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On the cone eigenvalue complementarity problem for higher-order tensors

Chen Ling[1] · **Hongjin He**[1] · **Liqun Qi**[2]

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Abstract In this paper, we consider the *tensor generalized eigenvalue complementarity problem* (TGEiCP), which is an interesting generalization of matrix *eigenvalue complementarity problem* (EiCP). First, we give an affirmative result showing that TGEiCP is solvable and has at least one solution under some reasonable assumptions. Then, we introduce two optimization reformulations of TGEiCP, thereby beneficially establishing an upper bound on cone eigenvalues of tensors. Moreover, some new results concerning the bounds on the number of eigenvalues of TGEiCP further enrich the theory of TGEiCP. Last but not least, an implementable projection algorithm for solving TGEiCP is also developed for the problem under consideration. As an illustration of our theoretical results, preliminary computational results are reported.

Keywords Higher order tensor · Eigenvalue complementarity problem · Cone eigenvalue · Optimization reformulation · Projection algorithm

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✉ Hongjin He
hehjmath@hdu.edu.cn

Chen Ling
cling_zufe@sina.com

Liqun Qi
maqilq@polyu.edu.hk

[1] Department of Mathematics, School of Science, Hangzhou Dianzi University, Hangzhou 310018, China

[2] Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

1 Introduction

The complementarity problem has become one of the most well-established disciplines within mathematical programming [10], in the last three decades. It is not surprising that the complementarity problem has received much attention of researchers, due to its widespread applications in the fields of engineering, economics and sciences. In the literature, many theoretical results and efficient numerical methods were developed, we refer the reader to [11] for an exhaustive survey on complementarity problems.

The *eigenvalue complementarity problem* (EiCP) not only is a special type of complementarity problems, but also extends the classical eigenvalue problem which can be traced back to more than 150 years (see [12,31]). EiCP first appeared in the study of static equilibrium states of mechanical systems with unilateral friction [8], and has been widely studied [1,9,14–16] in the last decade. Mathematically speaking, for two given square matrices $A, B \in \mathbb{R}^{n\times n}$, EiCP refers to the task of finding a scalar $\lambda \in \mathbb{R}$ and a vector $x \in \mathbb{R}^n\backslash\{0\}$ such that

$$0 \le x \perp w := (\lambda B - A)x \ge 0.$$

EiCPs are closely related to a class of differential inclusions with nonconvex processes defied by linear complementarity conditions, which serve as models for many dynamical systems, e.g., see a monograph [30]. Given a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^n$, we consider a dynamical system of the form:

$$\begin{cases} u(t) \ge 0, \\ \dot{u}(t) - Au(t) \ge 0, \\ \langle u(t), \dot{u}(t) - Au(t)\rangle = 0. \end{cases} \tag{1.1}$$

It is obvious that (1.1) is equivalent to $\dot{u}(t) \in F(u(t))$, where the process $F : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\mathrm{Gr}(F) := \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \ x \ge 0, y - Ax \ge 0, \langle x, y - Ax\rangle = 0\right\}$$

and is nonconvex. As noticed already by Rockafellar [26], the change of variable $u(t) = e^{\lambda t} v(t)$ leads to an equivalent system

$$\lambda v(t) + \dot{v}(t) \in F(v(t)).$$

This transformation efficiently utilizes the positive homogeneity of F. Therefore, if the pair (λ, x) satisfies $\lambda x \in F(x)$, then the trajectory $t \mapsto e^{\lambda t}x$ is a solution of dynamical system (1.1). Moreover, if such a trajectory is nonconstant, then x must be a nonzero vector, which further implies that (λ, x) is a solution of EiCP with $B := I$ (i.e., B is the identity matrix). The reader is referred to [8,27] for more details.

When B is symmetric positive definite and A is symmetric, EiCP is symmetric. In this case, it is well analyzed in [25] that EiCP is equivalent to finding a stationary point of a generalized Rayleigh quotient on a simplex. Generally speaking, the resulting equivalent optimization formulation is NP-complement [6,25] and very difficult to be solved efficiently, and in particular when the dimension of the problem is large.

In the current numerical analysis literature, considerable interests have arisen in extending concepts that are familiar from linear algebra to the setting of multilinear algebra. As a natural extension of the concept of matrix, a tensor, denoted by $\mathcal{A}$, is a multidimensional array, and its order is the number of dimensions. Let m and n be positive integers. We call $\mathcal{A} = (a_{i_1\cdots i_m})$, where $a_{i_1\cdots i_m} \in \mathbb{R}$ for $1 \le i_1, \ldots, i_m \le n$, a real m-th order n-dimensional square tensor, and it is further called *symmetric* if its entries are invariant under any permutation of its indices. The eigenvalues and eigenvectors of such a square tensor were introduced independently by Lim [18] and Qi [20].

For a vector $x = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$, $\mathcal{A}x^{m-1}$ is an n-vector with its i-th component defined by

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2} \ldots x_{i_m}, \quad \text{for} \quad i = 1, 2, \ldots, n,$$

and $\mathcal{A}x^m$ is the value at x of a homogeneous polynomial, defined by

$$\mathcal{A}x^m = \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1 i_2\ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m}.$$

For given tensors $\mathcal{A}$ and $\mathcal{B}$ with same structure, we say that $(\mathcal{A}, \mathcal{B})$ is an identical singular pair, if

$$\left\{x \in \mathbb{C}^n \backslash \{0\} : \mathcal{A}x^{m-1} = 0, \mathcal{B}x^{m-1} = 0\right\} \neq \emptyset.$$

Definition 1.1 ([5]) Let $\mathcal{A}$ and $\mathcal{B}$ be two m-th order n-dimensional tensors on $\mathbb{R}$. Assume that $(\mathcal{A}, \mathcal{B})$ is not an identical singular pair. We say $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is an eigenvalue–eigenvector pair of $(\mathcal{A}, \mathcal{B})$, if the n-system of equations:

$$(\mathcal{A} - \lambda\mathcal{B})x^{m-1} = 0, \tag{1.2}$$

that is,

$$\sum_{i_2,\ldots,i_m=1}^{n} (a_{ii_2\ldots i_m} - \lambda b_{ii_2\ldots i_m}) x_{i_2} \ldots x_{i_m} = 0, \quad i = 1, 2, \ldots, n,$$

possesses a nonzero solution. Here, λ is called a $\mathcal{B}$-eigenvalue of $\mathcal{A}$, and x is the corresponding $\mathcal{B}$-eigenvector of $\mathcal{A}$.

With the above definition, the classical higher order *tensor generalized eigenvalue problem* (TGEiP) is to find a pair of (λ, x) satisfying (1.2). It is obvious that if $\mathcal{B} = \mathcal{I}$, the unit tensor $\mathcal{I} = (\delta_{i_1\cdots i_m})$, where $\delta_{i_1\cdots i_m}$ is the Kronecker symbol

$$\delta_{i_1\cdots i_m} = \begin{cases} 1, & \text{if} \quad i_1 = \cdots = i_m, \\ 0, & \text{otherwise}, \end{cases}$$

then the resulting $\mathcal{B}$-eigenvalues reduce to the typical eigenvalues, and the real $\mathcal{B}$-eigenvalues with real eigenvectors are the H-eigenvalues, in the terminology of [20, 22]. In the literature, we have witnessed that tensors and eigenvalues/eigenvectors of tensors have fruitful applications in various fields such as magnetic resonance imaging [3,24], higher-order Markov chains [19] and best-rank one approximation in data analysis [23], whereby many nice properties such as the Perron-Frobenius theorem for eigenvalues/eigenvectors of nonnegative square tensor have been well established, see, e.g., [4,32].

In this paper, we consider the *tensor generalized eigenvalue complementarity problem* (TGEiCP), which can be mathematically characterized as finding a nonzero vector $\bar{x} \in \mathbb{R}^n$ and a scalar $\bar{\lambda} \in \mathbb{R}$ such that

$$\bar{x} \in K, \quad \bar{\lambda}\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1} \in K^*, \quad \left\langle \bar{x}, \bar{\lambda}\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1} \right\rangle = 0, \tag{1.3}$$

where $\mathcal{A}$ and $\mathcal{B}$ are two given m-th order n-dimensional higher tensors, K is a closed and convex cone in $\mathbb{R}^n$, and K^* is the positive dual cone of K, i.e., $K^* := \{w \in \mathbb{R}^n \;:\; \langle w, k \rangle \geq 0, \forall\, k \in K\}$. As EiCPs closely relate to differential inclusions with processes defined by linear complementarity conditions, TGEiCPs are also closely related to a class of differential inclusions with nonconvex processes H defined by

$$\begin{aligned} \mathrm{Gr}(H) := \Big\{&(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \; x \in K, \mathcal{B}y^{m-1} \\ &-\mathcal{A}x^{m-1} \in K^*, \left\langle x, \mathcal{B}y^{m-1} - \mathcal{A}x^{m-1} \right\rangle = 0 \Big\}. \end{aligned}$$

The scalar λ and the nonzero vector x satisfying system (1.3) are respectively called a K-eigenvalue of $(\mathcal{A}, \mathcal{B})$ and an associated K-eigenvector. In this situation, (λ, x) is also called a K-eigenpair of $(\mathcal{A}, \mathcal{B})$. The set of all eigenvalues is called the K-spectrum of $(\mathcal{A}, \mathcal{B})$, which is defined by

$$\sigma_K(\mathcal{A}, \mathcal{B}) := \left\{ \lambda \in \mathbb{R} \;:\; \exists x \in \mathbb{R}^n \backslash \{0\}, \; K \ni x \perp \lambda\mathcal{B}x^{m-1} - \mathcal{A}x^{m-1} \in K^* \right\}.$$

Throughout this paper, we assume the cone K is a *pointed cone*, i.e., $K \cap (-K) = \{0\}$. Moreover, we assume $\mathcal{B}x^m \neq 0$ for any $x \in K\backslash\{0\}$. Clearly, when $K = \mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$, then (1.3) reduces to

$$\bar{x} \geq 0, \quad \bar{\lambda}\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1} \geq 0, \quad \left\langle \bar{x}, \bar{\lambda}\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1} \right\rangle = 0, \tag{1.4}$$

which is a specialization of TGEiP. The scalar λ and the nonzero vector x satisfying system (1.4) are called a Pareto-eigenvalue of $(\mathcal{A}, \mathcal{B})$ and an associated Pareto-eigenvector, respectively. The set of all Pareto-eigenvalues, defined by $\sigma(\mathcal{A}, \mathcal{B})$, is called the Pareto-spectrum of $(\mathcal{A}, \mathcal{B})$. If in addition $m = 2$, the problem under consideration immediately reduces to the classical EiCP. If $\bar{x} \in \mathrm{int}(K)$ (respectively, $\bar{x} \in \{x \in$

$\mathbb{R}^n : x > 0\}$), then $\bar{\lambda}$ is called a strict K-eigenvalue (respectively, Pareto-eigenvalue) of $(\mathcal{A}, \mathcal{B})$. In particular, if $\mathcal{B} = \mathcal{I}$, then the K (Pareto)-eigenvalue/eigenvector of $(\mathcal{A}, \mathcal{B})$ is called the K (Pareto)-eigenvalue/eigenvector of $\mathcal{A}$, and the K (Pareto)-spectrum of $(\mathcal{A}, \mathcal{B})$ is called the K (Pareto)-spectrum of $\mathcal{A}$.

