On the Cone of Nonnegative Circuits

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Abstract. We discuss three equivalent formulations of a theorem of Seymour on nonnegative sums of circuits of a graph, and present a different (but not shorter) proof of Seymour's result.

1. Introduction

Let G be a connected bridgeless graph. If $X \subseteq V(G)$, $\partial(X)$ is the set of all edges e with exactly one end in X. If $e \in \partial(X)$, the vector $h(X, e)$, with coordinates indexed by $E(G)$, is

$$h(X, e)_f = \begin{cases} 1 & \text{if } f \in \partial(X) \backslash \{e\}, \\ -1 & \text{if } f = e, \\ 0 & \text{otherwise.} \end{cases}$$

A vector x *conforms* to a vector y if, for every j, $x_j > 0$ implies $y_j > 0$ and $x_j < 0$ implies $y_j < 0$. Denote by M the $(0, 1)$-matrix with rows corresponding to all circuits of G, columns to all edges of G, with

$$M_{Ce} = \begin{cases} 1 & \text{if } e \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Let $K(M) = \{z: Mz \geq 0\}$. Thus $K(M)$ is the set of all edge flows (assignments of weights to edges) for which the total flow of every circuit is nonnegative. Note that $h(X, e) \in K(M)$ for all X and e. For an edge flow z, set $N(z) = \{C: (Mz)_C = 0\}$, the set of circuits whose total flow is zero with respect to z. Such circuits will be called 0-circuits.

This paper originated from an attempt to find a short proof of a remarkable theorem of Seymour [2] (Theorem 1 below), which is the undirected analog of Hoffman's circulation theorem for directed networks [1]. We wanted to establish Theorem 1 by proving an equivalent result (Theorem 2) about generators of $K(M)$. But that program seemed to require a third result (Theorem 3) which, although an easy consequence of Theorem 1, is not so easy without it. Our discussion: Theorem 2 $\Rightarrow$ Theorem 1 $\Rightarrow$ Theorem 3 $\Rightarrow$ Theorem 2, together with a separate proof of a stronger result (Theorem 4) of which Theorem 3 is a corollary, yields a different proof of Theorem 1, but not a shorter one.

Theorem 1. *For any $c, d \in \mathbb{R}^{E(G)}$, the system of inequalities*

$$y \geq 0, \qquad c' \leq y'M \leq d' \tag{1.1}$$

is consistent if and only if

$$d \geq 0, \tag{1.2a}$$

$$d \geq c, \tag{1.2b}$$

and

for every $X \subseteq V(G)$, and every $e \in \partial(X)$, $\quad c_e \leq \sum_{f \in \partial(X) \setminus \{e\}} d_f.$ \tag{1.2c}

Note that y above is a vector of flows on circuits (an assignment of weights to circuits). This is the analog to flow conservation in directed networks. Then (1.1) corresponds to lower and upper bounds on the cumulative arc flows.

Theorem 2. *Every $z \in K(M)$ is a nonnegative sum of nonnegative vectors conforming to z and vectors $h(X, e)$ conforming to z.*

Theorem 3. *If $z \in K(M)$ and $E(G) = \bigcup_{C \in N(z)} E(C)$, then if $v_1, v_2 \in V(G)$, there is a path P joining v_1 and v_2 such that $z(P) = \sum_{e \in P} z_e \leq 0$.*

2.   Theorem 2 $\Rightarrow$ Theorem 1

The necessity of (1.2a)–(1.2c) is obvious. To show they are sufficient, rewrite (1.1) as

$$y'[M; -M] \leq [d'; -c'], \qquad y \geq 0. \tag{2.1}$$

By Farkas' theorem, (2.1) is consistent if and only if

$$s \geq 0, \qquad t \geq 0, \qquad M(s-t) \geq 0 \qquad \text{implies} \qquad d's - c't \geq 0. \qquad (2.2)$$

For any vector $z \in \mathbb{R}^{E(G)}$, let $z_e^+ = z_e$ if $z_e \geq 0$, otherwise $z_e^+ = 0$; $z_e^- = -z_e$ if $z_e \leq 0$, otherwise $z_e^- = 0$. Now consider $(s; t)$ satisfying $s \geq 0$, $t \geq 0$, $M(s-t) \geq 0$. Then $(s; t) = (q; q) + (z^+; z^-)$, where $q_e = \min(s_e, t_e)$ for each e, $z = s - t$, and $Mz \geq 0$.

