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On the Congruence Subgroup Problem: Determination of the "Metaplectic Kernel"

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Introduction

Let F be a global field (i.e. either a number field or a function field in one variable over a finite field) and let ∞ be the set of its archimedean places. Let $\mathfrak{S}$ be a finite non-empty set of places of F containing ∞. Let $\mathfrak{o}=\mathfrak{o}(\mathfrak{S})$ denote the ring of $\mathfrak{S}$-integers of F. Let $A(\mathfrak{S})$ denote the ring of $\mathfrak{S}$-adeles i.e. the restricted direct product of the completions F_v for $v \notin \mathfrak{S}$. Let $\mathscr{G}$ be an absolutely simple*, simply connected subgroup of SL_n defined over F. Recall that a subgroup Γ of $\mathscr{G}(F)$ is an $\mathfrak{S}$-*arithmetic* subgroup if $\Gamma \cap SL(n, \mathfrak{o})$ has finite index in Γ as well as in $\mathscr{G}(\mathfrak{o}) := \mathscr{G}(F) \cap SL(n, \mathfrak{o})$. An $\mathfrak{S}$-arithmetic subgroup Γ is a $\mathfrak{S}$-*congruence* subgroup if there exists a non-zero ideal $\mathfrak{a}$ in $\mathfrak{o}$ $(=\mathfrak{o}(\mathfrak{S}))$ such that

$$\Gamma \supset \{x \in \mathscr{G}(\mathfrak{o}) \mid x \equiv 1 \pmod{\mathfrak{a}}\}.$$

The family of $\mathfrak{S}$-arithmetic (resp. $\mathfrak{S}$-congruence) subgroups is a fundamental system of neighbourhoods of the identity for a topological group structure on $\mathscr{G}(F)$. We denote the respective completions by $\hat{\mathscr{G}}(\mathfrak{S})$ and $\bar{\mathscr{G}}(\mathfrak{S})$. There is evidently a (surjective) homomorphism

$$\hat{\mathscr{G}}(\mathfrak{S}) \to \bar{\mathscr{G}}(\mathfrak{S}).$$

The kernel, denoted $C(\mathfrak{S}, \mathscr{G})$ in the sequel, will be referred to as the *congruence subgroup kernel*. The determination of this kernel is the congruence subgroup problem. Long before this formulation (due to J-P. Serre) of the problem, R. Fricke and F. Klein, had exhibited examples of non-congruence subgroups in $SL_2(\mathbf{Z})$. Later T. Kubota showed that (for $\mathfrak{S}=\infty$) in SL_2 over a totally imaginary number field again non-congruence subgroups exist. The first computation of $C(\mathfrak{S}, \mathscr{G})$ was carried out by Mennicke and Bass-Lazard-Serre, independently, for $\mathscr{G}=SL_n/\mathbf{Q}$, $n \geqq 3$ and $\mathfrak{S}=\infty$; in this case $C(\mathfrak{S}, \mathscr{G})$ is trivial (or, equivalently, every arithmetic subgroup is a congruence subgroup). Later

* A connected algebraic group is said to be absolutely simple if it is semi-simple and it contains no proper connected normal subgroup defined over the algebraic closure of F

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for the split groups SL_n and Sp_{2n}, $C(\mathfrak{S},\mathcal{G})$ was determined by Bass-Milnor-Serre [3]. Matsumoto [14] then extended these results to cover all split groups using the work of Calvin Moore [20]. Deodhar [8] handles the case of quasi-split groups (using the centrality of $C(\mathfrak{S},\mathcal{G})$ proved in Raghunathan [24]).

The general strategy for the determination of $C(\mathfrak{S},\mathcal{G})$ has been the following. One first shows that $C(\mathfrak{S},\mathcal{G})$ is central in $\hat{\mathcal{G}}(\mathfrak{S})$. Once this is done, $\mathrm{Hom}(C(\mathfrak{S},\mathcal{G}),\mathbf{R}/\mathbf{Z})$ is shown (cf. Theorem 2.9 below) to be isomorphic to the *metaplectic kernel* $M(\mathfrak{S},\mathcal{G})$ viz. to the kernel of the restriction homomorphism:

$$H^2(\mathcal{G}(A(\mathfrak{S})),\mathbf{R}/\mathbf{Z})\to H^2(\mathcal{G}(F),\mathbf{R}/\mathbf{Z}),$$

where $\mathcal{G}(A(\mathfrak{S}))$ is equipped with the usual adelic topology and $H^2(\mathcal{G}(A(\mathfrak{S})),\mathbf{R}/\mathbf{Z})$ is the second cohomology of $\mathcal{G}(A(\mathfrak{S}))$, with coefficients in $\mathbf{R}/\mathbf{Z}$, defined in terms of Borel measurable cochains, while $H^2(\mathcal{G}(F),\mathbf{R}/\mathbf{Z})$ is the usual second cohomology (with coefficients in $\mathbf{R}/\mathbf{Z}$) of the abstract group $\mathcal{G}(F)$.

The centrality of $C(\mathfrak{S},\mathcal{G})$ is known to hold under the following conditions:

$$\mathcal{G} \text{ isotropic over } F \quad \text{and} \quad \sum_{v\in\mathfrak{S}} F_v\text{-rank}\,\mathcal{G}\geqq 2.$$

This was proved by Bass-Milnor-Serre for SL_n and SP_{2n}, by Serre for SL_2, by Matsumoto for all split groups, by Vaserstein for classical groups and by Raghunathan for all isotropic groups.

Kneser [13] has recently proved the centrality of $C(\mathfrak{S},\mathcal{G})$ for Spin groups of *anisotropic* quadratic forms over number fields.

The precise determination of $C(\mathfrak{S},\mathcal{G})$ has been carried hitherto only for the split groups (Bass-Milnor-Serre, C. Moore and Matsumoto) and for quasi-split groups (Deodhar). Raghunathan ([24]) showed that $C(\mathfrak{S},\mathcal{G})$ is finite if F-rank $\mathcal{G}\geqq 2$, for *F a number field.* In the present work we prove the finiteness by an entirely different argument which covers also the groups over function fields. The main thrust of the paper, however, is a precise determination of the metaplectic kernel $M(\mathfrak{S},\mathcal{G})$ (which, as we shall see, is closely related to the congruence subgroup kernel). One of the main results we prove is the following for *F-isotropic $\mathcal{G}$:*

$M(\mathfrak{S},\mathcal{G})$ is trivial if $\mathfrak{S}$ contains a nonarchimedean place; otherwise $M(\mathfrak{S},\mathcal{G})$ is isomorphic to a subgroup of the Pontrjagin dual $\hat{\mu}(F)=\mathrm{Hom}(\mu(F),\mathbf{R}/\mathbf{Z})$ of the group $\mu(F)$ of roots of unity in F.

(Actually our main theorem (Theorem 3.4) gives a more precise information in case $\mathfrak{S}$ contains a real place, but the formulation requires notation introduced in the main body of the paper.)

This work makes crucial use of the results we have obtained in [23] on topological central extensions of semi-simple groups over local fields and the results of Moore [20] on SL_2.

The results of this paper formed the topic of a course given by the first named author in the Summer of 1980 at the Universität Bielefeld, and he would like to thank A. Bak and Ulf Rehmann for their hospitality. At that time he learnt from Bak and Rehmann that they have independently and almost simultaneously computed $C(\mathfrak{S},\mathcal{G})$ for $\mathcal{G}=SL_{n,D}$, $n\geqq 3$ and D a central division algebra over a global field, using K-theoretic techniques (see [2]). Subsequently, Bak [1] announced (without proof) similar results for classical groups of F-rank $\geqq 2$, based on his work [2] with Rehmann.

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§1. A Review of the Local Results

1.1. Let k be a local (i.e., locally compact, nondiscrete) field. Let G be an absolutely simple, simply connected group defined and isotropic over k. We shall let $G(k)$ denote the group of k-rational points of G with the locally compact topology induced by the topology on k.

For a topological group H, $H^2(H, \mathbf{R}/\mathbf{Z})$ and $H^2(H, \mathbf{Q}/\mathbf{Z})$ will denote the second cohomology groups of H, defined in terms of the Borel measurable cochains, with coefficients in the trivial H-module $\mathbf{R}/\mathbf{Z}$ and $\mathbf{Q}/\mathbf{Z}$ respectively; here, as well as in the sequel, $\mathbf{R}/\mathbf{Z}$ is assumed to carry the usual compact topology, and $\mathbf{Q}/\mathbf{Z}$ the discrete topology. If on a group H no topology is prescribed, then for the purpose of defining $H^2(H, \mathbf{R}/\mathbf{Z})$ and $H^2(H, \mathbf{Q}/\mathbf{Z})$, we shall assume H endowed with the discrete topology.

We note that according to a result of D. Wigner ([30: Theorem 1]) if H is zero-dimensional, in particular for the group of rational points of an algebraic group over a nonarchimedean field, the cohomology groups based on continuous cochains and the cohomology groups defined in terms of Borel measurable cochains are identical.

In case H is a real analytic semisimple group with fundamental group $\pi_1(H)$, then it is well known that $H^2(H, \mathbf{R}/\mathbf{Z}) \cong \mathrm{Hom}(\pi_1(H), \mathbf{R}/\mathbf{Z})$.

Let Φ be the (k-)root system of G with respect to a maximal k-split torus, and let d be the dominant (or the highest) root with respect to a fixed ordering on Φ. Let G_{d} be the subgroup of G generated by the root subgroups U_{d} and $U_{-\mathsf{d}}$. Then G_{d} is an absolutely simple, simply connected k-subgroup of G of k-rank 1 (see §§ 3.1 and 3.2 below). Let a be a *long* k-root i.e. a k-root of length equal to that of d. Then a is conjugate to d under an element of the k-Weyl group. Let G_a be the subgroup of G generated by the root subgroups U_a and U_{-a}. Then G_a is conjugate to G_{d} under an element of $G(k)$.

1.2. **Theorem.** *The restriction homomorphism $H^2(G(k), \mathbf{R}/\mathbf{Z}) \to H^2(G_{\mathsf{d}}(k), \mathbf{R}/\mathbf{Z})$ is injective. Therefore, the restriction $H^2(G(k), \mathbf{R}/\mathbf{Z}) \to H^2(G_a(k), \mathbf{R}/\mathbf{Z})$ is injective for any long k-root a.*

In case k is nonarchimedean, this theorem is Theorem 9.5 of Prasad-Raghunathan [23]. On the other hand, if $k=\mathbf{R}$, then it is known (and can be proved, for example, by a case-by-case check using classification) that the map $\pi_1(G_{\mathsf{d}}(\mathbf{R})) \to \pi_1(G(\mathbf{R}))$, induced by the inclusion $G_{\mathsf{d}}(\mathbf{R}) \hookrightarrow G(\mathbf{R})$, is surjective. Now since $H^2(G(\mathbf{R}), \mathbf{R}/\mathbf{Z}) = \mathrm{Hom}(\pi_1(G(\mathbf{R})), \mathbf{R}/\mathbf{Z})$ and $H^2(G_{\mathsf{d}}(\mathbf{R}), \mathbf{R}/\mathbf{Z}) = \mathrm{Hom}(\pi_1(G_{\mathsf{d}}(\mathbf{R})), \mathbf{R}/\mathbf{Z})$, the assertion of the theorem follows. If $k=\mathbf{C}$, then both $G(\mathbf{C})$ and $G_{\mathsf{d}}(\mathbf{C})$ are simply connected and hence, $H^2(G(\mathbf{C}), \mathbf{R}/\mathbf{Z})$ as well as $H^2(G_{\mathsf{d}}(\mathbf{C}), \mathbf{R}/\mathbf{Z})$ is trivial.

