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ON THE COMPUTATION OF SPECIAL FUNCTIONS BY USING ASYMPTOTIC EXPANSIONS

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The use of asymptotic representations of some special functions is discussed. Instead of using the asymptotic series expansions of the functions we consider auxiliary functions which can be computed more efficiently than their corresponding asymptotic epansions. Applications are discussed for incomplete gamma functions and Bessel functions, with, respectively, error function and Airy functions as basic approximants.

1. INTRODUCTION

Many elementary functions can be computed by using polynomial or rational mini-max approximations. The methods that are based on this type of approximation are especially efficient when a single real argument has to be considered. For the more complicated higher transcendental functions of mathematical physics or mathematical statistics a different approach is usually needed. In many occasions these functions are multivariate, and sometimes we need to consider complex variables. In these circumstances analytical expansions and representations are preferred. We mention techniques based on recurrence relations, continued fractions, series expansions, etc. See, for instance, [3], [5], [6] for details on this topic. In the present paper we consider the use of asymptotic representations of functions of several variables, especially uniform representations. In the literature (see [7]), well-known results are available for a wide class of functions. For instance, Legendre functions, Bessel functions, (confluent) hypergeometric functions, Jacobi polynomials, Laguerre polynomials, etc. Essential in our approach is the fact that we do not use the asymptotic expansions of the functions involved. Instead, we consider a differential equation for auxilliary functions, and we base a numerical algorithm directly on this equation. As a result, we do not need the rather complicated coefficients of the asymptotic expansions. The method is demonstrated for two important special cases: the incomplete gamma function $P(a,x)$ and the Bessel function $J_a(x)$, and for related functions.

2. INCOMPLETE GAMMA FUNCTIONS

We consider the incomplete gamma functions in their normalized form:

$$P(a,x) = \frac{1}{\Gamma(a)} \int_0^a t^{a-1} e^{-t}\, dt, \tag{2.1}$$

$$Q(a,x) = \frac{1}{\Gamma(a)} \int_a^\infty t^{a-1} e^{-t}\, dt, \tag{2.2}$$

with $x \geq 0$ and $a > 0$. By this definition,

$$P(a,x) + Q(a,x) = 1.$$

So, in the computational problem we compute the one of P, Q that is less than $\frac{1}{2}$, and the complimentary relation gives the other one. With slight correction for small values of a and x, we have the following rule:

$$0 \leq a \leq x : \text{firstcompute}\, Q, \text{then}\, P = 1 - Q;$$

$$0 \leq x \leq a : \text{firstcompute}\, P, \text{then}\, Q = 1 - P.$$

This follows from the asymptotic relation

$$P(a,a),\, Q(a,a) \sim \frac{1}{2}, \quad a \to \infty.$$

A rather complete discussion of the computational problem for P and Q is given in [4]. Gautschi's algorithm is no longer efficient when the parameters a and x are both large and nearly equal. In a more recent paper [2] a method based on an asymptotic expansion described below is now available.

2.1. Uniform expansions

In [8] we obtained the following representations for the functions P and Q

$$P(a,x) = \frac{1}{2}\operatorname{erfc}(-\eta\sqrt{a/2}) - R_a(\eta), \qquad (2.3)$$

$$Q(a,x) = \frac{1}{2}\operatorname{erfc}(+\eta\sqrt{a/2}) + R_a(\eta); \qquad (2.4)$$

erfc is the error function defined by

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,dt. \qquad (2.5)$$

The real parameter η in the above representation is defined by

$$\frac{1}{2}\eta^2 = \lambda - 1 - \ln\lambda, \quad \lambda = \frac{x}{a}, \qquad (2.6)$$

where $\operatorname{sign}(\eta) = \operatorname{sign}(\lambda - 1)$. For the function $R_a(\eta)$ we derived an asymptotic expansion. Writing

$$R_a(\eta) = \frac{e^{-\frac{1}{2}\eta^2}}{\sqrt{2\pi a}} S_a(\eta), \qquad (2.7)$$

we have

$$S_a(\eta) \sim \sum_{n=0}^\infty \frac{C_n(\eta)}{a^n}, \quad \text{as } a \to \infty, \qquad (2.8)$$

uniformly with respect to $\eta \in (-\infty,\infty)$. This means, uniformly with respect to $x \in [0,\infty)$.

The first few coefficients are

$$C_0(\eta) = \frac{1}{\lambda - 1} - \frac{1}{\eta},$$

$$C_1(\eta) = \frac{1}{\eta^3} - \frac{1}{(\lambda-1)^3} - \frac{1}{(\lambda-1)^2} - \frac{1}{12(\lambda-1)}.$$

These two, and all higher coefficients have a removable singularity at $\eta = 0$. That is, at $x = a$.