By (1.2b), $d - c \geq 0$. But $s \geq 0$, $t \geq 0$ imply $q \geq 0$, so $d'q - c'q \geq 0$. By Theorem 2,

$$(z^+; z^-) = (p; 0) + \sum k_X (h^+(X, e); h^-(X, e)),$$

where $p \geq 0$, all $k_X \geq 0$. By (1.2a) $d'p - c'0 \geq 0$, and by (1.2c) $d'h^+(X, e) = \sum_{f \in \partial(X) \setminus \{e\}} d_f \geq c_e = c'h^-(X, e)$. This yields $d'h^+(X, e) - c'h^-(X, e) \geq 0$. So the implication (2.2) holds, and Theorem 1 is proved.   □

3. Theorem 1 ⇒ Theorem 3

Suppose v_1 and v_2 are ends of an edge $e \in E(G)$. By the hypothesis of Theorem 3, there is a 0-circuit containing e. If $z_e \leq 0$, the desired path is e. If $z_e > 0$, the other edges of the 0-circuit constitute the desired path.

Suppose v_1 and v_2 are not adjacent in G. Construct a graph G^* by creating a new edge f joining v_1 and v_2. We now describe vectors c and d for G. Let $c_f = 1$, $d_f = \infty$. For all other edges $e \in E(G^*)$, let $c_e = d_e =$ the number of circuits $C \in N(z)$ containing e. It is easy to see that (1.2a)–(1.2c) are satisfied. By the definition of c and d, it follows from Theorem 1 applied to G^* that in G there exist paths $P_1, \ldots, P_n$ joining v_1 and v_2 and positive numbers $\alpha_1, \ldots, \alpha_n$, together with a (possibly empty) set of circuits $C_1, \ldots, C_m$ and associated positive numbers $\beta_1, \ldots, \beta_m$ such that, designating the Cth row of M by M_C,

$$\sum \alpha_k P_k + \sum \beta_j C_j = c = \sum_{C \in N(z)} M_C. \qquad (3.1)$$

Note that in (3.1), as well as elsewhere in this paper, we may choose to abuse notation and identify paths, circuits or subgraphs with their incidence vectors. The above paths P_k correspond to circuits in G^* containing f. Because $c_f = 1$ we know $n \geq 1$.

Take the inner product of z with both the left- and right-hand sides of (3.1). Since $(C_j, z) \geq 0$ for all j and $(M_C, z) = 0$ for all $C \in N(z)$, it follows that $(P_k, z) \leq 0$ for at least one k.

4. Theorem 3 ⇒ Theorem 2

For any graph G and any vector $z \in \mathbb{R}^{E(G)}$ with $Mz \geq 0$, define G_z as follows: $V(G_z) = V(G)$, and v_1, v_2 are adjacent in G_z if and only if at least one of the following holds:

$$z_{(v_1, v_2)} \leq 0 \qquad (4.1)$$

or

$$\text{there is a 0-circuit containing } v_1 \text{ and } v_2. \tag{4.2}$$

It is easy to see that (4.1) and (4.2), using Theorem 3, imply:

$$\begin{array}{l}\text{if } v_1 \text{ and } v_2 \text{ are connected in } G_z, \text{ there is a nonpositive} \\ \text{path in } G \text{ joining } v_1 \text{ and } v_2. \end{array} \tag{4.3}$$

For, let G^z be obtained from G by removing all edges that are in no 0-circuit of z. Then Theorem 3 applies to each component of G^z. The components of G_z are obtained from the components of G^z by (possibly) connecting some of the latter by negative edges. Hence (4.3) holds.