We shall now in the rest of this section assume that k is nonarchimedean.

In the sequel $\hat{\mu}(k)$ will denote the Pontrjagin dual $\mathrm{Hom}(\mu(k), \mathbf{R}/\mathbf{Z})$ of the group $\mu(k)$ of roots of unity in k. Both $\mu(k)$ and $\hat{\mu}(k)$ are finite cyclic groups.

The following theorem was proved by C. Moore ([20]) for groups which split over k and then later by V. Deodhar ([8]) for quasi-split groups.

1.3. **Theorem.** *Let G be either split or quasi-split over k. Then $H^2(G(k), \mathbf{R}/\mathbf{Z})$ is isomorphic to a subgroup of $\hat{\mu}(k)$.*

1.4. *Remark.* If G splits over k, then the above theorem combined with a result of H. Matsumoto [14: Théorème 12.1] gives that $H^2(G(k), \mathbf{R}/\mathbf{Z})$ is actually isomorphic to $\hat{\mu}(k)$. The same holds also for quasi-split groups in view of an unpublished observation of P. Deligne (see Prasad-Raghunathan [23: § 5.10]).

The following theorem and two propositions (1.5, 1.6 and 1.8) are proved in Prasad-Raghunathan [23: §§ 9, 10].

1.5. **Theorem.** *Let G be an absolutely simple, simply connected group defined and isotropic over k. Then $G(k)$ admits a universal topological central extension and its topological fundamental group $\pi_1(G(k))$ is isomorphic to a quotient of $\mu(k)$. Moreover, $H^2(G(k), \mathbf{R}/\mathbf{Z})$ is isomorphic to $\mathrm{Hom}(\pi_1(G(k)), \mathbf{R}/\mathbf{Z})$.*

1.6. **Proposition.** *Let $G = SL_{n,D}$, where D is a central division algebra over k. Let K be an unramified extension of k of degree = degree $D/k = \sqrt{[D:k]}$ contained in D. Let H be the k-subgroup $SL_{n,\mathsf{K}}$ of $SL_{n,D}$ (H is k-isomorphic to $R_{\mathsf{K}/k}(SL_n)$). Then $G(k) = SL_n(D)$, $H(k) = SL_n(\mathsf{K})$, and the restriction*

$$H^2(SL_n(D), \mathbf{R}/\mathbf{Z}) \to H^2(SL_n(\mathsf{K}), \mathbf{R}/\mathbf{Z}) \quad \textit{is injective.}$$

1.7. Let D be a quaternion division algebra over k. Let K be a fixed unramified quadratic extension of k contained in D. Let $x \mapsto \bar{x} = \mathrm{Trd}\, x - x$ be the standard involution of D. Let π be an element of D such that for $x \in \mathsf{K}$, $\pi x \pi^{-1} = \bar{x}$. Then since $\pi \notin \mathsf{K}$ and π^2 commutes with K, we conclude that $\pi^2 \in k$ and hence, $\bar{\pi} = -\pi$. For $x \in D$, let $\sigma(x) = \pi \bar{x} \pi^{-1}$. Then σ is an involution of D; the space D^σ of elements fixed under σ is of dimension 3 and it contains K. Let $X = e_{-1} \cdot D + e_1 \cdot D$ be a right vector space over D of dimension 2 and let φ be the hyperbolic σ-antihermitian form on X defined by:

$$\varphi(e_{-1}, e_{-1}) = 0 = \varphi(e_1, e_1),$$
$$\varphi(e_{-1}, e_1) = 1 = -\varphi(e_1, e_{-1}).$$

Let $G = SU(\varphi)$. Then G is the simply connected k-rank 1 form of type C_2. We use the basis $\{e_{-1}, e_1\}$ to identify G with a k-subgroup of $SL_{2,D}$. It is easily checked that $SL_2(\mathsf{K}) \subset G(k) (\subset SL_2(D))$. Let H be the k-subgroup of G such that

$$H(k) = SL_2(\mathsf{K}) (\subset G(k)).$$

Then H is k-isomorphic to $R_{\mathsf{K}/k}(SL_2)$. Now with these notation we have:

1.8. **Proposition.** *The restriction homomorphism $H^2(G(k), \mathbf{R}/\mathbf{Z}) \to H^2(H(k), \mathbf{R}/\mathbf{Z})$ is injective.*

1.9. **Proposition.** *Let k be a nonarchimedean local field of characteristic zero. Let Q be a quadratic space over k and R be a nondegenerate subspace. We assume that the Witt index of R is at least 2. Let G and H be the spin groups associated with Q and R respectively. We identify H with a k-subgroup of G. Then $H^2(G(k), \mathbf{R}/\mathbf{Z})$ is isomorphic to $\hat{\mu}(k)$ and the restriction homomorphism $H^2(G(k), \mathbf{R}/\mathbf{Z}) \to H^2(H(k), \mathbf{R}/\mathbf{Z})$ is injective.*

Proof. It is obvious that to prove the proposition we may (and we shall) replace R by a quadratic subspace which is isomorphic to the orthogonal direct sum of two hyperbolic planes. Now let $R^{\perp}$ be the orthogonal complement of R in Q, and let V be the orthogonal direct sum $R \oplus R^{\perp} \oplus -R^{\perp}$; where $-R^{\perp}$ is the quadratic space with same underlying k-vector space as $R^{\perp}$ but whose quadratic form is the negative of the quadratic form on $R^{\perp}$. Then V is hyperbolic (i.e. it is the orthogonal direct sum of hyperbolic planes), and so the associated spin group $\mathscr{G}$ is a k-split form of type $\boldsymbol{D}$. It is easy to see that in the root system of $\mathscr{G}$, with respect to a suitable maximal k-split torus, there is a root such that the corresponding SL_2 is contained in H. Hence, according to a result of Moore (i.e. Theorem 1.2 for k-split groups), the restriction

$$H^2(\mathscr{G}(k), \mathbf{R}/\mathbf{Z}) \to H^2(H(k), \mathbf{R}/\mathbf{Z})$$

is injective. We also know (from 1.3, 1.4 above) that $H^2(\mathscr{G}(k), \mathbf{R}/\mathbf{Z})$ is isomorphic to (the finite group) $\hat{\mu}(k)$, whereas $H^2(G(k), \mathbf{R}/\mathbf{Z})$ is isomorphic to a subgroup of $\hat{\mu}(k)$ (Theorem 1.5). Therefore, from the following commutative triangle (in which all the maps are natural restrictions):

$$\begin{array}{ccc} H^2(\mathscr{G}(k), \mathbf{R}/\mathbf{Z}) & \longrightarrow & H^2(H(k), \mathbf{R}/\mathbf{Z}) \\ & \searrow \quad \nearrow & \\ & H^2(G(k), \mathbf{R}/\mathbf{Z}) & \end{array}$$

we conclude that $H^2(\mathscr{G}(k), \mathbf{R}/\mathbf{Z}) \to H^2(G(k), \mathbf{R}/\mathbf{Z})$ is an isomorphism; hence, we conclude that $H^2(G(k), \mathbf{R}/\mathbf{Z})$ is isomorphic to $\hat{\mu}(k)$ and the restriction $H^2(G(k), \mathbf{R}/\mathbf{Z}) \to H^2(H(k), \mathbf{R}/\mathbf{Z})$ is injective. This proves the proposition.

§ 2. Finiteness of the Metaplectic Kernel

2.1. Let F be a global field, $\mathfrak{o}$ its ring of integers. For a place v of F, F_v will denote the completion of F at v, and in case v is nonarchimedean, $\mathfrak{o}_v$ will denote the ring of integers of F_v, $\mathfrak{p}_v$ the unique maximum ideal of $\mathfrak{o}_v$, $\mathfrak{f}_v$ the residue field $\mathfrak{o}_v/\mathfrak{p}_x$, and p_v the characteristic of $\mathfrak{f}_v$.

A will denote the F-algebra of adeles of F, and for a finite set $\mathfrak{S}$ of places of F, $A(\mathfrak{S})$ will denote the F-algebra of $\mathfrak{S}$-adeles i.e. the restricted direct product of the completions F_v for $v \notin \mathfrak{S}$.

Let $\mathscr{G}$ be a connected, absolutely simple, simply connected F-subgroup of $SL_{n,F}$. In the following, we shall denote the schematic closure of $\mathscr{G}$ in the standard special linear $\mathfrak{o}$-group scheme $\mathscr{SL}_{n,\mathfrak{o}}$ again by $\mathscr{G}$.

$\mathscr{G}(A)$ (resp. $\mathscr{G}(A(\mathfrak{S}))$) is obviously a restricted direct product of $\mathscr{G}(F_v)$, v varying over all the places of F (resp. all $v \notin \mathfrak{S}$), with respect to the compact-open subgroups $\mathsf{K}_v = \mathscr{G}(\mathfrak{o}_v)$.

2.2. It is well known that for almost all places v, $\mathscr{G}$ is quasi-split over F_v, and $\mathscr{G} \otimes_{\mathfrak{o}} \mathfrak{o}_v$ reduces, mod $\mathfrak{p}_v$, to a smooth simply connected semi-simple group scheme $\mathsf{G}_v (= \mathscr{G} \otimes_{\mathfrak{o}} \mathfrak{f}_v)$ over the finite field $\mathfrak{f}_v$. According to the results of Steinberg [26] and Griess [10], the Schur multipliers of $\mathsf{G}_v(\mathfrak{f}_v)$ is a p_v-group (for a generators-and-relations-free proof of this result see Prasad [22]). Now as the homomorphism $\mathscr{G}(\mathfrak{o}_v) \to \mathsf{G}_v(\mathfrak{f}_v)$ is surjective for almost all v, and its kernel is a pro-p_v group, we conclude from the Hochschild-Serre spectral sequence that (for almost all v) every element of $H^2(\mathscr{G}(\mathfrak{o}_v), \mathbf{R}/\mathbf{Z})$ is of order a finite power of p_v. On the other hand, since F is absolutely unramified at almost all places, for almost all v, the completion F_v contains no nontrivial p_v-th root of unity and hence the finite group $H^2(\mathscr{G}(F_v), \mathbf{R}/\mathbf{Z})$ has no p_v-torsion (cf. 1.5). Therefore, the restriction homomorphism:

$$H^2(\mathscr{G}(F_v), \mathbf{R}/\mathbf{Z}) \to H^2(\mathscr{G}(\mathfrak{o}_v), \mathbf{R}/\mathbf{Z})$$

is trivial for almost all v.

2.3. We shall now show that for almost all v, $\mathsf{K}_v = \mathscr{G}(\mathfrak{o}_v)$ is perfect, i.e., $\mathsf{K}_v = (\mathsf{K}_v, \mathsf{K}_v)$. For this purpose, let $\mathfrak{S}$ be a finite set of places of F containing all the archimedean places, and all the nonarchimedean places v such that $\mathscr{G}$ is anisotropic over F_v, and such that $\sum_{v \in \mathfrak{S}} F_v\text{-rank}\,\mathscr{G} \geqq 2$. Let $\mathsf{K} = \prod_{v \notin \mathfrak{S}} \mathsf{K}_v$. Then K is a compact-open subgroup of $\mathscr{G}(A(\mathfrak{S}))$. As $\mathscr{G}$ is simply connected, for all v, the commutator subgroup $(\mathsf{K}_v, \mathsf{K}_v)$ is an open (and hence also closed) subgroup of K_v (Borel-Tits [5: 9.4(iii)]).