The main contributions of this paper are fourfold. As we have mentioned in above, TGEiCP is an essential extension of EiCP. Accordingly, a natural question is that whether TGEiCP has solutions like EiCP. In this paper, we first give an affirmative answer to this question, thereby discussing the existence of the solution of TGEiCP (1.3) under some conditions. Note that TGEiCP is also a special case of complementarity problem, and it is well documented in [10] that one of the most popular avenues to solve complementarity problems is reformulating them as optimization problems. Hence, we here also introduce two equivalent optimization reformulations of TGEiCP, which further facilitates the analysis of upper bound on cone eigenvalues of tensors. With the existence of the solution of TGEiCP, one may be interested in the number of such eigenvalues. Therefore, the third objective of this paper is to establish theoretical results concerning the bounds on the number of eigenvalues of TGEiCP. Finally, we develop a projection algorithm to solve TGEiCP, which is an easily implementable algorithm as long as the convex cone K is simple enough in the sense that the projection onto K has an explicit representation. As an illustration of our theoretical results, we implement our proposed projection algorithm to solve some synthetic examples and report the corresponding computational results.

The structure of this paper is as follows. In Sect. 2, the existence of solution for TGEiCP is discussed under some reasonable assumptions. Two optimization reformulations of TGEiCP are presented in Sect. 3, and the relationship of TGEiCP with the optimization of the Rayleigh quotient associated with tensors has been established. Moreover, based upon a reformulated optimization model, an upper bound on cone eigenvalues of tensor is also established. In Sect. 4, some theoretical results concerning the bounds on the number of eigenvalues of TGEiCP are presented. To solve TGEiCP, we develop the so-called *scaling-and-projection algorithm* (SPA) and conduct some numerical simulations to support our results of this paper. Finally, we complete this paper with drawing some concluding remarks in Sect. 6.

Notation Let $\mathbb{R}^n$ denote the real Euclidean space of column vectors of length n. Denote $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x > 0\}$. Let $\mathcal{A}$ be a tensor of order m and dimension n, and J be a subset of the index set $N := \{1, 2, \ldots, n\}$. We denote the principal sub-tensor of $\mathcal{A}$ by $\mathcal{A}_J$, which is obtained by homogeneous polynomial $\mathcal{A}x^m$ for all $x = (x_1, x_2, \ldots, x_n)^\top$ with $x_i = 0$ for $N \backslash J$. So, $\mathcal{A}_J$ is a tensor of order m and dimension $|J|$, where the symbol $|J|$ denotes the cardinality of J. For a vector $x \in \mathbb{R}^n$ and an integer $r \geq 0$, denote $x^{[r]} = (x_1^r, x_2^r, \ldots, x_n^r)^\top$.

2 Existence of the solution for TGEiCP

This section deals with the existence of the solution for TGEiCP. Let K be a closed and convex pointed cone in $\mathbb{R}^n$. Recall that a nonempty set $S \subset \mathbb{R}^n$ generates a cone K and write $K := \text{cone}(S)$ if $K := \{ts \ : \ s \in S, \ t \in \mathbb{R}_+\}$. If in addition S does not contain the zero vector and for each $k \in K \backslash \{0\}$, there exist unique $s \in S$ and

$t \in \mathbb{R}_+$ such that $k = ts$, then we say that S is a basis of K. Whenever S is a finite set, cone(conv(S)) is called a polyhedral cone, where conv(S) stands for the convex hull of S. Let K be a closed convex cone associated with a compact basis S. To study the existence of solution for TGEiCP, we first make the following assumption.

Assumption 2.1 It holds that $\mathcal{B}x^m \neq 0$ for every vector $x \in S$.

Remark 2.1 It is easy to see that Assumption 2.1 holds if and only if one of the tensors $\mathcal{B}$ (or $-\mathcal{B}$) is strictly K-positive, i.e., $\mathcal{B}x^m > 0$ (or $-\mathcal{B}x^m > 0$) for any $x \in K\backslash\{0\}$. In particular, when $K = \mathbb{R}^n_+$, the authors in [21,29] defined $\mathcal{B}$ as a strictly copositive tensor, i.e., $\mathcal{B}x^m > 0$ for any $x \in \mathbb{R}^n_+\backslash\{0\}$. It is easy to see that if $\mathcal{B}$ is nonnegative, i.e., $\mathcal{B} \geq 0$, and there is no index subset J of N such that $\mathcal{B}_J$ is a zero tensor, then $\mathcal{B}$ is a strictly copositive tensor, which immediately implies that Assumption 2.1 holds in this case.

From (1.3), one knows that if $(\bar{\lambda}, \bar{x}) \in \mathbb{R} \times (\mathbb{R}^n\backslash\{0\})$ is a K-eigenpair of $(\mathcal{A}, \mathcal{B})$, then necessarily

$$\bar{\lambda} = \frac{\mathcal{A}\bar{x}^m}{\mathcal{B}\bar{x}^m},$$

provided $\mathcal{B}\bar{x}^m \neq 0$. Consequently, by the second expression of (1.3), it holds that

$$\frac{\mathcal{A}\bar{x}^m}{\mathcal{B}\bar{x}^m}\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1} \in K^*.$$

We now present the existence theorem of TGEiCP, which is a particular instance of Theorem 3.3 in [17]. However, for the sake of completeness, here we still present its proof.

Theorem 2.1 *Let K be a cone associated with convex compact basis S. If Assumption 2.1 holds, then TGEiCP* (1.3) *has at least one solution.*

Proof Define $F : S \times S \to \mathbb{R}$ by

$$F(x, y) = \left\langle \mathcal{A}x^{m-1}, y \right\rangle - \frac{\mathcal{A}x^m}{\mathcal{B}x^m}\left\langle \mathcal{B}x^{m-1}, y \right\rangle. \tag{2.1}$$

Since $\mathcal{B}x^m \neq 0$ for any $x \in S$, it is obvious that $F(\cdot, y)$ is lower-semicontinuous on S for any fixed $y \in S$, and $F(x, \cdot)$ is concave on S for any fixed $x \in S$. By the well-known Ky Fan inequality [2], there exists a vector $\bar{x} \in S \subset K\backslash\{0\}$ such that

$$\sup_{y \in S} F(\bar{x}, y) \leq \sup_{y \in S} F(y, y). \tag{2.2}$$

Consequently, since $F(y, y) = 0$ for any $y \in S$, it follows from (2.2) that $F(\bar{x}, y) \leq 0$ for any $y \in S$. Let $\bar{\lambda} = \frac{\mathcal{A}\bar{x}^m}{\mathcal{B}\bar{x}^m}$. Then, by (2.1), one knows that $\langle \bar{\lambda}\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1}, y \rangle \geq 0$ for any $y \in S$, which implies

$$\bar{\lambda}\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1} \in K^*, \tag{2.3}$$

since for any $y \in K$ it holds that $y = ts$ for some $t \in \mathbb{R}_+$ and $s \in S$. Moreover, it is easy to observe that

$$\left\langle \bar{x}, \bar{\lambda}\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1} \right\rangle = 0,$$

which means, together with (2.3) and the fact that $\bar{x} \in K \setminus \{0\}$, that $(\bar{\lambda}, \bar{x})$ is a solution of (1.3). We obtain the desired result and complete the proof. □

From Theorem 2.1, we obtain the following corollary.

Corollary 2.1 *If $\mathcal{B}$ is strictly copositive, then* (1.4) *has at least one solution.*

Proof Let S be the standard simplex in $\mathbb{R}^n$ defined by

$$S := \left\{ x \in \mathbb{R}^n_+ \ : \ \sum_{i=1}^{n} x_i = 1 \right\}. \tag{2.4}$$

Clearly, S is a convex compact basis of $\mathbb{R}^n_+$. Thus, we immediately get the assertion of this corollary from Theorem 2.1. □

The following example shows that Assumption 2.1 is necessary to ensure the existence of the solution of TGEiCP.

Example 2.1 Let $m = 2$. Consider the case where

$$\mathcal{A} = \begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to see that Assumption 2.1 does not hold for the above two matrices. Since $\det(\lambda\mathcal{B} - \mathcal{A}) = -\lambda^2 - 11 \neq 0$ for any $\lambda \in \mathbb{R}$, we claim that the system of linear equations $(\lambda\mathcal{B} - \mathcal{A})x = 0$ has only one unique solution 0 for any $\lambda \in \mathbb{R}$, which means that $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^2_{++}$ satisfying (1.4) does not exist. Moreover, we may check that $(\lambda\mathcal{B} - \mathcal{A})x \geq 0$ does not hold for any $(\lambda, x) \in \mathbb{R} \times (\mathbb{R}^2_+ \setminus \{0\})$ with $x = (x_1, 0)^\top$ or $x = (0, x_2)^\top$. Therefore, problem (1.4) has no solution.

