Commun. Math. Phys. 99, 177–195 (1985)

Communications in Mathematical Physics

On the Concept of Attractor

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Abstract. This note proposes a definition for the concept of "attractor," based on the probable asymptotic behavior of orbits. The definition is sufficiently broad so that every smooth compact dynamical system has at least one attractor.

Attractors have played an increasingly important role in thinking about dynamical systems since their introduction some twenty years ago; yet there is no agreement as to the most useful definition. Section 1 of this note compares several definitions from the literature, and Sect. 2 proposes an alternative definition based on asymptotic behavior for almost every choice of initial point. The remaining two sections illustrate this definition by a number of examples, and discuss the stability and robustness of these attractors. There are three appendices. The first compares a closely related purely topological definition, the second studies real quadratic maps of their interval as a test case, and the third discusses strange attractors.

1. History

The following is quoted from Auslander, Bhatia, and Seibert (1964):

> "In the study of topological properties of ordinary differential equations, the stability theory of compact invariant sets (which may be regarded as generalizations of critical points and limit cycles) plays a central role. ... By Liapunov stability (or just stability) of the compact invariant set M, we mean that every orbit starting sufficiently close to M will remain in a given neighborhood of M. The set M is asymptotically stable if it is stable and is also an 'attractor' – that is, all orbits in a neighborhood ... of M approach M."

(Compare La Salle and Lefschetz, p. 31.) The word attractor, applied to a single invariant point for a smooth flow, had been used earlier by Coddington and

Levinson (1955) and by Mendelson (1960). However, attractors consisting of more than one point seem to have been first studied in this Auslander-Bhatia-Seibert paper. They explicitly considered the unstable case, in which an orbit which starts arbitrarily close to the attractor may wander far away before converging back towards the attractor. Compare Example 1 in Sect. 4 below. Although their definition is occasionally used by other authors (see La Salle, Hirsch, and perhaps Smale 1977), it has not been widely accepted, and most subsequent authors have required some form of stability as part of the definition. Smale (1967), in a widely read survey article, defined a more complicated object which might better be called an axiom A attractor. Smale's attractors, for a smooth map f from a compact manifold to itself, were "hyperbolic sets," containing a dense orbit, with periodic points everywhere dense, and satisfying the following rather awkward stability condition: The attractor A must have a neighborhood U so that A is equal to the intersection of the images $f^m(U)$ for $m>0$. Williams (1968) gave a related but simpler definition:

> "A subset A of $\Omega(f)$ is an *attractor* of f, provided it is indecomposable and has a neighborhood U such that $f(U) \subset U$ and $\bigcap_{i>0} f^i(U) = A$",

where $\Omega(f)$ is the non-wandering set of f, and where a closed f-invariant set is *indecomposable* if is not the union of two disjoint closed invariant subsets. The concept of attractor acquired great interest when Ruelle and Takens (1971) suggested that turbulent behavior in fluids might be caused by the presence of "strange" attractors. In fact, an explicit example in support of this idea had been worked out much earlier by Lorenz (1963). The definition of attractor used by Ruelle and Takens, like that of Smale, used an awkward form of stability:

> "A closed subset A of the non-wandering set Ω is an attractor if it has a neighborhood U such that $\bigcap_{t>0} D_{X,t}(U) = A$",

where the notation $D_{X,t}$ stands for the flow on a smooth manifold generated by a vector field X. (Note that this condition does not imply asymptotic stability. For example, according to Besicovitch, there is a homeomorphism of the plane, fixing the origin, such that the intersection of the successive images of the unit disk is the origin, even though every nonzero orbit is everywhere dense. See also Anosov and Katok.)

It is my contention that all of these definitions are too restrictive, since they exclude many interesting examples. Furthermore, in many cases they lead to an awkward situation in which there is no convenient language to describe where most points actually go when one follows a flow or iterates a mapping. In order to illustrate this point, Sect. 3 will describe a quadratic map of the interval which has no attractor at all according to the above definitions, although almost every orbit converges to a single uniquely defined compact set. A less restrictive definition has been given by Guckenheimer (1976), who requires only that an attractor must have:

> "... a fundamental system of neighborhoods, each of which is forward invariant under the flow generated by X".

In the language of Auslander, Bhatia, and Seibert, this is the condition of Liapunov stability (cf. Sect. 4). Sets having this property are important and useful; however, I believe that not every Liapunov stable set should be called an attractor. Here is an example. Consider a diffeomorphism or a flow on the plane which reduces to the identity map on a collection of concentric circles converging to the origin, but which pushes points slightly away from the origin otherwise. Then the origin is Liapunov stable, and hence would be called an attractor by Guckenheimer's definition, although it does not attract any other point.

Many other definitions of attractor can be found in the literature. The author's favorite reference is Collet and Eckmann (1980), which informally defines the attractor of a map f as

> "the set of points to which most points evolve under iterates of f".

This idea forms the basis for the present paper. For further discussion and other definitions the reader is referred to Conley, Guckenheimer and Holmes (Sect. 5.4), Kan, Ruelle (1981, 1983), as well as Zeeman.

2. Attractors, Minimal Attractors, and the Likely Limit Set

Let M be a smooth compact manifold, possibly with boundary, and let f be a continuous map from M into itself. The notation $f^n = f \circ \ldots \circ f$ will stand for the n^{th} iterate of f. Recall that the *omega limit set* $\omega(x)$ of a point $x \in M$ is the collection of all accumulation points for the sequence $x, f(x), f^2(x), \ldots$ of successive images of x. If we choose some metric for the topological space M, then $\omega(x)$ can also be described as the smallest closed set S such that the distance from $f^n(x)$ to the nearest point of S tends to zero as $n \to \infty$. The definition of omega limit set in the case of a continuous flow on M is completely analogous. Note that $\omega(x)$ is always closed and nonvacuous, with $f(\omega(x)) = \omega(x)$. Furthermore, $\omega(x)$ is always contained in the nonwandering set $\Omega(f)$.

Choose some measure μ on M which is equivalent to Lebesgue measure when restricted to any coordinate neighborhood. This can be constructed using a partition of unity, or using the volume form associated with a Riemannian metric. It doesn't really matter which particular measure we use, since we will usually only distinguish between sets of measure zero and sets of positive measure.