Now let $\Gamma = \mathscr{G}(F) \cap \mathsf{K}$. Then Γ is an $\mathfrak{S}$-arithmetic (in fact, an $\mathfrak{S}$-congruence) subgroup. Since $\mathscr{G}(F)$ is dense in $\mathscr{G}(A(\mathfrak{S}))$ ("Strong approximation", see Prasad [21: Theorem A] or Margulis [15: Theorem 4]), and K is an open subgroup of $\mathscr{G}(A(\mathfrak{S}))$, Γ is a dense subgroup of K. Since $\sum_{v \in \mathfrak{S}} F_v\text{-rank}\,\mathscr{G} \geqq 2$, according to a theorem of Margulis ([16, 17]), which completes the earlier results of Kazhdan [9] and Bernstein-Kazhdan, the commutator subgroup (Γ, Γ) is of finite index in Γ. Hence $(\mathsf{K}, \mathsf{K}) = \prod_{v \notin \mathfrak{S}} (\mathsf{K}_v, \mathsf{K}_v)$, which is a closed subgroup of K containing (Γ, Γ), is of finite index in K. This proves that for almost all v, $(\mathsf{K}_v, \mathsf{K}_v) = \mathsf{K}_v$, for otherwise $(\mathsf{K}, \mathsf{K}) = \prod_{v \notin \mathfrak{S}} (\mathsf{K}_v, \mathsf{K}_v)$ would be a subgroup of K of infinite index.

Let $\mathfrak{S}$ be an arbitrary finite set of places of F. Since $(\mathsf{K}_v, \mathsf{K}_v)$ is an open subgroup of K_v for all v, we now conclude that (K, K) is an open subgroup of K. This implies that the commutator subgroup $(\mathscr{G}(A(\mathfrak{S})), \mathscr{G}(A(\mathfrak{S})))$ is an open and hence closed subgroup of $\mathscr{G}(A(\mathfrak{S}))$. For any place v such that either $\mathscr{G}$ is isotropic over F_v or v is archimedean, $\mathscr{G}(F_v)$ is perfect ([23: § 6.15]), therefore if for every nonarchimedean place $v \notin \mathfrak{S}$, $\mathscr{G}$ is isotropic over F_v, then $\mathscr{G}(A(\mathfrak{S}))$ is perfect.

Now in view of the observations in 2.2 and 2.3, we conclude from Theorem 1.5 and a theorem of Moore ([20: Theorem 12.1]) the following at once:

2.4. **Theorem.** *Let $\mathfrak{S}$ be a finite set of places of F. Assume that for every nonarchimedean place $v \notin \mathfrak{S}$ of F, $\mathcal{G}$ is isotropic over F_v. Then $\mathcal{G}(A(\mathfrak{S}))$ admits a universal topological central extension, the topological fundamental group $\pi_1(\mathcal{G}(A(\mathfrak{S})))$ of $\mathcal{G}(A(\mathfrak{S}))$ is discrete and is isomorphic to the direct sum of the $\pi_1(\mathcal{G}(F_v))$, $v \notin \mathfrak{S}$; where $\pi_1(\mathcal{G}(F_v))$ is the topological fundamental group of $\mathcal{G}(F_v)$.*

As a consequence, $H^2(\mathcal{G}(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z})$ is isomorphic to the direct product of the $\mathrm{Hom}_{\mathbf{Z}}(\pi_1(\mathcal{G}(F_v)), \mathbf{R}/\mathbf{Z}) = H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z})$, $v \notin \mathfrak{S}$.

2.5. Now let $\mathfrak{S}$ be a finite set of places of F containing all the archimedean places. On $\mathcal{G}(F)$ we introduce two topologies: *The $\mathfrak{S}$-congruence subgroup topology* in which the family of $\mathfrak{S}$-congruence subgroups form a fundamental system of neighbourhoods of the identity; the completion of $\mathcal{G}(F)$ with respect to this topology shall be denoted by $\bar{\mathcal{G}}(\mathfrak{S})$. The other topology on $\mathcal{G}(F)$ is the $\mathfrak{S}$-arithmetic subgroup topology in which the family of $\mathfrak{S}$-arithmetic subgroups form a fundamental system of neighbourhoods of the identity; the completion of $\mathcal{G}(F)$ with respect to this topology shall be denoted by $\hat{\mathcal{G}}(\mathfrak{S})$. Since every $\mathfrak{S}$-congruence subgroup is $\mathfrak{S}$-arithmetic, the second topology is finer than the first, and hence there is a continuous surjective homomorphism $\hat{\mathcal{G}}(\mathfrak{S}) \to \bar{\mathcal{G}}(\mathfrak{S})$. The kernel of this homomorphism is called the *$\mathfrak{S}$-congruence subgroup kernel* and shall be denoted by $C(\mathfrak{S}, \mathcal{G})$. It is known that $C(\mathfrak{S}, \mathcal{G})$ is a second countable profinite (and hence compact) group; see Raghunathan [24]. The congruence subgroup problem is the problem of determination of $C(\mathfrak{S}, \mathcal{G})$ for a given $\mathfrak{S}$ and $\mathcal{G}$.

Let Z be the closure in $C(\mathfrak{S}, \mathcal{G})$ of the subgroup $(\mathcal{G}(F), C(\mathfrak{S}, \mathcal{G}))$ generated by $\{x y x^{-1} y^{-1} | x \in \mathcal{G}(F), y \in C(\mathfrak{S}, \mathcal{G})\}$. Then since $\mathcal{G}(F)$ is dense in $\hat{\mathcal{G}}(\mathfrak{S})$, Z is a closed normal subgroup of $\hat{\mathcal{G}}(\mathfrak{S})$. Let $\mathcal{C}(\mathfrak{S}, \mathcal{G}) = C(\mathfrak{S}, \mathcal{G})/Z$ and $\tilde{\mathcal{G}}(\mathfrak{S}) = \hat{\mathcal{G}}(\mathfrak{S})/Z$. Then clearly $\mathcal{C}(\mathfrak{S}, \mathcal{G})$ is contained in the center of $\tilde{\mathcal{G}}(\mathfrak{S})$.

We shall henceforth assume that $\mathfrak{S}$ contains all the archimedean places of F, and $\prod_{v \in \mathfrak{S}} \mathcal{G}(F_v)$ is non-compact (or, equivalently, $\sum_{v \in \mathfrak{S}} F_v\text{-rank}\, \mathcal{G} \geqq 1$). Then from the strong approximation (Prasad [21: Theorem A] or Margulis [15: Theorem 4]) it is immediate that $\bar{\mathcal{G}}(\mathfrak{S}) = \mathcal{G}(A(\mathfrak{S}))$. Thus we have a central extension:

$$(+) \qquad 1 \to \mathcal{C}(\mathfrak{S}, \mathcal{G}) \to \tilde{\mathcal{G}}(\mathfrak{S}) \to \mathcal{G}(A(\mathfrak{S})) \to 1$$

of $\mathcal{G}(A(\mathfrak{S}))$ by $\mathcal{C}(\mathfrak{S}, \mathcal{G}) = C(\mathfrak{S}, \mathcal{G})/Z$. Moreover, the natural imbedding of $\mathcal{G}(F)$ in $\mathcal{G}(A(\mathfrak{S}))$ and $\hat{\mathcal{G}}(\mathfrak{S})$ gives a splitting of the above extension over $\mathcal{G}(F) (\subset \mathcal{G}(A(\mathfrak{S})))$. We shall view $\mathcal{G}(F)$ as a subgroup of $\tilde{\mathcal{G}}(\mathfrak{S})$ using this splitting.

2.6. **Theorem.** *Assume that for every place $v \notin \mathfrak{S}$ of F, $\mathcal{G}$ is isotropic over F_v, and $\sum_{v \in \mathfrak{S}} F_v\text{-rank}\, \mathcal{G} \geqq 2$. Then $\mathcal{C}(\mathfrak{S}, \mathcal{G})$ is finite.*

Proof. According to Theorem 2.4, $\mathcal{G}(A(\mathfrak{S}))$ admits a universal topological central extension:

$$(*) \qquad 1 \to C \longrightarrow E \xrightarrow{\sigma} \mathcal{G}(A(\mathfrak{S})) \to 1,$$

and its topological fundamental group C is discrete. Now as

$$(+) \qquad 1 \to \mathcal{C}(\mathfrak{S}, \mathcal{G}) \to \tilde{\mathcal{G}}(\mathfrak{S}) \to \mathcal{G}(A(\mathfrak{S})) \to 1$$

is a topological central extension of $\mathscr{G}(A(\mathfrak{S}))$, there exists a homomorphism $\varphi\colon E\to\tilde{\mathscr{G}}(\mathfrak{S})$ which makes the following diagram commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \longrightarrow & E & \xrightarrow{\sigma} & \mathscr{G}(A(\mathfrak{S})) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 1 & \longrightarrow & \mathscr{C}(\mathfrak{S},\mathscr{G}) & \longrightarrow & \tilde{\mathscr{G}}(\mathfrak{S}) & \longrightarrow & \mathscr{G}(A(\mathfrak{S})) & \longrightarrow & 1. \end{array}$$

We claim that $\varphi(E)$ is a closed subgroup of $\tilde{\mathscr{G}}(\mathfrak{S})$ of finite index. Assuming the claim for a moment, it follows that $\varphi(C)$ is a closed subgroup of $\mathscr{C}(\mathfrak{S},\mathscr{G})$ of finite index, and hence (by Aren's lemma, see Bourbaki [7]) $\varphi(C)$, as a topological group, is a quotient of the discrete group C. But as any quotient of a discrete group is discrete and $C(\mathfrak{S},\mathscr{G})$, and so also $\mathscr{C}(\mathfrak{S},\mathscr{G})$, is compact, we conclude that $\varphi(C)$ is finite. Now since $\varphi(C)$ is of finite index in $\mathscr{C}(\mathfrak{S},\mathscr{G})$, $\mathscr{C}(\mathfrak{S},\mathscr{G})$ is also finite.

We shall now prove the claim. As C is discrete, E is locally isomorphic to $\mathscr{G}(A(\mathfrak{S}))$. Let U be a compact-open subgroup of E such that σ is injective on U. Then $\sigma(U)$ is a compact-open subgroup of $\mathscr{G}(A(\mathfrak{S}))$. Let $\Gamma=\mathscr{G}(F)\cap\sigma(U)$. Then Γ is an $\mathfrak{S}$-arithmetic (in fact, a $\mathfrak{S}$-congruence) subgroup of $\mathscr{G}(F)$. The restriction of σ to U is an isomorphism on to $\sigma(U)$, and hence its inverse gives a splitting of $(*)$ over $\sigma(U)$, and hence a splitting s on $\Gamma(\subset\sigma(U))$. The composite $\varphi\cdot s$ may be viewed as another splitting of $(+)$ over Γ. Now since the splitting $\varphi\cdot s$ of $(+)$ coincides on the commutator subgroup (Γ,Γ) with the restriction of the natural splitting (see 2.5) of $(+)$ on $\mathscr{G}(F)$, we conclude that $\varphi(U)$ contains (Γ,Γ). But $\varphi(U)$ is a compact subgroup of $\tilde{\mathscr{G}}(\mathfrak{S})$, so it contains also the closure of (Γ,Γ).

According to our hypothesis, $\sum_{v\in\mathfrak{S}} F_v\text{-rank}\,\mathscr{G}\geqq 2$, so a theorem of Margulis ([16, 17]) implies that the commutator subgroup (Γ,Γ) is of finite index in Γ. Therefore, (Γ,Γ) is an $\mathfrak{S}$-arithmetic subgroup of $\mathscr{G}(F)$. But it is obvious from the description of the $\mathfrak{S}$-arithmetic topology that the closure in $\hat{\mathscr{G}}(\mathfrak{S})$, and so also in $\tilde{\mathscr{G}}(\mathfrak{S})$, of any $\mathfrak{S}$-arithmetic subgroup is open. Hence, $\varphi(U)$ is open in $\tilde{\mathscr{G}}(\mathfrak{S})$, and so $\varphi(E)$ is an open subgroup of $\tilde{\mathscr{G}}(\mathfrak{S})$. In particular, $\varphi(E)$ contains a subgroup of finite index of the compact group $\mathscr{C}(\mathfrak{S},\mathscr{G})$. On the other hand, $\varphi(E)$ obviously projects on to $\mathscr{G}(A(\mathfrak{S}))$, so we conclude that $\varphi(E)$ is an open (and hence also a closed) subgroup of $\tilde{\mathscr{G}}(\mathfrak{S})$ of finite index. This completes the proof of the theorem.