Recall that z satisfies $Mz \geq 0$. If $z \geq 0$, we are done, so assume $E(z) = \{e: z_e < 0\}$ is not empty. We shall show there exist $X \subseteq V(G), f \in E(z) \cap \partial(X), t > 0$ so that:

$$h(X, f) \text{ conforms to } z; \tag{4.4}$$

$$z^* = z - th(X, f) \text{ conforms to } z, \text{ and satisfies } Mz^* \geq 0; \tag{4.5}$$

$$\operatorname{supp} z^* \subseteq \operatorname{supp} z; \tag{4.6}$$

$$N(z) \subseteq N(z^*); \tag{4.7}$$

with at least one of the inclusions in (4.6) and (4.7) proper. Clearly, these stipulations on z^* will suffice to prove Theorem 2.

For each $e = (v_i, v_j) \in E(z)$, consider the graph $(G - e)_z$ defined above, and let K_{iz}^e be the set of vertices of the component containing v_i in that graph. By (4.3),

$$K_{iz}^e \neq K_{jz}^e; \tag{4.8}$$

otherwise we would have a nonpositive path joining v_i and v_j which, together with e, would produce a negative circuit. Of all possible K_{iz}^e which could be so constructed, choose one with the fewest number of vertices, say K_{1z}^f. We first show

$$\text{if } e = (v_i, v_j) \in E(G) \quad \text{and} \quad v_i, v_j \in K_{1z}^f, \quad \text{then} \quad z_e \geq 0. \tag{4.9}$$

Assume otherwise. Let P be the (v_1, v_i)-path in $(G - f)_z \subseteq G_z$ whose existence is guaranteed by $v_i \in K_{1z}^f$. Suppose $v_1 \notin K_{1z}^e$. Let v_k be any vertex of K_{iz}^e and Q be a (v_i, v_k)-path in $(G - e)_z \subseteq G_z$. If $Q \not\subseteq (G - f)_z$ then Q contains f or some edge derived via (4.2) from a 0-circuit containing f. In this case we can see that v_1 can also be reached from v_i in $(G - e)_z$, contradicting $v_1 \notin K_{iz}^e$. Hence $Q \subseteq (G - f)_z$. But then we can get from v_1 to v_k in $(G - f)_z$ by first following P and then following Q. Thus $v_k \in K_{1z}^f$ and K_{iz}^e is properly contained in K_{1z}^f, contradicting the minimality of K_{1z}^f. Therefore $v_1 \in K_{iz}^e$. The same reasoning shows that $v_1 \in K_{jz}^e$. But this means $K_{iz}^e = K_{jz}^e$, violating (4.8). So (4.9) is true.
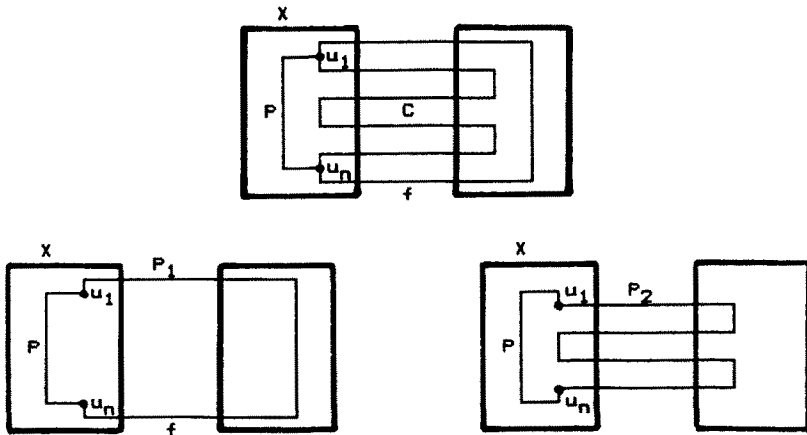
Fig. 1

Next, let $X = K_{12}^f$. The vector $h(X, f)$ satisfies (4.4), by (4.1). We now claim

$$N(z) \subseteq N(h(X, f)). \tag{4.10}$$

Since $N(z^*)$ differs from $N(z)$ only on circuits crossing X, (4.10) would imply that (4.7) is true.