Definition. A closed subset $A \subset M$ will be called an *attractor* if it satisfies two conditions:

(1) the *realm of attraction* $\varrho(A)$, consisting of all points $x \in M$ for which $\omega(x) \subset A$, must have strictly positive measure; and

(2) there is no strictly smaller closed set $A' \subset A$ so that $\varrho(A')$ coincides with $\varrho(A)$ up to a set of measure zero.

The first condition says that there is some positive possibility that a randomly chosen point will be attracted to A, and the second says that every part of A plays an essential role.

Note. In the literature, the set $\varrho(A)$ is usually called the "basin of attraction" if it is an open set, and the "stable manifold" if it is a lower dimensional smooth manifold. I have avoided both terminologies since our sets $\varrho(A)$ are not open in general (cf. Sect. 3), and are certainly not lower dimensional manifolds. For any closed set A, it is not difficult to check that $\varrho(A)$ is necessarily a Borel set (or more precisely a countable intersection of σ-compact sets), and hence is measurable.

Basic properties of attractors are the following. An attractor A is necessarily closed, nonvacuous, and contained in the nonwandering set $\Omega(f)$, with $f(A)=A$. Any finite union of attractors is again an attractor; and more generally the closure of an arbitrary union of attractors is an attractor. Proofs are easily supplied. In order to show that attractors always exist, we will first construct one particularly important attractor.

Definition. The *likely limit set* $\Lambda=\Lambda(f)$ is the smallest closed subset of M with the property that $\omega(x)\subset\Lambda$ for every point $x\in M$ outside of a set of measure zero.

Lemma 1. *This likely limit set Λ is well defined and is an attractor for f. In fact, Λ is the unique maximal attractor, which contains all others.*

Sketch of Proof. Let $\{U_i\}$ be a countable basis for the open subsets of M, and let U be the union of those U_i such that $U_i\cap\omega(x)=\emptyset$ for almost every x. Then it follows that $U\cap\omega(x)=\emptyset$ for almost every x. The complement of U is the required likely limit set Λ. Further details of the argument are straightforward. □

If f is a measure preserving transformation, that is if $\mu(S)=\mu(f^{-1}(S))$ for every measurable set S, then $\Lambda(f)$ will be equal to the entire manifold M. Furthermore, every attractor will coincide with its realm of attraction, up to a set of measure zero. In this case, these constructions are probably not too interesting. However, when Λ is a proper subset of M, it provides a usefull tool for studying asymptotic behavior for almost all orbits.

Here is a more general construction for attractors. If $S\subset M$ is any subset of positive measure, then we define $\Lambda(f,S)$ to be the smallest closed subset of M which contains $\omega(x)$ for almost every point x of S. It is easy to check that $\Lambda(f,S)$ is well defined, and is an attractor. Note that its realm of attraction necessarily contains S, up to a set of measure zero. Evidently every attractor can be obtained by this construction. One special case is of particular interest.

Lemma 2. *If S is a compact set of positive measure with the property that $f(S)\subset S$, then S necessarily contains at least one attractor.*

For $\Lambda(f,S)$ is an attractor, and is clearly contained in S. □

We will be particularly interested in *minimal attractors*, that is, attractors for which no proper subset is an attractor. Evidently a closed set $A\subset M$ is a minimal attractor if and only if

(1′) its realm of attraction $\varrho(A)$ has positive measure, and

(2′) there is no strictly smaller closed set $A'\subset A$ for which $\varrho(A')$ has positive measure.

The number of distinct minimal attractors for f is at most countably infinite. If A is a minimal attractor, note that $\omega(x)$ is precisely equal to A for almost every x in $\varrho(A)$. There are many interesting cases in which the union of the minimal attractors for f is equal to the entire likely limit set Λ, or at least is everywhere dense in Λ.

Lemma 3. *Suppose that the likely limit set Λ is a union of finitely many disjoint minimal attractors $A_1, \ldots, A_n$. Then the corresponding realms of attraction $\varrho(A_i)$ form a partition of M into disjoint sets of positive measure, up to a set of measure zero. Every attractor is a union of minimal attractors, and for almost every point x of M the limit set $\omega(x)$ is precisely equal to some A_i.*

Proof. Choose neighborhoods U_i of the A_i so that $f(U_i)$ is disjoint from U_j for $i \neq j$, and let U be the union of the U_i. Then almost every orbit $x_0 \mapsto x_1 \mapsto \ldots$ will satisfy $x_t \in U$ for large t; hence x_t must belong to just one U_i for large t. The proof is now straightforward. □

In cases where this lemma applies, the collection of minimal attractors forms a very useful tool. However, the minimality condition may be very hard to verify, and there is no guarantee that a map f has any minimal attractors at all. Counterexamples are provided by the identity map, the concentric circle example of Sect. 1, or by more interesting examples such as the polynomial map

$$(c, y) \mapsto (c, y^2 - c)$$

in two variables (see Fig. 1). In such cases where there are not enough minimal attractors, or in cases where the minimal attractors are not known, it is necessary to fall back on the more general concept of attractor.

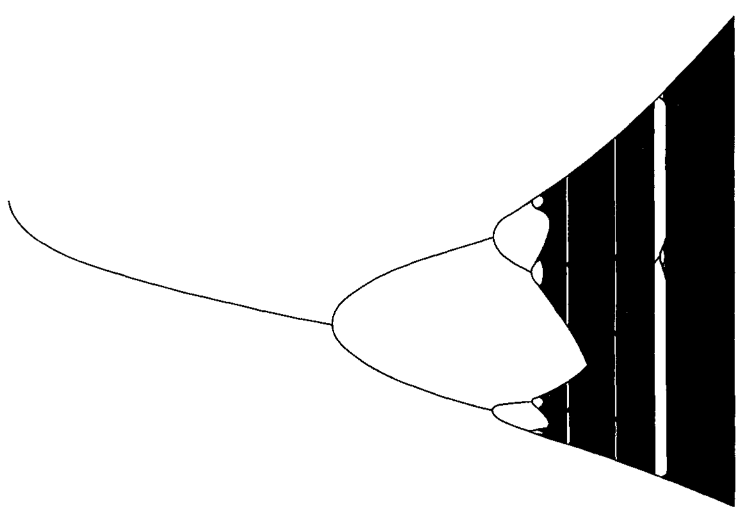

Fig. 1. Likely limit set (in the finite plane) for $(c, y) \mapsto (c, y^2 - c)$

3. The Feigenbaum Attractor

This section will illustrate these ideas by describing a smooth map which has a unique attractor according to the definitions in Sect. 2, but no attractor at all according to most of the definitions in Sect. 1. Let $c = 1.401155189\ldots$ be the smallest real number for which the real quadratic map $x \mapsto x^2 - c$ has infinitely many distinct periodic orbits. This map $f(x) = x^2 - c$ has been studied by Feigenbaum, since it represents the point of transition from stable periodic behavior to chaotic behavior. In order to have a compact domain of definition, let us choose some interval I containing the origin with $f(I) \subset I$, and think of f as a map from I into itself (Fig. 2).

