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ON THE CONCURRENT VECTOR FIELDS OF IMMERSED MANIFOLDS

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Let R^m be an m-dimensional Riemannian manifold[1] with covariant derivative D and let $x: M^n \to R^m$ be an immersion of an n-dimensional manifold M^n into R^m. A vector field X in R^m over M^n is called a *concurrent vector field*[2] if we have $dx+DX=0$, where dx denotes the differential of the immersion x. In particular, if X is a normal vector field of M^n in R^m, then the vector field X is called a *concurrent normal vector field.*

The main purpose of this paper is to study the behavior of the concurrent vector fields of immersed manifolds and also find a characterization of the concurrent vector fields with constant length.

§1. Preliminaries.

Let R^m be an m-dimensional Riemannian manifold with covariant derivative D. By a frame $e_1, \cdots, e_m$, we mean an ordered set of m orthonormal vectors $e_1, \cdots, e_m$ in the tangent space at a point of R^m. The frame $e_1, \cdots, e_m$ defines uniquely a dual coframe $\bar{\omega}_1, \cdots, \bar{\omega}_m$ in the cotangent space and vice versa. The fundamental theorem of local Riemannian geometry says that in a neighborhood U of a point p there exists uniquely a set of linear differential forms $\bar{\omega}_{AB}$ satisfying the conditions:

$$\bar{\omega}_{AB}+\bar{\omega}_{BA}=0, \tag{1}$$

and

$$d\bar{\omega}_A=\sum\bar{\omega}_B\wedge\bar{\omega}_{BA}, \tag{2}$$

where here and in the sequel the indices $A, B, \cdots$ run over the range $\{1, \cdots, m\}$. The linear differential forms $\bar{\omega}_{AB}$ are called the connection forms and the covariant derivative DX of a vector field $X=\sum X_A e_A$, is given by

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1) Manifolds, mappings, tensor fields and other geometric objects are assumed to be differentiable and of class C^∞.

2) In [3], a vector field X is called a concurrent vector field if there exists a function f such that $dx+D(fX)=0$, but in this paper, we adopt the above definition.

(3) $$DX=\sum DX_A\otimes e_A,$$

where

(4) $$DX_A=dX_A+\sum X_B\bar{\omega}_{BA}.$$

The vector field X is said to be *parallel* if $DX=0$. For the vectors e_A themselves equation (3) gives

(5) $$De_A=\sum\bar{\omega}_{AB}\otimes e_B.$$

The structure equations of R^m are given by (2) and

(6) $$d\bar{\omega}_{AB}=\sum\bar{\omega}_{AC}\wedge\bar{\omega}_{CB}+\bar{\Omega}_{AB},\qquad \bar{\Omega}_{AB}=\sum\frac{1}{2}\bar{R}_{ABCD}\bar{\omega}_C\wedge\bar{\omega}_D.$$

The tensor field $\bar{R}_{ABCD}$ is called the Riemann-Christoffel tensor. From $\bar{R}_{ABCD}$ the *Ricci tensor* and the *scalar curvature* are defined respectively by

(7) $$\bar{R}_{AB}=\bar{R}_{BA}=\sum\bar{R}_{CABC},$$

(8) $$\bar{S}=\sum\bar{R}_{AA}.$$

Let $x\colon M^n\to R^m$ be an immersion of an n-dimensional manifold M^n into R^m, and let B be the set of all elements $b=(p, e_1, \cdots, e_n, e_{n+1}, \cdots, e_m)$ such that $p\in M^n$, $e_1, \cdots, e_n$ are tangent vectors, $e_{n+1}, \cdots, e_m$ are normal vectors to $x(M^n)$ at $x(p)$ and $(x(p), e_1, \cdots, e_m)$ is a frame in R^m, where we have identified $dx(e_i)$ with e_i; $i=1, \cdots, n$. Let ω_A, ω_{AB} be the forms previously denoted by $\bar{\omega}_A$, $\bar{\omega}_{AB}$ relative to this particular frame field. Then we have