To show (4.10) it is sufficient to show that $C \in N(z)$ implies that C either misses $\partial(X)$ or intersects $\partial(X)$ in f and exactly one (positive) edge. If C intersects $\partial(X)$, but does not contain f, then we have a contradiction of (4.2). Further, by (4.1), every edge in $\partial(X) \setminus \{f\}$ is positive. Suppose C contains more than one (positive) edge of $\partial(x)$. From (4.3) and (4.9), any two vertices of C in X can be connected by a path of 0-edges. It follows that, if C contains more than one positive edge of $\partial(X)$, there is a path of 0-edges, all vertices of which are in X, say $P = (u_1, u_2), (u_2, u_3), \ldots, (u_{n-1}, u_n)$, where u_1 and u_n are in C, all others are not, and C is the mod 2 sum of circuits $P + P_1$ and $P + P_2$ (P_1 and P_2 being paths whose union is C), and $P + P_1$, $P + P_2$ both contain edges of $\partial(X)$. (See Fig. 1.) But $\sum_{e \in P + P_1} z_e \geq 0$ and $\sum_{e \in P + P_2} z_e \geq 0$. Since P is a 0-path, and C is a 0-circuit, it follows that $P + P_1$ and $P + P_2$ are both 0-circuits. But only one can contain f, so the other violates (4.2).

Let t be the largest positive number such that z^* satisfies (4.5). By (4.4) and (4.10), such a positive number exists, and at least one of (4.6) and (4.7) is proper inclusion.

5. Proof of Theorem 3

Theorem 3 follows as a corollary of the following stronger result:

Theorem 4. *If $z \in K(M)$ and $E(G) = \bigcup_{C \in N(z)} E(C)$, then if $v_1, v_2 \in V(G)$ there*

is a pair P_1, P_2 of (v_1, v_2)-paths and a (possibly empty) set of circuits S such that (as incidence vectors)

$$P_1 + P_2 + \sum_{C \in S} C = \sum_{C \in N(z)} M_C. \tag{5.1}$$

Once we have established Theorem 4, we can take the inner product of z with both sides of (5.1). Then, just as in Section 3, since $(C, z) \geq 0$ for all $C \in S$ and $(M_C, z) = 0$ for all $C \in N(z)$, it follows that at least one of the two paths is nonpositive. Hence Theorem 3 holds.

Proof. Consider all subsets $T \subseteq N(z)$ for which v_1 and v_2 are connected in the subgraph $\bar{G}$ with edge set $E(\bar{G}) = \bigcup_{C \in T} E(C)$. Determine the minimum of $|E(\bar{G})|$ over all such subsets T. Now, of those subsets T which achieve this minimum, select one containing the fewest number of circuits, say $\{C_1, \ldots, C_n\}$.

If $n = 1$ then v_1 and v_2 are contained in a common 0-circuit. In this case we choose P_1 and P_2 to be the two (v_1, v_2)-paths determined by this circuit and S to be all of the other 0-circuits of G. Then (5.1) holds, and we are done. So assume that $n > 1$.

From the minimality of n we may assume (after appropriate relabeling if necessary) that

$$v_1 \in V(C_1) \backslash V(C_2), \qquad v_2 \in V(C_n) \backslash V(C_{n-1}), \tag{5.2}$$

and

$V(C_i) \cap V(C_j) \neq \varnothing$ if and only if $|i - j| = 0$ or 1 (and hence no $\quad$ (5.3) vertex of $\bar{G}$ is in more than two of the C_i).