The orbit of zero under f is an almost periodic sequence which can be described as follows (cf. Misiurewicz). The numbers $0 = a_0 \mapsto a_1 \mapsto a_2 \ldots$ are all distinct. However, the difference $a_m - a_n$ is very small whenever $m - n$ is divisible by a high power of two. Thus the closure A of this orbit is a Cantor set, homeomorphic to the ring $\varprojlim (Z/2^k Z)$ of 2-adic integers in such a way that each $a_n = f^n(0)$ corresponds to the 2-adic integer n. Note that f restricted to A corresponds to the homeomorphism which adds one to each 2-adic integer. It follows that there are no periodic points in A. However, it can be shown that there are periodic points arbitrarily close to every point of A. These periodic points are all unstable, with period equal to a power of two.

The dynamic structure of $f: I \to I$ is as follows. For almost every initial point x_0 in I the successive images $x^n = f^n(x_0)$ converge towards the Cantor set A. Hence $A = A(f)$ is the unique attractor for f. But there are a countable infinity of exceptional points whose successive images do not converge towards A. Furthermore, these exceptional points are everywhere dense in I. For according to Guckenheimer (1979), since f has no periodic attractor, any arbitrarily small open interval $U \subset I$ has some forward image $f^n(U)$ which contains the origin. Choosing a point x in U which maps to a periodic point in this neighborhood of the origin, we see that $\omega(x)$ is a periodic orbit disjoint from A. Thus A is certainly not asymptotically stable.

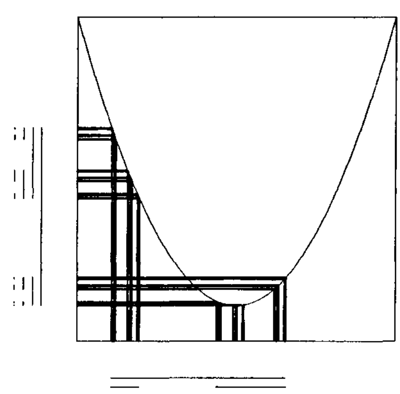

Fig. 2. The unique attractor for the map $x \mapsto x^2 - 1.4011\ldots$ on an appropriate interval is the indicated Cantor set

However, A is Liapunov stable, and in fact is equal to the intersection of a nested sequence of asymptotically stable sets. In spite of the dense set of exceptional points, the tendency to converge towards A is extremely strong. If U is an arbitrarily small neighborhood of A, let us choose a smaller neighborhood V so that $f(V) \subset V$. Then if we start at any point x_0 in I and follow the successive images $x_n = f^n(x_0)$ there are just two possibilities. Either this orbit eventually hits the open set V and remains trapped within $V \subset U$ forever after, or else it manages to precisely hit one of the finitely many periodic points which lie outside of V, and remains trapped in a finite periodic orbit outside of V thereafter. The first possibility occurs for a dense open set of initial points x_0, while the second possibility occurs only for a countable nowhere dense set of x_0.

Further details of this argument will be given in Appendix 2. For readers who prefer to work with flows, Kan has described somewhat analogous examples of smooth flows on the 3-dimensional sphere so that almost every orbit converges towards a "solenoid," locally homeomorphic to the product of a Cantor set and a 1-manifold, which contains no periodic orbits, but can be approximated arbitrarily closely by periodic orbits. These examples are also Liapunov stable, but not asymptotically stable. (See also Grebogi, Ott, Pelikan, and Yorke. Related but less differentiable examples have been described by Bowen and Franks, and by Franks and Young.) It seems likely that this lack of asymptotic stability may be shared by many other fractal attractors.

4. Stability and Robustness

Let $f: M \to M$ be a fixed smooth map. As in previous sections, we will say that a closed subset $A \subset M$ with $f(A) = A$ is *Liapunov stable* (also called "orbitally stable") if it has arbitrarily small neighborhoods U with $f(U) \subset U$; and *asymptotically stable* if it is Liapunov stable and also satisfies the Auslander-Bhatia-Seibert condition that its realm of attraction $\varrho(A)$ is an open set. In the asymptotically stable case, if we choose U with closure contained in $\varrho(A)$, then it follows easily that A is equal to the intersection of the sequence of forward images $U \supset f(U) \supset f^2(U) \supset \dots$.

Note that we have not imposed any stability requirement at all as part of the definition of attractor. Of course, many interesting attractors are asymptotically stable, or at least Liapunov stable. However, if we required some form of stability as part of the definition, then it would be awkward to talk about transitional case, and it would no longer be true that every smooth map on a compact manifold has at least one attractor. Here is a simple minded example to illustrate this point.

Example 1. Let M be the circle of real numbers modulo 2π, and let

$$f(\theta) \equiv \theta + 1 - \cos(\theta) \pmod{2\pi}.$$

Then the fixed point $\theta = 0$ is the unique attractor. This attractor is not Liapunov stable, since no nontrivial neighborhood is mapped into itself by f. (Compare Fig. 3. We will describe such a fixed point as "one-sided stable.") A completely analogous example on the real projective line $R \cup \infty$, is the map $f(x) = x + 1$, with

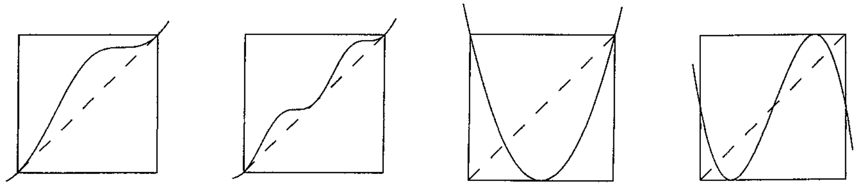

Fig. 3. Graphs of Examples 1–4

the point at infinity as unique attractor. There is a well known classical example of a fractal attractor with similar stability properties, namely the Denjoy C^1-diffeomorphism of the circle, which has a Cantor set as unique attractor (cf. Schweitzer). Here is a really pathological example. Let $f(x)=4x(1-x^2)^2$ for $|x|\leqq 1$, with $f(x)=0$ otherwise. Then surely the only attractor is the origin, which is a strictly repulsive fixed point.