2.2. A numerical approach

In [2] Gautschi's algorithm is modified by using (2.8) for the case of large parameters. In order to compute the coefficients near the point $\eta = 0$, the coefficients $C_n(\eta)$ are expanded in Maclaurin series. In [9] we have given the following alternative approach, in which the tabulation of many coefficients can be avoided.

Differentiating one of (2.3), (2.4) with respect to η gives

$$\frac{1}{a}\frac{d}{d\eta}S_a(\eta) = \eta S_a(\eta) - \frac{1}{\Gamma^*(a)}f(\eta) + 1 \qquad (2.9)$$

with

$$f(\eta) = \frac{1}{\lambda}\frac{d\lambda}{d\eta} = \frac{\eta}{\lambda - 1},$$

$$\Gamma^*(a) = \sqrt{a/2\pi}\,e^a a^{-a}\Gamma(a).$$

The functions $f(\eta)$ and $S_a(\eta)$ are analytic in a large domain of the complex η-plane. Singularities nearest to the origin are $\eta_\pm = 2\sqrt{\pi}\exp(\pm 3\pi i/4)$. So we can expand

$$S_a(\eta) = \frac{1}{\Gamma^*(a)}\sum_{m=0}^\infty b_m(a)\eta^m,$$

$$f(\eta) = 1 + \sum_{m=1}^\infty f_m\eta^m;$$

both expansions converge for $|\eta| < 2\sqrt{\pi}$. Substituting the expansion for f into (2.9) and comparing equal powers of η, one obtains

$$b_1(a) = a[\Gamma^*(a) - 1],$$

$$(m+1)b_{m+1}(a) = a[b_{m-1}(a) - f_m], \quad m \geq 1. \quad (2.10)$$

Furthermore, we have

$$b_0(a) = \Gamma^*(a)S_a(0) = \sqrt{2\pi a}\,\Gamma^*(a)[\frac{1}{2} - P(a,a)].$$

Recursion (2.10) can be viewed as a first order inhomogeneous recursion relation for $b_{2m}(a)$, $b_{2m+1}(a)$ of which the first values b_0, b_1 are defined. However, the recursion in the forward direction is not stable. This especially holds when a is large. Another point is that the computation of b_0, b_1 is not a straightforward numerical problem, again when a is large. By using (2.10) in backward direction with guessed values of B_{2N}, b_{2N+1}, where N is sufficiently large, these difficulties can be avoided.

With succes we have constructed an algorithm in which we only need the tabulation of $f_1, \ldots, f_N$, with $N = 26$, and we can use it in the sector $a > 20, 0.7a < x < 1.4x$ to obtain 18d accuracy. In this way the algorithms in both [2] and [4] can be made considerably more efficient.

3. ORDINARY BESSEL FUNCTIONS

For large values of the parameters, the Bessel functions $J_a(x)$ and $Y_a(x)$ can be expanded in terms of Airy functions. From [4, p. 425] we obtain

$$J_\nu(\nu z) \sim \frac{\phi(\zeta)}{\nu^{1/3}}\Big[Ai(\eta)\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{\nu^{2s}} + \frac{Ai'(\eta)}{\nu^{4/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{\nu^{2s}}\Big],$$

$$Y_\nu(\nu z) \sim -\frac{\phi(\zeta)}{\nu^{1/3}}\Big[Bi(\eta)\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{\nu^{2s}} + \frac{Bi'(\eta)}{\nu^{4/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{\nu^{2s}}\Big],$$

as $\nu \to \infty$, uniformly with respect to $z \in [0,\infty)$. The expansions are valid in a much larger z−domain, but here we concentrate on real values. The coefficients A_s, B_s are given in [7], with $A_0 = 1$. The parameter ζ is defined by

$$\frac{2}{3}\zeta^{3/2} = \ln\frac{1+\sqrt{1-z^2}}{z} - \sqrt{1-z^2}, \quad 0 \le z \le 1,$$

$$\frac{2}{3}(-\zeta)^{3/2} = \sqrt{z^2-1} - \arccos\frac{1}{z}, \quad z \ge 1.$$

Furthermore,

$$\eta = \nu^{2/3}\zeta, \quad \phi(\zeta) = \Big(\frac{4\zeta}{1-z^2}\Big)^{1/4}.$$

