x \in S \mid x = v^0 + \sum_{(j,k) \in T} \lambda(j,k) D q(j,k) \text{ for nonnegative numbers } \lambda(j,k) \right\}$$

where $q(j,k) = q(k + \sum_{i=1}^{j-1} m_i)$. Clearly, the relative interiors of all $A(T)$, $T \in \mathfrak{T}^1$, partition S. Note that $A(T)$ depends on the point v^0.

LEMMA 3.2. (a) *The product space S is the union of the subsets $A(T)$, $T \in \mathfrak{T}^2$.*
(b) $A(T_1) \cap A(T_2) = A(T_1 \cap T_2)$ for all $T_1, T_2 \in \mathfrak{T}^2$.

PROOF. To prove part (a) let x be an arbitrary point of S. Then there is a unique $T_1 \in \mathfrak{T}^1$ such that $x \in A(T_1)$ and $\lambda(j,k) > 0$ for all $(j,k) \in T_1$. For all $T_2 \subset \mathfrak{T}^2$ such that $T_1 \subset T_2$, we have now that $x = v^0 + \sum_{(j,k) \in T_2} \lambda(j,k) D q(j,k)$ with $\lambda(j,k) = 0$ for all $(j,k) \in T_2 \setminus T_1$, i.e., x is an element of $A(T_2)$. The proof of part (b) follows immediately from the definition of $A(T)$ and the fact that the $\lambda(j,k)$'s are unique.

Let t be the number of elements of $T \in \mathfrak{T}^1$ and let $\sigma(w^0, \gamma^T) = \sigma(w^0, \ldots, w^t)$ be a t-simplex in S with vertices $w^0, \ldots, w^t$ such that

$$w^0 = v^0 + \sum_{(j,k) \in T} \mu(j,k) D q(j,k) \quad \text{for nonnegative integers } \mu(j,k)$$

and

$$w^i = w^{i-1} + D q(\gamma_i^T), \qquad i = 1, \ldots, t,$$

where $\gamma^T = (\gamma_1^T, \ldots, \gamma_t^T)$ is a permutation of the elements of T. Observe that all vertices are elements of $A(T)$. Hence, $\sigma(w^0, \gamma^T)$ is a simplex of $A(T)$. Moreover, let x be an element of $A(T)$. So there are unique nonnegative numbers $\lambda(j,k)$, $(j,k) \in T$, such that

$$x = v^0 + \sum_{(j,k) \in T} \lambda(j,k) Dq(j,k).$$

Let $w^0 = v^0 + \sum_{(j,k) \in T} \mu(j,k) Dq(j,k)$ with $\mu(j,k) =$ entier $(\lambda(j,k))$ nonnegative integers, and let γ^T be a permutation of the elements of T such that

$$z(\gamma_1^T) \geqslant z(\gamma_2^T) \geqslant \cdots \geqslant z(\gamma_t^T)$$

where $z(j,k) = \lambda(j,k) - \mu(j,k)$. Then x is a point in $\sigma(w^0, \gamma^T)$. Hence, the collection of simplices $\sigma(w^0, \gamma^T)$ cover $A(T)$. Furthermore, let $\beta_0 = 1 - z(\gamma_1^T)$, $\beta_i = z(\gamma_i^T) - z(\gamma_{i+1}^T)$, $i = 1, \ldots, t-1$, and $\beta_t = z(\gamma_t^T)$. Then $x = \sum_{i=0}^t \beta_i w^i$ and x lies in the open face of σ with vertices w^i having positive β_i. Analogously to the proof in Todd [15, pp. 30, 31] this face is the only open face in $A(T)$ containing x. Consequently for any $T \in \mathfrak{T}^1$ we have the following corollary.

COROLLARY 3.3. *The collection of all t-simplices $\sigma(w^0, \gamma^T)$ triangulates $A(T)$.*

THEOREM 3.4. *The product space S is triangulated by the union of the simplices induced by the triangulation of $A(T)$, $T \in \mathfrak{T}^2$.*

PROOF. We have to prove that for any pair $T_1, T_2 \in \mathfrak{T}^2$ the triangulation of $A(T_1 \cap T_2)$ induced by the triangulation of $A(T_1)$ is the triangulation of $A(T_1 \cap T_2)$ induced by the triangulation of $A(T_2)$.

Let T_1 and T_2 be two different elements of $\mathfrak{T}^2$ and let σ_1 be a simplex of $A(T_1)$ with vertices $w_1^0, \ldots, w_1^{M-n}$. Since $w_1^0 \in A(T_1)$, we have $w_1^0 = v^0 + \sum_{(j,k) \in T_1} \mu(j,k) Dq(j,k)$ for unique nonnegative integers $\mu(j,k)$ and $w_1^h = w_1^0 + \sum_{i=1}^h Dq(\gamma_i^{T_1})$, $h = 1, \ldots, M-n$, for some permutation $\gamma_i^{T_1}$ of the elements of T_1.

Clearly, if $w_1^0 \notin A(T_2)$, then $w_1^h \notin A(T_2), h = 1, \ldots, M-n$, and $\sigma_1 \cap A(T_2)$ is empty. Hence if $\sigma_1 \cap A(T_2)$ is not empty, $w_1^0 \in A(T_2)$. Furthermore, if for some h, $0 < h \leqslant M-n$, $w_1^h \in A(T_1) \cap A(T_2) = A(T_1 \cap T_2)$, then

(a) $\mu(j,k) = 0$, for all $(j,k) \notin T_1 \cap T_2$.