2.7. *Remark.* Under the hypothesis of the preceding theorem, as $\mathscr{C}(\mathfrak{S},\mathscr{G})$ is finite, $\tilde{\mathscr{G}}(\mathfrak{S})$ is locally isomorphic to $\mathscr{G}(A(\mathfrak{S}))$, and therefore (see 2.3), the commutator subgroup $(\tilde{\mathscr{G}}(\mathfrak{S}),\tilde{\mathscr{G}}(\mathfrak{S}))$ is an open and hence also a closed subgroup of $\tilde{\mathscr{G}}(\mathfrak{S})$.

2.8. *Remark.* It is known that if $\mathscr{G}$ is isotropic over F and $\sum_{v\in\mathfrak{S}} F_v\text{-rank}\,\mathscr{G}\geqq 2$, then the congruence subgroup kernel $C(\mathfrak{S},\mathscr{G})$ is central in $\hat{\mathscr{G}}(\mathfrak{S})$ (and hence Z is trivial, $\mathscr{C}(\mathfrak{S},\mathscr{G})=C(\mathfrak{S},\mathscr{G})$ and $\tilde{\mathscr{G}}(\mathfrak{S})=\hat{\mathscr{G}}(\mathfrak{S})$). This was first proved by Bass-Milnor-Serre [3] for the groups SL_n, Sp_{2n}; then by Matsumoto [14] for all F-

split groups and later by Vaserstein for all classical isotropic groups. Raghunathan ([24]) has proved the centrality of $C(\mathfrak{S},\mathscr{G})$ for all groups of F-rank>1, by a uniform argument, and has also proven the finiteness of $C(\mathfrak{S},\mathscr{G})$ if F is a number field ([24]). Recently M. Kneser [13] has proved that $C(\mathfrak{S},\mathscr{G})$ is central for certain F-anisotropic $\mathscr{G}$ too, provided that $\sum_{v\in\mathfrak{S}} F_v\text{-rank}\,\mathscr{G}\geqq 2$.

We note that Raghunathan's proof in [24] of finiteness of $C(\mathfrak{S},\mathscr{G})$, in case F is a number field, is completely different from the proof of Theorem 2.6 given above.

2.9. **Theorem.** *Assume that for every place $v\notin\mathfrak{S}$ of F, $\mathscr{G}$ is isotropic over F_v, and $\sum_{v\in\mathfrak{S}} F_v\text{-rank}\,\mathscr{G}\geqq 2$. Then the transgression from* $\mathrm{Hom}(\mathscr{C}(\mathfrak{S},\mathscr{G}),\mathbf{R}/\mathbf{Z})$ *to the metaplectic kernel* $M(\mathfrak{S},\mathscr{G})=\mathrm{Ker}(H^2(\mathscr{G}(A(\mathfrak{S})),\mathbf{R}/\mathbf{Z})\xrightarrow{\text{rest}} H^2(\mathscr{G}(F),\mathbf{R}/\mathbf{Z}))$, *induced by the central extension:*

$$(+)\qquad 1\to\mathscr{C}(\mathfrak{S},\mathscr{G})\to\tilde{\mathscr{G}}(\mathfrak{S})\to\mathscr{G}(A(\mathfrak{S}))\to 1,$$

(which splits over $\mathscr{G}(F)$), is surjective. If $\mathscr{G}(F)$ is perfect, then the transgression is bijective.

Proof. According to Theorem 2.4 and Theorem 1.5, $\mathscr{G}(A(\mathfrak{S}))$ admits a universal topological central extension and its topological fundamental group $\pi_1(\mathscr{G}(A(\mathfrak{S})))$ is a direct sum of certain finite cyclic groups with discrete topology. Hence, for any locally compact second countable topological $\mathscr{G}(A(\mathfrak{S}))$-module M, with trivial $\mathscr{G}(A(\mathfrak{S}))$ action on M, the transgression map:

$$\mathrm{Hom}(\pi_1(\mathscr{G}(A(\mathfrak{S}))),M)\to H^2(\mathscr{G}(A(\mathfrak{S})),M)$$

is an isomorphism.

Since $\pi_1(\mathscr{G}(A(\mathfrak{S})))$ is a torsion group, the natural inclusion

$$\mathrm{Hom}(\pi_1(\mathscr{G}(A(\mathfrak{S}))),\mathbf{Q}/\mathbf{Z})\to\mathrm{Hom}(\pi_1(\mathscr{G}(A(\mathfrak{S}))),\mathbf{R}/\mathbf{Z})$$

is an isomorphism, and hence $H^2(\mathscr{G}(A(\mathfrak{S})),\mathbf{Q}/\mathbf{Z})\to H^2(\mathscr{G}(A(\mathfrak{S})),\mathbf{R}/\mathbf{Z})$ is an isomorphism. On the other hand, since $(\mathscr{G}(F),\mathscr{G}(F))$ is of finite index in $\mathscr{G}(F)$ (Prasad [21: Theorem C]), $H^1(\mathscr{G}(F),\mathbf{R}/\mathbf{Q})=\mathrm{Hom}(\mathscr{G}(F),\mathbf{R}/\mathbf{Q})=\{0\}$, and so $H^2(\mathscr{G}(F),\mathbf{Q}/\mathbf{Z})\to H^2(\mathscr{G}(F),\mathbf{R}/\mathbf{Z})$ is injective. Therefore, the metaplectic kernel $M(\mathfrak{S},\mathscr{G})$ is isomorphic to

$$\mathrm{Ker}(H^2(\mathscr{G}(A(\mathfrak{S})),\mathbf{Q}/\mathbf{Z})\xrightarrow{\text{rest}} H^2(\mathscr{G}(F),\mathbf{Q}/\mathbf{Z})).$$

Thus to prove the theorem, we only need to prove that the transgression from $\mathrm{Hom}(\mathscr{C}(\mathfrak{S},\mathscr{G}),\mathbf{Q}/\mathbf{Z})$ to

$$\mathrm{Ker}(H^2(\mathscr{G}(A(\mathfrak{S})),\mathbf{Q}/\mathbf{Z})\xrightarrow{\text{rest}} H^2(\mathscr{G}(F),\mathbf{Q}/\mathbf{Z}))$$

is surjective. For this, we shall adopt an argument of Bass-Milnor-Serre [3: §15].

Let $x \in \operatorname{Ker}(H^2(\mathscr{G}(A(\mathfrak{S})), \mathbf{Q}/\mathbf{Z}) \xrightarrow{\text{rest}} H^2(\mathscr{G}(F), \mathbf{Q}/\mathbf{Z}))$, and let

$$0 \to \mathbf{Q}/\mathbf{Z} \longrightarrow \mathsf{E} \xrightarrow{\tau} \mathscr{G}(A(\mathfrak{S})) \to 1$$

be a topological central extension of $\mathscr{G}(A(\mathfrak{S}))$ representing x. Since the restriction of x to $\mathscr{G}(F)$ is trivial, there is a section $s\colon \mathscr{G}(F) \to \mathsf{E}$. Since $\mathbf{Q}/\mathbf{Z}$ is discrete, E is locally isomorphic to $\mathscr{G}(A(\mathfrak{S}))$. Let U be an open compact subgroup of E such that τ is injective on U, and let $\Gamma = \tau(U) \cap \mathscr{G}(F)$. Then since $\tau(U)$ is an open-compact subgroup of $\mathscr{G}(A(\mathfrak{S}))$, Γ is an $\mathfrak{S}$-congruence subgroup. Since the restriction of τ to U is an isomorphism on to $\tau(U)$, the restriction of its inverse to Γ gives an splitting of the above extension over Γ with image in U. Since any two splittings over Γ coincide on the commutator subgroup $\Delta = (\Gamma, \Gamma)$, we conclude that $s(\Delta) \subset U$. But Δ is of finite index in Γ (Margulis [16, 17]) and hence Δ is an $\mathfrak{S}$-arithmetic group.

Since U is a profinite group, the homomorphism $s|\Delta$ extends to a continuous homomorphism from the profinite completion $\hat{\Delta}$ of Δ to U. Since Δ is an open neighbourhood of the identity in the $\mathfrak{S}$-arithmetic topology on $\mathscr{G}(F)$, we see that $s\colon \mathscr{G}(F) \to \mathsf{E}$ is continuous in the $\mathfrak{S}$-arithmetic topology. Since E is complete, s extends to a continuous homomorphism $\hat{\mathscr{G}}(\mathfrak{S}) \to \mathsf{E}$. Since the square

$$\begin{array}{ccc}
\mathsf{E} & \longrightarrow & \mathscr{G}(A(\mathfrak{S})) \\
\uparrow & & \| \\
\hat{\mathscr{G}}(\mathfrak{S}) & \longrightarrow & \mathscr{G}(A(\mathfrak{S}))
\end{array}$$

commutes on the dense subgroup $\mathscr{G}(F)$ of $\hat{\mathscr{G}}(\mathfrak{S})$, it is commutative. Thus we have a morphism of the group extensions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Q}/\mathbf{Z} & \longrightarrow & \mathsf{E} & \longrightarrow & \mathscr{G}(A(\mathfrak{S})) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & C(\mathfrak{S}, \mathscr{G}) & \longrightarrow & \hat{\mathscr{G}}(\mathfrak{S}) & \longrightarrow & \mathscr{G}(A(\mathfrak{S})) & \longrightarrow & 1.
\end{array}$$

But since the first extension is a central extension, Z (introduced in 2.5) lies in the kernel of the homomorphism $\hat{\mathscr{G}}(\mathfrak{S}) \to \mathsf{E}$. Thus we have the induced morphism of the central extensions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Q}/\mathbf{Z} & \longrightarrow & \mathsf{E} & \longrightarrow & \mathscr{G}(A(\mathfrak{S})) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & \mathscr{C}(\mathfrak{S}, \mathscr{G}) & \longrightarrow & \tilde{\mathscr{G}}(\mathfrak{S}) & \longrightarrow & \mathscr{G}(A(\mathfrak{S})) & \longrightarrow & 1.
\end{array}$$

This proves that the transgression from $\operatorname{Hom}(\mathscr{C}(\mathfrak{S}, \mathscr{G}), \mathbf{Q}/\mathbf{Z})$ to

$$\operatorname{Ker}(H^2(\mathscr{G}(A(\mathfrak{S})), \mathbf{Q}/\mathbf{Z}) \xrightarrow{\text{rest}} H^2(\mathscr{G}(F), \mathbf{Q}/\mathbf{Z})),$$

induced by the central extension:

$$1 \to \mathscr{C}(\mathfrak{S}, \mathscr{G}) \to \tilde{\mathscr{G}}(\mathfrak{S}) \to \mathscr{G}(A(\mathfrak{S})) \to 1$$

is surjective.