Here is a simple variant of Example 1.

Example 2. On the circle of real numbers modulo 2π, let $f(\theta)=\theta+\sin^2(\theta)$. Then A consists of the two points 0 and π. Each is a minimal attractor, but neither is stable since there are points arbitrarily close to either one whose successive images converge to the other.

Note that an attractor which has positive measure need not attract anything outside of itself. For example an irrational rotation of the circle has the entire circle as minimal attractor.

Example 3. The Ulam von Neumann Attractor. Consider the quadratic map $f(x) =2x^2-1$ from $R\cup\infty$ to itself. Then for almost every point x in the interval $I=[-1,1]$ the limit set $\omega(x)$ is equal to the entire interval I. A proof can be based on the corresponding property for the squaring map $z\mapsto z^2$ on the unit circle, making use of the observation that f restricted to I is equal to the correspondence $\mathrm{Re}(z)\mapsto \mathrm{Re}(z^2)$, where z ranges over the unit circle. On the other hand, for every point x outside of I the orbit of x diverges to infinity. It follows that $A(f)$ consists of the interval I, which is an unstable minimal attractor, together with the point at infinity which is a stable attractor. Every point outside of I is repelled by I.

By a linear change of coordinate, this map can be put into the form $x\mapsto x^2-2$. More generally, for any parameter c we can consider the map $x\mapsto x^2-c$. If c belongs to the interval $[-1/4, 2]$, this map has at least one bounded attractor. It may be true that it has only one bounded attractor, which is either a periodic orbit, a finite union of intervals, or (in uncountably many exceptional cases) an almost periodic Cantor set as in Sect. 3 (cf. Lemma 4 in Appendix 2). Presumably, for almost all parameter values this bounded attractor is asymptotically stable. However, in the almost periodic case it is only Liapunov stable, and there are at least a countable infinity of cases which are not even Liapunov stable. These include the two extreme values $c=-1/4$ and $c=2$, and the value $c=7/4$ which corresponds to a one-sided stable orbit of period 3.

Note that two distinct minimal attractors need not be disjoint from each other. However, the intersection of two minimal attractors must be too small to attract points from any set of positive measure.

Example 4. The likely limit set for the cubic map $f(x)=3\sqrt{3}(x-x^3)/2$ consists of two unstable minimal attractors $[-1,0]$ and $[0,1]$, together with the stable attractor $\{\infty\}$. Here the coefficient $3\sqrt{3}/2$ is chosen so that each of these intervals will map precisely onto itself. In this case, it is interesting to note that the union $[-1,1]$ of the two unstable attractors is asymptotically stable. (The proof that $[0,1]$ is a minimal attractor can be outlined as follows. One can first check that the change of variable $x=\sin^2(\theta)$ converts f, restricted to this interval, to an expanding map. It then follows from Li and Yorke that it has an ergodic invariant measure which is equivalent to Lebesgue measure.)

It is often convenient to describe instability properties of such examples by constructing an associated graph.

Definition. The *stability diagram* associated with a map f is the 1-dimensional complex with one vertex a_i for each minimal attractor A_i, and with edges as follows. There is a directed edge from vertex a_i to the distinct vertex a_j whenever there exist points arbitrarily close to A_i whose limit set $\omega(x)$ intersects A_j; and also there is a loop joining a_i to itself whenever there exist points arbitrarily close to A_i whose successive images temporarily wander away, outside of some fixed neighborhood of A_i, and yet have limit set $\omega(x)$ which intersects A_i.

If A_i is Liapunov stable, note that no edge can lead away from the corresponding vertex. The stability diagrams corresponding to the four examples are shown in Fig. 4.

Next let us study the behavior of the set $\Lambda=\Lambda(f)$ as we perturb the map f. Recall that the *Hausdorff distance* between two closed sets is the smallest number δ such that each closed ball of radius δ centered at a point of either set necessarily contains a point of the other set.

Definition. We will say that the likely limit set Λ for a given map f is *robust* if the Hausdorff distance between $\Lambda(f)$ and $\Lambda(g)$ tends to zero whenever g tends to f in the C^∞-topology (roughly speaking, whenever the k^{th} derivative of g converges uniformly to the k^{th} derivative of f for every $k\geqq 0$). Similarly, an attractor A for f is *robust* if any g which is C^∞-close to f has an attractor A' which is Hausdorff-close to A.

Note that this attractor A' may be qualitatively quite different from A. The condition of Hausdorff closeness guarantees only that these two sets will be rather hard to distinguish by a computer experiment. For example, it follows from Lemma 2 that any asymptotically stable periodic orbit is necessarily robust.

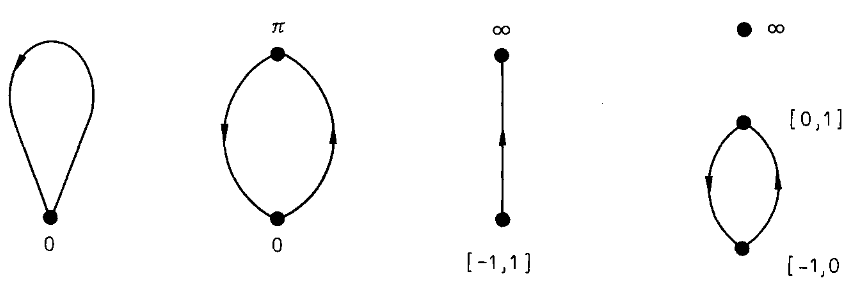

Fig. 4. Stability diagrams for Examples 1–4

However, the attractor A' which Hausdorff approximates this orbit A may have period equal to some multiple of the period of A, or may even be infinite. The most common possibility is bifurcation. For example, each of the maps $x \mapsto x - x^3$ and $x \mapsto x^2 - 3/4$ has a unique stable finite fixed point, which bifurcates under perturbation. The unique attractor $\Lambda(f)$ for the Feigenbaum map f of Sect. 3 is also robust. Again, for g close to f the likely limit set $\Lambda(g)$ may be qualitatively quite different from $\Lambda(f)$. This Cantor set may simplify into a stable periodic orbit, or complicate into a chaotic attractor.