(b) $\gamma_i^{T_1} \in T_1 \cap T_2$, for all $i = 1, \ldots, h$.

This implies that $w_1^i \in A(T_1 \cap T_2)$ for all $i = 0, \ldots, h$.

Because $T_1 \neq T_2$, $w_1^{M-n} \notin A(T_1 \cap T_2)$. So there exists an integer h^*, $0 \leqslant h^* \leqslant M - n - 1$, such that $w_1^0, \ldots, w_1^{h^*}$ are points of $A(T_1) \cap A(T_2)$, whereas $w_1^{h^*+1}, \ldots, w_1^{M-n}$ are not. Hence, $w_1^0 = v^0 + \sum_{(j,k) \in T_2} \mu(j,k) Dq(j,k)$ for unique integers $\mu(j,k)$, and $w_1^i = w_1^{i-1} + Dq(\gamma_i^{T_2})$ for $i = 1, \ldots, h^*$ for any permutation γ^{T_2} of the elements of T_2, such that $\gamma_i^{T_2} = \gamma_i^{T_1}$i for $i = 1, \ldots, h^*$. For all permutations γ^{T_2} satisfying the last condition, the simplex $\sigma(w_2^0, \ldots, w_2^{m-n})$ with $w_2^i = w_1^i, i = 0, \ldots, h^*$ and $w_2^i = w_2^{i-1} + Dq(\gamma_i^{T_2}), i = h^* + 1, \ldots, M-n$, is a simplex induced by the triangulation of $A(T_2)$.

So, the triangulations of $A(T_1 \cap T_2)$ induced by the triangulation of $A(T_1)$ is consistent with that induced by the triangulation of $A(T_2)$, for all T_1 and $T_2 \in \mathfrak{T}^2$, which proves the theorem.

From the proof it follows directly that the triangulation of $A(T_1 \cap T_2)$ induced by those of $A(T_1)$ (and $A(T_2)$) is the triangulation of $A(T_1 \cap T_2)$ introduced above.

Theorem 3.4 is clarified by Figure 3.1.Observe that the dimension of S is $M - n$, which is also the dimension of each region $A(T)$, $T \in \mathfrak{T}^2$. In §4, every point in S receives a label induced by the function $y(x)$, which is an element of $\bigcup_{j=1}^n I(j, m_j)$. Note that this set contains M elements.

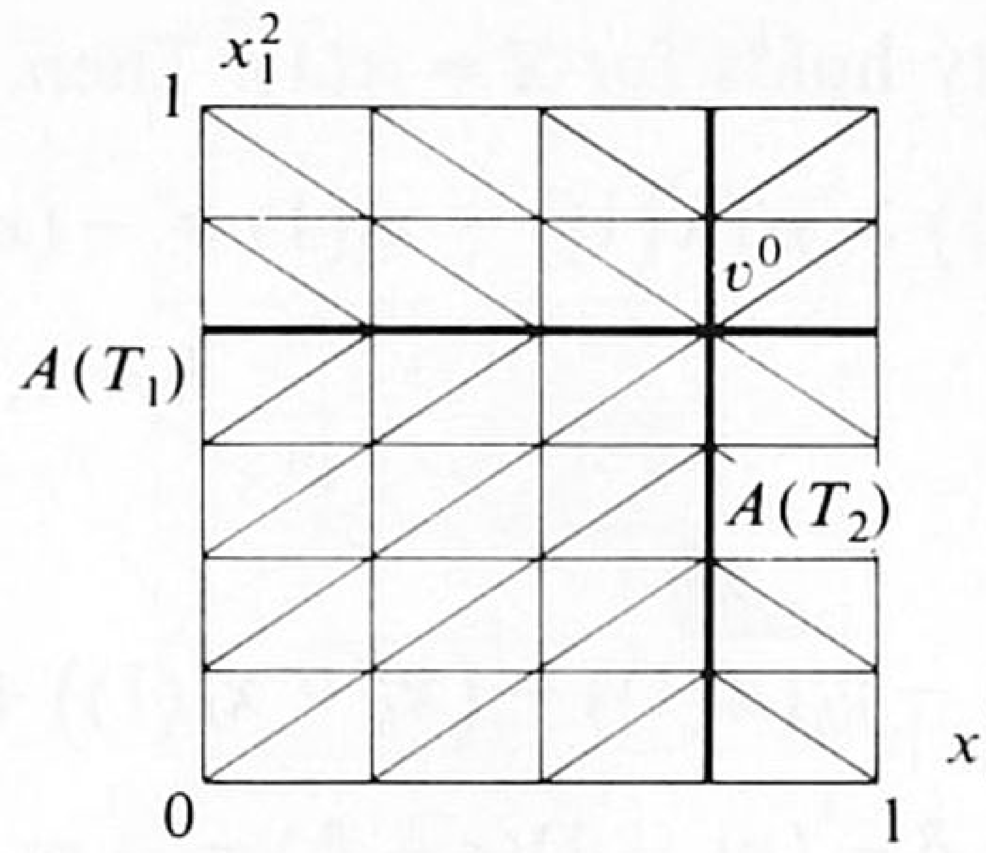

FIGURE 3.1. $n = 2$; $m_1 = m_2 = 2$; $d_1 = 4$; $d_2 = 6$;
$v^0 = (\frac{3}{4}, \frac{1}{4}; \frac{2}{3}, \frac{1}{3})$;
$T_1 = \{(1, 1)\}$; $T_2 = \{(1, 2); (2, 1)\}$.