Suppose there is a vertex v of $\bar{G}$ of degree larger than three. Then v must be common to C_i and C_{i+1} for some i, and by (5.2) must in fact be of degree four. Reduce the degree of v by splitting it, introducing a new edge with weight zero common to both C_i and C_{i+1}. This splitting operation cannot create any negative circuit. For suppose C is a circuit in the new graph. If the new edge is shrunk, C will become either one or two nonnegative circuits whose total weight equals that of C. This argument shows also that if C is a 0-circuit in the new graph, the resulting circuits must be 0-circuits in the shrunken graph.

Repeating the splitting procedure with other vertices, if necessary, we may assume

$$\text{every vertex of } \bar{G} \text{ is of degree two or three.} \tag{5.4}$$

As one consequence of the above assumptions, C_i and C_{i+1} share at least one edge, $1 \leq i \leq n - 1$.

Define the graphs

$$G_j = \bigcup_{i=1}^{j} C_i, \qquad j = 1, \ldots, n,$$

$$F_j = \begin{cases} C_j \cap C_{j+1}, & j = 1, \ldots, n-1, \\ \{v_2\}, & j = n, \end{cases}$$

$$D_j = \begin{cases} \varnothing, & j = 0, \\ \bigcup_{i=1}^{j} F_i, & j = 1, \ldots, n. \end{cases}$$

Let $G_j^{(e)}$ be the graph obtained from G_j by duplicating the edges of G_j that are in $D_{j-1}, j = 1, \ldots, n$; let $G_j^{(d)}$ be the subgraph of G_j with edge set $E(G_j) \backslash E(D_{j-1})$, $j = 1, \ldots, n$; and let B_j be the component of $G_j^{(d)}$ that contains $v_1, j = 1, \ldots, n$. Note that $G_j^{(e)}$ is Eulerian and that $G_j^{(d)}$ is a vertex-disjoint union of circuits. Observe also that $z(G_j^{(e)}) = \sum_{i=1}^{j} z(C_i) = 0$.

Now let $x_0 = v_1$ and x_1 be the last vertex of F_1 encountered while traversing C_1 from v_1 to v_1. Define R_1 to be the (x_0, x_1)-path whose edges are contained in $E(B_1) \backslash E(C_2) = E(C_1) \backslash E(C_2)$, and H_1 to be the other (x_0, x_1)-path contained in C_1. Then the following three conditions hold for $i = 1$:

$$x_i \text{ is a vertex of } F_i \text{ of degree three in } \bar{G}, \tag{5.5}$$

$$R_i \text{ is an } (x_0, x_i)\text{-path whose edges are contained in} \tag{5.6}$$
$$E(B_i) \backslash E(C_{i+1}),$$

and

$$H_i \text{ is a vertex-disjoint union of an } (x_0, x_i)\text{-path and} \tag{5.7}$$
$$\text{(possibly) some circuits, such that } D_i \subseteq H_i \subseteq G_i.$$

Assume (5.5)–(5.7) hold for $i = k$, for some $1 \le k \le n-1$. Note that $R_k \subseteq B_{k+1}$. Define Q_{k+1} to be the (x_0, x_k)-path with edge set $E(B_{k+1}) \backslash E(R_k)$. Now Q_{k+1} contains at least one edge f in $E(C_{k+1}) \backslash E(C_k)$, one of the three edges incident to x_k. Observe that (5.3) implies that f cannot be in C_{k+2}.

There are two cases to consider (see Figs. 2 and 3):