It is conjectured that any robust attractor must be Liapunov stable. To illustrate this conjecture, we have noted that Examples 1, 2, 3, and 4 are not stable, and it is easy to show that they also are not robust.

An attractor may fail to be robust in at least three different ways. It may simply evaporate, or explode into a much bigger set, or implode into a much smaller set. (In the case of an asymptotically stable attractor, only the last possibility can occur, by Lemma 2.) Here are examples to illustrate these three possibilities. The map $x \mapsto x + x^2$, with a one-sided stable attractor at the origin, can be approximated arbitrarily closely by a map for which the only attractor is the point at infinity. The translation $x \mapsto x + 1$ of Example 1, with the infinite point as unique attractor, can be approximated arbitrarily closely by a fractional linear transformation $x \mapsto (1+x)/(1-\varepsilon x)$, for which every orbit is dense, so that the only attractor is the entire set $R \cup \infty$. Finally, let $f(x) = x^2 - c$, where $c = 1.543689\ldots$ is the real root of the equation $c^3 - 2c^2 + 2c - 2 = 0$, so that $f^4(0) = f^3(0) = -f^2(0)$. Then f has a unique finite attractor $A = [f(0), f^2(0)]$, which is asymptotically stable. (The proof is similar to that in Example 4.) Yet f is equal to the limit of a sequence $f_3, f_4, \ldots$ of quadratic maps, where the unique finite attractor A_n for f_n is a superstable orbit of period n consisting of zero and $n-2$ negative numbers, but only one positive number $f_n^2(0)$. Thus A_n is substantially smaller than A for all n.

Note that the likely limit set may fail to be robust even if every attractor is robust, since new attractors may suddenly appear. As an example, the unique attractor for the map $f(x) = x + x^3$ is the stable fixed point at infinity; yet maps arbitrarily close to f have an additional stable fixed point at the origin.

Even when an attractor disappears under perturbation of a map, it leaves behind a "resonance," that is a region where successive images of a point may get trapped for many iterations before escaping (cf. Grebogi, Ott, and Yorke). In practical applications, if the perturbation is very small so that the expected escape time is very large, it may be difficult to tell the difference between such a resonance and an actual attractor.

Appendix 1. The Generic Limit Set

The following variation on the likely limit set Λ of Sect. 2 is sometimes easier to work with. Let us say that a property of a point $x \in M$ is true for *generic* x if it is true for almost all x in the sense of Baire category theory, i.e., for all x outside of a countable union of nowhere dense subsets of M. Define the *generic limit set* $\Gamma \subset M$ to be the smallest closed subset of M with the property that, for generic x, the limits set $\omega(x)$ is contained in Γ. There is an analogous concept of "generic-attractor." The definition will be left to the reader.

In many cases the two limit sets Γ and Λ are equal to each other. However, this is not always true, and I do not know any good criterion for equality. The set Γ is likely to be easier to compute since its definition is purely topological, whereas the definition of Λ uses both topology and measure theory. On the other hand, the set Λ is more closely related to the probable asymptotic behavior which one would like to study in applications. Also, the set Λ is closer to something which one can actually draw pictures of in numerical experiments.

The only general relation between these two sets which is known to me is the statement that the intersection $\Gamma \cap \Lambda$ is always non-vacuous. This follows from the more precise statement that, for generic x, the intersection $\omega(x) \cap \Lambda$ is non-vacuous. In fact, for any neighborhood U of Λ the set of x whose full orbit is disjoint from U is closed and of measure zero, hence nowhere dense. Therefore, the orbit of a generic x intersects every such neighborhood, and hence satisfies $\omega(x) \cap \Lambda \neq \emptyset$.

Following are two rather pathological examples.

Example 5. The likely limit set Λ can be strictly larger than Γ. Consider the gradient flow associated with a smooth real valued function on the 2-dimensional sphere which takes its minimum along a Cantor set of positive measure, but has only non-degenerate critical points otherwise, with just one local maximum x_0. Then $\Gamma = \{x_0\}$, yet Λ contains an entire Cantor set. A similar example involving a C^1-horseshoe has been given by Bowen (1975).

Example 6. The set Λ can be strictly smaller than Γ, at least in the case of a non-differentiable map f. More precisely, there is a continuous map of the interval for which a generic orbit is dense, so that Γ is the entire interval, and for which Λ is a Cantor set. In fact, with suitable modification of the example, Λ will split up into two or more minimal attractors, which may be disjoint or not according to taste. I don't know whether there exists a smooth map with these properties.

Let us start with the continuous measure-preserving map $F(x) = 2 \cdot |x| - 1$ on the interval $[-1, 1]$. Then it is not difficult to show that, for generic x, the limit set $\omega(x)$ is equal to the entire interval $[-1, 1]$ (cf. Appendix 2). Hence $\Gamma = [-1, 1]$. However, we will construct a Borel probability measure μ on M so that, with μ-probability equal to one, the limit set $\omega(x)$ avoids some fixed open subset of $[-1, 1]$. Let C be the closed set consisting of all points x whose full orbit is disjoint from the interval $(1/2, 1]$. It is not difficult to check that C is a Cantor set. In fact, the "kneading sequence" $\mathrm{sgn}(x_0)$, $\mathrm{sgn}(x_1), \ldots$ associated with an orbit $x_0 \mapsto x_1 \mapsto \ldots$ in C can be any sequence of signs which does not have two $+1$'s in a row. Therefore, we can choose a probability measure μ_0 with support equal to C, and so that points have measure zero. Now define a sequence of probability measures inductively by setting

$$2\mu_{n+1}(S) = \mu_n(F(S \cap [-1, 0])) + \mu_n(F(S \cap [0, 1])),$$

so that the support of μ_n is equal to the full t-fold inverse image $F^{-n}(C)$. Then the measure $\sum \mu_n / 2^{n+1}$ has support equal to the entire interval $[-1, 1]$. Yet, with μ-probability equal to one, the limit set $\omega(x)$ is contained in C. [If μ_0 is constructed with more care, then $\omega(x)$ will actually be equal to C with probability one.] Note that the continuous change of coordinate $y = \int_{-1}^{x} d\mu$ transform μ to the standard

Lebesgue measure on the unit interval. Therefore, F is topologically conjugate to a continuous map for which Γ is strictly larger than Λ.