Normally, in applying fixed point algorithms, the number of labels is one more than the dimension of the problem, while the algorithm can only terminate with a full-dimensional completely labelled simplex. The algorithm to be described in §5, however, will terminate with a simplex with at least $\min_i m_i$ and with at most $M - n + 1$ labels, to yield a good approximation.

4. Integer labelling and approximation. To apply an algorithm based on simplicial subdivision we have to label every point of the product space S.

A point $x = (x^1, \ldots, x^n)$ of S receives an integer label $l(x)$ by the following rule: $l(x) = (j, k)$ if (j, k) is the lexicographic least index such that $x_k^j > 0$ and $y_k^j(x) - x_k^j \leqslant y_h^i(x) - x_h^i$ for all $(i, h) \in I(i, m_i)$, $i = 1, \ldots, n$.

Note that the labelling rule is proper in the sense that $l(x) \neq (j, k)$ if $x_k^j = 0$. A face will be called completely labelled if all its vertices have a different label.

A completely labelled face with set of labels L is a j-stopping face, if, for all $k = 1, \ldots, m_j$, (j, k) is an element of L, whereas at least one $(i, h) \in I(i, m_i)$ is not in L, for all $i \neq j$.

Let the norm of a vector z in R^n be the supremum norm, i.e., $\|z\| = \max_{i=1,\ldots,n}(|z_i|)$. Let $\|x - \bar{x}\| < \delta$ imply $\|y(x) - y(\bar{x})\| \leqslant \epsilon$ and let the grid be so fine that $\|x - \bar{x}\| < \delta$ if x and $\bar{x}$ are points in the same face. Then we have the following theorem.

THEOREM 4.1. *Let $\bar{x}$ be any point in a j-stopping face τ. Then*

$$\|y(\bar{x}) - \bar{x}\| \leqslant (\epsilon + \delta)m_j \cdot \max_{i \neq j}(m_i - 1).$$

PROOF. Let $x(1), x(2), \ldots, x(m_j)$ be the m_j vertices of τ such that $l(x(k)) = (j, k)$. Since $\sum_{k=1}^{m_j} y_k^j(x) = \sum_{k=1}^{m_j} x_k^j$ $(= 1)$, we must have by definition of the labelling rule

$$y_k^j(x(k)) - x_k(k) \leqslant 0, \qquad k = 1, \ldots, m_j.$$

Hence, for any $\bar{x}$ in the face,

$$\begin{aligned} y_k^j(\bar{x}) - \bar{x}_k^j &= y_k^j(\bar{x}) - y_k^j(x(k)) - (\bar{x}_k^j - x_k^j(k)) \\ &\quad + y_k^j(x(k)) - x_k^j(k) \\ &\leqslant \epsilon + \delta, \qquad k = 1, \ldots, m_j. \end{aligned}$$

On the other hand, we have,

$$y_k^j(\bar{x}) - \bar{x}_k^j = -\sum_{\substack{i=1 \\ i \neq k}}^{m_j} \{y_i^j(\bar{x}) - \bar{x}_i^j\} \geqslant -(m_j - 1)(\epsilon + \delta), \qquad k = 1, \ldots, m_j.$$

In particular, this inequality holds for $\bar{x} = x(1)$. Then, again by the labelling rule,

$$y_h^i(x(1)) - x_h^i(1) \geq y_1^j(x(1)) - x_1^j(1) \geq -(m_j - 1)(\epsilon + \delta), \quad h = 1, \ldots, m_i; i \neq j.$$

So, for any $\bar{x}$ in the face,

$$y_h^i(\bar{x}) - \bar{x}_h^i = y_h^i(\bar{x}) - y_h^i(x(1)) - (\bar{x}_h^i - x_h^i(1)) + y_h^i(x(1)) - x_h^i(1) \geq -\epsilon - \delta - (m_j - 1)(\epsilon + \delta) = -m_j(\epsilon + \delta)$$

and

$$y_h^i(\bar{x}) - \bar{x}_h^i = -\sum_{\substack{k=1 \\ k \neq h}}^{m_i} \{ y_k^i(\bar{x}) - \bar{x}_k^i \} \leq (m_i - 1)m_j(\epsilon + \delta), \quad h = 1, \ldots, m_i; i \neq j.$$

Combining all of this together, we obtain for any $\bar{x}$ in the face,

$$\| y^j(\bar{x}) - \bar{x}^j \| \leq (m_j - 1)(\epsilon + \delta)$$

and

$$\| y^i(\bar{x}) - \bar{x}^i \| \leq (m_i - 1)m_j(\epsilon + \delta), \qquad i \neq j.$$

Therefore,

$$\| y(\bar{x}) - \bar{x} \| \leq (\epsilon + \delta)m_j \cdot \max_{i \neq j}(m_i - 1).$$

5. The application of van der Laan and Talman's algorithm. In a particular grid, let v^0 be the starting point. Assume that for the triangulation of §3 the algorithm generates a face $\tau(w^0, \ldots, w^t)$ and a set T of t labels such that there is a permutation γ^T of the elements of T and an M-dimensional nonnegative integer vector R with the following properties:

(1) the $(k + \sum_{h=1}^{i-1} m_h)$th component of R is zero if $(i,k) \notin T$;

(2) $w^0 = v^0 + \sum_{(j,k) \in T} R(j,k)Dq(j,k)$, where $R(j,k)$ is the $(k + \sum_{h=1}^{m_j - 1} m_h)$th component of R;

(3) $w^i = w^{i-1} + Dq(\gamma_i^T)$, for $i = 1, 2, \ldots, t$;

(4) all elements of T are a label of one of the vertices of the face and two vertices, say w^{i_1} and w^{i_2}, have the same label; one of them must be just found, say w^{i_2}.