The coefficients A_s, B_s of the asymptotic series are complicated expressions. Especially, they are rather difficult to compute near the "turning point" $z = 1$, or equivalently, $\zeta = 0$. In [1] the relevant coefficients are expanded in series. As in the previous section we propose a different approach, in which the direct computation of A_s, B_s can be circumvented. To this end we define two functions F, G which represent the asymptotic series in the above representations of the Bessel functions. Thus we proceed with the exact representation

$$J_\nu(\nu z) = \frac{\phi(\zeta)}{\nu^{1/3}}\Big[Ai(\eta)F_\nu(\zeta) + \frac{Ai'(\eta)}{\nu^{4/3}}G_\nu(\zeta)\Big], \quad (3.1)$$

$$Y_\nu(\nu z) = -\frac{\phi(\zeta)}{\nu^{1/3}}\Big[Bi(\eta)F_\nu(\zeta) + \frac{Bi'(\eta)}{\nu^{4/3}}G_\nu(\zeta)\Big]. \quad (3.2)$$

Using the Wronskian relation for the Airy functions, viz.

$$Ai(z)Bi'(z) - Ai'(z)Bi(z) = \frac{1}{\pi},$$

we can invert the relations (3.1) and (3.2), and we obtain for the new functions F and G the relations

$$F_\nu(\zeta) = \frac{\pi\nu^{-1/3}}{\phi(\zeta)}\Big[J_\nu(\nu z)Bi'(\eta) + Y_\nu(\nu z)Ai'(\eta)\Big],$$

$$G_\nu(\zeta) = -\frac{\pi\nu}{\phi(\zeta)}\Big[J_\nu(\nu z)Bi(\eta) + Y_\nu(\nu z)Ai(\eta)\Big].$$

Olver's approach for deriving the Airy-type expansions for the Bessel functions is based on the differential equation

$$\frac{d^2W}{d\zeta^2} = [\nu^2\zeta + f(\zeta)]W, \quad (3.3)$$

where

$$f(\zeta) = \frac{5}{16\zeta^2} + \frac{\zeta z^2(z^2+4)}{4(z^2-1)^3}.$$

This equation is obtained from the well-kown Bessel equation by using a Liouville-Green transformation. From equation (3.3) it is not difficult to obtain the following differential equation defining the functions $F_\nu(\zeta), G_\nu(\zeta)$:

$$\begin{aligned} &F'' + G + 2\zeta G' - f(\zeta)F = 0, \\ &G'' + 2\nu^2F' - f(\zeta)G = 0. \end{aligned} \quad (3.4)$$

This system is equivalent with a (4 by 4)-system of first order equations, admitting four independent solutions. The solution that we need satisfies regularity and boundary conditions at, say, $\zeta = 0$. Using well-know estimates of the Bessel functions at the turning point $z = 1$ we derive

$$F_\nu(0) = 1 + \mathcal{O}(\nu^{-1}), \quad G_\nu(0) = \mathcal{O}(\nu^{-1/3}),$$

as $\nu \to \infty$. On the other hand, $F_\nu(+\infty) = 1$ and $G_\nu(+\infty) = 0$. It is also possible to give similar results for the derivatives of these functions.

The functions $F_\nu(\zeta)$ and $G_\nu(\zeta)$ are the analogues of the function $S_a(\eta)$ of the previous section. These functions are the "slowly varying" parts in the representations (3.1) and (3.2). A possible approach to solve the system of differential equations (3.4) is to use analytical techniques. Just as in the previous case we can expand the solution in MacLaurin series. Then we also need an expansion for $f(\zeta)$. This function is singular at

$$\zeta = (3\pi/2)^{2/3}e^{\pm\pi i/3}.$$

Hence the radius of convergence of the series of $F_\nu(\zeta)$ and $G_\nu(\zeta)$ in powers of ζ equals 2.81....

Our first computer exeriments show that again this method is very promising. The resulting code is rather efficient and we only need the storage of Taylor coefficients of the function f. We will present more numerical details in a future paper.

The same methods can be used for Airy-type asymptotic expansions for other special functions. We mention as interesting cases parabolic cylinder functions, Coulomb wave functions, and other members of the class of Whittaker functions.

To stay in the class of Bessel functions, we mention the modified Bessel function of the third kind $K_{i\nu}(z)$ of imaginary order, which plays an important role in the diffraction theory of pulses and in the study of certain hydrodynamical studies. Moreover, this function is the kernel of the Lebedev transform. The equation (3.4) is exactly the same for this modified Bessel function. The function f is slightly different for this case. It seems that there is no published code for the numerical evaluation of the function $K_{i\nu}(z)$.

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