(9) $$\omega_r=0,\qquad r, s, t, \cdots=n+1, \cdots, m.$$

Taking the exterior derivative and using (2), we get

(10) $$\sum\omega_i\wedge\omega_{ir}=0,\qquad i, j, k, \cdots=1, \cdots, n.$$

By Cartan's lemma we have

(11) $$\omega_{ir}=\sum A_{rij}\omega_j,\qquad A_{rij}=A_{rji}.$$

The *mean curvature vector* H is defined by

(12) $$H=\frac{1}{n}\sum A_{rii}e_r.$$

M^n is called a *minimal submanifold* of R^m if $H=0$. If $e=\sum\cos\theta_r e_r$ is a unit normal vector field, then the second fundamental form at e is given by

(13) $$A(e)=\sum\cos\theta_r A_{rij}\omega_i\omega_j.$$

The second fundamental form at a normal vector $N\neq 0$ is defined as the second fundamental form at the unit direction of N. If the second fundamental form $A(N)$ at a normal vector N does not vanish and proportional to the first fundamental form $I=\sum\omega_i\omega_i$, then we say that M^n is *umbilical with respect to* N. In particular, if M^n is umbilical with respect to the mean curvature vector H at every point of M^n, then M^n is called a *pseudo-umbilical submanifold* of R^m.

For an immersion $x: M^n\to R^m$ if there does not exist a nowhere vanishing normal vector field N such that $DN=0$, then x is called a *substantial immersion.* If R^m is a euclidean m-space E^m, then x is substantial if and only if there does not exist an $(m-1)$-dimensional linear subspace of E^m containing $x(M^n)$.

For a normal vector field X in R^m over M^n, the covariant derivative DX can be decomposed into two parts:

$$DX=(DX)^t+D^*X, \tag{14}$$

where $(DX)^t$ is tangent to M^n and D^*X is normal to M^n. If the normal part D^*X vanishes, then X is called a *parallel vector field in the normal bundle.*

§2. Some results on concurrent normal vector fields.

Suppose that N is a concurrent normal vector field, i.e., N is a normal vector field and

$$dx+DN=0. \tag{15}$$

Thus, if we put $N=he$, $h=(N\cdot N)^{1/2}$, then we have

$$dx+(Dh)e+hDe=0. \tag{16}$$

Since dx is tangent to $x(M^n)$ and De is perpendicular to e, we get $Dh=0$. Thus N has constant length. Furthermore, by taking the scalar product of dx with (15), we get

$$dx\cdot dx+dx\cdot DN=0. \tag{17}$$

Thus, if we put $e=\sum\cos\theta_r e_r$, then, by (5), (11) and (17), we get

$$\sum\omega_i\omega_i-h\sum\cos\theta_r A_{rij}\omega_i\omega_j=0. \tag{18}$$

Moreover, by (14) and (15), we have

$$D^*N=0. \tag{19}$$

Therefore we have

PROPOSITION 1. *Let $x: M^n\to R^m$ be an immersion of M^n into R^m. If N is a concurrent normal vector field of M^n in R^m, then N has constant length, N is parallel in the normal bundle and M^n is umbilical in the direction of N.*

REMARK 1. Conversely, if N is a nowhere vanishing normal vector field parallel in the normal bundle and M^n is umbilical in the direction of N, then there exists a concurrent vector field parallel to the normal vector field N.

In the following, we denote the length of the mean curvature vector H by α, and we call it the *mean curvature* of M^n in R^m.

From proposition 1, we have

PROPOSITION 2. *Let x: $M^n \to R^m$ be an immersion of M^n into R^m. Then x is pseudo-umbilical and the mean curvature vector H is parallel in the normal bundle if and only if H/α^2 is concurrent.*

By a result of the authors [4] and proposition 2, we have

THEOREM 3. *Let x: $M^n \to E^m$ be an immersion of M^n into a euclidean space E^m of dimension m. Then the vector field H/α^2 is concurrent if and only if M^n is a minimal submanifold of a hypersphere of E^m.*

If N and N' are two concurrent normal vector fields, then, by (15), we get $D(N \cdot N')=0$. Hence the normal vector fields N and N' make a constant angle. Therefore $N-N'$ is a nowhere zero normal vector field with $D(N-N')=0$. Thus we have

PROPOSITION 4. *Let x: $M^n \to R^m$ be a substantial immersion of M^n into R^m. Then there exists at most one concurrent normal vector field.*

REMARK 2. In [2], Schouten and one of the authors proved that every invariant submanifold of an almost Kähler manifold is minimal. Therefore every invariant submanifold of an almost Kähler manifold does not admit a concurrent normal vector field.