Observe that such a face is a simplex of $A(T)$.

Now the algorithm replaces the vertex w^{i_1} according to Table 5.1 by a new vertex, producing a new simplex of the triangulation of $A(T)$, and its label is computed.

As we will show below, when continuing the algorithm by replacing the vertex having the same label as that of the new vertex, one of the following cases must occur:

(a) the algorithm finds a label (i,k) not in T;

(b) $R(j,k)$ becomes negative for some (j,k) in T.

TABLE 5.1

s is the index of the vector which must be replaced

	w^0 becomes	γ^T becomes	R becomes
$s = 0$	$w^0 + q(\gamma_1)$	$(\gamma_2, \ldots, \gamma_t, \gamma_1)$	$R + e(\gamma_1)$
$1 \leq s \leq t-1$	w^0	$(\gamma_1, \ldots, \gamma_{s-1}, \gamma_{s+1}, \gamma_s, \gamma_{s+2}, \ldots, \gamma_t)$	R
$s = t$	$w^0 - q(\gamma_t)$	$(\gamma_t, \gamma_1, \ldots, \gamma_{t-1})$	$R - e(\gamma_t)$

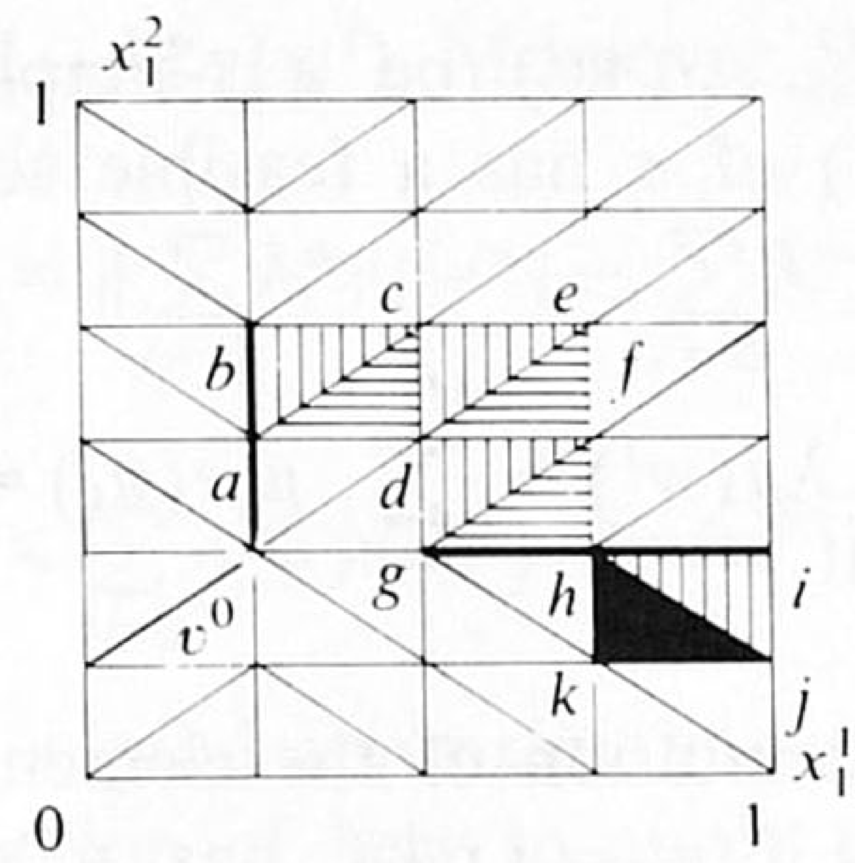

FIGURE 5.1. $n = m_1 = m_2 = 2$; $d_1 = 4$; $d_2 = 6$;
$v^0 = (\frac{1}{4}, \frac{3}{4}; \frac{1}{3}, \frac{2}{3})$; $l(v^0) = l(a) = l(d) = l(g) = (2, 2)$;
$l(b) = l(c) = l(e) = l(f) = l(h) = (1, 2)$;
$l(i) = l(j) = (2, 1)$; $l(k) = (1, 1)$ or $(2, 2)$.

In case (a) the current simplex $\sigma(\tilde{w}^0, \ldots, \tilde{w}^t)$ of $A(T)$ is completely labelled and the algorithm terminates if σ is a j-stopping face for some j. Otherwise the algorithm continues with a $(t + 1)$-dimensional simplex of $A(T \cup \{(i, k)\})$ with vertices the $t + 1$ current points $\tilde{w}^0, \ldots, \tilde{w}^t$ and a new point $\tilde{w}^{t+1}$, obtained by summing up $\tilde{w}^t$ and $Dq(i, k)$, and the labelset $T \cup \{(i, k)\}$. Clearly, this extension is always feasible. After computing the label of the new point the algorithm continues as before. In case (b) the replacement step is not performed but the algorithm continues with the $(t - 1)$-dimensional simplex $\tau(\tilde{w}^0, \ldots, \tilde{w}^{t-1})$ of $A(T \backslash \{(j, k)\})$ and the labelset $T \backslash \{(j, k)\}$ and removes the vertex with label (j, k). The algorithm starts with the one dimensional simplex $\sigma(v^0, v^1)$ of $A(l(v^0))$, where $v^1 = v^0 + Dq(l(v^0))$, with R the zero vector and with $T = \{l(v^0)\}$ by computing $l(v^1)$ and comparing it with $l(v^0)$. As proved in van der Laan and Talman [6], both the start and the replacement steps are unique. So, if all replacement steps are feasible, the algorithm generates a path of adjacent simplices of variable dimension such that the common facets of two adjacent simplices of $A(T)$ carry all the labels of T. Therefore a stopping face will be always found, since the number of faces in the product space S is finite and cycling cannot occur. The algorithm is illustrated in Figure 5.1.