3 Optimization reformulations of TGEiCP

In this section, we study two optimization reformulations of (1.4). We begin with introducing the so-called generalized Rayleigh quotient related to tensors. For two given m-th order n dimensional tensors $\mathcal{A}$ and $\mathcal{B}$, the related Rayleigh quotient is defined by

$$\lambda(x) = \frac{\mathcal{A}x^m}{\mathcal{B}x^m}, \tag{3.1}$$

where $\mathcal{B}x^m \neq 0$. If $m = 2$, then $\lambda(x)$ defined by (3.1) reduces to the result presented in [25]. When $\mathcal{A}$ is symmetric and $\mathcal{B}$ is symmetric and strictly copositive, it is easy to see that the gradient of $\lambda(x)$ is

$$\nabla\lambda(x) = \frac{m}{\mathcal{B}x^m}\left[\mathcal{A}x^{m-1} - \lambda(x)\mathcal{B}x^{m-1}\right]. \tag{3.2}$$

Notice that the gradient formula (3.2) of the Rayleigh quotient is only valid when $\mathcal{A}$ and $\mathcal{B}$ are both symmetric. Moreover, in this case, the stationary points of $\lambda(x)$ correspond to solutions of (1.4). However, if either $\mathcal{A}$ or $\mathcal{B}$ is not symmetric, (3.2) is incorrect, and the relationship between stationary points and solutions of the TGEiCP with $K = \mathbb{R}^n_+$ ceases to hold.

The following lemma presents two fundamental properties of the generalized Rayleigh quotient $\lambda(x)$ in (3.1), whose matrix version has been proposed in [25]. Its proof is straightforward and skipped here.

Lemma 3.1 *For all $x \in \mathbb{R}^n\backslash\{0\}$, the following statements hold:*

(i). $\lambda(\tau x) = \lambda(x), \quad \forall \tau > 0$;
(ii). $x^\top \nabla\lambda(x) = 0$.

We first consider the following optimization problem

$$\rho(\mathcal{A}, \mathcal{B}) := \max_x \ \{\, \lambda(x) : \ x \in S \,\}, \tag{3.3}$$

where $\lambda(x)$ is defined in (3.1), and the constraint set S is determined by (2.4).

We generalize the result of symmetric EiCP studied in [25] to TGEiCP as the following proposition.

Proposition 3.1 *Assume that the tensors $\mathcal{A}$ and $\mathcal{B}$ are symmetric and $\mathcal{B}$ is strictly copositive. Let $\bar{x}$ be a stationary point of (3.3). Then $(\lambda(\bar{x}), \bar{x})$ is a solution of TGEiCP with $K = \mathbb{R}^n_+$.*

Proof Since $\bar{x}$ is a stationary solution of (3.3), from the structure of S, there exist $\bar{\alpha} \in \mathbb{R}^n$ and $\bar{\beta} \in \mathbb{R}$, such that

$$\begin{cases} -\nabla\lambda(\bar{x}) = \bar{\alpha} + \bar{\beta}e, \\ \bar{\alpha} \geq 0, \bar{x} \geq 0, \\ \bar{\alpha}^\top \bar{x} = 0, \\ e^\top \bar{x} = 1, \end{cases} \tag{3.4}$$

where $e \in \mathbb{R}^n$ is a vector of ones. By (3.4), we know $-\bar{x}^\top \nabla\lambda(\bar{x}) = \bar{\beta}$, which implies, together with Lemma 3.1 (ii), that $\bar{\beta} = 0$. Consequently, from (3.2), the first two expressions of (3.4) and the fact that $\mathcal{B}\bar{x}^m > 0$, it holds that $\lambda(\bar{x})\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1} \geq 0$. This means, together with the fact that $\bar{x} \geq 0$ and $\bar{x}^\top(\lambda(\bar{x})\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1}) = 0$, that $(\lambda(\bar{x}), \bar{x})$ is a solution of TGEiCP with $K = \mathbb{R}^n_+$. We complete the proof. □

In what follows, we denote

$$\lambda^{\max}_{\mathcal{A},\mathcal{B}} = \max\left\{\lambda \ : \ \exists\, x \in \mathbb{R}^n_+\backslash\{0\} \text{ suct that } (\lambda, x) \text{ is a solution of } (1.4)\right\}$$

for notational simplicity. Then, the following theorem characterizes the relationship between problem (3.3) and TGEiCP with $K := \mathbb{R}^n_+$.

Theorem 3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two m-th order n-dimensional symmetric tensors. If $\mathcal{B}$ is strictly copositive, then $\lambda^{\max}_{\mathcal{A},\mathcal{B}} = \rho(\mathcal{A}, \mathcal{B})$.*

Proof It is obvious that the constraint set Ω of (3.3) is compact, and hence there exists a vector $\bar{x} \in \Omega$ such that $\rho(\mathcal{A}, \mathcal{B}) = \lambda(\bar{x})$. It is clear that $\{e\} \cup \{e_i \ : \ i \in I(\bar{x})\}$ is linearly independent because of $\bar{x} \neq 0$, where $I(\bar{x}) := \{i \in N \ : \ \bar{x}_i = 0\}$. Consequently, the first order optimality condition of (3.3) holds, which means that $\bar{x}$ is stationary point of (3.3). By Proposition 3.1, we know that $(\lambda(\bar{x}), \bar{x})$ is a solution of TGEiCP with $K = \mathbb{R}^n_+$. Hence, it holds that $\rho(\mathcal{A}, \mathcal{B}) \leq \lambda^{\max}_{\mathcal{A},\mathcal{B}}$.

On the other hand, let (λ, x) be a solution of TGEiCP with $K := \mathbb{R}^n_+$, then $\lambda = \mathcal{A}x^m/\mathcal{B}x^m$. Taking $y = x/(e^\top x)$ implies that $y \in \Omega$. By Lemma 3.1 (i), we know $\lambda = \mathcal{A}y^m/\mathcal{B}y^m$, which implies that $\lambda \leq \rho(\mathcal{A}, \mathcal{B})$ from the definition of $\rho(\mathcal{A}, \mathcal{B})$. So, we have $\lambda^{\max}_{\mathcal{A},\mathcal{B}} \leq \rho(\mathcal{A}, \mathcal{B})$.

Therefore, we obtain the desired result and complete the proof. $\square$

We now study another optimization reformulation of TGEiCP with $K := \mathbb{R}^n_+$. We consider the following optimization problem

$$\phi(\mathcal{A}, \mathcal{B}) = \max_x \left\{\mathcal{A}x^m \ : \ x \in \Sigma\right\}, \tag{3.5}$$

where $\Sigma := \{x \in \mathbb{R}^n_+ \ : \ \mathcal{B}x^m = 1\}$ is assumed to be compact.

Remark 3.1 If $\mathcal{B}$ is strictly copositive, then we claim that Σ is compact. Indeed, if Σ is not compact, then there exists a sequence $\{x^{(k)}\} \subset \Sigma$ such that $\|x^{(k)}\| \to \infty$ as $k \to \infty$. Taking $y^{(k)} := x^{(k)}/\|x^{(k)}\|$ clearly shows $y^{(k)} \in \mathbb{R}^n_+$ and $\|y^{(k)}\| = 1$. Without loss of generality, we may assume that there exists a vector $\bar{y} \in \mathbb{R}^n_+$ satisfying $\|\bar{y}\| = 1$, such that $y^{(k)} \to \bar{y}$ as $k \to \infty$. On the other hand, we have $\mathcal{B}(y^{(k)})^m = 1/\|x^{(k)}\|^m$, which implies $\mathcal{B}\bar{y}^m = 0$. It contradicts the fact that $\mathcal{B}\bar{y}^m > 0$, because of $\bar{y} \in \mathbb{R}^n_+\backslash\{0\}$.

For TGEiCP with $K := \mathbb{R}^n_+$ and (3.5), we have the following theorem which can be proved in a similar way that used in [28].

Theorem 3.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two m-th order n dimensional symmetric tensors. If $\mathcal{B}x^m > 0$ for any $x \in \mathbb{R}^n_+\backslash\{0\}$, then $\lambda^{\max}_{\mathcal{A},\mathcal{B}} = \phi(\mathcal{A}, \mathcal{B})$.*

It follows from Theorems 3.1 and 3.2 that solving the largest Pareto eigenvalue of TGEiCP is an NP-hard problem in general, i.e., there are no polynomial-time algorithm for solving the largest Pareto eigenvalue of TGEiCP. In the rest of this section, based upon Theorem 3.2, we further study the bound of Pareto eigenvalue of TGEiCP with $\mathcal{B} := \mathcal{I}$ and $K := \mathbb{R}^n_+$.

We denote by Ω^* the solution set of (1.4) with $\mathcal{B} := \mathcal{I}$ and let

$$|\lambda|^{\max}_{\mathcal{A}} = \max\left\{ |\lambda| \ : \ \exists\, x \in \mathbb{R}^n_+\backslash\{0\} \text{ suct that } (\lambda, x) \in \Omega^* \right\}.$$

Theorem 3.3 *Suppose $\mathcal{B} := \mathcal{I}$. It holds that*

$$|\lambda|^{\max}_{\mathcal{A}} \leq \min\left\{n^{\frac{m-2}{2}}\|\mathcal{A}\|_F, \ \bar{a} \cdot n^{m-1}\right\},$$

where $\bar{a} := \max\left\{ |a_{i_1 i_2 \ldots i_m}| \ : \ 1 \leq i_1, i_2, \ldots, i_m \leq n \right\}$.

