

ON THE CONJECTURE OF JEŚMANOWICZ
CONCERNING PYTHAGOREAN TRIPLES

MOUJIE DENG AND G.L. COHEN

Let a, b, c be relatively prime positive integers such that $a^2 + b^2 = c^2$. Jeśmanowicz conjectured in 1956 that for any given positive integer n the only solution of $(an)^x + (bn)^y = (cn)^z$ in positive integers is $x = y = z = 2$. Building on the work of earlier writers for the case when $n = 1$ and $c = b + 1$, we prove the conjecture when $n > 1$, $c = b + 1$ and certain further divisibility conditions are satisfied. This leads to the proof of the full conjecture for the five triples $(a, b, c) = (3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41)$ and $(11, 60, 61)$.

1. INTRODUCTION

Let a, b, c be relatively prime positive integers such that $a^2 + b^2 = c^2$, and let n be a positive integer. Clearly, the Diophantine equation

$$(1) \quad (na)^x + (nb)^y = (nc)^z,$$

has the solution $x = y = z = 2$. Whether there are other solutions in positive integers when $n = 1$ has been investigated by a number of writers. Sierpiński [6] showed there were no other solutions when $n = 1$ and $(a, b, c) = (3, 4, 5)$, and Jeśmanowicz [2] that there were no others when $n = 1$ and $(a, b, c) = (5, 12, 13), (7, 24, 25), (9, 40, 41)$ and $(11, 60, 61)$. He conjectured that the equation (1) has no positive integer solutions for any n other than $x = y = z = 2$.

The general solution of $a^2 + b^2 = c^2$ in relatively prime positive integers is of course well known to be $a = u^2 - v^2, b = 2uv, c = u^2 + v^2$, where $u > v > 0$, $\gcd(u, v) = 1$ and one of u, v is even, the other odd. A number of other special cases of Jeśmanowicz's conjecture have since been settled. Lu [5] proved it when $v = n = 1$. In 1965, Dem'janenko [1] extended earlier results in several papers by proving the conjecture to be true whenever $n = 1$ or 2 and $u = v + 1$. Takakuwa and Asaeda (see [7]) have proved the conjecture in a number of special cases in which, in particular,

Received 5th January, 1998

The first author would like to thank the second author and UTS for their hospitality and excellent working conditions during his stay at UTS.

We are grateful to the referee for several suggestions which led to improvements in the presentation of these proofs.

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$n = 1$ and $v \equiv 1 \pmod{4}$, and Takakuwa [8] has proved it when u is exactly divisible by 2, $v = 3, 7, 11$ or 15, and $n = 1$. More recently, Le has verified the conjecture if $n = 1$ and 2 exactly divides uv and, in [3], c is a prime power, in [4], $v \equiv 3 \pmod{4}$ and $u \geq 81v$.

A more general conjecture has been considered by Terai (see [9]). He asks whether the equation (1) with $n = 1$ and $a^p + b^q = c^r$, has any positive integer solutions other than $(x, y, z) = (p, q, r)$. In particular, he has considered $(p, q, r) = (2, 2, 3)$ and $(p, q, r) = (2, 2, 5)$.

Some authors and reviewers have stipulated that $n = 1$ in (1), but this is apparently not part of Jeśmanowicz's conjecture. Nor is it a particularly easy case when $n > 1$. In this paper, we shall take $a = 2k + 1$, $b = 2k(k + 1)$, $c = 2k(k + 1) + 1$, where k is a positive integer, and will obtain by completely elementary means certain conditions on n under which the only positive integer solution of the equation (1) is $x = y = z = 2$. This will lead us to prove Jeśmanowicz's conjecture in full for this generalisation of the original five cases settled by Sierpiński and Jeśmanowicz, that is, for $k \in \{1, 2, 3, 4, 5\}$.

For any integer $N > 1$ with prime factorisation $\prod_{i=1}^t p_i^{\alpha_i}$, we write $C(N) = \prod_{i=1}^t p_i$. All Greek and Roman letters in this paper denote positive integers unless specified otherwise.

The following two theorems will be proved.