Finally, we will show that all replacement steps are feasible. Clearly, a replacement step is not feasible if and only if the vertex to be replaced is not on some boundary face of S, whereas the remaining points are all on that boundary face. So, let $\tilde{w}^s$ be the vector to be replaced, for some $s, 0 \leqslant s \leqslant t$, and let $\tilde{w}^0, \ldots, \tilde{w}^{s-1}, \tilde{w}^{s+1}, \ldots, \tilde{w}^t$ be the remaining vertices of the face on, say the (j, k)th boundary face of S, i.e., the $(k + \sum_{h=1}^{j-1} m_h)$th component of $\tilde{w}^s$ is equal to d_j and the same component of the other vertices is equal to zero. Let $T, \tilde{\gamma}^T$ and $\tilde{R}$ be the current ones of Table 5.1. If the starting point v^0 is not on the (j, k)th boundary face, (j, k) must be an element of T. Hence, $\tilde{\gamma}_{s+1} = (j, k), s \leqslant t - 1$, and at least one of the vertices of face τ must have label (j, k). Since we have a proper labelling, $\tilde{w}^s$ is the only one and can therefore not be replaced. In the case that v^0 is on the (j, k)th boundary face of S, either one of the vertices has label (j, k) which is identical to the case just mentioned, or (j, k) is not an element of T. In that case $s = t$ and the $\tilde{\gamma}_t$th component of $\tilde{R}$ becomes negative, since $\tilde{w}^t$ is the only vertex of τ in the interior of $A(T)$. Consequently, the replacement step is not performed but $\tilde{w}^t$ is deleted and the algorithm continues with a $(t - 1)$-dimensional simplex of $A(T \backslash \{(j, k)\}$. This together proves that every replacement step must be feasible.

6. Vector labelling. In this section we generalize the algorithm for vector labelling. A point $x = (x^1, \ldots, x^n)$ of S receives a vector label $l(x) \in R^M$ by the following rule

$$l(x) = x - y(x) + e,$$

where e is the vector $(1, \ldots, 1)'$.

Let for some $T \in \mathfrak{T}^1, \sigma(w^0, \ldots, w^t)$ be a t-simplex of $A(T)$. Then the facet $\tau(w^0, \ldots, w^{s-1}, w^{s+1}, \ldots, w^t)$ of σ has a feasible solution if the system of linear equations

$$\sum_{\substack{i=0 \\ i \neq s}}^{t} \lambda_i l(w^i) + \sum_{h=1}^{M-t} \mu_h e(\pi_h) = e,$$

where $(\pi_1, \ldots, \pi_{M-t})$ is a permutation of the elements of $\cup_j I(j, m_j)$ not in T and where $e(\pi_h) = e(k + \sum_{i=1}^{j-1} m_i)$ if $\pi_h = (j,k)$, has a nonnegative solution λ_i^*, $i = 1, \ldots, s-1, s+1, \ldots, t$ and μ_h^*, $h = 1, \ldots, M-t$.

LEMMA 6.1. *If a facet $\tau(w^0, \ldots, w^{s-1}, w^{s+1}, \ldots w^t)$ in $A(T)$ has a feasible solution, then*

$$M \sum_{\substack{i=0 \\ i \neq s}}^{t} \lambda_i^* + \sum_{h=1}^{M-t} \mu_h^* = M.$$

PROOF. The system of linear equations is

$$\sum_{i \neq s} \lambda_i^* \left(w^i - y(w^i) + e\right) + \sum \mu_h^* e(\pi_h) = e.$$

Summing up over all components we obtain

$$\sum_{i \neq s} \lambda_i^* \left\{ \sum_{j=1}^{n} \sum_{k=1}^{m_j} \left(w_k^{ij} - y_k^j(w^i) + 1\right) \right\} + \sum \mu_h^* = M.$$

Since for all i, $\sum_{k=1}^{m_j} w_k^{ij} = \sum_{k=1}^{m_j} y_k^j(w^i) = 1$, for $j = 1, \ldots, n$, the lemma follows immediately.

A t-dimensional simplex $\sigma(w^0, \ldots, w^t)$ of $A(T)$ for some $T \in \mathfrak{T}^1$ will be called completely labelled if the set of linear equations

$$\sum_{i=0}^{t} \lambda_i l(w^i) + \sum_{h=1}^{M-t-1} \mu_h e(\pi_h) = e$$

has a nonnegative solution λ_i^* and μ_h^* for some permutation $(\pi_1, \ldots, \pi_{M-t-1})$ of $M-t-1$ elements of the $M-t$ elements of $\cup_j I(j, m_j)$ not in T.

Clearly, following the proof of Lemma 6.1, this solution has the property that $M \sum_{i=0}^{t} \lambda_i^* + \sum_{h=1}^{M-t-1} \mu_h^* = M$.

A face of $A(T)$, $T \in \mathfrak{T}^1$, having a feasible solution such that $\mu_h^* = 0$ for all h, is called a stopping face. Observe that a stopping face is either a $(t-1)$-dimensional facet of a simplex of $A(T)$ or it is completely labelled t-simplex of $A(T)$.