Similarly, let $\hat{C}$ be the Cantor set consisting of all points whose orbits are disjoint from the interval $(-1/2, 0)$, and let $\hat{\mu}$ be an associated measure, so that with $\hat{\mu}$-probability equal to one the limit set $\omega(x)$ must be equal to $\hat{C}$. If we combine these two constructions by using the probability measure $(\mu + \hat{\mu})/2$, then $\omega(x)$ will be equal to C with probability $1/2$, and to $\hat{C}$ with probability $1/2$. Thus in this case there are two minimal attractors, which intersect only along the periodic orbit $C \cap \hat{C} = \{-3/5, 1/5\}$. We can modify the example so that the two minimal attractors will be disjoint, by using the slightly larger interval $(-3/4, 0)$ for the construction of $\hat{C}$.

Appendix 2. Maps of the Interval

The following discussion will depend strongly on Guckenheimer [17].

Let $x \mapsto f(x)$ be a real quadratic mapping which carries some finite interval I into itself. If we choose this interval I to be maximal, then $f: I \to I$ will belong to the class of maps studied by Guckenheimer. That is: f is smooth of class C^3, with a single interior critical point, with negative Schwarzian derivative

$$2(Df)(D^3f) - 3(D^2f)^2 < 0,$$

and f maps both endpoints of I to a single endpoint x_0 at which the derivative satisfies $Df(x_0) \geqq 1$. (This last condition is essential, but was inadvertently omitted in [17].)

Lemma 4. *For $f: I \to I$ satisfying these conditions, the generic limit set Γ is either a finite periodic orbit, a Cantor set equal to the closure of the orbit of the critical point, or a finite union of say n intervals, bounded by the first $2n$ forward images of the critical point, and containing the critical point in its interior. Furthermore, for generic x, the omega limit set $\omega(x)$ is precisely equal to Γ.*

In the first two cases, for every x outside of a set of measure zero we have $\omega(x) = \Gamma$. Hence the likely limit set Λ is equal to Γ, and is a minimal attractor. It is quite possible that this statement is true in the third case also, but I have not been able to decide this question (cf. Example 6 above).

Proof. First suppose that f has a periodic attractor, that is either a stable or a one-sided stable periodic orbit A. Then there exists a neighborhood (or one-sided neighborhood) U consisting of points whose orbits converge uniformly to A. According to [17, p. 135 and 2.8, 3.1], for every point x outside of a closed set of measure zero the orbit of x eventually falls into this neighborhood U, and hence converges to A. Clearly, it follows that $\Gamma = \Lambda = A$.

Henceforth, let us assume that f has no stable or one-sided stable periodic orbit. In order to handle this case we will need the following.

Definition. We will say that the map f is *reducible* if there exists a closed interval J about the origin and an integer $n \geqq 2$ so that the successive images $J, f(J), f^2(J), \ldots, f^{n-1}(J)$ have disjoint interiors, and so that $f^n(J) \subset J$.

The idea is that whenever this is the case we can reduce the study of f to the study of the n-fold iterate $g=f^n$, considered a map from the subinterval J into itself. In fact, if f is reducible, [17, 2.6 and 3.1] proves that, for every x outside of a closed set of measure zero, the orbit of x eventually meets the interior of J. It follows that the likely limit set $\Lambda=\Lambda(f)$ is equal to the n-fold union $\Lambda(g)\cup f(\Lambda(g)) \cup \dots f^{n-1}(\Lambda(g))$; with a similar description for $\Gamma(f)$.

We may always assume that the interval J is chosen so as to be maximal. It then follows easily that $g=f^n$ maps the boundary of J to one boundary point. Since there are no stable periodic orbits, the derivative of g at this boundary fixed point, must be at least one. Since the condition of negative Schwarzian derivative is preserved under composition, we see that this new map $g: J \to J$ satisfies all of Guckenheimer's conditions.

If this new map $g: J \to J$ is also reducible, then we iterate the construction. We must distinguish between the "finitely reducible case" in which the reduction process terminates with a non-reducible map after finitely many steps, and the "infinitely reducible case" in which it continues ad infinitum (Fig. 2). Evidently, in order to understand the finitely reducible case, we need only understand the case of a non-reducible map.

Suppose then that f is not reducible. We will prove that f is topologically conjugate to a piecewise linear map of the form

$$F(y)=s\cdot|y|-1 \quad \text{for} \quad |y|\leqq 1/(s-1),$$

where $\log(s)>0$ is the topological entropy [17, 4.5]). Since we have excluded the trivial case where f has a stable or one-sided stable fixed point, it is not difficult to check that f must have strictly positive topological entropy. Hence, by Milnor and Thurston, f is topologically semi-conjugate to F, say $h\circ f=F\circ h$, where h is a monotone map. Thus every inverse image $h^{-1}(y)$ is either a point or an interval J. If some $h^{-1}(y)$ were an interval, then the orbit of y would have to contain the critical point 0, say $F^m(y)=0$. For otherwise any two points of J would have the same kneading sequence, which is impossible by [17, 2.6]. In fact, this orbit would have to contain 0 more than once, since the interval $f^{m+1}(J)$ also maps to a single point under h. Hence $F^n(0)$ would be 0 for some n, necessarily greater than one. Therefore, f^n would map the interval $h^{-1}(0)$ into itself, and hence f would be reducible, contradicting our hypothesis. This contradiction proves that h must in fact be a homeomorphism, so that f is topologically conjugate to F.

Thus we are reduced to studying the piecewise linear map F. We may assume that the growth number s is greater than $\sqrt{2}$, since otherwise it is easy to check that F is reducible. Then we will prove that the generic limit set of F is equal to the interval

$$A=[-1,s-1]=[F(0),F^2(0)].$$

In fact, it is easy to check that $F(A)=A$, and that the orbit of any y in the open interval $|y|<1/(s-1)$ eventually gets trapped in this subinterval A.

The map F from A to itself has a strong transitivity property as follows. *For any subinterval $J\subset A$ there exists a positive integer n so that $F^n(J)$ is equal to the entire interval A.* For if J is small enough so that the two sets J and $F(J)$ do not both contain zero, then $F^2(J)$ is longer than J by a factor of at least $s^2/2>1$. Thus the successive images of J expand exponentially until say $F^{m-1}(J)$ and $F^m(J)$ both contain zero; and hence $F^{m+1}(J)=A$.