THEOREM 1. *Let $a = 2k + 1$, $b = 2k(k + 1)$, $c = 2k(k + 1) + 1$, for some positive integer k . Suppose that a is a prime power, and that the positive integer n is such that either $C(b) \mid n$ or $C(n) \nmid b$. Then the only solution of the Diophantine equation $(an)^x + (bn)^y = (cn)^z$ is $x = y = z = 2$.*

THEOREM 2. *For each of the Pythagorean triples $(a, b, c) = (3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, $(9, 40, 41)$, $(11, 60, 61)$, and for any positive integer n , the only solution of the Diophantine equation $(an)^x + (bn)^y = (cn)^z$ is $x = y = z = 2$.*

Theorem 2 is the confirmation of Jeśmanowicz's conjecture in the five cases stated, corresponding to $k = 1, 2, 3, 4, 5$, respectively, in Theorem 1. The first case, when $k = 1$, is an immediate corollary of Theorem 1. The remainder of the proof of Theorem 2 uses Theorem 1 and special arguments in each of the cases $k = 2, 3, 4, 5$, with no pattern apparent. It is plausible that similar approaches will be successful for the next permissible cases $k = 6, 8, 9, 11, \dots$, but the details have not been carried out.

Three lemmas will be required.

LEMMA 1. *Let $a = 2k + 1$, $b = 2k(k + 1)$, $c = 2k(k + 1) + 1$, for some positive integer k . The only solution of the Diophantine equation $a^x + b^y = c^z$ is $x = y = z = 2$.*

This is Dem'janenko's result, mentioned above, when $u = v + 1$ and $n = 1$.

LEMMA 2. *If $z \geq \max\{x, y\}$, then the Diophantine equation $a^x + b^y = c^z$, where a, b and c are any positive integers (not necessarily relatively prime) such that $a^2 + b^2 = c^2$, has no solution other than $x = y = z = 2$.*

PROOF: If $z = 1$ then $x = y = 1$ and $(a + b)^2 > a^2 + b^2 = c^2$, so $a + b > c$. Suppose $z \geq 2$ and, without loss of generality, that $x \leq y$. If $y = 1$, then $a^x + b^y = a + b < a^2 + b^2 = c^2 \leq c^z$. If $y \geq 2$, then

$$a^x + b^y \leq (a^2)^{y/2} + (b^2)^{y/2} \leq (a^2 + b^2)^{y/2} = c^y \leq c^z,$$

and there is strict inequality unless $x = y = z = 2$. □

LEMMA 3. *If p is an odd prime and $\gcd(a, b) = 1$, then*

$$\gcd\left(a + b, \frac{a^p + b^p}{a + b}\right) = 1 \text{ or } p.$$

PROOF: Let q be a prime divisor of $a + b$, so that $q \nmid a$ and $b \equiv -a \pmod{q}$. Then

$$\frac{a^p + b^p}{a + b} = a^{p-1} - a^{p-2}b + \dots + b^{p-1} \equiv pa^{p-1} \pmod{q}.$$

It follows, if q is a divisor of $(a^p + b^p)/(a + b)$, that $q = p$ and that p is an exact divisor of $(a^p + b^p)/(a + b)$. □

2. PROOF OF THEOREM 1

By Lemma 1, we may suppose $n > 1$, and by Lemma 2 that $z < \max\{x, y\}$. Of course, $a^2 + b^2 = c^2$. Notice also that $a^2 = b + c$, $c = b + 1$, $b = k(a + 1)$, $c = k(a - 1) + a$, and a, b, c are relatively prime in pairs. We also suppose that equation (1) holds, and will show that this leads to a contradiction. There are two main cases to the proof, depending on whether $\gcd(n, c) = 1$ or $\gcd(n, c) > 1$, and numerous subcases in each case, indexed by a decimal numbering system.

1 Suppose $\gcd(n, c) = 1$. We cannot have $x = y$, since then $z < x$ and, from (1), we may write $n^{x-z}(a^x + b^x) = c^z$. Then $\gcd(n, c) > 1$, a contradiction.

1.1 Suppose $x > y$, so that we may write $n^y(n^{x-y}a^x + b^y) = n^z c^z$. Then clearly $z \geq y$, so that also $z < x$.