THEOREM 5. *Let x: $M^n \to R^m$ be an immersion of M^n into a Riemannian manifold R^m of constant sectional curvature K. If N is a concurrent normal vector field, then the scalar curvature S of M^n, the mean curvature α and the length $|N|$ of N satisfy the following inequality*:

$$S \leqq n^2(\alpha^2+K)-n(K+|N|^{-2}). \tag{20}$$

If the dimension of M^n is >2, then the equality sign holds when and only when M^n is a pseudo-umbilical submanifold of R^m with constant sectional curvature $K+\alpha^2$.

Proof. Since R^m has constant sectional curvature K, we have, by (6),

$$d\omega_{ij}-\sum \omega_{ik} \wedge \omega_{kj}=\sum \omega_{ir} \wedge \omega_{rj}-K\omega_i \wedge \omega_j. \tag{21}$$

Thus, by putting $d\omega_{ij}-\sum \omega_{ik} \wedge \omega_{kj}=\Omega_{ij}=\sum(1/2)R_{ijkh}\omega_k \wedge \omega_h$, we have

$$R_{ijkh}=-K(\delta_{ik}\delta_{jh}-\delta_{ih}\delta_{jk})-\sum(A_{rik}A_{rjh}-A_{rih}A_{rjk}). \tag{21}$$

Therefore, we get

(23) $$R_{jk}=(n-1)K\delta_{jk}-\sum A_{rik}A_{rji}+\sum A_{rii}A_{rjk}.$$

Now, suppose that N is a concurrent normal vector field, and choose e_{n+1} in the direction of N, then, by proposition 1, we can verify that

(24) $$A_{n+1ij}=(|N|)^{-1}\delta_{ij}.$$

Substituting (24) into (23), we get

(25) $$R_{jk}=(n-1)K\delta_{jk}-\frac{\delta_{jk}}{N\cdot N}+\sum A_{rii}A_{rjk}-\sum_{s=n+2}^{m}(\sum_{i=1}^{m}A_{sik}A_{sji}).$$

Hence we have

(26) $$S=n(n-1)K-n(N\cdot N)^{-1}+n^2\alpha^2-\sum_{s=n+2}^{m}(\sum_{i=1}^{m}A_{rij}^2).$$

Therefore, by (26), we get the inequality (20).

If the equality of (20) holds, then, by (26), we get

$$A_{sij}=0, \qquad s=n+2, \cdots, m; \qquad i, j=1, \cdots, n.$$

Hence the mean curvature vector H is parallel to N and the mean curvature $\alpha=|N|^{-1}$. Therefore, by the fact that R^m has constant sectional curvature, we know that M^n has sectional curvature $K+\alpha^2$. If the dimension of M^n is greater than 2, then, by a well-known theorem of Shur, we know that $K+\alpha^2$ is a constant. Therefore M^n is a pseudo-umbilical submanifold and has constant sectional curvature provided $n>2$. The converse of this is trivial. This completes the proof of the theorem.

3. Concurrent vector fields with constant length.

PROPOSITION 6. *Let x: $M^n\to E^m$ be an immersion of M^n into a euclidean space E^m of dimension m. Then there exists a concurrent normal vector field if and only if $x(M^n)$ is contained in a hypersphere of E^m.*

Proof. Suppose that there exists a concurrent normal vector field N. Then, by proposition 1, N has constant length. On the other hand, by (15), we get

$$x(p)+N=c=\text{constant}.$$

Hence we have $(x-c)\cdot(x-c)=N\cdot N=$constant. Thus $x(M^n)$ is contained in a hypersphere of E^m centered at c. Conversely, if $x(M^n)$ is contained in a hypersphere of E^m centered at c, then the vector field $c-x(p)$ is a concurrent normal vector field. This completes the proof of the proposition.

From proposition 1, we know that every concurrent normal vector field has constant length. In the following, we want to find a necessary and sufficient condition that an arbitrary concurrent vector field has constant length.

Let X be a vector field in R^m over M^n. We define an $(n-1)$-form Θ_X by

$$\Theta_X=\sum(-1)^{i-1}(X\cdot e_i)\omega_1\wedge\cdots\wedge\hat{\omega}_i\wedge\cdots\wedge\omega_n, \tag{27}$$

where $\hat{\omega}_i$ denotes that the term ω_i is omitted. The form Θ_X is a well-defined $(n-1)$-form on M^n.

