

# ON THE CONNECTEDNESS OF DEGENERACY LOCI AND SPECIAL DIVISORS

BY

W. FULTON<sup>(1)</sup> and R. LAZARFELD

*Brown University  
Providence, U.S.A.*

*Harvard University  
Cambridge, U.S.A.*

## Introduction

Let  $C$  be a smooth complex projective curve of genus  $g$ , and let  $J$  be the Jacobian of  $C$ . Upon choosing a base-point in  $C$ ,  $J$  may be identified with the set of linear equivalence classes of divisors of degree  $d$  on  $C$ . Denote by  $W_d^r$  the algebraic subvariety of  $J$  parametrizing divisors which move in a linear system of dimension at least  $r$ . A fundamental theorem of Kempf [9] and Kleiman and Laksov [11, 12] asserts that these loci are non-empty when their expected dimension

$$\rho = g - (r+1)(g-d+r)$$

is non-negative. We complement this existence theorem with two results on the global structure of  $W_d^r$  when  $\rho > 0$ . First of all, for an arbitrary curve  $C$ , we prove

**THEOREM I.** *If  $\rho > 0$ , then  $W_d^r$  is connected.*

When  $C$  is generic (in the sense of moduli), deep results about the local geometry of  $W_d^r$  have been obtained by Griffiths and Harris [5] and by Gieseker [4]. Combining these with Theorem I, we deduce the

**COROLLARY.** *For a generic curve  $C$ ,  $W_d^r$  is irreducible when  $\rho > 0$ .*

By a standard construction,  $W_d^r$  may be realized as the locus where a certain homomorphism of vector bundles on  $J$  drops rank. Theorem I then becomes a simple consequence of a general result—of independent interest—on the connectivity of such degeneracy loci.

---

<sup>(1)</sup> Partially supported by the J. S. Guggenheim Foundation and by NSF Grant MCS 78-04008.

Specifically, let  $X$  be an irreducible complex projective variety of dimension  $n$ , and let

$$\sigma: E \rightarrow F$$

be a homomorphism of vector bundles of ranks  $e$  and  $f$  on  $X$ . Put

$$D_k(\sigma) = \{x \in X \mid \text{rank } \sigma(x) \leq k\}.$$

If non-empty, the degeneracy locus  $D_k(\sigma)$  has codimension  $\leq (e-k)(f-k)$  in  $X$ . Under a suitable positivity hypothesis on the bundles in question, we prove that  $D_k(\sigma)$  is connected as soon as its expected dimension is positive:

**THEOREM II.** *Assume that the vector bundle  $E^* \otimes F = \text{Hom}(E, F)$  is ample. Then:*

(a)  $D_k(\sigma)$  is non-empty when  $n \geq (e-k)(f-k)$ ,

and

(b)  $D_k(\sigma)$  is connected when  $n > (e-k)(f-k)$ .

It was suggested in [3, § 10] that such a result should hold. Note that it is not assumed that  $D_k(\sigma)$  actually has the expected dimension.

The present paper is divided into two parts. § 1 is devoted to the proof of Theorem II, and to a simple consequence concerning the singularities of finite mappings to projective space. The application to loci of special divisors occupies § 2. Our construction of  $W'_d$  follows the well-known approach of Grothendieck, Mattuck, Schwartzenberger, Kempf, Kleiman and Laksov. Since only an elementary part of their work is needed, we have included details for the convenience of the reader. We remark that statement (a) of Theorem II gives rise to a simple proof of the Kempf–Kleiman–Laksov existence theorem for special divisors, bypassing the Chern class computations of the original proofs. On the other hand, those calculations lead to a formula for the cohomology class of  $W'_d$ , which is important in enumerative questions. We recommend Chapter 3 of the forthcoming book [1], whose notation we follow, for an account of results along these lines.