Proof Let (λ, x) be an arbitrary solution of (1.4) with $\mathcal{B} := \mathcal{I}$. Then it holds that

$$\lambda = \frac{\mathcal{A}x^m}{\sum_{i=1}^n x_i^m},$$

which implies

$$|\lambda| = \frac{|\mathcal{A}x^m|}{\sum_{i=1}^n x_i^m} \leq \frac{\|\mathcal{A}\|_F \|x^m\|_F}{\sum_{i=1}^n x_i^m},$$

where $x^m := \left(x_{i_1}x_{i_2}\ldots x_{i_m}\right)_{1\leq i_1,\ldots,i_m\leq n}$, which is an m-th order n-dimensional tensor. Since

$$\|x^m\|_F^2 = \sum_{i_1,i_2,\ldots,i_m=1}^{n} \left(x_{i_1}x_{i_2}\ldots x_{i_m}\right)^2 = \left(\sum_{i=1}^n x_i^2\right)^m \leq n^{m-2}\left(\sum_{i=1}^n x_i^m\right)^2,$$

we obtain

$$|\lambda| \leq n^{\frac{m-2}{2}}\|\mathcal{A}\|_F.$$

On the other hand, we have

$$|\lambda| = \frac{|\mathcal{A}x^m|}{\sum_{i=1}^n x_i^m} \leq \frac{\bar{a}\left(\sum_{i=1}^n x_i\right)^m}{\sum_{i=1}^n x_i^m} \leq \bar{a}\cdot n^{m-1}.$$

Hence we know

$$|\lambda| \leq \min\left\{n^{\frac{m-2}{2}}\|\mathcal{A}\|_F, \bar{a}\cdot n^{m-1}\right\}.$$

Since λ is arbitrary, we obtain the desired result and complete the proof. □

For the case where $\mathcal{B}$ is strict copositive but $\mathcal{B} \neq \mathcal{I}$, similarly, we may obtain

$$|\lambda_{\mathcal{A},\mathcal{B}}^{\max}| \leq \frac{1}{N_{\min}(\mathcal{B})}\min\left\{n^{\frac{m-2}{2}}\|\mathcal{A}\|_F, \bar{a}\cdot n^{m-1}\right\},$$

where $N_{\min}(\mathcal{B}) := \min\left\{\mathcal{B}x^m \ : \ x \in \mathbb{R}_+^n,\ \sum_{i=1}^n x_i^m = 1\right\} > 0$ (see [21, Theorem 5]). Here, we notice that the computation of $N_{\min}(\mathcal{B})$ is also NP-hard itself.

4 Bounds for the number of Pareto eigenvalues

In this section, we study the estimation of the numbers of Pareto-eigenvalue of $(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are two given m-th order n-dimensional tensors. We begin this section with some basic concepts and properties of eigenvalue/eigenvector of tensors.

It is well known that, on the left-hand side of (1.2), $(\mathcal{A} - \lambda\mathcal{B})x^{m-1}$ is indeed a set of n homogeneous polynomials with n variables, denoted by $\{P_i^\lambda(x) : 1 \leq i \leq n\}$, of degree $(m-1)$. In the complex field, in order to study the solution set of a system of n homogeneous polynomials $(P_1, \ldots, P_n)$, in n variables, the concept of the resultant $\mathrm{Res}(P_1, \ldots, P_n)$ is well defined and introduced in algebraic geometry literature, e.g., see [7]. Actually, $\mathrm{Res}(P_1, \ldots, P_n)$ is a polynomial in the coefficients $a_{i_1 i_2 \ldots i_m} - \lambda b_{i_1 i_2 \ldots i_m}$, whose total degree is $n(m-1)^{n-1}$ in the considered case here. Moreover, (1.2) has nonzero solutions if and only if $\mathrm{Res}(P_1, \ldots, P_n) = 0$, we refer the reader to [7] for more details. Applying it to our current problem, $\mathrm{Res}(P_1, \ldots, P_n)$ has the following properties.

Proposition 4.1 *We have the following results:*

(i). $\mathrm{Res}(P_1, \ldots, P_n) = 0$, *if and only if there exists* $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ *satisfying* (1.2).
(ii). *The degree of* λ *in* $\mathrm{Res}(P_1, \ldots, P_n)$ *is at most* $n(m-1)^{n-1}$.

For the special case where $m = 2$, we notice that the resultant $\mathrm{Res}(P_1, \ldots, P_n)$ coincides with the ordinary determinant $\det(\mathcal{A}-\lambda\mathcal{B})$. Accordingly, Proposition 4.1 reduces to the corresponding classical properties of matrix eigenvalue/eigenvector problems.

For TGEiCP with $K = \mathbb{R}^n_+$, we present the following proposition which fully characterizes the Pareto-spectrum of TGEiCP.

Proposition 4.2 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two* m*-th order* n*-dimensional tensors. A real number* λ *is Pareto-eigenvalue of* $(\mathcal{A}, \mathcal{B})$, *if and only if there exists a nonempty subset* $J \subseteq N$ *and a vector* $w \in \mathbb{R}^{|J|}_{++}$ *such that*

$$\mathcal{A}_J w^{m-1} = \lambda \mathcal{B}_J w^{m-1} \tag{4.1}$$

and

$$\sum_{i_2,\ldots,i_m \in J} (\lambda b_{i i_2 \ldots i_m} - a_{i i_2 \ldots i_m}) w_{i_2} \cdots w_{i_m} \geq 0, \quad \text{for every } i \in N \backslash J. \tag{4.2}$$

In such a case, the vector $x \in \mathbb{R}^n_+$ *defined by*

$$x_i = \begin{cases} w_i, & i \in J, \\ 0, & i \in N \backslash J \end{cases}$$

is a Pareto-eigenvector of $(\mathcal{A}, \mathcal{B})$, *associated with the real number* λ.

Proof It can be proved in a similar way that used in [28] and we skip it here. □

Remark 4.1 It is obvious that, in the case where $\mathcal{B} := \mathcal{I}$, (4.1) and (4.2) turn out to be

$$\mathcal{A}_J w^{m-1} = \lambda w^{[m-1]} \tag{4.3}$$

and

$$\sum_{i_2,\ldots,i_m\in J} a_{ii_2\ldots i_m} w_{i_2}\ldots w_{i_m} \leq 0, \quad \text{for every} \quad i \in N\backslash J, \tag{4.4}$$

respectively. The corresponding conclusions of Pareto-eigenvalues of $\mathcal{A}$ have been studied in [28].

By Proposition 4.2, if λ is Pareto-eigenvalue of $(\mathcal{A}, \mathcal{B})$, then there exists a nonempty subset $J \subseteq N$ such that λ is a strict eigenvalue of $(\mathcal{A}_J, \mathcal{B}_J)$. Motivated by the works on estimating the cardinality of the Pareto-spectrum of matrices [27], we now state and prove the main results in this section.

Theorem 4.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two given m-th order n-dimensional tensors. Assume that $(\mathcal{A}, \mathcal{B})$ is not an identical singular pair. Then there are at most $\varrho_{m,n} := nm^{n-1}$ Pareto-eigenvalues of $(\mathcal{A}, \mathcal{B})$.*

Proof It is obvious that, for every $k = 0, 1, \ldots, n-1$, there are $\binom{n}{n-k}$ corresponding principal sub-tensors pair of order m dimension $n-k$. Moreover, by Proposition 4.1, we know that every principal sub-tensors pair of order m dimension $n-k$ can have at most $(n-k)(m-1)^{n-k-1}$ strict Pareto-eigenvalues. By Proposition 4.2, we obtain the upper bound

$$\varrho_{m,n} = \sum_{k=0}^{n-1} \binom{n}{n-k} (n-k)(m-1)^{n-k-1} = nm^{n-1}.$$

Hence proved. □

Now we extend the above result to a more general case where K is a polyhedral convex cone. A closed convex cone K in $\mathbb{R}^n$ is said to be finitely generated if there is a linearly independent collection $\{c_1, c_2, \ldots, c_p\}$ of vectors in $\mathbb{R}^n$ such that

$$K = \text{cone}\{c_1, c_2, \ldots, c_p\} = \left\{ \sum_{i=1}^{p} \alpha_j c_j \ : \ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)^\top \in \mathbb{R}^p_+ \right\}. \tag{4.5}$$

Apparently, $K = \{C^\top \alpha : \ \alpha \in \mathbb{R}^p_+\}$, where $C = [c_1, c_2, \ldots, c_p]^\top$. Moreover, we can see that the dual cone of K, denoted by K^*, is equivalent to $\{w \in \mathbb{R}^n : \ Cw \geq 0\}$.

Theorem 4.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two given m-th order n-dimensional tensors. If the closed convex cone K admits representation (4.5), then $(\mathcal{A}, \mathcal{B})$ has at most $\varrho_{m,p} := pm^{p-1}$ K-eigenvalues.*

Proof We first prove that problem (1.3) with K defined by (4.5) is equivalent to finding a vector $\bar{\alpha} \in \mathbb{R}^p\backslash\{0\}$ and $\bar{\lambda} \in \mathbb{R}$ such that

$$\bar{\alpha} \geq 0, \quad \bar{\lambda}\mathcal{D}\bar{\alpha}^{m-1} - \mathcal{G}\bar{\alpha}^{m-1} \geq 0, \quad \left\langle \bar{\alpha}, \bar{\lambda}\mathcal{D}\bar{\alpha}^{m-1} - \mathcal{G}\bar{\alpha}^{m-1} \right\rangle = 0, \tag{4.6}$$

where $\mathcal{D}$ and $\mathcal{G}$ are two m-th order p-dimensional tensors, whose elements are denoted by

$$d_{i_1 i_2 \ldots i_m} = \sum_{j_1, j_2, \ldots, j_m = 1}^{n} b_{j_1 j_2 \ldots j_m} c_{i_1 j_1} c_{i_2 j_2} \ldots c_{i_m j_m}$$

and

$$g_{i_1 i_2 \ldots i_m} = \sum_{j_1, j_2, \ldots, j_m = 1}^{n} a_{j_1 j_2 \ldots j_m} c_{i_1 j_1} c_{i_2 j_2} \ldots c_{i_m j_m},$$

for $1 \leq i_1, i_2, \ldots, i_m \leq p$, respectively.