1.1.1 Suppose $n \nmid b^y$. Since we may write $n^{x-y}a^x + b^y = n^{z-y}c^z$, then we cannot have $z > y$, so $z = y$ in this case, and $n^{x-z}a^x + b^z = c^z$. Modulo a , we have $k^z \equiv (-k)^z$, and $\gcd(a, k) = 1$, so z is even. Write $z = 2z_1$, so that

$$n^{x-z}a^x = c^z - b^z = (c^{z_1} + b^{z_1})(c^{z_1} - b^{z_1}).$$

The factors on the right cannot both be divisible by a . Since $a^x > a^z = a^{2z_1} = (c + b)^{z_1} \geq c^{z_1} + b^{z_1} > c^{z_1} - b^{z_1}$, we have a contradiction.

1.1.2 Suppose $n \mid b^y$. Then it is not the case that $C(n) \nmid b$, so necessarily $C(b) \mid n$. We may take $b = \prod_{i=1}^s r_i^{\nu_i}$ (prime factorisation), so $n = \prod_{i=1}^s r_i^{\nu_i}$ with $\nu_i \geq 1$ for each $i = 1, \dots, s$. By the division algorithm, we write $\gamma_i y = t_i \nu_i + l_i$, say, where $t_i \geq 1$ and $0 \leq l_i < \nu_i$, for $i = 1, \dots, s$.

1.1.2.1 If $x > y + t_i$ for all $i = 1, \dots, s$, then we may write

$$(2) \quad \prod_{i=1}^s r_i^{\nu_i(y+t_i)} \left(\prod_{i=1}^s r_i^{\nu_i(x-y-t_i)} \cdot a^x + \prod_{i=1}^s r_i^{l_i} \right) = \prod_{i=1}^s r_i^{\nu_i z} \cdot c^z.$$

Since we cannot have $r_i \mid c$ for any $i = 1, \dots, s$, because $\gcd(n, c) = 1$, and since $l_i < \nu_i$ for each i , it follows that we must have $z = y + t_1 = \dots = y + t_s$, so $t_1 = \dots = t_s = t$, say, and (2) reduces to $\prod_{i=1}^s r_i^{\nu_i(x-y-t)} \cdot a^x + \prod_{i=1}^s r_i^{l_i} = c^z$. It is then apparent that $l_i = 0$ for $i = 1, \dots, s$, so that

$$(3) \quad \frac{\nu_1}{\gamma_1} = \dots = \frac{\nu_s}{\gamma_s} = \frac{y}{t} = \frac{y'}{t'},$$

say, where $\gcd(y', t') = 1$. Also, (2) further reduces to

$$(4) \quad n^{x-z} a^x + 1 = c^z.$$

If z is even, then, writing $z = 2z_1$, we have $n^{x-z} a^x = (c^{z_1} + 1)(c^{z_1} - 1)$. However, a cannot divide both factors on the right, and

$$a^x > a^z = a^{2z_1} = (b + c)^{z_1} > c^{z_1} + 1 > c^{z_1} - 1,$$

so this is impossible.

Suppose now that z is odd. Using (3), we have $n = b^{y'/t'}$ so that, from (4), $b^{y'(x-z)} a^{xt'} = (c^z - 1)^{t'} = ((b + 1)^z - 1)^{t'}$. Since b is even, then $b \nmid z$, so $(b + 1)^z - 1$ is divisible by b exactly. Hence $y'(x - z) = t'$. Since $\gcd(y', t') = 1$, then $y' = 1$, $x = z + t'$ and, from (3), $yt' = t$. Since $z = y + t = y(1 + t')$, then t' is even and x is odd. Write $x = 2x_1 + 1$. We have $n^{x-z} = n^{t'} = b^{y'} = b$, so that, from (4),

$$c^z - 1 = ba^x = a(c - 1)(b + c)^{x_1} = a(c - 1)(2c - 1)^{x_1}.$$

Modulo c , we have $a(-1)^{x_1} \equiv 1$, from which $c \mid (a + 1)$ or $c \mid (a - 1)$. But this is impossible, since $c > a + 1$.