In the following theorem let $\sigma(w^0, \ldots, w^t)$ be a simplex of $A(T)$ such that either σ is a (completely labelled) stopping face with solution λ_i^*, $i = 0, \ldots, t$, or σ has for some $s, 0 \leqslant s \leqslant t$, a facet $\tau(w^0, \ldots, w^{s-1}, w^{s+1}, \ldots, w^t)$ which is a stopping face with solution λ_i^*, $i = 0, \ldots, s-1, s+1, \ldots, t$. In the latter case we define $\lambda_s^* = 0$.

THEOREM 6.2. *Let $\|x - \bar{x}\| < \delta$ imply $\| y(x) - y(\bar{x})\| \leqslant \epsilon$ and let the grid be so fine that $\|x - \bar{x}\| < \delta$ if x and $\bar{x}$ are points in the same simplex of the triangulation of S. Then $\| y(w^*) - w^*\| \leqslant \epsilon$, where $w^* = \sum_{i=0}^{t} \lambda_i^* w^i$.*

PROOF. Since for all h, $\mu_h^* = 0$ we can rewrite the set of linear equations as

$$\sum_{i=0}^{t} \lambda_i^* \left(w^i - y(w^i)\right) = 0.$$

Hence $w^* = \sum_{i=0}^{t} \lambda_i^* w^i = \sum_{i=0}^{t} \lambda_i^* y(w^i)$. Moreover $\sum_{i=0}^{t} \lambda_i^* = 1$. Therefore

$$\|y(w^*) - w^*\| = \|\sum_{i=0}^{t} \lambda_i^* y(w^*) - \sum_{i=0}^{t} \lambda_i^* y(w^i)\|$$

$$\leqslant \sum_{i=0}^{t} \lambda_i^* \|y(w^*) - y(w^i)\| \leqslant \epsilon \sum_{i=0}^{t} \lambda_i^* = \epsilon.$$

The theorem means that w^* is a good approximation of a fixed point of y.

We now give a short description of how to apply the algorithm, described in the previous section, for vector labelling.

Starting in a point v^0, with grid size vector d, the algorithm computes $l(v^0)$ and makes a pivot step with $l(v^0)$ in the set $I\mu = e$, where I is the M-identity matrix. If the (i,h)th unit column is eliminated, the point $v^1 = v^0 + Dq(i,h)$ is calculated, its vector label $l(v^1)$ is computed and a pivot step is made in the new system of equations. If the vector $l(v^0)$ is eliminated, v^0 is replaced according to Table 5.1. and a pivot step is made with the label of a new vertex etc. As soon as a unit vector is eliminated again, say $e(j,k)$, $(j,k) \neq (i,h)$, a completely labelled simplex of $A(\{(i,h)\})$ is found and, analogously to the case of integer labelling, the algorithm proceeds with a one higher dimensional simplex of $A(\{(i,h)\} \cup \{(j,k)\})$ by adding $Dq(j,k)$ to the last vertex of the current face. The algorithm continues with alternating pivot and replacement steps. In general, if a unit column is eliminated, i.e., a completely labelled simplex is found, the dimension is increased in the same way. If, however, in a replacement step a component of R, say the hth, becomes negative the replacement step is not performed, the last vertex w^t of the current simplex $\sigma(w^0, \ldots, w^t)$ is deleted while a pivot step is made with the hth unit vector since $w^t = w^{t-1} + Dq(h)$. The algorithm terminates if a stopping face is found. Since the set of all feasible solutions is bounded (see Lemma 6.1), the pivot steps can be always carried out and they are also unique assuming nondegeneracy. To resolve degeneracy, e.g., by lexicographic rules, we refer to Eaves [1] or Todd [15]. In the following we will assume that degeneracy does not occur. Of course if a stopping face is found, the solution is degenerated, but then a good approximation is found (see Theorem 6.2.).

Again, see [6] and also [7], the start and all replacement steps are unique whereas the extension to a higher dimensional simplex is always feasible and unique. So, if the replacement steps are also feasible, the algorithm generates a path of adjacent simplices of variable dimension such that the common facets of two adjacent simplices of $A(T)$ have a feasible solution.

As argued in §5, an infeasible replacement step would occur if in a certain face $\sigma(\tilde{w}^0, \ldots, \tilde{w}^t)$, for some (j,k), the $(k + \sum_{i=1}^{j-1} m_i)$th component of $\tilde{w}^s$, for some $s, 0 \leqslant s \leqslant t$ is d_j and if the same component is zero for the other vertices, while $\tilde{w}^s$ has to be replaced. However, if the starting point v^0 is not on the (j,k)th boundary face of S, we have the following lemma.

Lemma 6.3. *If the set of linear equations*

$$\sum_{\substack{i=0 \\ i \neq s}}^{t} \lambda_i l(w^i) + \sum_{h=1}^{M-t} \mu_h e(\pi_h) = e$$

has a feasible solution, then $\tau(\tilde{w}^1, \ldots, \tilde{w}^{s-1}, \tilde{w}^{s+1}, \ldots, \tilde{w}^t)$ *is a stopping face.*

Proof. If a set of linear equations has a nonnegative solution λ_i^* and μ_h^*, then from Lemma 6.1.

$$M \sum_{\substack{i=0 \\ i \neq s}}^{t} \lambda_i^* + \sum_{h=1}^{M-t} \mu_h^* = M,$$

which implies that either $\sum_{i \neq s}\lambda_i^* = 1$ or $\sum_{i \neq s}\lambda_i^* < 1$. In the first case $\mu_h^* = 0$, $h = 1, \ldots, M - t$ implying that τ is a stopping face. In the latter case, observe that $(j,k) \in T$, since v^0 is not on the (j,k)th boundary face. Hence $\pi_h \neq (j,k)$ for all h and the $(k + \sum_{i=1}^{j-1} m_i)$th component of the set of linear equations is

$$\sum_{i \neq s} \lambda_i^* \left(w_k^{ij} - y_k^j(w^t) + 1\right) = 1.$$

Since $w_k^{ij} = 0$, for $i \neq s$ it follows that $\sum_{i \neq s}\lambda_i^* \geqslant 1$ which contradicts $\sum\lambda_i^* < 1$.