In case $\mathscr{G}(F)$ is perfect, then so is $\tilde{\mathscr{G}}(\mathfrak{S})$, since $\mathscr{G}(F)$ is dense in $\tilde{\mathscr{G}}(\mathfrak{S})$ and the commutator subgroup of $\tilde{\mathscr{G}}(\mathfrak{S})$ is closed (2.7). From this it is easy to deduce that (in case $\mathscr{G}(F)$ is perfect), the map

$$\mathrm{Hom}(\mathscr{C}(\mathfrak{S},\mathscr{G}),\mathbf{R}/\mathbf{Z})\to M(\mathfrak{S},\mathscr{G})$$

is injective. This proves the theorem.

Now Theorem 2.6 gives the following:

2.10. **Theorem.** *Assume that for every place $v\notin\mathfrak{S}$ of F, $\mathscr{G}$ is isotropic over F_v and $\sum_{v\in\mathfrak{S}} F_v$-rank $\mathscr{G}\geqq 2$. Then the metaplectic kernel $M(\mathfrak{S},\mathscr{G})$ is finite.*

§ 3. Isotropic Groups: The Main Theorem and some Reductions

3.1. Let G be an absolutely simple, simply connected, algebraic group defined over F. Let S be a maximal F-split torus of G and let T be a maximal F-torus containing S. Assume that F-rank $G=\dim S=1$, and let a be a generator of the character group of S. We assume that $\{\pm a, \pm 2a\}$ is the set of roots of G with respect to S. For $b=\pm a$ or $\pm 2a$, let U_b be the root subgroup corresponding to b (i.e., the subgroup denoted by $U_{(b)}$ in Borel-Tits [4: 5.2]); it is a connected unipotent subgroup defined over F. Let $Z(S)$ be the centralizer of S in G and let $P=Z(S)\cdot U_a$. Then P is a parabolic F-subgroup.

Let $\Phi=\Phi(G,T)$ be the root system of G with respect to T. We fix a Borel subgroup of G containing T, contained in P, and defined over the separable closure of F. This gives an ordering on the root system Φ such that if the restriction to S of a root in Φ is either a or $2a$, then that root is positive. Let $\Phi^+(\subset\Phi)$ be the set of positive roots, Δ be the set of simple roots and $d(\in\Phi^+)$ be the dominant (or the highest) root.

Let G_{2a} be the subgroup of G generated by the root subgroups U_{2a} and U_{-2a}. Then G_{2a} is a semisimple group of F-rank 1 and it is normalized by $Z(S)$ $(\supset T)$. It is not hard to see that the Dynkin diagram of G_{2a} is obtained from the Tits index (Tits [27: 2.3]) of G/F as follows: To the Tits index of G/F appropriately adjoin a vertex corresponding to $-d$, delete the unique distinguished vertex or orbit (since G is of F-rank 1, the Tits index has either a unique distinguished vertex or a unique distinguished orbit), and all the edges containing the deleted vertex (or vertices). Then the Dynkin diagram of G_{2a} is just the connected component containing the vertex corresponding to $-d$ in the residual diagram. We see that since the Dynkin diagram of G_{2a} is connected, G_{2a} is *absolutely simple.*

From the above description of the (absolute) Dynkin diagram of G_{2a}, we know a set of simple (absolute) roots, and also the corresponding coroots (as 1-parameter subgroups of T). A case-by-case check, using the Tits' classification ([27]) of absolutely simple groups of F-rank 1 (F a global field) in terms of the index, shows that G_{2a} is simply connected (recall that G is simply connected) – *in fact if F is an arbitrary field, then G_{2a} fails to be simply connected only for a rank 1 form of type $\mathbf{E}_8$, namely the form $\mathbf{E}_{8,1}^{91}$; but this form does not exist over any local or global field.*

3.2. Now let $\mathscr{G}$ be, as before, a simply connected absolutely simple group defined over a global field F. In the rest of this paper *we shall assume that* $\mathscr{G}$ *is isotropic over* F. Let $\mathscr{S}$ be a maximal F-split torus of $\mathscr{G}$ and let $\mathscr{T}$ be a maximal F-torus containing $\mathscr{S}$. Let Φ be the set of roots (the F-roots!) of $\mathscr{G}$ with respect to $\mathscr{S}$. We fix a minimal parabolic F-subgroup $\mathscr{P}$ containing $\mathscr{T}$. This determines an ordering on Φ; let Φ^+ be the set of positive roots, Δ the set of simple roots and d be the dominant root with respect to this ordering.

For a root $b \in \Phi$, let $\mathscr{U}_b$ be the root subgroup corresponding to b; it is a connected unipotent F-subgroup normalized by the centralizer $Z(\mathscr{S})$ of $\mathscr{S}$. Let $\mathscr{G}_b$ be the subgroup generated by $\mathscr{U}_b$ and $\mathscr{U}_{-b}$. Then $\mathscr{G}_b$ is a connected semi-simple F-subgroup of F-rank 1, and it is simply connected if b is a nondivisible root.

Using the Tits index of $\mathscr{G}$ it is seen that there is a simple root a $(\in \Delta)$ such that $\mathscr{G}_a$ is absolutely simple, and a is multipliable (i.e. $2a$ is a root) in case Φ is nonreduced, and in case Φ is reduced, a is long. Since a is a simple root, $\mathscr{G}_a$ is simply connected. Moreover, from the observations in 3.1 it is clear that if $2a$ is a root, $\mathscr{G}_{2a}$ is an absolutely simple, simply connected group. Now since the dominant root d is conjugate (under the F-Weyl group) to a if Φ is reduced, and to $2a$ if Φ is nonreduced, we conclude that $\mathscr{G}_d$ is always absolutely simple and simply connected.

3.3. Let v be a place of F. Let $\mathscr{S}_v$ be a maximal F_v-split torus of $\mathscr{G}$ containing $\mathscr{S}$ and let Φ_v be the root system of $\mathscr{G}$ with respect to $\mathscr{S}_v$. For any root $b_v \in \Phi_v$, let $\mathscr{U}_{b_v}$ be the corresponding root subgroup and $\mathscr{G}_{b_v}$ the connected semi-simple F_v-subgroup generated by $\mathscr{U}_{b_v}$ and $\mathscr{U}_{-b_v}$. We fix a minimal parabolic subgroup of $\mathscr{G}$ defined over F_v and contained in the minimal parabolic F-subgroup $\mathscr{P}$ fixed above in 3.2. This gives an ordering on the root system Φ_v compatible with the ordering on Φ. Let d_v be the positive dominant root in Φ_v with respect to this ordering. Then it is obvious that the restriction of d_v to $\mathscr{S}$ is the positive dominant root d of Φ. Hence, $\mathscr{G}_{d_v} \subset \mathscr{G}_d$.

The following is the main theorem of this paper.

3.4. **Theorem.** *Let* $\mathfrak{S}$ *be a finite set of places of* F *(not necessarily containing all the archimedean places), and let* $M(\mathfrak{S}, \mathscr{G})$ *be the metaplectic kernel:*

$$\operatorname{Ker}(H^2(\mathscr{G}(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z}) \xrightarrow{\text{rest}} H^2(\mathscr{G}(F), \mathbf{R}/\mathbf{Z})).$$

Then

(i) $M(\mathfrak{S}, \mathscr{G})$ *is trivial if* $\mathfrak{S}$ *contains a nonarchimedean place.*

(ii) $M(\mathfrak{S}, \mathscr{G})$ *is trivial also if there is a real place* $r \in \mathfrak{S}$ *such that* G_{d_r} *is isomorphic to* SL_2.

(iii) $M(\mathfrak{S}, \mathscr{G})$ *is isomorphic to a subgroup of* $\hat{\mu}(F)$ *if every place in* $\mathfrak{S}$ *is archimedean.*

3.5. We shall prove first that the assertion (iii) of the preceding theorem is a consequence of (i). So assume (i) and also that every place in $\mathfrak{S}$ is archimedean ($\mathfrak{S}$ may, for example, be empty). For a nonarchimedean place v, let $\mathfrak{S}_v = \mathfrak{S} \cup \{v\}$. Then by (i) $M(\mathfrak{S}_v, \mathscr{G})$ is trivial. On the other hand, it is obvious that

$\mathcal{G}(A(\mathfrak{S}))$ is a direct product of $\mathcal{G}(A(\mathfrak{S}_v))$ and $\mathcal{G}(F_v)$, and hence

$$H^2(\mathcal{G}(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z}) = H^2(\mathcal{G}(A(\mathfrak{S}_v)), \mathbf{R}/\mathbf{Z}) \oplus H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z}),$$

since $\mathcal{G}(F_v)$ is perfect (see, for example, Prasad-Raghunathan [23: § 6.15]). Now let $p: \mathcal{G}(A(\mathfrak{S})) \to \mathcal{G}(A(\mathfrak{S}_v))$ be the natural projection. Then we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{G}(A(\mathfrak{S})) & \longleftarrow & \mathcal{G}(F) \\ \Big\downarrow p & & \Big\| \\ \mathcal{G}(A(\mathfrak{S}_v)) & \longleftarrow & \mathcal{G}(F) \end{array}$$

which induces the following commutative diagram in the cohomology:

$$\begin{array}{ccc} H^2(\mathcal{G}(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z}) & \xrightarrow{\text{rest}} & H^2(\mathcal{G}(F), \mathbf{R}/\mathbf{Z}) \\ \Big\uparrow p^* & & \Big\| \\ H^2(\mathcal{G}(A(\mathfrak{S}_v)), \mathbf{R}/\mathbf{Z}) & \xrightarrow{\text{rest}} & H^2(\mathcal{G}(F), \mathbf{R}/\mathbf{Z}). \end{array}$$

Under the natural identification of $H^2(\mathcal{G}(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z})$ with

$$H^2(\mathcal{G}(A(\mathfrak{S}_v)), \mathbf{R}/\mathbf{Z}) \oplus H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z}),$$

the projection $p: \mathcal{G}(A(\mathfrak{S})) \to \mathcal{G}(A(\mathfrak{S}_v))$ induces the natural inclusion of $H^2(\mathcal{G}(A(\mathfrak{S}_v)), \mathbf{R}/\mathbf{Z})$ in

$$H^2(\mathcal{G}(A(\mathfrak{S}_v)), \mathbf{R}/\mathbf{Z}) \oplus H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z}).$$

Now since $M(\mathfrak{S}_v, \mathcal{G})$ is trivial, the restriction map

$$H^2(\mathcal{G}(A(\mathfrak{S}_v)), \mathbf{R}/\mathbf{Z}) \to H^2(\mathcal{G}(F), \mathbf{R}/\mathbf{Z})$$

is injective, and we conclude that under the natural projection of

$$H^2(\mathcal{G}(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z}) \ \ (= H^2(\mathcal{G}(A(\mathfrak{S}_v)), \mathbf{R}/\mathbf{Z}) \oplus H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z}))$$

on to $H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z})$, the metaplectic kernel $M(\mathfrak{S}, \mathcal{G})$ maps injectively in to $H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z})$.

Now let $\mathfrak{Q}$ be the set of nonarchimedean places v of F such that $\mathcal{G}$ is quasi-split over F_v; it is known that $\mathfrak{Q}$ contains almost all places of F. Then by the results of Moore and Deodhar (1.3), for $v \in \mathfrak{Q}$, $H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z})$ is isomorphic to a subgroup of $\hat{\mu}(F_v)$. Thus for every $v \in \mathfrak{Q}$, the metaplectic kernel $M(\mathfrak{S}, \mathcal{G})$ is isomorphic to a subgroup of the finite cyclic group $\hat{\mu}(F_v)$. But since $\mathfrak{Q}$ contains all but finitely many places of F, it follows easily from Theorem 9 of Heilbronn [11]* that $\bigcap_{v \in \mathfrak{Q}} \mu(F_v) = \mu(F)$. From this we conclude that $M(\mathfrak{S}, \mathcal{G})$ is isomorphic to a subgroup of $\hat{\mu}(F)$. This proves that assertion (i) of Theorem 3.4 implies assertion (iii).