It follows from this property that, for a generic point y, the limit set $\omega(y)$ is equal to the entire set A. For if $\{U_i\}$ is a basis for the open subsets of A, then the set of y whose orbit intersects U_i is dense and open. Thus the orbit of a generic y intersects every U_i, and hence is dense in A. Therefore, the generic limit set $\Gamma(F)$ equals A, and it follows that the generic limit set $\Gamma(f)$ for the smooth map f is the corresponding interval $h^{-1}(A)$. More explicitly, if c is the critical point of the non-reducible map f, then $\Gamma(f)$ is equal to the closed interval bounded by $f(c)$ and $f^2(c)$. Similarly, the generic limit set for a finitely reducible map f is a finite union of intervals, bounded by appropriate forward images of the critical point.

The likely limit set Λ seems much harder to determine. It follows easily from this discussion that Λ is a subset of Γ, but I have not been able to prove that $\Lambda=\Gamma$ in the non-reducible or finitely reducible case.

Finally, we must consider the case where the reduction process does not terminate after finitely many steps, but rather continues ad infinitum. In this infinitely reducible case it is not difficult to check that there is a unique attractor $A=\Lambda=\Gamma$ which is a Cantor set, homeomorphic to an inverse limit of finite cyclic groups Z/nZ. Furthermore, the map f is almost periodic on A, and corresponds to the adding machine map $\alpha\mapsto\alpha+1$ from this limit group to itself. Any such inverse limit of finite cyclic groups can occur, so there are uncountably many distinct examples. Details will be omitted. $\square$

It follows easily from this discussion that f has sensitive dependence on initial conditions (as defined in [17]) if and only if Γ is a finite union of intervals. For the class of maps studied by Guckenheimer, there are just two qualitatively different forms of behavior. If f has a periodic or almost periodic attractor A, then $A=\Lambda(f)=\Gamma(f)$ is a set of measure zero, f does not have sensitive dependence on initial conditions, and the topological entropy of $f|A$ is zero. In all other cases, the generic limit set is an interval or union of intervals, with positive measure, with sensitive dependence on initial conditions, and the entropy of $f|\Gamma(f)$ is strictly positive. In the latter case, the dynamics of f must be described as *chaotic*.

Concluding Remarks. There are many unsolved problems, even for real quadratic maps of the interval. For example, in the chaotic case, I don't know how to decide whether $\Lambda=\Gamma$, or to decide whether there can be more than one attractor. Equivalently, is it true that almost every orbit in the union of intervals Γ is everywhere dense in Γ? More generally, I don't know how to answer the following basic question for maps of $R\cup\infty$ to itself. Suppose either that $f(x)$ is a rational function of degree at least two, or that f has negative Schwarzian derivative. Does it follow that the likely limit set $\Lambda(f)$ is equal to $\Gamma(f)$, and can be expressed as the union of finitely many minimal attractors? Similarly, I have no idea how to decide when the set $\Gamma(f)$ [much less $\Lambda(f)$] is robust. Here is an explicit guess for the class of maps considered by Guckenheimer: The interval $[f(0), f^2(0)]$ is conjectured to be a robust necessarily minimal attractor if and only if the orbit of the critical point is everywhere dense in this interval. (Compare the discussion of $x\mapsto x^2-1.543689\ldots$ in Sect. 4.)

Appendix 3. Strange Attractors

The concept of a *strange attractor*, as described by Ruelle and Takens, is perhaps intentionally not precisely defined. The original discussion emphasized wild topology. An excellent introduction to this subject can be found in Ruelle's survey article (1979–80), which rather emphasizes chaotic dynamical behavior. Guckenheimer and Holmes define a strange attractor as one which has a transversal homoclinic point, and hence has chaotic behavior.

It is important to note that there is no necessary connection between wild topology or fractal geometry and strange dynamical behavior. For example, the fractal attractor of Sect. 3 is topologically wild, but has very sedate almost periodic dynamics, with topological entropy equal to zero. On the other hand, the map $(\theta, t) \mapsto (2\theta, t/2)$ from the cylinder $S^1 \times [-1, 1]$ to itself has a smooth manifold $S^1 \times 0$ as unique attractor; yet the dynamical behavior is completely chaotic. In fact, this map is ergodic and mixing with respect to the standard invariant measure on the circle, and has positive entropy. The behavior of this example under perturbation will be studied below. For an analogous example using a diffeomorphism or flow in place of an iterated map, we could substitute an Anosov diffeomorphism of the torus, or the geodesic flow on the unit tangent bundle of a manifold of negative curvature, in place of the squaring map on the circle.

Here is an example of an attractor which is chaotic and topologically wild, yet simple enough to analyze almost completely (cf. Kaplan and Yorke). Let us start with the map $f_0(e^{i\theta}, x) = (e^{2i\theta}, cx)$ from the cylinder $S^1 \times R$ to itself, where c is a constant in the range $0 < c < 1/2$. Clearly, f_0 has a unique attractor which is asymptotically stable, since all orbits tend towards the circle $S^1 \times 0$, and since almost very orbit on the circle is everywhere dense. This attractor $S^1 \times 0$ is a smooth manifold, but is dynamically chaotic. Now make a small perturbation of f_0 to the map

$$f_\varepsilon(e^{i\theta}, x) = (e^{2i\theta}, cx + \varepsilon \cos\theta).$$

Then the smooth attractor $S^1 \times 0$ will be replaced by a fractal attractor (Fig. 5).

To see this, first note that a scale change in the x-coordinate will replace f_ε by the map

$$f_1(e^{i\theta}, x) = (e^{2i\theta}, cx + \cos\theta).$$

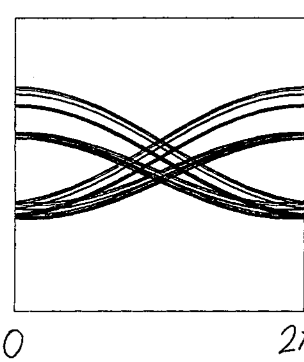

Fig. 5. The unique attractor for the map $(\theta, x) \mapsto (2\theta, 0.4x + \cos\theta)$

We can further simplify by introducing a third coordinate y, and extending f_1 to the smooth map

$$(e^{i\theta}, x, y) \mapsto (e^{2i\theta}, cx + \cos\theta, cy + \sin\theta)$$

from $S^1 \times R \times R$ to itself. Setting $w = e^{i\theta}$ and $z = x + iy$, we can write this map as

$$F(w, z) = (w^2, cz + w)$$

from the 3-dimensional manifold $S^1 \times C$ to itself.