1.1.2.2 If $x \leq y + t_i$ for at least one $i = 1, \dots, s$, then we can quickly obtain a contradiction. The approach may be illustrated by taking $x \leq y + t_1$ and $x > y + t_i$ for $i = 2, \dots, s$ (if $s \geq 2$). Then, adjusting (2), we may write

$$r_1^{\nu_1 x} \prod_{i=2}^s r_i^{\nu_i(y+t_i)} \left(\prod_{i=2}^s r_i^{\nu_i(x-y-t_i)} \cdot a^x + r_1^{\nu_1(y+t_1-x)} \prod_{i=1}^s r_i^{l_i} \right) = \prod_{i=1}^s r_i^{\nu_i z} \cdot c^z.$$

But since $x > z$, this implies that $r_1 \mid \prod_{i=2}^s r_i^{\nu_i z} \cdot c^z$, which is the desired contradiction.

1.2 Suppose $x < y$ and write (1) as $n^x(a^x + n^{y-x}b^y) = n^z c^z$. Then clearly $y > z \geq x$.

1.2.1 If $n \nmid a^x$, then we cannot have $z > x$, so $z = x$ and we have $n^{y-z}b^y = c^z - a^z$. Consider this equation modulo 4 if $k = 1$, in which case $a = 3, b = 4$ and $c = 5$, and modulo $k + 1$ if $k > 1$. In both cases, we conclude that z must be even. Write $z = 2z_1$.

If $k = 1$, then $n^{y-z}4^y = 5^z - 3^z = (5^{z_1} + 3^{z_1})(5^{z_1} - 3^{z_1})$. The factors on the right are both even but cannot both be divisible by 4. Hence one of them is divisible by 2^{2y-1} . But

$$2^{2y-1} > 2^{2z-1} = 2^{4z_1-1} \geq 2^{3z_1} = (5 + 3)^{z_1} \geq 5^{z_1} + 3^{z_1} > 5^{z_1} - 3^{z_1}.$$

We have a contradiction.

Suppose $k > 1$. We have $n^{y-z}b^y = (c^{z_1} + a^{z_1})(c^{z_1} - a^{z_1})$, and we observe that $b = 2k(k + 1), k \mid (c - a) \mid (c^{z_1} - a^{z_1})$ and $\gcd(c^{z_1} + a^{z_1}, c^{z_1} - a^{z_1}) = 2$. If z_1 is even, or if z_1 is odd and k is even (in which case, $a \equiv c \equiv 1 \pmod{4}$), then $c^{z_1} + a^{z_1}$ is divisible by 2 but not by 4, so that $2^{y-1}k^y \mid (c^{z_1} - a^{z_1})$. However,

$$2^{y-1}k^y = \frac{(2k)^y}{2} \geq \frac{(2k)^{z+1}}{2} = k(4k^2)^{z_1} > (2k^2 + 2k + 1)^{z_1} = c^{z_1} > c^{z_1} - a^{z_1},$$

which is also a contradiction. If z_1 and k are both odd, then, since $c \equiv -a \equiv 1 \pmod{k + 1}$, we have $(k + 1) \mid (c^{z_1} + a^{z_1})$ and $4 \nmid (c^{z_1} - a^{z_1})$. Hence $2^{y-1}(k + 1)^y \mid (c^{z_1} + a^{z_1})$. But

$$\begin{aligned} 2^{y-1}(k + 1)^y &> \frac{1}{2}(2(k + 1))^z = \frac{1}{2}(4k^2 + 8k + 4)^{z_1} \\ &\geq (2k^2 + 4k + 2)^{z_1} = (c + a)^{z_1} \geq c^{z_1} + a^{z_1}, \end{aligned}$$

our final contradiction in this case.

1.2.2 Suppose $n \mid a^x$. Write $a = p^\alpha$, where p is prime, and $n = p^\nu$. Also, write $\alpha x = \nu t + l$, where $0 \leq l < \nu$.

Suppose $y > x + t$, and write (1) as $n^{x+t}(p^l + n^{y-x-t}b^y) = n^z c^z$. From this, it follows that $z = x + t$ and $l = 0$, so that $n^{y-z}b^y = c^z - 1$. If z is odd, then, as in the last paragraph of 1.1.2.1, $c^z - 1$ is exactly divisible by b . But $y > z$, so $y \geq 2$ and $b^2 \mid (c^z - 1)$. Then z must be even. Write $z = 2z_1$. We have $c^{z_1} + 1 \equiv 2 \pmod{b}$, from which $(c^{z_1} + 1, b) = 2$. Since $n^{y-z}b^y = (c^{z_1} + 1)(c^{z_1} - 1)$, we must then have $b^{y/2} \mid (c^{z_1} - 1)$. But

$$\frac{b^y}{2} > \frac{b^{2z_1}}{2} = \frac{1}{2}(c - a)^{z_1}(c + a)^{z_1} \geq c^{z_1} + a^{z_1} > c^{z_1} - 1.$$

This is a contradiction.