Let $(\bar{x}, \bar{\lambda}) \in (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}$ be a solution of (1.3) with K defined by (4.5). Since $\bar{x} \in K$, by the definition of K, there exists a nonzero vector $\bar{\alpha} \in \mathbb{R}_+^p$ such that $\bar{x} = C^\top \bar{\alpha}$. Consequently, from $\bar{\lambda}\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1} \in K^*$ and the expression of K^*, it holds that $C(\bar{\lambda}\mathcal{B}\bar{x}^{m-1} - \mathcal{A}\bar{x}^{m-1}) \geq 0$, which implies

$$C\left(\bar{\lambda}\mathcal{B}\left(C^\top \bar{\alpha}\right)^{m-1} - \mathcal{A}\left(C^\top \bar{\alpha}\right)^{m-1}\right) \geq 0. \tag{4.7}$$

By the definitions of $\mathcal{D}$ and $\mathcal{G}$, we know that (4.7) can be equivalently written as

$$\bar{\lambda}\mathcal{D}\bar{\alpha}^{m-1} - \mathcal{G}\bar{\alpha}^{m-1} \geq 0.$$

Moreover, it is easy to verify that $\langle \bar{\alpha}, \bar{\lambda}\mathcal{D}\bar{\alpha}^{m-1} - \mathcal{G}\bar{\alpha}^{m-1} \rangle = 0$. Conversely, if $(\bar{\alpha}, \bar{\lambda}) \in (\mathbb{R}^p \backslash \{0\}) \times \mathbb{R}$ satisfies (4.6), then we can prove that $(\bar{x}, \bar{\lambda})$ with $\bar{x} = C^\top \bar{\alpha}$ satisfies (1.3) in a similar way.

Consequently, by applying Theorem 4.1 to the problem (4.6), we know that $(\mathcal{A}, \mathcal{B})$ has at most $\varrho_{m,p} = pm^{p-1}$ K-eigenvalues. The proof is completed. □

The above theorem shows that $\sigma_K(\mathcal{A}, \mathcal{B})$ has finitely many elements in case K is a polyhedral convex cone. However, in the nonpolyhedral case the situation can be worse. For instance, Iusem and Seeger [13] constructed a symmetric matrix A (i.e., 2-th order n dimensional tensor) and a nonpolyhedral convex cone K such that $\sigma_K(A, I_n)$ behaves like the Cantor ternary set, i.e., it is uncountable and totally disconnected.

In the rest of this section, we discuss the case where $\mathcal{B} := \mathcal{I}$. We first present the following lemmas.

Lemma 4.1 *Let $\mathcal{A}$ be an m-th order n-dimensional nonnegative tensor, i.e., $a_{i_1 \ldots i_m} \geq 0$ for $1 \leq i_1, \ldots, i_m \leq n$. If $\mathcal{A}$ has two eigenvectors in $\mathbb{R}_{++}^n$, then, the corresponding eigenvalues are equal.*

Proof Let λ_1 and λ_2 be two Pareto-eigenvalues of $\mathcal{A}$, and let $x, y \in \mathbb{R}_{++}^n$ be the corresponding Pareto-eigenvectors, which means

$$\mathcal{A}x^{m-1} = \lambda_1 x^{[m-1]} \quad \text{and} \quad \mathcal{A}y^{m-1} = \lambda_2 y^{[m-1]}.$$

Since $\mathcal{A}$ is nonnegative tensor, we know that λ_1, λ_2 are nonnegative. Without loss of generality, assume $\lambda_1 \geq \lambda_2$. If $\lambda_1 = 0$, then $\lambda_2 = 0$. Now we assume $\lambda_1 > 0$. Denote

$$t_0 = \min\{t > 0 \ : \ ty - x \in \mathbb{R}^n_+\}, \tag{4.8}$$

which must exist since $y \in \mathbb{R}^n_{++}$. It is obvious that $t_0 y - x \in \mathbb{R}^n_+$, which immediately implies that $t_0 y_i \geq x_i$ for all i. Consequently, since $a_{i_1 \ldots i_m} \geq 0$ for $1 \leq i_1, \ldots, i_m \leq n$, by the definitions of $\mathcal{A}x^{m-1}$ and $\mathcal{A}(t_0 y)^{m-1}$, one knows that

$$t_0^{m-1} \lambda_2 y^{[m-1]} - \lambda_1 x^{[m-1]} = \mathcal{A}(t_0 y)^{m-1} - \mathcal{A}x^{m-1} \in \mathbb{R}^n_+,$$

which implies

$$t_0 (\lambda_2/\lambda_1)^{\frac{1}{m-1}} y - x \in \mathbb{R}^n_+.$$

By (4.8), we know that $t_0 \leq t_0 (\lambda_2/\lambda_1)^{\frac{1}{m-1}}$, which implies $\lambda_1 \leq \lambda_2$. Therefore, we obtain $\lambda_1 = \lambda_2$ and complete the proof. □

Let $\mathcal{A}$ be an m-th order n-dimensional tensor. We say that $\mathcal{A}$ is a Z-tensor, if all off-diagonal entries of $\mathcal{A}$ are nonpositive.

Lemma 4.2 *Let $\mathcal{A}$ be an m-th order n-dimensional tensor satisfying any of the following conditions: (i) $-\mathcal{A}$ is a Z-tensor; (ii) $\mathcal{A}$ is a Z-tensor. Then, $\mathcal{A}$ admits at most one strict eigenvalue, i.e., its corresponding eigenvector has positive entries.*

Proof We first consider case (i). Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be two strict eigenvalues of $\mathcal{A}$, i.e., there are vectors $x, y \in \mathbb{R}^n_{++}$ such that $\mathcal{A}x^{m-1} = \lambda_1 x^{[m-1]}$ and $\mathcal{A}y^{m-1} = \lambda_2 y^{[m-1]}$. Hence,

$$(\mathcal{A} + \mu\mathcal{I})x^{m-1} = (\lambda_1 + \mu)x^{[m-1]} \quad \text{and} \quad (\mathcal{A} + \mu\mathcal{I})y^{m-1} = (\lambda_2 + \mu)y^{[m-1]},$$

where μ is any real number. Since $-\mathcal{A}$ is a Z-tensor, $\mathcal{A} + \mu\mathcal{I}$ is nonnegative for μ sufficiently large. By Lemma 4.1, we obtain the equality $\lambda_1 + \mu = \lambda_2 + \mu$, which implies the desired conclusion.

For case (ii), the conclusion can be proved in a similar way. □

Proposition 4.3 *Let $\mathcal{A}$ be an m-th order n-dimensional tensor satisfying any of the following conditions: (i) $-\mathcal{A}$ is a Z-tensor; (ii) $\mathcal{A}$ is a Z-tensor. Then, $\mathcal{A}$ has at most $\rho_n := 2^n - 1$ Pareto eigenvalues.*

Proof We only consider case (i). The conclusion for case (ii) can be proved in a similar way. For every $k = 0, 1, \ldots, n-1$, there are $\binom{n}{n-k}$ principal sub-tensors of order m dimension $n - k$. Since $-\mathcal{A}$ is a Z-tensor, it is clear that any principal sub-tensors of $-\mathcal{A}$ are also Z-tensors. Consequently, by Lemma 4.2, we know that, every principal sub-tensor can have at most one strict eigenvalue. Therefore, by Proposition 4.2, one gets the upper bound

$$\rho_n = \sum_{k=0}^{n-1} \binom{n}{n-k} \cdot 1 = 2^n - 1.$$

We obtain the desired result and complete the proof. $\square$

It is easy to see that, if $\mathcal{A}$ is a nonnegative tensor, then $-\mathcal{A}$ is a Z-tensor. Hence, by Proposition 4.3, we know that any m-th order n dimensional nonnegative tensor can have at most $(2^n - 1)$ Pareto eigenvalues. The following example shows that the bound ρ_n is sharp within the second class mentioned in Proposition 4.3. This is what we call the exponential growth phenomenon.

Example 4.1 Consider a 3-rd order n-dimensional tensor $\mathcal{A} = (a_{i_1i_2i_3})_{1\le i_1,i_2,i_3\le n}$ with $a_{i_1i_2i_3} = -a^{i_1+i_2+i_3}$ and $a > \sqrt[3]{4}$. Given an arbitrary index set $J = \{l_1, l_2, \ldots, l_r\}$ with $1 \le l_1 < l_2 < \cdots < l_r \le n$, the principal sub-tensor $\mathcal{A}_J = (c_{j_1j_2j_3})_{1\le j_1,j_2,j_3\le r}$ has $c_{j_1j_2j_3} = -a^{l_{j_1}+l_{j_2}+l_{j_3}}$. Take vector $\xi = (a^{\frac{l_1}{2}}, a^{\frac{l_2}{2}}, \ldots, a^{\frac{l_r}{2}})^\top$. It is obvious that $\xi \in \mathbb{R}^r_{++}$ and

$$\begin{aligned}\left(\mathcal{A}_J\xi^2\right)_j = \sum_{j_2,j_3=1}^{r} c_{jj_2j_3}\xi_{j_2}\xi_{j_3} &= -\sum_{j_2,j_3=1}^{r} a^{l_j+l_{j_2}+l_{j_3}} a^{\frac{l_{j_2}}{2}} a^{\frac{l_{j_3}}{2}} \\ &= -\left(\sum_{s\in J} a^{\frac{3}{2}s}\right)^2 a^{l_j} = \lambda_J \xi_j^2,\end{aligned}$$

where $\lambda_J = -\left(\sum_{s\in J} a^{\frac{3}{2}s}\right)^2$. This means that (4.3) holds. Since $a_{i_1i_2i_3} < 0$ and $\xi > 0$, we do not worry about the condition (4.4). By Remark 4.1, we know that λ_J is a Pareto-eigenvalue of $\mathcal{A}$. Now, let us proceed to check that $\lambda_{J_1} \ne \lambda_{J_2}$ whenever $J_1 \ne J_2$. Take $J_1, J_2 \subseteq \{1, 2, \ldots, n\}$ with $J_1 \ne J_2$. Since $J_1 \triangle J_2 = (J_1 \backslash J_2) \cup (J_2 \backslash J_1) \ne \emptyset$, one can define $k = \max\{k \in \{1, 2, \ldots, n\}, k \in J_1 \triangle J_2\}$. Without loss of generality, we assume that $k \in J_2$, which implies $k \notin J_1$. In this case, we have

$$\sqrt{\lambda_{J_1}} - \sqrt{\lambda_{J_2}} = \sum_{s\in J_2} a^{\frac{3}{2}s} - \sum_{s\in J_1} a^{\frac{3}{2}s} = \sum_{s\in J_2, s\le k} b^s - \sum_{s\in J_1, s\le k-1} b^s.$$

where $b = a^{\frac{3}{2}}$. This implies that

$$\begin{aligned}\sqrt{\lambda_{J_1}} - \sqrt{\lambda_{J_2}} &= \sum_{s\in J_2, s\le k} b^s - \sum_{s\in J_1, s\le k-1} b^s \ge b^k - \sum_{s=1}^{k-1} b^s \\ &= \frac{b^{k+1} - 2b^k + b}{b-1} \ge \frac{b}{b-1} > 0,\end{aligned}$$

where the last inequality comes from the fact $b > 2$ with a given condition $a > \sqrt[3]{4}$. Therefore, we conclude that $\lambda_{J_1} \neq \lambda_{J_2}$. Since there are $2^n - 1$ ways of choosing the index set J, many elements exist in the Pareto spectrum of this special tensor $\mathcal{A}$.