The main purpose of this section is to prove the following theorems:

THEOREM 7. *Let $x: M^n\to R^m$ be an immersion of an oriented closed manifold M^n into R^m. Then a concurrent vector field X in R^m over M^n has constant length if and only if the $(n-1)$-form Θ_X is closed.*

Proof. If X is a concurrent vector field with constant length, then we have

$$d(X\cdot X)=2X\cdot DX=0.$$

Hence, by the definition of the concurrent vector fields and the above equation we have

$$\sum(X\cdot e_i)\omega_i=0. \tag{28}$$

Thus, by taking the Hodge star operator on both sides of (28), we get $\Theta_X=0$. In particular, Θ_X is closed.

Conversely, suppose that the $(n-1)$-form Θ_X is closed. By a direct computation of the exterior derivative of (27) we get

$$-X\cdot H=1, \tag{29}$$

where H denotes the mean curvature vector of M^n in R^m. On the other hand, the Laplacian $\Delta(X\cdot X)$ of $X\cdot X$ is given by

$$\Delta(X\cdot X)=2n(1+X\cdot H). \tag{30}$$

Hence we get $\Delta(X\cdot X)=0$. Therefore the concurrent vector field X has constant length. This completes the proof of the theorem.

If R^m is euclidean, then by theorem 7 we have

COROLLARY [1]. *Let $x: M^n\to E^m$ be an immersion of an oriented closed manifold M^n into E^m, and X be the position vector field of M^n in E^m with respect to the origin of E^m. Then $x(M^n)$ is contained in a hypersphere of E^m centered at the origin of E^m if and only if the form $\Theta_X=0$.*

THEOREM 8. *Let $x: M^n\to R^m$ be an immersion of M^n into R^m with a concurrent normal vector field X, and let the first normal vector e_{n+1} be in the direction* *of the mean curvature vector H, that is,* $H=\alpha e_{n+1}$. *Then the immersion x is pseudo-umbilical if and only if*

$$\sum_{s=n+2}^{m}(X\cdot e_s)\omega_{si}=0, \qquad i=1, \cdots, n. \tag{31}$$

Proof. Since X is a concurrent normal vector field, the length of X is constant. Thus, by (30), we have

$$\alpha(X\cdot e_{n+1})=1. \tag{32}$$

Put $X=fe_{n+1}+\sum_{s=n+2}^{m}(X\cdot e_s)e_s$. Then, by taking covariant derivative, we have

$$DX=(df)e_{n+1}+\sum f\omega_{n+1i}e_i+\sum f\omega_{n+1r}e_r+\sum_{s=n+2}^{m}(d(X\cdot e_s))e_s$$

$$+\sum_{s=n+2}^{m}(X\cdot e_s)\omega_{si}e_i+\sum_{s=n+2}^{m}(X\cdot e_s)\omega_{sr}e_r.$$

Comparing the tangential parts of the above equation we get

$$\omega_i+f\omega_{n+1i}=\sum_{s=n+2}^{m}(X\cdot e_s)\omega_{is}, \qquad i=1, \cdots, n. \tag{33}$$

Suppose that the immersion x is pseudo-umbilical. Then we have

$$\omega_{n+1i}=-\alpha\omega_i=-f^{-1}\omega_i, \qquad i=1, \cdots, n. \tag{34}$$

Thus, by (33) and (34), we get (31). Conversely, if (31) holds, then, by (33), we get

$$\omega_i=-f\omega_{n+1i}, \qquad i=1, \cdots, n.$$

Hence the immersion x is pseudo-umbilical. This completes the proof of the theorem.

REMARK. Let X be a concurrent normal vector field and let e_{n+1} be in the direction of the mean curvature vector H as in theorem 8. Let $\bar{e}$ be the unit normal vector field in the direction of $X-(X\cdot e_{n+1})e_{n+1}$, i.e.,

$$X=(X\cdot e_{n+1})e_{n+1}+(X\cdot\bar{e})\bar{e}.$$

Then the condition (31) means

$$(X\cdot\bar{e})A(\bar{e})=0, \tag{32}$$

where $A(\bar{e})$ denotes the second fundamental form at $\bar{e}$.

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