$$F_{k+1} \not\subseteq Q_{k+1}, \tag{5.8}$$

or

$$F_{k+1} \subseteq Q_{k+1}. \tag{5.9}$$

Case (5.8). Consider G'_{k+1}, the mod 2 sum of H_k and Q_{k+1}, a subgraph of G_{k+1}. From (5.8) we have $F_{k+1} \not\subseteq G'_{k+1}$. Note that G'_{k+1} contains f, since Q_{k+1} does and $H_k \subseteq G_k$ so H_k contains no edge in $E(C_{k+1}) \backslash E(C_k)$. Let G''_{k+1} be the subgraph of $G_{k+1}^{(e)}$ with edge set $E(G_{k+1}^{(e)}) \backslash E(G'_{k+1})$, regarding G'_{k+1} as a subgraph of $G_{k+1}^{(e)}$.
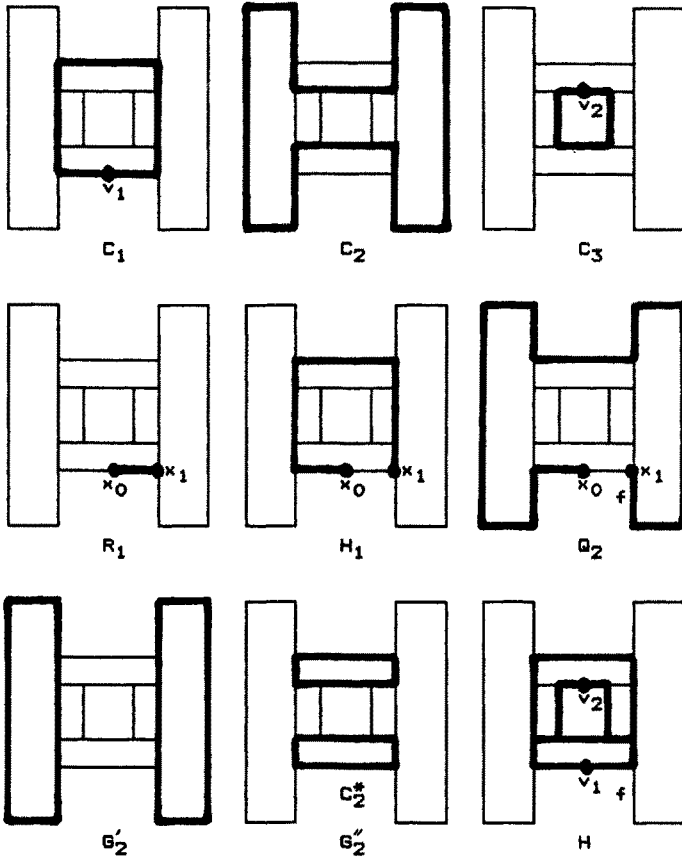
Fig. 2. Case (5.8).

For every duplicated edge in $G_{k+1}^{(e)}$, one edge of the pair is in G_{k+1}', while the other is in G_{k+1}''. This is true because $D_k \subseteq G_{k+1}'$, since $D_k \subseteq H_k$ and no edge of D_k is in B_{k+1}. Also, every vertex of G_{k+1}' is of even degree, as is every vertex of G_{k+1}''. Thus, both G_{k+1}' and G_{k+1}'' are vertex-disjoint unions of circuits of $G_{k+1}^{(e)}$, and in fact of circuits of G_{k+1}.

Then $z(G_{k+1}') \geq 0$, $z(G_{k+1}'') \geq 0$ and $z(G_{k+1}') + z(G_{k+1}'') = z(G_{k+1}^{(e)}) = 0$ force $z(G_{k+1}'') = 0$, and in fact we have G_{k+1}'' written as a vertex-disjoint union of 0-circuits. By (5.8) there is a circuit C_{k+1}^* of G_{k+1}'' that contains a vertex of F_{k+1}. Now C_{k+1}^* also contains at least one edge of C_k, otherwise (5.3) forces $E(C_{k+1}^*) \subseteq E(C_{k+1}) \setminus E(C_k)$, which is impossible. However, we know also that $C_{k+1}^* \neq C_{k+1}$, because C_{k+1}^* cannot contain f.