Proposition 4.4 *Suppose that there exists an index subset $J_0 \subseteq N$ with $|J_0| = l$ such that $a_{ii_2\ldots i_m} > 0$ for any $i \in J_0$ and $i_2, \ldots, i_m \in N \backslash \{i\}$. Then $\mathcal{A}$ has at most $\gamma^l_{m,n} := [n(m-1)+l](m-1)^{l-1}m^{n-l-1}$ Pareto-eigenvalues. In particular, if $J_0 = N$, then $\mathcal{A}$ has at most $\varrho_{m-1,n} := n(m-1)^{n-1}$ Pareto-eigenvalues.*

Proof Under the given condition, we only need to consider the principal sub-tensor $\mathcal{A}_J$ with $J_0 \subseteq J$, which is due to the condition (4.2). Among the principal sub-tensors of order m dimension k, there are $\binom{n-l}{k-l}$ of them satisfying that property. This leads to the upper bound

$$\begin{aligned}\gamma^l_{m,n} &= \sum_{k=l}^{n} \binom{n-l}{k-l} k(m-1)^{k-1} \\ &= (m-1)^l \sum_{s=0}^{n-l} \binom{n-l}{s} (s+l)(m-1)^{s-1} \\ &= [n(m-1)+l](m-1)^{l-1}m^{n-l-1}.\end{aligned}$$

In particular, if $J_0 = N$, we immediately obtain the desired result. The proof is completed. ◻

A similar type of argument leads to the following result:

Proposition 4.5 *Suppose that there exists an index set $J_0 \subseteq N$ with $|J_0| = l$ such that $a_{ii_2\ldots i_m} > 0$ for any $i \in J_0$ and $i_2, \ldots, i_m \in N \backslash \{i\}$. Moreover, suppose that $-\mathcal{A}$ is a Z-tensor. Then, $\mathcal{A}$ has at most $\alpha^l_n := 2^{n-l}$ Pareto-eigenvalues.*

Proof This time one has to compute

$$\alpha^l_n = \sum_{k=l}^{n} \binom{n-l}{k-l} \cdot 1 = \sum_{s=0}^{n-l} \binom{n-l}{s} \cdot 1 = 2^{n-l}.$$

We obtain the desired result and complete the proof. ◻

Theorems 4.1–4.2 and Propositions 4.3–4.5 extend the corresponding results for bounds on Pareto eigenvalues of square matrices, which have been studied in [27], to the case higher order tensors. For the square matrix case, i.e., $m = 2$, it is well established in [27] that

$$\alpha^1_n \leq \rho_n \leq \gamma^1_{2,n} \leq \varrho_{2,n}.$$

Considering the tensor case, i.e., $m \geq 3$, it is obvious that $\alpha^l_n \leq \rho_n$ and $\gamma^l_{m,n} \leq \varrho_{m,n}$ for any $1 \leq l \leq n$. Moreover, it is not difficult to verify that, if $l = n$ then $\gamma^l_{m,n} =$

$n(m-1)^{n-1} \geq n2^{n-1} \geq \rho_n$; if $1 \leq l \leq n-1$, then $\gamma_{m,n}^l \geq [n(m-1)+1](m-1)^{n-2} \geq (2n+1)2^{n-2} \geq \rho_n$. Therefore, it always holds that

$$\alpha_n^l \leq \rho_n \leq \gamma_{m,n}^l \leq \varrho_{m,n}$$

for any $1 \leq l \leq n$.

5 Numerical algorithm and simulations

In this section, we first introduce an implementable algorithm for solving the TGEiCP. Then, we conduct some numerical results to verify the existence of the solution of TGEiCP and the reliability of our proposed algorithm.

5.1 Numerical algorithm

It well known that the general nonlinear complementarity problem can also be transformed into a system of equations. Therefore, it is of course possible to apply the *semismooth* and *smoothing* Newton methods to solve the problem under consideration in this paper. However, TGEiCP is more complicated than the classical EiCP due to the high-dimensional structure of tensors, thereby making such second-order algorithms difficult to be implemented. Motivated by the recent work in [9] for solving matrix cone constrained eigenvalue problem, in this section, we extend the so-called *scaling-and-projection algorithm* (SPA), developed in [9], to solve (1.3) and follow the same name for TGEiCP. The corresponding algorithm can be described in Algorithm 1. Throughout this section, we assume that $\mathcal{B}$ is strictly K-positive, i.e., $\mathcal{B}x^m > 0$ for any $x \in K\backslash\{0\}$.

Algorithm 1 A Scaling-and-Projection Algorithm (SPA).

1: Take any starting point $u^{(0)} \in K\backslash\{0\}$, and define $x^{(0)} = u^{(0)}/\sqrt[m]{\mathcal{B}(u^{(0)})^m}$.
2: **for** $k = 0, 1, 2, \cdots$ **do**
3: One has a current point $x^{(k)} \in K\backslash\{0\}$. Compute

$$\lambda_k = \frac{\mathcal{A}\left(x^{(k)}\right)^m}{\mathcal{B}\left(x^{(k)}\right)^m} \quad \text{and} \quad y^{(k)} = \mathcal{A}\left(x^{(k)}\right)^{m-1} - \lambda_k \mathcal{B}\left(x^{(k)}\right)^{m-1}. \tag{5.1}$$

4: If $\|y^{(k)}\| = 0$, then stop. Otherwise, let $s_k := \|y^{(k)}\|$, and compute

$$u^{(k)} = \Pi_K\left[x^{(k)} + s_k y^{(k)}\right] \quad \text{and} \quad x^{(k+1)} = \frac{u^{(k)}}{\sqrt[m]{\mathcal{B}(u^{(k)})^m}}. \tag{5.2}$$

5: **end for**

It is easy to verify that iterative scheme (5.1) always ensures $\langle x^{(k)}, y^{(k)}\rangle = 0$. As a consequence, $y^{(k)} \in K^*$ clearly means that $(x^{(k)}, y^{(k)})$ is a solution of problem

(1.3). However, for the sake of convenience, we often use $\|y^{(k)}\| = 0$ as the stopping condition in algorithmic framework instead of $y^{(k)} \in K^*$.

As we have mentioned, our proposed algorithm is a straightforward extension of [9], we can easily get the following convergence theorem. For sake of simplicity, we skip the proof of convergence of Algorithm 1; the interested reader is referred to [9] for a similar proof.

Theorem 5.1 *Let the sequence $\{x^{(k)}\}$ be generated by Algorithm 1 and further satisfy $\mathcal{B}(x^{(k)})^m = 1$. Assume convergence of $\{x^{(k)}\}$ toward some limit that one denotes by $\bar{x}$. Then,*

$$\lim_{k\to\infty} \lambda_k = \bar{\lambda} := \frac{\mathcal{A}\bar{x}^m}{\mathcal{B}\bar{x}^m}, \qquad \lim_{k\to\infty} y^{(k)} = \bar{y} := \mathcal{A}\bar{x}^{m-1} - \bar{\lambda}\mathcal{B}\bar{x}^{m-1}, \tag{5.3}$$

and $(\bar{\lambda}, \bar{x})$ is a solution of (1.3).

Remark 5.1 As mentioned in [9], if K has a complicated structure, then computing $u^{(k)}$ in Algorithm 1 is not an easy task. However, there are many interesting cones for which the projection map admits an explicit and easily computable formula. This is true, for instance, for the Pareto cone, for the Loewner cone of positive semidefinite symmetric matrices, for the Lorentz cone and, more generally, for any revolution cone. Therefore Algorithm 1 is easily implemented as long as the projection onto K is easy enough to be computed explicitly.

Remark 5.2 The tensors $\mathcal{A}$ and $\mathcal{B}$ considered above are not necessarily symmetric. If $K = \mathbb{R}^n_+$ and the tensors $\mathcal{A}$ and $\mathcal{B}$ are both symmetric, then the symmetric TGEiCP can be solved by computing a stationary point of the nonlinear program (3.3). The constraint set of this program is the simplex S defined by (2.4). The special structure of this set S makes the computation of projections of vectors over S very easy. On the other hand, the objective function of the required nonlinear program has Hessian whose computation is quite complicated. These features lead to our decision of investigating first order algorithms that are based on gradients and projections.

5.2 Numerical simulations

We have theoretically discussed the existence of the solution of TGEiCP in Sect. 2 and introduced an implementable projection method to solve the problem under consideration in Sect. 5.1. Thus, in this section, we aim at verifying that our theoretical results are true, in addition to demonstrating the reliability of the proposed algorithm. We implement Algorithm 1 by MATLAB R2012b and conduct the numerical simulations on a Lenovo notebook with Inter(R) Core(TM) i5-2410M CPU 2.30GHz and 4GB RAM running on Windows 7 Home Premium operating system.

In our experiments, we concentrate on three concrete TGEiCPs with symmetric structure and only list the details of tensors $\mathcal{A}$ and $\mathcal{B}$ in the ensuing examples.