The lemma means that if the system has a solution, the algorithm terminates with the stopping face $\tau(\tilde{w}^0, \ldots, \tilde{w}^{s-1}, \tilde{w}^{s+1}, \ldots, \tilde{w}^t)$, yielding an approximation of a fixed point on the (j,k)th boundary face of S. Otherwise there is no feasible solution, excluding that $\tilde{w}^s$ has to be replaced. If v^0 is on the (j,k)th boundary face, $\tilde{w}^s$ can also not be replaced, using the same argument as in the previous section, combined with Lemma 6.3.

By the unicity of the pivot and the replacement steps it cannot happen that the algorithm generates a simplex visited already before. Therefore we can conclude that a stopping face must be always found, since all replacement steps are feasible and the number of faces of the triangulation of S is finite.

Finally we have the following lemma.

LEMMA 6.4. *For some j, let $\sigma(w^0, \ldots, w^t)$ be a completely labelled simplex such that for $h = 1, \ldots, M - t - 1$, $\pi_h \neq (j,k)$, $k = 1, \ldots, m_j$. Then σ is a stopping face.*

PROOF. Since, for all h, $\pi_h \neq (j,k)$ for all k, we obtain by summing up the equations over the m_j components (j,k)

$$\sum_{i=0}^{t} \lambda_i^* \sum_{k=1}^{m_j} \left(w_k^{ij} - y_k^j(w^i) + 1\right) = m_j.$$

Since $\sum_{k=1}^{m_j} w_k^{ij} = \sum_{k=1}^{m_j} y_k^j(w^i) = 1$ for $i = 0, \ldots, t$, we have that $\sum_{i=0}^{t}\lambda_i^* = 1$ and hence, from Lemma 6.1., we get $\mu_h^* = 0$ for all h.

Analogously to the case of integer labelling, we call such a face a j-stopping face. In general, the algorithm will terminate with such a face.

7. The application of the algorithm to noncooperative n person games. The noncooperative n person game can be characterized by n players, indexed by $j = 1, \ldots, n$, and by m_j pure strategies for player j, indexed by $k = 1, \ldots, m_j$, for all j. Denote by the strategy vector $\underline{i} = (\underline{i}_1, \ldots, \underline{i}_n) \in I = I_{m_1} \times \cdots \times I_{m_n}$ that player j uses his $\underline{i}_j$th pure strategy, where $I_m = \{1, \ldots, m\}$. Let $a^j(\underline{i})$ be the loss to player j if strategy $\underline{i}$ is played. Without loss of generality we can assume that all losses are positive. Further we assume that $m_j < \infty$, all j. For $j = 1, \ldots, n$ let S^{m_j} denote the set of mixed strategies of player j, i.e., $x^j \in S^{m_j}$ is the vector of probabilities that the jth player uses his strategies. Then $S = \prod_{j=1}^{n} S^{m_j}$ is the product space of the mixed strategies of the players.

If $x = (x^1, \ldots, x^n)$ is an element of S, then the expected loss to player j is given by

$$p^j(x) = \sum_{\underline{i} \in I} a^j(\underline{i}) \prod_{k=1}^{n} x_{\underline{i}_k}^k.$$

The marginal loss to player j, when he plays his hth pure strategy, is defined by

$$m_h^j(x) = \sum_{\underline{i}} a^j(\underline{i}) \prod_{\substack{k=1 \\ k \neq j}}^{n} x_{\underline{i}_k}^k$$

where the sum is taken over all $\underline{i} \in I$ with $\underline{i}_j = h$.

DEFINITION 7.1. An equilibrium point is a vector $x^* \in S$ such that

$$m_h^j(x^*) \geqslant p^j(x^*), \qquad h = 1, \ldots, m_j; j = 1, \ldots, n.$$

To prove there exists always an equilibrium point we define a continuous function y from S to itself such that a fixed point of y is an equilibrium. To construct y, denote $\max(p^j(x) - m_h^j(x), 0)$ by $b_h^j(x)$ for $h = 1, \ldots, m_j; j = 1, \ldots, n$; and define $y_h^j(x)$ by

$$y_h^j(x) = \frac{x_h^j + b_h^j(x)}{1 + \sum_{k=1}^{m_j} b_k^j(x)}, \qquad h = 1, \ldots, m_j; j = 1, \ldots, n.$$

Then $y(x) = (y^1(x), \ldots, y^n(x))$ where $y^j(x) = (y_1^j(x), \ldots, y_{m_j}^j(x))$.