* We thank M. Ram Murty for this reference

Since

$$H^2(\mathscr{G}(F_v), \mathbf{R}/\mathbf{Z}) \xrightarrow{\text{rest}} H^2(\mathscr{G}_{d_v}(F_v), \mathbf{R}/\mathbf{Z})$$

is injective (Theorem 1.2), we conclude that for all v,

$$H^2(\mathscr{G}(F_v), \mathbf{R}/\mathbf{Z}) \xrightarrow{\text{rest}} H^2(\mathscr{G}_d(F_v), \mathbf{R}/\mathbf{Z})$$

is injective.

Let

$$M(\mathfrak{S}, \mathscr{G}_d) = \operatorname{Ker}(H^2(\mathscr{G}_d(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z}) \xrightarrow{\text{rest}} H^2(\mathscr{G}_d(F), \mathbf{R}/\mathbf{Z}))$$

Then as (see 2.4) $H^2(\mathscr{G}(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z}) = \prod_{v \notin \mathfrak{S}} H^2(\mathscr{G}(F_v), \mathbf{R}/\mathbf{Z})$, and

$$H^2(\mathscr{G}_d(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z}) = \prod_{v \notin \mathfrak{S}} H^2(\mathscr{G}_d(F_v), \mathbf{R}/\mathbf{Z}),$$

the restriction

$$H^2(\mathscr{G}(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z}) \to H^2(\mathscr{G}_d(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z})$$

is injective, and hence it provides an imbedding of $M(\mathfrak{S}, \mathscr{G})$ in $M(\mathfrak{S}, \mathscr{G}_d)$. Thus to prove the first two assertions of Theorem 3.4, we need only show that $M(\mathfrak{S}, \mathscr{G}_d)$ is trivial if $\mathfrak{S}$ contains either a nonarchimedean place, or a real place r such that $\mathscr{G}_{d_r}$ is isomorphic to SL_2. In other words, to prove Theorem 3.4, after replacing $\mathscr{G}$ by $\mathscr{G}_d$ we may (and we shall) assume that $\mathscr{G} = \mathscr{G}_d$. Then $\mathscr{G}$ is of F-rank 1 and its F-root system is $\{-d, d\}$; thus the F-root system of $\mathscr{G}$ is reduced.

§4. A Result of Moore and the Proof of the Main Theorem

The following theorem is a simple consequence of (in fact it is equivalent to) a result of C. Moore [20: Theorem 12.3].

4.1. **Theorem.** *Let $\mathfrak{F}$ be a finite separable extension of F. Let $\mathfrak{S}$ be a finite set of places of F. Then the metaplectic kernel*

$$M(\mathfrak{S}, R_{\mathfrak{F}/F}(SL_2)) = \operatorname{Ker}(H^2(R_{\mathfrak{F}/F}(SL_2)(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z}) \xrightarrow{\text{rest}} H^2(R_{\mathfrak{F}/F}(SL_2)(F), \mathbf{R}/\mathbf{Z}))$$

is trivial if $\mathfrak{S}$ contains a nonarchimedean place. If $\mathfrak{S}$ contains a real place r which has an extension as a real place of $\mathfrak{F}$, then again the metaplectic kernel is trivial. In all the other cases the metaplectic kernel is isomorphic to $\hat{\mu}(\mathfrak{F}) = \operatorname{Hom}(\mu(\mathfrak{F}), \mathbf{R}/\mathbf{Z})$; where $\mu(\mathfrak{F})$ is the group of roots of unity in $\mathfrak{F}$.

We have shown in §3 that to prove the main theorem (Theorem 3.4), it is enough to prove its first two assertions for absolutely simple, simply connected groups of F-rank 1 whose F-root system (i.e. the root system relative to a maximal F-split torus) is reduced. Accordingly, we shall assume now that $\mathscr{G}$ is an absolutely simple, simply connected F-group of F-rank 1 whose F-root system is reduced, and $\mathfrak{S}$ contains either a nonarchimedean place v, or a real place r such that $\mathscr{G}_{d_r}$ is isomorphic to SL_2. Now to prove that $M(\mathfrak{S}, \mathscr{G})$ is trivial we shall make use of the following:

4.2. **Theorem.** *Let v be either a nonarchimedean or a real place of F. Then there is a finite separable extension $\mathfrak{F}$ of F and a F-subgroup $\mathcal{H}$ of $\mathcal{G}$, $\mathcal{H}$ isomorphic to $R_{\mathfrak{F}/F}(SL_2)$, such that the restriction homomorphism*

$$H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z}) \to H^2(\mathcal{H}(F_v), \mathbf{R}/\mathbf{Z})$$

is injective. Moreover, if there is a real place r such that $\mathcal{G}_{d_r}$ is isomorphic to SL_2, then $\mathfrak{F}$ may be chosen such that every extension of r to $\mathfrak{F}$ is real.

Assuming this theorem for a moment, we shall prove the triviality assertions of Theorem 3.4.

According to Theorem 2.4 $H^2(\mathcal{G}(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z})$ is a direct product $\prod_{v \notin \mathfrak{S}} H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z})$. If $M(\mathfrak{S}, \mathcal{G})$ is non-trivial, let $\alpha = (\alpha_v)_{v \notin \mathfrak{S}}$ ($\alpha_v \in H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z})$) be a nonzero element of the metaplectic kernel $M(\mathfrak{S}, \mathcal{G})$. Let v_0 be a place such that $\alpha_{v_0} \neq 0$. Then v_0 is either a nonarchimedean or a real place, because if v is a complex place, $F_v = \mathbf{C}$, and as $\mathcal{G}(\mathbf{C})$ is simply connected, $H^2(\mathcal{G}(F_v), \mathbf{R}/\mathbf{Z}) = \{0\}$. Now let $\mathcal{H}$ be a F-subgroup of $\mathcal{G}$, isomorphic to $R_{\mathfrak{F}/F} SL_2$, where $\mathfrak{F}$ is a finite separable extension of F, such that the restriction homomorphism $H^2(\mathcal{G}(F_{v_0}), \mathbf{R}/\mathbf{Z}) \to H^2(\mathcal{H}(F_{v_0}), \mathbf{R}/\mathbf{Z})$ is injective, and if there is a real place r of F such that $\mathcal{G}_{d_r}$ is isomorphic to SL_2, then every extension of r to $\mathfrak{F}$ is real. Now since the restriction $H^2(\mathcal{G}(F_{v_0}), \mathbf{R}/\mathbf{Z}) \to H^2(\mathcal{H}(F_{v_0}), \mathbf{R}/\mathbf{Z})$ is injective, the image of $\alpha = (\alpha_v)_{v \in \mathfrak{S}}$ in

$$H^2(\mathcal{H}(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z}) = \prod_{v \notin \mathfrak{S}} H^2(\mathcal{H}(F_v), \mathbf{R}/\mathbf{Z}),$$

under the restriction homomorphism, is nonzero, and it is clearly contained in the metaplectic kernel

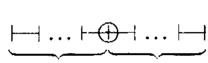
$$M(\mathfrak{S}, \mathcal{H}) = \mathrm{Ker}(H^2(\mathcal{H}(A(\mathfrak{S})), \mathbf{R}/\mathbf{Z}) \xrightarrow{\text{rest}} H^2(\mathcal{H}(F), \mathbf{R}/\mathbf{Z})).$$

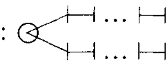
But since $\mathcal{H}$ is isomorphic to $R_{\mathfrak{F}/F}(SL_2)$, according to Theorem 4.1 (in view of our hypothesis on $\mathfrak{S}$), the metaplectic kernel $M(\mathfrak{S}, \mathcal{H})$ is trivial. This implies that the restriction of α to $\mathcal{H}(A(\mathfrak{S}))$ is zero, this is a contradiction which proves that the metaplectic kernel $M(\mathfrak{S}, \mathcal{G})$ is trivial.

Now to prove Theorem 4.2 we shall make use of explicit description of the groups $\mathcal{G}$. The following is a complete list. It is extracted from the classification given in Tits [27]; we have also taken into account the result of G. Harder on the vanishing of the Galois cohomology of simply connected groups over global function fields.

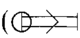
In the sequel we shall assume that v is either a nonarchimedean or a real place of F.

(i) *Inner forms of type A*: $\mathcal{G} = SL_{2,D}$, where D is a central simple division algebra over F of degree $d \geq 1$. The Tits index is:

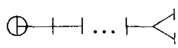
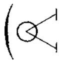


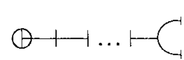
(ii) *Outer forms of type* A: $\mathscr{G}=SU(h)$; where D is a central simple division algebra of degree $n \geqq 1$ over a quadratic Galois extension $\mathscr{F}$ of F, and h is the hyperbolic hermitian form in 2 variables defined in terms of an involution σ of D of the second kind such that the subfield of $\mathscr{F}$ pointwise fixed by σ is F. We shall denote $SU(h)$ by $SU_{2,D/\sigma}$. The Tits index of $\mathscr{G}/F$ is: 

(iii) *Forms of type* $\boldsymbol{B}$ or $\boldsymbol{D}$: $\mathscr{G}$ is the spin group of a nondegenerate quadratic form in $n(\geqq 5)$ variables and of Witt indes 1; if F is of positive characteristic, then $n=5$ or 6; if F is of characteristic 2 and $n=6$, the form is assumed to be nondefective, if $n=5$, the form is of defect 1. The Tits index is:

 if n is odd ( if $n=5$),

and

 if n is even ( if $n=6$) and

the descriminant of the quadratic form is $(-1)^{n/2}$, or  if n is even ( if $n=6$) and the descriminant of the quadratic form is $\neq(-1)^{n/2}$.

Proof of Theorem 4.2. (i) We first take up the case where $\mathscr{G}=SL_{2,D}$, for a central simple division algebra D over F:

There exists a central simple division algebra D_v over F_v such that $D \otimes_F F_v$ is isomorphic to the matrix algebra $M_n(D_v)$ for some $n \geqq 1$ ([6: § 5, Théorème 2]). In case v is nonarchimedean, let $\mathfrak{F}_v$ be an unramified extension of F_v of degree = degree $D_v/F_v = \sqrt{[D_v:F_v]}$, contained in D_v ([25: Chapitre XII]). In case v is a real place, let $\mathfrak{F}_v = \mathbf{R}$ or $\mathbf{C}$ according as degree $D_v/F_v=1$ or 2; we identify $\mathfrak{F}_v$ with a field contained in D_v. Let

$$A_v=\left\{\left.\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right| x_i \in \mathfrak{F}_v \text{ for } 1 \leqq i \leqq n\right\} \subset M_n(D_v).$$

Let $a_v \in A_v$ be a *regular* element (i.e., the roots of the reduced characteristic polynomial of a_v are all distinct). We choose an element a of D "sufficiently" close to a_v in the topology on $D \otimes_F F_v$ induced by the local field topology on F_v (note that in this topology D is dense in $D \otimes_F F_v$). Then the field $\mathfrak{F}(\subset D)$, generated by a over F, is a Galois extension of F of degree = degree $D/F = d = n \cdot [\mathfrak{F}_v : F_v]$, and as a F_v-algebra $\mathfrak{F} \otimes_F F_v$ is isomorphic to $A_v(\cong(\mathfrak{F}_v)^n)$. If there exists a real place r such that $\mathscr{G}_{d_r}$ is isomorphic to SL_2, then since $\mathscr{G}/F$ is an inner form of type A, $\mathscr{G}$, and hence D, splits over F_r. Now since in $D \otimes_F F_r$ $(=M_d(\mathbf{R}))$ the set of elements which have all the eigenvalues real and distinct is nonempty and open, and D in its diagonal imbedding in $(D \otimes_F F_v) \times (D \otimes_F F_r)$ is dense (weak approximation), we may (and we shall) further assume that the element a of D is so chosen that $\mathfrak{F} \otimes_F F_r$ is isomorphic to $\mathbf{R}^d$. This implies that all the places of $\mathfrak{F}$ extending r are real. Now using the theorem of Skölem-Noether and the simple fact that given a subalgebra B_v of the matrix algebra $M_n(\mathfrak{F}_v)$, B_v isomorphic to A_v, there is an (inner) automorphism of $M_n(\mathfrak{F}_v)$ which takes B_v on to A_v, we see that there is an F_v-algebra isomorphism of $D\otimes_F F_v$ with $M_n(D_v)$ which maps $\mathfrak{F}\otimes_F F_v$ on to A_v.