Example 5.1 We consider two 4-th order 2-dimensional symmetric tensors $\mathcal{A}$ and $\mathcal{B}$, where the tensor $\mathcal{A}$ is specified as

$$\mathcal{A}(:,:,1,1)=\begin{pmatrix}0.8147 & 0.5164\\ 0.5164 & 0.9134\end{pmatrix},\quad \mathcal{A}(:,:,1,2)=\begin{pmatrix}0.4218 & 0.8540\\ 0.8540 & 0.9595\end{pmatrix},$$
$$\mathcal{A}(:,:,2,1)=\begin{pmatrix}0.4218 & 0.8540\\ 0.8540 & 0.9595\end{pmatrix},\quad \mathcal{A}(:,:,1,2)=\begin{pmatrix}0.6787 & 0.7504\\ 0.7504 & 0.3922\end{pmatrix},$$

and the tensor $\mathcal{B}$ is specified as

$$\mathcal{B}(:,:,1,1)=\begin{pmatrix}1.6324 & 1.1880\\ 1.1880 & 1.5469\end{pmatrix},\quad \mathcal{B}(:,:,1,2)=\begin{pmatrix}1.6557 & 1.4424\\ 1.4424 & 1.9340\end{pmatrix},$$
$$\mathcal{B}(:,:,2,1)=\begin{pmatrix}1.6557 & 1.4424\\ 1.4424 & 1.9340\end{pmatrix},\quad \mathcal{B}(:,:,1,2)=\begin{pmatrix}1.6555 & 1.4386\\ 1.4386 & 1.0318\end{pmatrix}.$$

Example 5.2 This example considers two 4-th order 3-dimensional symmetric tensors $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are specified as follows:

$$\mathcal{A}(:,:,1,1)=\begin{pmatrix}0.6229 & 0.2644 & 0.3567\\ 0.2644 & 0.0475 & 0.7367\\ 0.3567 & 0.7367 & 0.1259\end{pmatrix},\ \mathcal{A}(:,:,1,2)=\begin{pmatrix}0.7563 & 0.5878 & 0.5406\\ 0.5878 & 0.1379 & 0.0715\\ 0.5406 & 0.0715 & 0.3725\end{pmatrix},$$
$$\mathcal{A}(:,:,1,3)=\begin{pmatrix}0.0657 & 0.4918 & 0.9312\\ 0.4918 & 0.7788 & 0.9045\\ 0.9312 & 0.9045 & 0.8711\end{pmatrix},\ \mathcal{A}(:,:,2,1)=\begin{pmatrix}0.7563 & 0.5878 & 0.5406\\ 0.5878 & 0.1379 & 0.0715\\ 0.5406 & 0.0715 & 0.3725\end{pmatrix},$$
$$\mathcal{A}(:,:,2,2)=\begin{pmatrix}0.7689 & 0.3941 & 0.6034\\ 0.3941 & 0.3577 & 0.3465\\ 0.6034 & 0.3465 & 0.4516\end{pmatrix},\ \mathcal{A}(:,:,2,3)=\begin{pmatrix}0.8077 & 0.4910 & 0.2953\\ 0.4910 & 0.5054 & 0.5556\\ 0.2953 & 0.5556 & 0.9608\end{pmatrix},$$
$$\mathcal{A}(:,:,3,1)=\begin{pmatrix}0.0657 & 0.4918 & 0.9312\\ 0.4918 & 0.7788 & 0.9045\\ 0.9312 & 0.9045 & 0.8711\end{pmatrix},\ \mathcal{A}(:,:,3,2)=\begin{pmatrix}0.8077 & 0.4910 & 0.2953\\ 0.4910 & 0.5054 & 0.5556\\ 0.2953 & 0.5556 & 0.9608\end{pmatrix},$$
$$\mathcal{A}(:,:,3,3)=\begin{pmatrix}0.7581 & 0.7205 & 0.9044\\ 0.7205 & 0.0782 & 0.7240\\ 0.9044 & 0.7240 & 0.3492\end{pmatrix},\ \mathcal{B}(:,:,1,1)=\begin{pmatrix}0.6954 & 0.4018 & 0.1406\\ 0.4018 & 0.9957 & 0.0483\\ 0.1406 & 0.0483 & 0.0988\end{pmatrix},$$
$$\mathcal{B}(:,:,1,2)=\begin{pmatrix}0.6730 & 0.5351 & 0.4473\\ 0.5351 & 0.2853 & 0.3071\\ 0.4473 & 0.3071 & 0.9665\end{pmatrix},\ \mathcal{B}(:,:,1,3)=\begin{pmatrix}0.7585 & 0.6433 & 0.2306\\ 0.6433 & 0.8986 & 0.3427\\ 0.2306 & 0.3427 & 0.5390\end{pmatrix},$$
$$\mathcal{B}(:,:,2,1)=\begin{pmatrix}0.6730 & 0.5351 & 0.4473\\ 0.5351 & 0.2853 & 0.3071\\ 0.4473 & 0.3071 & 0.9665\end{pmatrix},\ \mathcal{B}(:,:,2,2)=\begin{pmatrix}0.3608 & 0.3914 & 0.5230\\ 0.3914 & 0.6822 & 0.5516\\ 0.5230 & 0.5516 & 0.7091\end{pmatrix},$$
$$\mathcal{B}(:,:,2,3)=\begin{pmatrix}0.4632 & 0.2043 & 0.2823\\ 0.2043 & 0.7282 & 0.7400\\ 0.2823 & 0.7400 & 0.9369\end{pmatrix},\ \mathcal{B}(:,:,3,1)=\begin{pmatrix}0.7585 & 0.6433 & 0.2306\\ 0.6433 & 0.8986 & 0.3427\\ 0.2306 & 0.3427 & 0.5390\end{pmatrix},$$
$$\mathcal{B}(:,:,3,2)=\begin{pmatrix}0.4632 & 0.2043 & 0.2823\\ 0.2043 & 0.7282 & 0.7400\\ 0.2823 & 0.7400 & 0.9369\end{pmatrix},\ \mathcal{B}(:,:,3,3)=\begin{pmatrix}0.8200 & 0.5914 & 0.4983\\ 0.5914 & 0.0762 & 0.2854\\ 0.4983 & 0.2854 & 0.1266\end{pmatrix}.$$

Example 5.3 This example also considers two 4-th order 3-dimensional symmetric tensors $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ take their components as follows:

$$\mathcal{A}(:,:,1,1)=\begin{pmatrix}0.4468&0.4086&0.5764\\0.4086&0.8176&0.5867\\0.5764&0.5867&0.8116\end{pmatrix},\ \mathcal{A}(:,:,1,2)=\begin{pmatrix}0.2373&0.5028&0.7260\\0.5028&0.5211&0.4278\\0.7260&0.4278&0.6791\end{pmatrix},$$

$$\mathcal{A}(:,:,1,3)=\begin{pmatrix}0.0424&0.0841&0.6220\\0.0841&0.8181&0.4837\\0.6220&0.4837&0.6596\end{pmatrix},\ \mathcal{A}(:,:,2,1)=\begin{pmatrix}0.2373&0.5028&0.7260\\0.5028&0.5211&0.4278\\0.7260&0.4278&0.6791\end{pmatrix},$$

$$\mathcal{A}(:,:,2,2)=\begin{pmatrix}0.3354&0.7005&0.3154\\0.7005&0.1068&0.7164\\0.3154&0.7164&0.7150\end{pmatrix},\ \mathcal{A}(:,:,2,3)=\begin{pmatrix}0.1734&0.5972&0.6791\\0.5972&0.0605&0.4080\\0.6791&0.4080&0.6569\end{pmatrix},$$

$$\mathcal{A}(:,:,3,1)=\begin{pmatrix}0.0424&0.0841&0.6220\\0.0841&0.8181&0.4837\\0.6220&0.4837&0.6596\end{pmatrix},\ \mathcal{A}(:,:,3,2)=\begin{pmatrix}0.1734&0.5972&0.6791\\0.5972&0.0605&0.4080\\0.6791&0.4080&0.6569\end{pmatrix},$$

$$\mathcal{A}(:,:,3,3)=\begin{pmatrix}0.4897&0.6299&0.6104\\0.6299&0.0527&0.5803\\0.6104&0.5803&0.5479\end{pmatrix},\ \mathcal{B}(:,:,1,1)=\begin{pmatrix}2.5328&2.6133&2.7630\\2.6133&2.5502&2.4151\\2.7630&2.4151&2.3012\end{pmatrix},$$

$$\mathcal{B}(:,:,1,2)=\begin{pmatrix}2.3955&2.2026&2.8921\\2.2026&2.8852&2.5060\\2.8921&2.5060&2.2619\end{pmatrix},\ \mathcal{B}(:,:,1,3)=\begin{pmatrix}2.5186&2.8867&2.7372\\2.8867&2.4538&2.2579\\2.7372&2.2579&2.1332\end{pmatrix},$$

$$\mathcal{B}(:,:,2,1)=\begin{pmatrix}2.3955&2.2026&2.8921\\2.2026&2.8852&2.5060\\2.8921&2.5060&2.2619\end{pmatrix},\ \mathcal{B}(:,:,2,2)=\begin{pmatrix}2.9037&2.7948&2.5391\\2.7948&2.1978&2.2653\\2.5391&2.2653&2.4799\end{pmatrix},$$

$$\mathcal{B}(:,:,2,3)=\begin{pmatrix}2.6280&2.1537&2.2689\\2.1537&2.9841&2.2698\\2.2689&2.2698&2.1981\end{pmatrix},\ \mathcal{B}(:,:,3,1)=\begin{pmatrix}2.5186&2.8867&2.7372\\2.8867&2.4538&2.2579\\2.7372&2.2579&2.1332\end{pmatrix},$$

$$\mathcal{B}(:,:,3,2)=\begin{pmatrix}2.6280&2.1537&2.2689\\2.1537&2.9841&2.2698\\2.2689&2.2698&2.1981\end{pmatrix},\ \mathcal{B}(:,:,3,3)=\begin{pmatrix}2.9427&2.3596&2.7611\\2.3596&2.7011&2.6822\\2.7611&2.6822&2.6665\end{pmatrix}.$$

Note that the stopping criterion in Algorithm 1 is $\|y^{(k)}\| = 0$ for exactly solving TGEiCP. In practical implementation, we usually use

$$\text{RelErr} := \left\|y^{(k)}\right\| := \left\|\mathcal{A}(x^{(k)})^{m-1} - \lambda_k \mathcal{B}(x^{(k)})^{m-1}\right\| \leq \text{Tol} \tag{5.4}$$

as the termination criterion to pursue an approximate solution with a preset tolerance 'Tol'. Now, we test three scenarios of 'Tol' by setting $\text{Tol} := \left\{5 \times 10^{-3}, 10^{-3}, 5 \times 10^{-4}\right\}$. We consider two cases of the starting point $u^{(0)}$, where the first case is a vector of ones, i.e., $u^{(0)} = (1, \ldots, 1)^\top$, and the second one is a random vector uniformly distributed in (0, 1), (the corresponding MATLAB script is `rand(n,1)`). To demonstrate the reliability of Algorithm 1, we report the number of iterations ('Iter.'), computing time in seconds ('Time'), the relative error ('RelErr') defined by (5.4), eigenvalue ('EigValue') and the corresponding eigenvector ('EigVector'). The computational results with respect to different initial points are summarized in Tables 1 and 2, respectively.