If $y \leq x + t$, then write (1) as $n^y(n^{x+t-y}p^l + b^y) = n^z c^z$. Since $y > z$, we have $n \mid c^z$, a contradiction.

2 In the second main case, we suppose $\gcd(n, c) > 1$. Write $c = \prod_{i=1}^t q_i^{\alpha_i}$ (prime factorisation).

2.1 Suppose first that $C(n) \mid c$, so that we may write $n = \prod_{i=1}^s q_i^{\beta_i}$, say, with $s \leq t$ and $\beta_i \geq 1$ for $i = 1, \dots, s$.

2.1.1 Suppose $x = y$, so $z < x$. From (1), we have

$$(a^x + b^x) \prod_{i=1}^s q_i^{\beta_i x} = \prod_{i=1}^s q_i^{\beta_i z} \cdot \prod_{i=1}^s q_i^{\alpha_i z} \cdot \prod_{i=s+1}^t q_i^{\alpha_i z},$$

so that

$$(5) \quad a^x + b^x = \prod_{i=1}^s q_i^{\alpha_i z - \beta_i(x-z)} \cdot \prod_{i=s+1}^t q_i^{\alpha_i z}.$$

It is clear from this that $\alpha_i z - \beta_i(x - z) \geq 0$ for each $i = 1, \dots, s$.

We shall show that $\alpha_1 z - \beta_1(x - z) > \alpha_1$. Suppose this is not true. If $t = 1$ then $s = 1$ and $q_1^{\alpha_1 z - \beta_1(x-z)} \leq q_1^{\alpha_1} = c < a^x + b^x$, contradicting (5). If $t > 1$ then, since $q_1^{\alpha_1} \leq c/q_2 < c - 1 = b$ and $\prod_{i=2}^t q_i^{\alpha_i} \leq c/q_1 < c - 1 = b$, we have

$$\prod_{i=1}^s q_i^{\alpha_i z - \beta_i(x-z)} \cdot \prod_{i=s+1}^t q_i^{\alpha_i z} < q_1^{\alpha_1} \prod_{i=2}^t q_i^{\alpha_i z} < b^{z+1} \leq b^x < a^x + b^x,$$

another contradiction. Thus $\alpha_1 z - \beta_1(x - z) > \alpha_1$, and similarly $\alpha_i z - \beta_i(x - z) > \alpha_i$ for $i = 2, \dots, s$. It follows from (5) that

$$(6) \quad a^x + b^x \equiv 0 \pmod{c}.$$

If x is odd, say $x = 2x_1 + 1$, then

$$a^x + b^x = aa^{2x_1} + bb^{2x_1} \equiv a(-1)^{x_1} - 1 \pmod{c}.$$

By (6), then $c \mid (a - 1)$ or $c \mid (a + 1)$, which is impossible since $c > a + 1$.

If x is even, say $x = 2x_1$, then, from (6), $(-1)^{x_1} + 1 \equiv 0 \pmod{c}$, so x_1 is odd. In that case, $a^x + b^x = (a^2)^{x_1} + (b^2)^{x_1}$ is divisible by $a^2 + b^2$, and, since $z < x$ implies $x > 2$, the quotient must exceed 1. Furthermore, by (5), $(a^x + b^x)/(a^2 + b^2)$ is divisible by q_j , say, for some $j = 1, \dots, t$. Since $a^2 \equiv -1 \equiv b^2 \pmod{c}$, we have

$$\frac{a^x + b^x}{a^2 + b^2} = a^{2(x_1-1)} - a^{2(x_1-2)}b^2 + \dots + b^{2(x_1-1)} \equiv x_1 \equiv 0 \pmod{q_j},$$

that is, $q_j \mid x_1$. Then $a^{2q_j} + b^{2q_j}$ divides $a^{2x_1} + b^{2x_1}$. Furthermore, $(a^{2q_j} + b^{2q_j})/(a^2 + b^2)$ divides $a^{2x_1} + b^{2x_1}$, and, from (5), must be a product of primes in $\{q_1, \dots, q_t\}$. It follows then from Lemma 3 that $\gcd(a^2 + b^2, (a^{2q_j} + b^{2q_j})/(a^2 + b^2)) = q_j$. However, it is clear that $(a^{2q_j} + b^{2q_j})/(a^2 + b^2) > q_j$, and $\prod_{i=1}^t q_i^2 \mid (a^2 + b^2)$, so we have a contradiction.