Consider the graph $H = C_1 \cup \cdots \cup C_k \cup C_{k+1}^* \cup C_{k+2} \cup \cdots \cup C_n$, a proper subgraph of $\bar{G}$ since it does not contain f. The vertices v_1 and v_2 are connected in this graph. Start with $\bar{G}$, take all of the edges that were introduced by splitting vertices near the beginning of the proof, and shrink them. Do the same thing with H. Note that f is contained in only one of the C_i and hence is not one of those edges to be shrunk. The resulting shrunken H will then be a proper subgraph
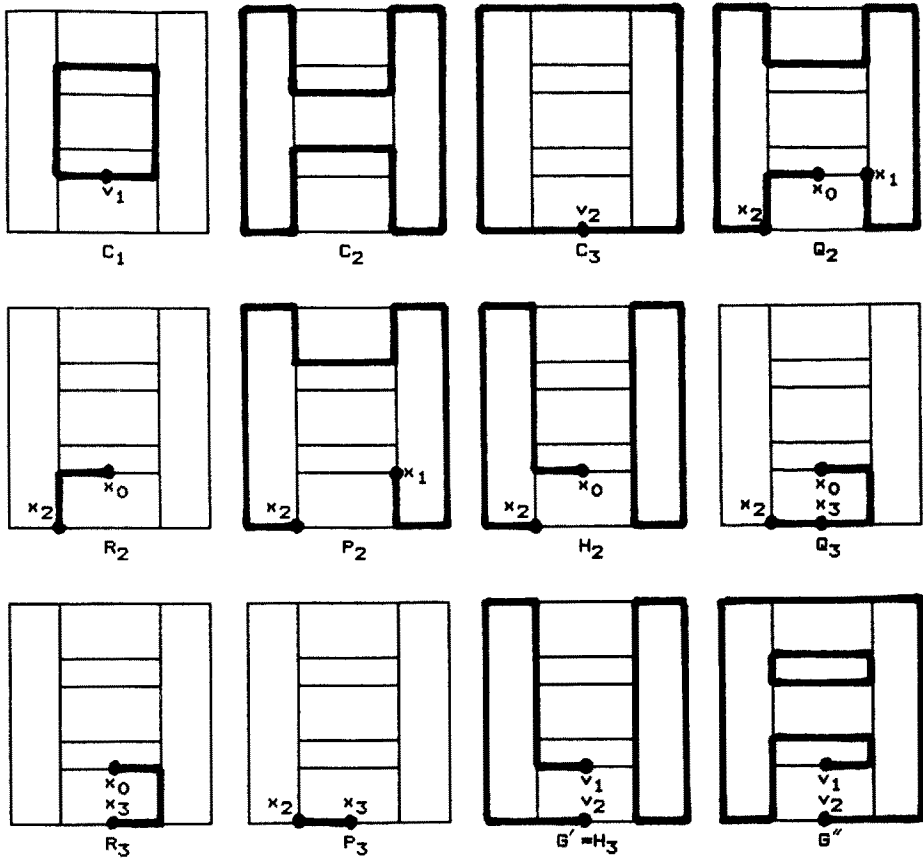
Fig. 3. Case (5.9).

of the shrunken $\bar{G}$. Every edge of H is contained in a 0-circuit of H and v_1, v_2 are connected in H. Yet $|E(H)| < |E(\bar{G})|$, contradicting our initial assumption about the minimality of $|E(\bar{G})|$. Therefore case (5.8) cannot occur.

Case (5.9). Let x_{k+1} be the first vertex of F_{k+1} encountered while traveling from x_0 to x_k in Q_{k+1}. Define R_{k+1} to be the portion of the above path from x_0 to x_{k+1}, P_{k+1} to be the portion of the above path from x_{k+1} to x_k, and H_{k+1} to be the mod 2 sum of H_k and P_{k+1}, a subgraph of G_{k+1}. By (5.9), $F_{k+1} \subseteq P_{k+1}$, implying that $D_{k+1} \subseteq H_{k+1}$, so (5.5)–(5.7) hold for $i = k+1$.

Since case (5.8) is never encountered, we will eventually construct H_n. First note that $x_n = v_2$. Let $G' = H_n$, and G'' be the subgraph of $G_n^{(e)}$ with edge set $E(G_n^{(e)}) \backslash E(G')$, regarding G' as a subgraph of $G_n^{(e)}$. For every duplicated edge in $G_n^{(e)}$, one edge of the pair is in G', while the other is in G'', because $D_n \subseteq H_n$. Also, every vertex of G' is of even degree, except $x_0 = v_1$ and $x_n = v_2$, which are of degree one, and the same is true of G''. Thus G' and G'' each consists of a (v_1, v_2)-path, P' and P'', respectively, and a vertex-disjoint union of circuits of $G_n^{(e)}$, and in fact of circuits of $G_n = \tilde{G}$.