From the data reported in Tables 1 and 2, it is clear that our Algorithm 1 can successfully solve the TGEiCP, even though it seems that the number of iterations

Table 1 Computational results with starting point $(1, \dots, 1)^\top$

Example	Tol	Iter.	Time	RelErr	EigValue	EigVector
Example 5.1	5.0e−03	657	0.17	5.007e−03	0.4859	$(0.2697, 0.6407)^\top$
Example 5.2	5.0e−03	231	0.06	5.006e−03	1.5609	$(0.2168, 0.1532, 0.8774)^\top$
Example 5.3	5.0e−03	536	0.14	5.005e−03	0.2189	$(0.0630, 0.0000, 0.7236)^\top$
Example 5.1	1.0e−03	3211	0.78	1.000e−03	0.4850	$(0.2601, 0.6512)^\top$
Example 5.2	1.0e−03	1703	0.44	1.001e−03	1.5512	$(0.2194, 0.1576, 0.8683)^\top$
Example 5.3	1.0e−03	2584	0.65	1.000e−03	0.2173	$(0.0542, 0.0000, 0.7319)^\top$
Example 5.1	5.0e−04	6367	1.61	5.001e−04	0.4849	$(0.2589, 0.6525)^\top$
Example 5.2	5.0e−04	2929	0.77	5.002e−04	1.5514	$(0.2199, 0.1575, 0.8678)^\top$
Example 5.3	5.0e−04	5293	1.52	5.000e−04	0.2171	$(0.0530, 0.0000, 0.7330)^\top$

Table 2 Computational results with a random starting point

Example	Tol	Iter.	Time	RelErr	EigValue	EigVector
Example 5.1	5.0e−03	277	0.09	5.005e−03	0.4859	$(0.2697, 0.6407)^\top$
Example 5.2	5.0e−03	291	0.09	5.003e−03	1.5464	$(0.2172, 0.1600, 0.8673)^\top$
Example 5.3	5.0e−03	519	0.14	5.003e−03	0.2189	$(0.0623, 0.0008, 0.7234)^\top$
Example 5.1	1.0e−03	3218	0.80	1.000e−03	0.4850	$(0.2601, 0.6512)^\top$
Example 5.2	1.0e−03	1613	0.43	1.001e−03	1.5511	$(0.2195, 0.1577, 0.8680)^\top$
Example 5.3	1.0e−03	2636	0.70	1.000e−03	0.2173	$(0.0542, 0.0000, 0.7319)^\top$
Example 5.1	5.0e−04	6071	1.54	5.000e−04	0.4847	$(0.2565, 0.6551)^\top$
Example 5.2	5.0e−04	2341	0.64	5.002e−04	1.5510	$(0.2203, 0.1576, 0.8672)^\top$
Example 5.3	5.0e−04	5341	1.41	5.001e−04	0.2171	$(0.0530, 0.0000, 0.7330)^\top$

Table 3 Computational results with starting point $(1, \ldots, 1)^\top$ and $\alpha = 5$ in (5.5)

Example	Tol	Iter.	Time	RelErr	EigValue	EigVector
Example 5.1	5.0e−03	130	0.04	5.012e−03	0.4859	$(0.2696, 0.6407)^\top$
Example 5.2	5.0e−03	62	0.02	5.006e−03	1.5472	$(0.2168, 0.1597, 0.8682)^\top$
Example 5.3	5.0e−03	105	0.03	5.010e−03	0.2189	$(0.0629, 0.0000, 0.7236)^\top$
Example 5.1	1.0e−03	639	0.17	1.001e−03	0.4850	$(0.2601, 0.6512)^\top$
Example 5.2	1.0e−03	230	0.07	1.001e−03	1.5501	$(0.2204, 0.1580, 0.8664)^\top$
Example 5.3	1.0e−03	513	0.13	1.001e−03	0.2173	$(0.0542, 0.0000, 0.7319)^\top$
Example 5.1	5.0e−04	1270	0.32	5.002e−04	0.4849	$(0.2589, 0.6525)^\top$
Example 5.2	5.0e−04	549	0.14	5.003e−04	1.5513	$(0.2207, 0.1574, 0.8669)^\top$
Example 5.3	5.0e−04	1054	0.28	5.003e−04	0.2171	$(0.0530, 0.0000, 0.7330)^\top$
Example 5.1	1.0e−04	6297	1.55	1.000e−04	0.4848	$(0.2579, 0.6536)^\top$
Example 5.2	1.0e−04	3227	0.84	1.000e−04	1.5520	$(0.2203, 0.1571, 0.8679)^\top$
Example 5.3	1.0e−04	6332	1.65	1.000e−04	0.2170	$(0.0518, 0.0005, 0.7337)^\top$

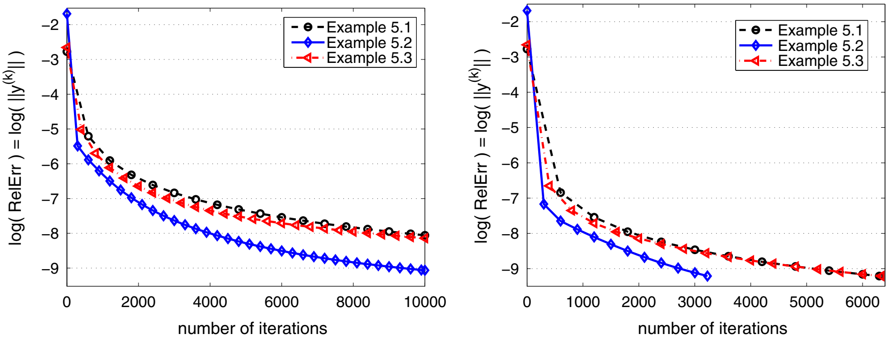

Fig. 1 Evolutions of 'RelErr' defined by (5.4) with respect to iterations. The *left plot* corresponds to the original projection scheme, i.e., $\alpha = 1$. The *right one* is corresponding to (5.5) with $\alpha = 5$

increases significantly as the decrease of tolerance 'Tol'. Actually, we tested a series of random starting points, and observed that random starting points often perform better than the deterministic vector of ones in terms of taking less iterations as reported in Table 2. However, all experiments show that Algorithm 1 is reliable for solving TGEiCP.

Taking a revisit on Algorithm 1, the iterative scheme (5.2) plays an significant role in the whole algorithm. In other words, the projection step given in (5.2) dominates the main task of Algorithm 1. As we know, the typical projection methods consist of two important components, i.e., step size and search direction. In Algorithm 1, s_k and $y^{(k)}$ serve as the step size and search direction, respectively. It is well known that good choices of step size and search direction may lead to promising numerical performance. Turning our attention to (5.2), it can be easily seen that step size s_k approaches to zero as the sequence $\{x^{(k)}\}$ gets close to a solution of TGEiCP, thereby reducing the speed of convergence of Algorithm 1. A naturally simple idea is to increase s_k by attaching a larger constant α to it, that is, the projection step in (5.2) turns out to be

$$u^{(k)} = \Pi_K \left[x^{(k)} + \alpha s_k y^{(k)} \right]. \tag{5.5}$$

In our experiments, we observe that Algorithm 1 could be accelerated greatly when we set $\alpha \in (1, 8)$. We also report some computational results in Table 3.

By comparing the results in Tables 1 and 3, it is apparent that the refined projection step (5.5) outperforms the original one in (5.2) in terms of taking much less iterations. In Fig. 1, we further consider two different projection steps, and graphically plot the evolutions of the relative error defined by (5.4) in the logarithmic sense, i.e., $\log(\|y^{(k)}\|)$, with respect to iterations, where the stopping tolerance 'Tol' is set to be $\mathrm{Tol} := 10^{-4}$.

It is clear from the above results that attaching a relaxation factor α in (5.5) is necessary to improve the numerical performance of our algorithm. In future work, we will introduce a self-adaptive strategy to adjust the relaxation factor α for an acceleration of the proposed method.

6 Conclusions

This paper considers the TGEiCP with symmetric structure, which is an interesting generalization of matrix eigenvalue complementarity problem. To the best of our knowledge, the development of TGEiCP is in its infancy and such a problem has been received much less attention. In this paper, we discuss the existence of the solution of TGEiCP under some conditions, in addition to presenting two equivalent optimization reformulations for the purpose of analyzing the upper bound on cone eigenvalues of tensors. The bounds on the number of eigenvalues of TGEiCP are also presented. Finally, we develop a first-order projection method which might be a better candidate for TGEiCP than second-order solvers. Note that we only consider the optimization reformulations of symmetric TGEiCP, and many problems lack such a symmetric structure. Hence, our future work will further study TGEiCPs in absence of the symmetric property. On the other hand, our numerical simulations show us that the attached α in (5.5) is important for algorithmic acceleration. Then, how to improve the numerical performance of Algorithm 1 is also one of our future concerns.

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