Let $\mathscr{H}$ be the F-subgroup $SL_{2,\mathfrak{F}}(\cong R_{\mathfrak{F}/F}SL_2)$ of $SL_{2,D}$. It is obvious that there is an isomorphism of $SL_{2,D}(F_v)\,(=SL_2(D\otimes_F F_v))$ with $SL_{2n}(D_v)$ which maps $\mathscr{H}(F_v)$ onto the subgroup:

$$(SL_2(\mathfrak{F}_v))^n=\begin{pmatrix} SL_2(\mathfrak{F}_v) & & & \\ & SL_2(\mathfrak{F}_v) & & \\ & & \ddots & \\ & & & SL_2(\mathfrak{F}_v)\end{pmatrix}\subset SL_{2n}(\mathfrak{F}_v)\subset SL_{2n}(D_v).$$

If v is nonarchimedean, according to Proposition 1.6, the restriction

$$H^2(SL_{2n}(D_v),\mathbf{R}/\mathbf{Z})\to H^2(SL_{2n}(\mathfrak{F}_v),\mathbf{R}/\mathbf{Z})$$

is injective. Also a result of Moore (Theorem 1.2 for split groups) implies that the restriction $H^2(SL_{2n}(\mathfrak{F}_v),\mathbf{R}/\mathbf{Z})\to H^2(SL_2(\mathfrak{F}_v),\mathbf{R}/\mathbf{Z})$ is injective, where $SL_2(\mathfrak{F}_v)$ is assumed to be imbedded in $SL_{2n}(\mathfrak{F}_v)$ as $\begin{pmatrix}\square & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}$. Hence, a fortiori, the restriction

$$H^2(SL_{2n}(\mathfrak{F}_v),\mathbf{R}/\mathbf{Z})\to H^2((SL_2(\mathfrak{F}_v))^n,\mathbf{R}/\mathbf{Z})$$

is injective. Now it is obvious that the restriction

$$H^2(\mathscr{G}(F_v),\mathbf{R}/\mathbf{Z})\to H^2(\mathscr{H}(F_v),\mathbf{R}/\mathbf{Z})$$

is injective. On the other hand, if v is real, and D_v is the quaternion division algebra, then $\forall m$, $SL_m(D_v)$ is simply connected and hence $H^2(SL_m(D_v),\mathbf{R}/\mathbf{Z})=\{0\}$, and if $D_v=\mathbf{R}$, then Theorem 1.2 implies that the restriction $H^2(\mathscr{G}(F_v),\mathbf{R}/\mathbf{Z})\to H^2(\mathscr{H}(F_v),\mathbf{R}/\mathbf{Z})$ is injective. This proves Theorem 4.2 in case $\mathscr{G}=SL_{2,D}$.

(ii) Now we take up the case where $\mathscr{G}=SU_{2,D/\sigma}$; D is a central simple division algebra of degree $n\geqq 1$ over a quadratic Galois extension $\mathscr{F}$ of F, σ is an involution of D of the second kind such that the subfield of $\mathscr{F}$ fixed pointwise by σ is F, and $SU_{2,D/\sigma}$ is the special unitary group of the hyperbolic hermitian form in 2 variables defined in terms of σ.

Let D^σ be the F-vector subspace of D consisting of elements fixed under σ.

We first consider the case where $\mathscr{F}$ is linearly disjoint from F_v. Then $\mathscr{F}\otimes_F F_v=:\mathscr{F}_v$ is a field ($\mathscr{F}_v=\mathbf{C}$ in case v is real), $D\otimes_F F_v$ is isomorphic to the matrix algebra $M_n(\mathscr{F}_v)$; where $n=$degree $D/\mathscr{F}$; $\mathscr{G}$ is quasi-split over F_v and its F_v-rank$=n$; in fact, $\mathscr{G}/F_v$ is the special unitary group of the direct sum of n hyperbolic planes (the form defined in terms of the nontrivial automorphism of

$\mathscr{F}_v/F_v$). There is an isomorphism of $D \otimes_F F_v$ with $M_n(\mathscr{F}_v)$ such that the diagonal matrices of $M_n(\mathscr{F}_v)$, with entries in F_v, are fixed under the involution of $M_n(\mathscr{F}_v)$ induced by the involution σ of D (existence of such an isomorphism is obvious from the fact that any hermitian form is diagonalizable and the involutions of the second kind of $M_n(\mathscr{F}_v)$ correspond to the hermitian forms on $\mathscr{F}_v^n$). Now since the set of diagonal matrices of $M_n(\mathscr{F}_v)$, with coefficients in F_v, is a F_v-subalgebra isomorphic to $(F_v)^n$, we see that $D^\sigma \otimes_F F_v$ contains an F_v-algebra isomorphic to $(F_v)^n$. From this we conclude that the set of elements of $D^\sigma \otimes_F F_v$ ($\subset D \otimes_F F_v$) whose characteristic polynomial has distinct roots ($\in F_v$) is nonempty; it is not difficult to see that this subset is open in $D^\sigma \otimes_F F_v$. (We note that the characteristic polynomial of any element in $D^\sigma \otimes_F F_v$ has coefficients in F_v.)

If there exists a real place r such that F_r is linearly disjoint from $\mathscr{F}$, then $\mathscr{F}_r = \mathscr{F} \otimes_F F_r$ is the field of complex numbers, $D \otimes_F F_r$ is isomorphic to the matrix algebra $M_n(\mathscr{F}_r)$, and $\mathscr{G}/F_r$ is the special unitary group of the direct sum of n hyperbolic planes (the form defined in terms of the nontrivial automorphism of $\mathscr{F}_r/F_r$), hence $\mathscr{G}_{d_r}$ is isomorphic to SL_2. As above, we see that $D^\sigma \otimes_F F_r$ contains a F_r-subalgebra isomorphic to $(F_r)^n$. Therefore, the set of elements of $D^\sigma \otimes_F F_r$ ($\subset D \otimes_F F_r$) whose characteristic polynomial has distinct real roots is nonempty (note that the characteristic polynomial of every element in $D^\sigma \otimes_F F_r$ has coefficients in $F_r = \mathbf{R}$); it is obviously an open subset.

We shall now consider the case where $\mathscr{F} \subset F_v$. In this case, $D \otimes_F F_v = M_m(D_v) \otimes \sigma(M_m(D_v))$, where D_v is a central division algebra with center F_v and $M_m(D_v)$ is the matrix algebra of $m \times m$ matrices with coefficients in D_v ($m \geqq 1$). Let $\mathfrak{F}_v$ be an extension of F_v of degree = degree D_v/F_v, contained in D_v, $\mathfrak{F}_v$ is assumed to be an unramified extension of F_v if v is nonarchimedean, and let

$$A_v = \left\{ \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_m \end{pmatrix} \middle| \, x_i \in \mathfrak{F}_v \text{ for } 1 \leqq i \leqq m \right\}.$$

Let

$$\mathsf{A}_v = \{a + \sigma(a) \,|\, a \in A_v\}.$$

Then it is obvious that $\mathscr{F} \cdot \mathsf{A}_v = A_v \oplus \sigma(A_v)$ is a maximal abelian subalgebra of $D \otimes_F F_v = M_m(D_v) \oplus \sigma(M_m(D_v))$.

If there exists a real place r of F such that $\mathscr{G}_{d_r}$ is isomorphic to SL_2 and $\mathscr{F} \subset F_r$ (the case where $\mathscr{F}$ is linearly disjoint from F_r has already been considered above), then $D \otimes_F F_r = M_n(F_r) \oplus \sigma(M_n(F_r))$, where $n = \deg D/\mathscr{F}$. Let

$$A_r = \left\{ \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \middle| \, x_i \in F_r \text{ for } 1 \leqq i \leqq n \right\}$$

and let

$$\mathsf{A}_r = \{a + \sigma(a) \,|\, a \in A_r\}.$$

Then A_r is a F_r-subalgebra isomorphic to $(F_r)^n$ and it is contained in $D^\sigma \otimes_F F_r$. From this we conclude once again that the set of elements in $D^\sigma \otimes_F F_r$ whose reduced characteristic polynomial has distinct real roots is a nonempty subset of $D^\sigma \otimes_F F_r$; it is clearly open.

Now by an approximation argument (note that D^σ is dense in $D^\sigma \otimes_F F_v$), we see that there is a Galois extension $\mathfrak{F}_0$ of F, of degree n, contained in D^σ, such that $\mathfrak{F}_0 \otimes_F F_v$ is isomorphic to $(F_v)^n$ in case $\mathscr{F}$ is linearly disjoint from F_v, and is isomorphic to $(\mathfrak{F}_v)^m$ if $\mathscr{F} \subset F_v$, where, as before, $\mathfrak{F}_v$ is an extension of F_v of maximal degree contained in D_v, $\mathfrak{F}_v$ is assumed to be an unramified extension of F_v if v is nonarchimedean. If moreover, there exists a real place r such that $\mathscr{G}$ is isomorphic to SL_2, then we may (and we shall), using the density of the diagonal imbedding of D^σ in $(D^\sigma \otimes_F F_v) \times (D^\sigma \otimes_F F_r)$, assume that $\mathfrak{F}_0$ is such that $\mathfrak{F}_0 \otimes_F F_r$ is isomorphic to $F_r^n\,(=\mathbf{R}^n)$, from this it is obvious that all the places of $\mathfrak{F}_0$ extending r are real.

Now let $\mathfrak{F}$ be the field spanned by $\mathfrak{F}_0$ and $\mathscr{F}$. It is obvious that $\mathscr{F}$ is stable under σ. Using the theorem of Skölem-Noether we see that in case $\mathscr{F}$ is linearly disjoint from F_v, there is an F_v-algebra isomorphism of $D \otimes_F F_v$ with $M_n(\mathscr{F}_v)$ which maps $\mathfrak{F} \otimes_F F_v$ $(\subset D \otimes_F F_v)$ on to the subalgebra of the diagonal matrices in $M_n(\mathscr{F}_v)$; whereas in case $\mathscr{F} \subset \mathrm{F}_v$, there is an F_v-algebra isomorphism of $D \otimes_F F_v$ with $M_n(D_v) \oplus \sigma(M_m(D_v))$ which takes $\mathfrak{F} \otimes_F F_v$ onto the subalgebra $A_v \oplus \sigma(A_v)$.