Using Brouwer's fixed point theorem, Owen [11] proved that there exists a fixed point of y and that this point must be an equilibrium point. So, with the algorithm described in the previous section, an equilibrium point can be computed. However, we will apply the algorithm with a labelling rule, based on the following complementarity conditions, which are equivalent with the equilibrium condition (cf. [3]),

$$x_k^j\left[m_k^j(x) - v^j(x)\right] = 0 \quad \text{where } v^j(x) = \min_k m_k^j(x),$$

$$x_k^j \geqslant 0 \quad \text{and} \quad \sum_{k=1}^{m_j} x_k^j = 1 \quad \text{for } k = 1, \ldots, m_j \text{ and } j = 1, \ldots, n.$$

Therefore we define $l(x) = (j,k)$ if (j,k) is the lexicographic least index with $x_k^j < 0$ and $m_k^j(x) - v^j(x) \geqslant m_h^i(x) - v^i(x)$ for all i and h in case of integer labelling and $l(x) \in R^M$ with

$$l_k^j(x) = m_k^j(x) - v^j(x) + 1 \quad \text{if } x_k^j > 0,$$
$$= 1 \quad \text{if } x_k^j = 0,$$

in case of vector labelling.

Clearly, a j-stopping face yields a good approximation for an equilibrium point.

8. Computational experiences and conclusions. We applied the algorithm for two noncooperative three-person games each person having three strategies. For the first game the data of game 2 of Garcia, Lemke and Lüthi [3] were used. The data of the second example are given in Table 8.2. Instead of the standard triangulation of S^{m_j} we used a triangulation of T^{m_j}, the affine hull of S^{m_j}, as proposed in van der Laan and Talman [8]. This triangulation is induced by the $m_j \times m_j$ matrix Q_j, with diagonal elements $1 - m_j$ and off-diagonal elements 1. A point outside S received the label of its projection (per player) on the boundary of S. For both examples, the algorithm was started with grid sizes $d_j = 3$[1] for $j = 1, 2, 3$. The factor of incrementation was set equal to two and the algorithm was terminated with d_j equal to 384.This is equal to a grid size of 665 in the algorithm of Garcia, Lemke and Lüthi [3]. In the first stage the starting point was chosen as $(1/3, \ldots, 1/3)$ for both games. If the algorithm terminated with a stopping face $\tau(w^0, \ldots, w^t)$, the new starting point in the next stage was chosen to be

$$v^0 = \frac{1}{t+1} \sum_{i=0}^{t} \tilde{w}^i \quad \text{for integer labelling}$$

and

$$v^0 = \sum_{i=0}^{t} \lambda_i^* \tilde{w}^i \Big/ \sum_{i=0}^{t} \lambda_i^* \quad \text{for vector labelling}$$

[1] In the first example $d_j = 6$ for the case with vector labelling.

TABLE 8.1

Game 1. The approximate fixed point was
$x^* = (0.3904, 0.2959, 0.3138; 0.3889, 0.2970, 0.3141;$
$0.9690, 0.0310, 0)$ *in case of integer labelling; and*
$x^* = (1, 0, 0; 0, 1, 0; 0, 1, 0)$ *in case of vector labelling*

	Integer labelling		Vector labelling	
Gridsize	Iterations	Accuracy	Iterations	Accuracy
6	68	0.25	43	0.18
24	191	0.075	150	0
96	275	0.012	–	–
384	322	0.003	–	–

TABLE 8.2

The number in the (j, k)th row and the (i_{k_1}, i_{k_2})th column is the loss of player j if j uses his kth pure strategy and if for $h = 1, 2$, player k_h uses his i_{k_h}th pure strategy, with $k_1, k_2 \neq j$ and $k_1 < k_2$

	(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)	(2, 3)	(3, 1)	(3, 2)	(3, 3)
(1, 1)	2	3	4	2	3	3	4	1	5
(1, 2)	1	1	4	3	4	1	6	8	2
(1, 3)	4	7	2	4	5	5	3	6	4
(2, 1)	5	6	7	4	8	9	3	5	1
(2, 2)	1	1	3	3	2	1	2	2	4
(2, 3)	2	3	6	5	3	6	7	5	8
(3, 1)	1	3	5	1	6	2	1	2	4
(3, 2)	2	6	5	3	3	7	8	5	5
(3, 3)	5	2	2	4	6	5	8	1	3

TABLE 8.3

Game 2.
$x^* = (0.4286, 0.5714, 0; 0, 1, 0; 0, 0.6650, 0.3350),$
in case of integer labelling; and
$x^* = (0.4284, 0.5716, 0; 0, 1, 0; 0, 0.6667, 0.3333),$
in case of vector labelling

	Integer labelling		Vector labelling	
Gridsize	Iterations	Accuracy	Iterations	Accuracy
6	27	0.2	40	0.06
24	53	0.07	55	0.005
96	84	0.02	71	0.001
384	113	0.004	87	0.0005

where $\tilde{w}^i$ is the projection of w^i on the boundary of S, if w^i is not in S. Note that $\sum_i \lambda_i^*$ converges to one if the d_j's go to infinity. The results are shown in Tables 8.1 and 8.3. (The tables show the accumulated number of iterations.) The accuracy is defined by $\max_{j,k} x_k^{*j}(m_k^j(x^*) - v^j(x^*))$, where x^* is the approximate fixed point.

In comparison with the algorithm of Garcia, Lemke and Lüthi [3], the algorithm proposed here takes significantly fewer iterations; to achieve a higher accuracy the algorithm proposed in [3] must be restarted in a corner of the $M - 1$ dimensional unit simplex with a larger grid size, whereas our algorithm can restart in the last found approximation. Moreover in [3] two different points of S^M can represent the same strategy vector. In the product space S, however, a strategy corresponds to just one point in S and reversely.

By applying other restart algorithms ([16], [6], [10], [15]) or homotopy algorithms ([1], [2]) the number of labels is equal to the number of variables, which implies that they must always operate in S^M.

Concerning the labelling rule it seemed to us that labelling on $y(x)$, the function defined in §7, does not work very well. More research to find better labelling rules needs to be done.

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