2.1.2 Now suppose $x > y$. From (1), we may write

$$\prod_{i=1}^s q_i^{\beta_i y} (n^{x-y} a^x + b^y) = \prod_{i=1}^s q_i^{\beta_i z} \cdot \prod_{i=1}^t q_i^{\alpha_i z}.$$

If $z \geq y$ then $q_1 \mid b$, contradicting $\gcd(b, c) = 1$, so $z < y$ and we write

$$(7) \quad n^{x-y} a^x + b^y = \prod_{i=1}^s q_i^{\alpha_i z - \beta_i (y-z)} \cdot \prod_{i=s+1}^t q_i^{\alpha_i z}.$$

Again we have a contradiction if $\alpha_j z - \beta_j (y - z) > 0$ for some $j = 1, \dots, s$, since then $q_j \mid b$, so $\prod_{i=1}^s q_i^{\alpha_i z - \beta_i (y-z)} = 1$. It follows that $s < t$ but, since $\prod_{i=s+1}^t q_i^{\alpha_i} < c/q_1 < b$, we have

$$\prod_{i=s+1}^t q_i^{\alpha_i z} < b^z < b^y < n^{x-y} a^x + b^y,$$

which is then a contradiction of (7).

Similarly, we cannot have $x < y$.

2.2 If $C(n) \nmid c$, then we may write $n = n_1 n_2$, where $n_1 > 1$ and $\gcd(n_1, n_2) = \gcd(n_1, c) = 1$.

2.2.1 If $x = y$ then (1) becomes $n_1^x n_2^x (a^x + b^x) = n_1^z n_2^z c^z$. Since $z < x$, this implies that $n_1 \mid n_2^z c^z$, a contradiction.

2.2.2 Suppose $x > y$, and write (1) as $n_1^y n_2^y (n^{x-y} a^x + b^y) = n_1^z n_2^z c^z$. If $z \geq y$ then $\gcd(b, c) > 1$, since $\gcd(n, c) > 1$, and this is a contradiction. If $z < y$ then $n_1 \mid c^z$, and this is also impossible.

Similarly, we cannot have $x < y$.

This completes the proof of Theorem 1. □

3. PROOF OF THEOREM 2

In the notation of Theorem 1, we must extend that proof for the special cases $k = 2, 3, 4$ and 5 , without the restriction that $C(b) \mid n$ or $C(n) \nmid b$. In effect, we need only look to the case **1.1.2** in the proof of Theorem 1, so that we may assume $\gcd(n, c) = 1, x > y, n \mid b^y$ and $C(b) \nmid n$. The four values of k must be considered in turn. For this purpose, we continue the previous decimal indexing.

3 Take $k = 2$, so $(a, b, c) = (5, 12, 13)$. The relevant assumptions are: $x > y, n \mid 12^y$ and $6 \nmid n$. Then n is either a power of 3 or a power of 2.

3.1 Let $n = 3^r$, and write $y = tr + l$ where $0 \leq l < r$. If $x > y + t$, then we may write (1) as $n^{y+t} (n^{x-y-t} 5^x + 3^l 4^y) = n^z 13^z$. It follows that $z = y + t$, so that $n^{x-z} 5^x + 3^l 4^y = 13^z$, and then that $l = 0$ since $x > z$. Then, modulo 5, $(-1)^y \equiv 3^z$, from which z must be even. Write $z = 2z_1$, so that $n^{x-z} 5^x = (13^{z_1} + 2^y)(13^{z_1} - 2^y)$. The factors on the right cannot both be divisible by 5, and, noting that $z = y + t > y$,

$$5^x > 5^z = 25^{z_1} > 13^{z_1} + 4^{z_1} > 13^{z_1} + 2^y > 13^{z_1} - 2^y,$$

so we have a contradiction. If $x \leq y + t$, then we may write (1) as

$$n^x (5^x + 3^l n^{y+t-x} 4^y) = n^z 13^z.$$

This is clearly impossible, since $x > z$.