We now have (as incidence vectors) $P' + P'' + \sum_{C \in S'} C + \sum_{C \in S''} C = \sum_{i=1}^{n} C_i$. Shrink those edges that were used to split vertices near the beginning of the proof. The result will be two (v_1, v_2)-paths P_1, P_2 and a set T of circuits such that $P_1 + P_2 + \sum_{C \in T} C = \sum_{i=1}^{n} C_i$ in this shrunken $\bar{G}$. (Note that each path will shrink into a path, together with (possibly) some circuits.) Finally, we arrive at the desired set S of 0-circuits by adding to T all 0-circuits not in $\{C_1, \ldots, C_n\}$. Then (5.1) holds. $\square$

5.1. Examples

In Fig. 2 we have three circuits C_1, C_2, and C_3. Tracing C_1 clockwise from $x_0 = v_1$ to v_1 the last vertex of C_2 we encounter is x_1. The path R_1 is the (x_0, x_1)-path lying only in C_1; H_1 is the other (x_0, x_1)-path. We now construct Q_2 by leaving x_0 in the opposite direction of R_1, staying on edges lying in exactly one of C_1 or C_2, until we arrive at x_1. Observe that F_2, the intersection of C_2 and C_3 is not contained in Q_2 (in fact, it is missed entirely). Thus we are in case (5.8). We let G_2' be the mod 2 sum of H_1 and Q_2, and choose G_2'' to be the graph having all edges that are in $C_1 \cap C_2$ as well as those edges of C_1 and C_2 that are missed by G_2'. Pick a circuit C_2^* of G_2'' that meets F_2 (either will do). Then $H = C_1 \cup C_2^* \cup C_3$ is a union of 0-circuits in which v_1 and v_2 are connected, and H has fewer edges than $\bar{G} = C_1 \cup C_2 \cup C_3$.

If, however, we keep C_1 and C_2 as in Fig. 2 but alter C_3 and v_2 to get Fig. 3, we will construct the same R_1, H_1, and Q_2 as before. But now F_2 is completely contained in Q_2 so we proceed as in case (5.9). Following Q_2 from x_0 to x_1, the first vertex of C_3 we encounter is x_2. The part of Q_2 between x_0 and x_2 is R_2; the rest of Q_2 is P_2. We set H_2 to be the mod 2 sum of H_1 and P_2. Construct Q_3 by leaving x_0 in the opposite direction of R_2, staying on edges lying in exactly one of C_1, C_2 or C_3, until we reach x_2. Note that F_3, the point v_2 itself which we now call x_3, is contained in Q_3. So again we are in case (5.9). The part of Q_3 between x_0 and x_3 is R_3; the rest of Q_3 is P_3. The mod 2 sum of H_2 and P_3 is H_3, which is also G'. We get G'' by choosing all edges of $C_1 \cup C_2 \cup C_3$ that lie in exactly two of these three circuits, together with all other edges missed by G'. Each of G' and G'' has a (v_1, v_2)-path, and at least one of these must be nonpositive.

6. Remarks

One of the referees kindly provided the following information: A. Sebö (Budapest) independently arrived at Theorem 3 through his work on t-joins, which are closely related to sums of circuits. Among the results he obtained is that under the conditions of Theorem 3 (or the slightly weaker condition that every positive edge is in a 0-circuit and all circuits are nonnegative) the distance $d(x, y)$ between any two vertices x, y is nonpositive, the relation $x \sim y$ if and only if $d(x, y) = 0$

is an equivalence relation, and that properties of the equivalence classes can be used to obtain a Kotzig-type theorem for t-joins.

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