Lemma 5. *There is a unique attractor $A \subset S^1 \times C$, homeomorphic to the dyadic solenoid, which is asymptotically stable, and in fact, attracts every orbit. More precisely, for any compact neighborhood K of A the successive images $F^n(K)$ converge towards A, and for almost every point (w, z) of $S^1 \times C$ the omega limit set $\omega(x)$ is equal to A. This set A supports a measure so that F restricted to A is measure theoretically equivalent to the full Bernoulli shift of entropy* $\log 2$.

Here by the dyadic solenoid $\hat{S}$ we mean the inverse limit of the sequence of squaring homomorphisms

$$S^1 \leftarrow S^1 \leftarrow S^1 \leftarrow \dots .$$

Thus, by definition, a point of $\hat{S}$ is a sequence $\hat{w} = (w_0, w_1, w_2, \dots)$ of points on the unit circle so that

$$w_1 = \pm\sqrt{w_0}, \qquad w_2 = \pm\sqrt{w_1}, \dots .$$

Define $A = g(\hat{S})$ to be the image of $\hat{S}$ under the continuous map

$$g(\hat{w}) = (w_0, w_1 + cw_2 + c^2 w_3 + \dots).$$

Since we have assumed that $|c| < 1/2$, it is quite easy to check that g is a topological embedding. (With more work, one could check this for somewhat larger c.) Note that F maps A homeomorphically onto itself. In fact,

$$F(w_0, w_1 + cw_2 + c^2 w_3) = (w_0^2, w_0 + cw_1 + c^2 w_2 \dots),$$

so F restricted to A corresponds to the "shift automorphism"

$$(w_0, w_1, w_2, \dots) \mapsto (w_0^2, w_0, w_1, \dots)$$

from the topological group $\hat{S}$ to itself.

Next we will see that, for any starting point (w, z) in $S^1 \times C$, the distance between $F^n(w, z)$ and the set A converges geometrically (and uniformly on any compact set) to zero. In fact, choose any point $\hat{w} = (w_0, w_1, \dots)$ in $\hat{S}$ with $w_0 = w$, and let $g(\hat{w}) = (w_0, z_0) \in A$. Then by inspecting the definition of F we see that the distance between $F^n(w, z)$ and the point $F^n(w_0, z_0) \in A$ is exactly $c^n|z - z_0|$, which converges to zero as $n \to \infty$.

More geometrically, let k be some constant in the open interval from $1/(1-c)$ to $1/c$, and let T be the solid torus consisting of all (w, z) with $|z| \leq k$. Then it is not hard to see that F maps T homeomorphically into its own interior, with degree two, and that A is equal to the intersection of the forward images $F^n(T)$. In this form, the example has been studied by Smale (1967), by Ruelle and Takens, and by Ruelle (1979–80).

To study the ergodic behavior for F restricted to A, or equivalently for the shift automorphism of $\hat{S}$, write each point of $\hat{S}$ as $(w_0, w_1, \ldots)$, with $w_n = \exp(\pi i x_n)$. Expressing the number x_0 in binary notation as $a_0 + a_1/2 + a_2/4 + \ldots$, it follows that x_1 has the form $a_{-1} + a_0/2 + a_1/4 \ldots$, with similar expressions for the higher x_n. Thus the shift automorphism of $\hat{S}$ corresponds to the Bernoulli shift on a doubly infinite sequence $\ldots a_{-1}, a_0, a_1, a_2, \ldots$ of zeros and ones.

The Hausdorff dimension of $A \subset S^1 \times C$ can be computed as $1 - \log(2)/\log(c)$. This lies strictly between 1 and 2, so A is a fractal set, as defined by Mandelbrodt.

Now let A' be the image of A under the projection map from $S^1 \times C$ to $S^1 \times R$. Then it follows easily from the discussion above that A' is an attractor for the map f. In fact, for any compact neighborhood N of A', the successive images $f^k(N)$ converge to A'. Furthermore, A' supports an invariant measure, with entropy $\log 2$, so that almost every orbit in A' is dense. This attractor A' is nowhere dense, since it is a compact set of Hausdorff dimension $d \leqq 1 - \log(2)/\log(c) < 2$. In fact, A' can be described as the closure of the union of the graphs of the smooth almost periodic functions

$$g_n(\theta) = \cos((\theta + 2\pi n)/2) + c\cos((\theta + 2\pi n)/4) + c^2\cos((\theta + 2\pi n)/8) + \ldots,$$

where n ranges over all integers. Since g_n is uniformly close to g_m whenever n is close to m in the 2-adic topology, the function g_N is also defined and real analytic for any 2-adic integer N. With this convention, we can say simply that A' is the union of the graphs of the uncountably many distinct functions g_N. Thus A' is certainly not a manifold. It seems likely that its Hausdorff dimension is equal to $1 - \log(2)/\log(c) > 1$. (Compare Farmer, Ott, and Yorke.) On the other hand, if we modify this example by choosing c sufficiently close to 1, then it can be shown that the corresponding attractor A' contains an entire open subset of the cylinder. (See Frederickson, Kaplan, Yorke, and Yorke.)

If we carry out an analogous construction starting out with the Anosov diffeomorphism $(\phi, \psi) \mapsto (2\phi + \psi, \phi + \psi)$ of the torus in place of the squaring map on the circle, then we obtain an attractor which is topologically a torus $S^1 \times S^1$. Let $\lambda = (\sqrt{5} - 1)/2$ be the smaller eigenvalue of the associated integer matrix. Then this torus is embedded in a nowhere differentiable manner into the manifold $S^1 \times S^1 \times R$ if $\lambda < c < 1$, but is n times continuously differentiable if $0 < c < \lambda^n$ (see Kaplan, Mallet-Paret, and Yorke).

Acknowledgements. I am grateful to R. Butler, J. Guckenheimer, W. Thurston, R. F. Williams, and J. Yorke for a number of helpful suggestions.

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Communicated by O. E. Lanford

Received December 5, 1984; in revised form December 26, 1984