3.2 Now let $n = 2^s$, and write $2y = ts + l$, where $0 \leq l < s$. As in **3.1**, we easily show that we cannot have $x \leq y + t$, so $x > y + t$ and we may write $n^{y+t} (n^{x-y-t} 5^x + 2^l 3^y) = n^z 13^z$. This implies that $z = y + t$, and then that $l = 0$, so

$$(8) \quad n^{x-z} 5^x + 3^y = 13^z.$$

Then $3^y \equiv 3^z \pmod{5}$, so y and z are both even or both odd. If $4 \mid n^{x-z}$, then (8), considered modulo 4, shows that y is even. If $n^{x-z} = 2$, then (8), considered modulo 3, shows that x is odd so that $z = x - 1$ is even. Thus we may put $z = 2z_1$ and $y = 2y_1$, and then $n^{x-z} 5^x = (13^{z_1} + 3^{y_1})(13^{z_1} - 3^{y_1})$. As in **3.1**, we may show this to be impossible.

4 Now take $k = 3$, so $(a, b, c) = (7, 24, 25)$. We are assuming that $x > y, n \mid 24^y$ and $6 \nmid n$, so that again n is a power of 3 or a power of 2.

4.1 Suppose $n = 3^r$, and $y = tr + l$ where $0 \leq l < r$. As in 3.1, we see that we must have $x > y + t$, and, as before, that $z = y + t$ and $l = 0$. Then $n^{x-z}7^x + 8^y = 25^z$. Considering this equation modulo 3, this implies that we may write $y = 2y_1$ so that $n^{x-z}7^x = (5^z + 8^{y_1})(5^z - 8^{y_1})$. However, 7 cannot divide both factors on the right and

$$7^x > 7^z = 7^t 49^{y_1} > 5^t (25^{y_1} + 8^{y_1}) \geq 5^{y+t} + 8^{y_1} = 5^z + 8^{y_1} > 5^z - 8^{y_1},$$

so we have a contradiction.

4.2 If $n = 2^s$, then, very much as in 3.2, we again obtain a contradiction.

5 Next, take $k = 4$, so $(a, b, c) = (9, 40, 41)$. We are assuming that $x > y$, $n \mid 40^y$ and $10 \nmid n$, so that n is a power of 5 or a power of 2.

5.1 Suppose $n = 5^r$, and $y = tr + l$ where $0 \leq l < r$. Again, we must have $x > y + t$, so that, from (1), $n^{y+t}(n^{x-y-t}9^x + 5^l 8^y) = n^z 41^z$, and this implies that $z = y + t$, and then that $l = 0$. The equation $n^{x-z}9^x + 8^y = 41^z$, considered modulo 5, shows that y is even, and then, considered modulo 3, that z is even. Write $y = 2y_1$ and $z = 2z_1$, so that we have $n^{x-z}9^x = (41^{z_1} + 8^{y_1})(41^{z_1} - 8^{y_1})$. The factors on the right cannot both be divisible by 3, and $9^x > 9^z = 81^{z_1} > 41^{z_1} + 8^{z_1} > 41^{z_1} + 8^{y_1} > 41^{z_1} - 8^{y_1}$, so we have a contradiction.

5.2 If $n = 2^s$, then, again as in 3.2, we obtain a contradiction.

6 In our final case, take $k = 5$, so $(a, b, c) = (11, 60, 61)$. We assume that $x > y$, $n \mid 60^y$ and $30 \nmid n$. We need to consider the possibilities $n = 3^{r_1}$, $n = 5^{r_2}$, $n = 2^s$, $n = 3^{r_1} 5^{r_2}$, $n = 2^s 3^{r_1}$, $n = 2^s 5^{r_2}$, and the proofs in each of these cases follow lines similar to the above.

This completes the proof of Theorem 2. □

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Heilongjiang Nongken Teachers' College
A Cheng City
People's Republic of China

School of Mathematical Sciences
University of Technology, Sydney
PO Box 123
Broadway NSW 2007
Australia
e-mail: g.cohen@maths.uts.edu.au