Clearly,

$$SU_{2,D/\sigma}(F) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{bmatrix} {}^\sigma a & {}^\sigma c \\ {}^\sigma b & {}^\sigma d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};\ a, b, c, d \in D \right\}.$$

Let $\mathscr{H}$ be the F-subgroup of $SU_{2,D/\sigma}$ such that

$$\mathscr{H}(F) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU_{2,D/\sigma}(F);\ a, b, c, d \in \mathfrak{F} \right\}.$$

Then $\mathscr{H}$ is F-isomorphic to $R_{\mathfrak{F}_0/F} SL_2$. Assume first that $\mathscr{F}$ is linearly disjoint from F_v, then F_v-rank $\mathscr{H} = F_v$-rank $\mathscr{G} = n$; there is an identification of $\mathscr{G}$ (over F_v) with the special unitary group of the direct sum $\boldsymbol{H} \oplus \ldots \oplus \boldsymbol{H}$ of n hyperbolic planes, in such a way that the subgroup $\mathscr{H}/F_v$ gets identified with the subgroup $SU(\boldsymbol{H})^n$. Note that $SU(\boldsymbol{H})$ is isomorphic to SL_2 over F_v. It is easy to see that $\mathscr{H}$ contains the SL_2 corresponding to a suitable long root in the root system of $\mathscr{G}$ with respect to a maximal F_v-split torus contained in $\mathscr{H}$. Hence, it follows from a result of Deodhar (i.e., Theorem 1.2 for quasi-split groups) that the restriction:

$$H^2(\mathscr{G}(F_v), \mathbf{R}/\mathbf{Z}) \to H^2(\mathscr{H}(F_v), \mathbf{R}/\mathbf{Z})$$

is injective.

We shall now consider the case where $\mathscr{F} \subset F_v$. In this case $D \otimes_F F_v$ is isomorphic to $M_m(D_v) \oplus \sigma(M_m(D_v))$. It is easily seen that $\mathscr{G}/F_v$ is isomorphic to SL_{2m,D_v}, $\mathscr{H}$ is F-isomorphic to $R_{\mathfrak{F}_0/F} SL_2$, and we can choose an isomorphism of $\mathscr{G}(F_v)$ with $SL_{2m}(D_v)$ so that under this isomorphism $\mathscr{H}(F_v)$ corresponds to the subgroup

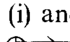
$$(SL_2(\mathfrak{F}_v))^m = \begin{pmatrix} SL_2(\mathfrak{F}_v) & & & \\ & SL_2(\mathfrak{F}_v) & & \\ & & \ddots & \\ & & & SL_2(\mathfrak{F}_v) \end{pmatrix} \subset SL_{2m}(\mathfrak{F}_v) \subset SL_{2m}(D_v).$$

Now, as in the part (i) of the proof, we conclude that the restriction $H^2(\mathscr{G}(F_v), \mathbf{R}/\mathbf{Z}) \to H^2(\mathscr{H}(F_v), \mathbf{R}/\mathbf{Z})$ is injective.

(iii) Finally we take up the case where $\mathscr{G}$ is a F-form of type $\boldsymbol{B}$ or $\boldsymbol{D}$. Let $V = \boldsymbol{H} \oplus V_0$ be a quadratic space over F of dimension n and of Witt index 1; where $\boldsymbol{H}$ is a hyperbolic plane and V_0 is an anisotropic quadratic subspace of dimension $n-2 \geqq 3$. Then $\mathscr{G} = \mathrm{Spin}(V)$. In case v is a real place and V_0 is anisotropic over $F_v(=\mathbf{R})$, $\mathscr{G}(=\mathrm{Spin}(V))$ is of rank 1 over $\mathbf{R}$ and $\mathscr{G}(\mathbf{R})$ is simply connected, hence $H^2(\mathscr{G}(\mathbf{R}), \mathbf{R}/\mathbf{Z}) = \{0\}$, and Theorem 4.2 is obvious. So we assume that if v is real, V_0 is isotropic over F_v.

Now we consider the case where $n > 6$. In this case F is of characteristic zero, and $\dim V_0 > 4$. Hence $V_0 \otimes_F F_v$ is isotropic. Using a continuity argument (see Kneser [12: §2], his proof works also for forms over $\mathbf{R}$) we can find a nondegenerate quadratic subspace W of V_0 of dimension 3 or 4 according as n is odd or even, such that $W \otimes_F F_v$ is isotropic and if there exists a real place r such that $\mathscr{G}_{d_r}$ is isomorphic to SL_2 (in this case the Witt index of V over F_r is at least 2, since F_r-rank $\mathscr{G} \geqq 2$), then $W \otimes_F F_r$ is also isotropic. Let $W^\perp$ be the orthogonal complement of W in V_0. Then $V = \boldsymbol{H} \otimes W \otimes W^\perp$. Let $\mathscr{G}_0$ be the spin group of the quadratic space $\boldsymbol{H} \oplus W$. Then in its natural imbedding in $\mathscr{G}(=\mathrm{Spin}\, V))$, $\mathscr{G}_0$ is a F-subgroup of $\mathscr{G}$. Moreover, if there is a real place r of F such that $\mathscr{G}_{d_r}$ is isomorphic to SL_2, then since the Witt index of $\boldsymbol{H} \oplus W$ over F_r is at least 2, $\mathscr{G}_0$ is quasi-split over F_r. Since $(\boldsymbol{H} \oplus W) \otimes_F F_v$ is of Witt index $\geqq 2$, by Proposition 1.9 the restriction homomorphism

$$H^2(\mathscr{G}(F_v), \mathbf{R}/\mathbf{Z}) \to H^2(\mathscr{G}_0(F_v), \mathbf{R}/\mathbf{Z})$$

is injective if v is nonarchimedean. On the other hand, if v is a real place, $F_v = \mathbf{R}$ and $\pi_1(\mathscr{G}_0(\mathbf{R})) \to \pi_1(\mathscr{G}(\mathbf{R}))$ is easily seen to be surjective and hence the restriction $H^2(\mathscr{G}(F_v), \mathbf{R}/\mathbf{Z}) \to H^2(\mathscr{G}_0(F_v), \mathbf{R}/\mathbf{Z})$ is again injective. In view of this, it is obvious that to prove the theorem in case $\mathscr{G}$ is a F-form of type $\boldsymbol{B}$ or $\boldsymbol{D}$, we may replace $\mathscr{G}$ by $\mathscr{G}_0$ $(=\mathrm{Spin}(\boldsymbol{H} \oplus W)$. Thus we may (and we shall) assume that n is either 5 or 6. Now in case $n=6$, $\mathscr{G}$ is a F-form of (inner or outer) type A_3, and the theorem for groups of type A has already been proved in (i) and (ii) above. So we shall assume that $n=5$. Then the Tits index of $\mathscr{G}/F$ is , and $\mathscr{G}$ can be thought of as a F-rank 1 form of type $\boldsymbol{C}_2$.

There is a bijective correspondence between the set of isomorphism classes of absolutely simple, simply connected F-rank 1 groups of type $\boldsymbol{C}_2$ and the set of isomorphism classes of quaternion division algebras over F: To any quaternion division algebra D we associate the special unitary group of a anti-hermitian hyperbolic form in 2 variables defined in terms of an involution (of the first kind) of D such that the F-subspace of elements in D fixed by the involution is 3-dimensional. Any other involution of D of this type determines an isomorphic algebraic group.

Now to describe the F-rank 1 simply connected group of type $\boldsymbol{C}_2$ associated with a quaternion division algebra D, in a form convenient for the present purpose, we fix an involution of D as follows:

Let $\mathscr{F}(\subset D)$ be a quadratic Galois extension of F such that $\mathscr{F} \otimes_F F_v$ is an extension of F_v (assumed to be unramified, if v is nonarchimedean) of degree 2 in case $D\otimes_F F_v$ is a division algebra, otherwise ($D\otimes_F F_v$ is isomorphic to the matrix algebra $M_2(F_v)$) $\mathcal{F}\otimes_F F_v$ is isomorphic to $F_v\oplus F_v$. Moreover, if there exists a real place r such that $\mathcal{G}_{d_r}$ is isomorphic to SL_2, then $\mathcal{G}$ being a form of type C_2, actually splits over F_r and hence, D splits over F_r, and we may (and we shall assume further that $\mathcal{F}\otimes_F F_r$ is isomorphic to $(F_r)^2$; this implies that the extensions of r to $\mathcal{F}$ are real. Let $x\mapsto\bar{x}=\mathrm{Tr}d\,x-x$ be the standard involution of D. Let π be an element of D such that for $x\in\mathcal{F}$, $\pi x\pi^{-1}=\bar{x}$. Then since π^2 commutes with $\mathcal{F}$ and $\pi\notin\mathcal{F}$, $\pi^2\in F$, and hence $\bar{\pi}=-\pi$. For $x\in D$, let $\sigma(x)=\pi\bar{x}\pi^{-1}$. Then σ is an involution of D and the space D^σ, of elements fixed under σ, is of dimension 3, and it contains $\mathcal{F}$.

Now let $X=e_{-1}\cdot D+e_1\cdot D$ be a right vector space over D of dimension 2, and let φ be the hyperbolic σ-antihermitian form on X determined by:

$$\varphi(e_{-1},e_{-1})=0=\varphi(e_1,e_1),$$
$$\varphi(e_{-1},e_1)=1=-\varphi(e_1,e_{-1}).$$

Then $\mathcal{G}$ is F-isomorphic to $SU(\varphi)$. We shall use the basis $\{e_{-1},e_1\}$ to identify $\mathcal{G}$ with a F-subgroup of $SL_{2,D}$. Let $\mathcal{H}$ be the F-subgroup such that

$$\mathcal{H}(F)=SL_2(\mathcal{F})\,(\subset\mathcal{G}(F)\subset SL_2(D)).$$

Then $\mathcal{H}$ is F-isomorphic to $R_{\mathcal{F}/F}SL_2$. Now in case $D\otimes_F F_v$ is a division algebra, F_v-rank $\mathcal{G}=1$, thus in this case $\mathcal{G}/F_v$ is a relative rank 1 form of type C_2; if v is real, then $\mathcal{G}(F_v)$ is simply connected and hence $H^2(\mathcal{G}(F_v),\,\mathbf{R}/\mathbf{Z})=\{0\}$, on the other hand, if v is nonarchimedean, then according to Proposition 1.8, the restriction

$$H^2(\mathcal{G}(F_v),\mathbf{R}/\mathbf{Z})\to H^2(\mathcal{H}(F_v),\mathbf{R}/\mathbf{Z})$$

is injective. If D splits over F_v, then both $\mathcal{G}$ and $SL_{2,D}$ split over F_v. Now if v is real, then $F_v=\mathbf{R}$ and it is easily seen that the natural map $\pi_1(\mathcal{H}(\mathbf{R}))\to\pi_1(\mathcal{G}(\mathbf{R}))$ is surjective, and hence the restriction

$$H^2(\mathcal{G}(F_v),\mathbf{R}/\mathbf{Z})\to H^2(\mathcal{H}(F_v),\mathbf{R}/\mathbf{Z})$$

is injective. If v is nonarchimedean, then as the restriction

$$H^2(SL_{2,D}(F_v),\mathbf{R}/\mathbf{Z})\to H^2(\mathcal{H}(F_v),\mathbf{R}/\mathbf{Z})$$

is injective (this follows from Theorem 1.2), the image is of order $\#\mu(F_v)$. This implies that (since $SL_{2,D}\supset\mathcal{G}\supset\mathcal{H}$) the image of the restriction homomorphism

$$H^2(\mathcal{G}(F_v),\mathbf{R}/\mathbf{Z})\to H^2(\mathcal{H}(F_v),\mathbf{R}/\mathbf{Z})$$

is of order $\geqq\#\mu(F_v)$. But as $\mathcal{G}$ splits over F_v, $H^2(\mathcal{G}(F_v),\mathbf{R}/\mathbf{Z})$ is isomorphic to $\hat{\mu}(F_v)$, and from this it is obvious that the restriction

$$H^2(\mathcal{G}(F_v),\mathbf{R}/\mathbf{Z})\to H^2(\mathcal{H}(F_v),\mathbf{R}/\mathbf{Z})$$

is injective. This completes the proof of Theorem 4.2.

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