Finally, a word on the proof of Theorem II may prove helpful. The strategy is to reduce the problem to proving the vanishing of certain singular cohomology groups, and then to draw on Lefschetz-type theorems to establish these vanishings. Consider for example the special case when  $E$  is a trivial line bundle and  $k=0$ , so that  $D_0(\sigma) = Z(\sigma)$ , the zero-locus of the section  $\sigma$  of the ample vector bundle  $F$ . If  $F$  is a line bundle, then  $X - Z(\sigma)$  is affine; hence  $H^i(X - Z(\sigma)) = 0$  for  $i \geq \dim X + 1$ , and this easily leads to a proof that  $Z(\sigma)$  is connected if  $\dim X \geq 2$ . When  $f = \text{rk}(F) > 1$ ,  $\sigma$  determines a section  $\sigma^*$  of the line bundle  $O(1)$  on  $\mathbf{P}(F^*)$ . Since  $\mathbf{P}(F^*) - Z(\sigma^*)$  fibres over  $X - Z(\sigma)$  with fibres  $\mathbf{C}^{f-1}$ , and since  $O(1)$  is

ample on  $\mathbf{P}(F^*)$  by the ampleness of  $F$ , one deduces that  $H^i(X - Z(\sigma)) = 0$  for  $i \geq \dim X + f$ ; the connectedness of  $Z(\sigma)$  follows when  $\dim X \geq f + 1$ . This argument was used by Sommese [15], and an elaboration of this construction plays an important role in the proof of Theorem II.

*Acknowledgements.* We have benefitted from conversations with E. Arbarello, D. Eisenbud and K. Vilonen. We are particularly grateful to J. Harris for his suggestions regarding the application to special divisors.

### § 1. Connectedness of degeneracy loci

We start with some notation. If  $A$  is a vector bundle on a variety  $X$ ,  $A(x)$  denotes the fibre of  $A$  at a point  $x \in X$ .  $\mathbf{P}(A)$  is the projective bundle whose fibre over  $x \in X$  is the projective space of one-dimensional subspaces of  $A(x)$ . Finally, recall that by definition,  $A$  is *ample* if the line bundle  $\mathcal{O}_{\mathbf{P}(A^*)}(1)$  on  $\mathbf{P}(A^*)$  is ample [7].

This section is devoted to the proof of

**THEOREM 1.1.** *Let  $X$  be an irreducible complex projective variety of dimension  $n$ , and let*

$$\sigma: E \rightarrow F$$

*be a homomorphism of vector bundles on  $X$  of ranks  $e$  and  $f$ . Assume that the vector bundle*

$$E^* \otimes F = \text{Hom}(E, F)$$

*is ample, and let*

$$D_k(\sigma) = \{x \in X \mid \text{rank } \sigma(x) \leq k\}.$$

*Then:*

(a)  $D_k(\sigma)$  is non-empty if  $n \geq (e - k)(f - k)$ ,

*and*

(b)  $D_k(\sigma)$  is connected when  $n > (e - k)(f - k)$ .

*Proof.* To begin with, note that it suffices to prove the theorem for normal varieties. For if  $\nu: \tilde{X} \rightarrow X$  is the normalization of  $X$ , then  $\nu$  gives rise to a homomorphism  $\bar{\sigma}: \nu^*(E) \rightarrow \nu^*(F)$  of vector bundles on  $\tilde{X}$ , and  $D_k(\bar{\sigma})$  surjects onto  $D_k(\sigma)$ . Moreover  $\nu^*(E^*) \otimes \nu^*(F)$  is the pull-back of an ample vector bundle under a finite morphism, and hence is ample. Thus, we may assume that  $X$  is normal. Furthermore, replacing  $\sigma$  if necessary by its transpose, we may suppose that  $f \geq e$ .

Put  $k' = e - k$ , and let

$$\pi: \mathbf{G} = G_{k'}(E) \rightarrow X$$

be the Grassmannian bundle of  $k'$ -planes in  $E$ . Denote by  $S$  the rank  $k'$  tautological sub-bundle of  $\pi^*E$  on  $\mathbb{G}$ . Then there is a natural homomorphism

$$\tau: S \rightarrow \pi^*F,$$

defined as the composition  $S \hookrightarrow \pi^*E \xrightarrow{\pi^*\sigma} \pi^*F$ . Thus  $\tau$  takes a  $k'$ -plane in  $E(x)$  to its image under  $\sigma(x)$ . Let

$$Y \subseteq \mathbb{G}$$

denote the zero-locus of  $\tau$  (i.e. the zero-set of the corresponding section of  $S^* \otimes \pi^*F$ ). Then  $Y$  surjects onto  $D_k(\sigma)$ , so it is enough to show that  $Y$  is non-empty and connected in the appropriate range of dimensions. To this end, we will study the cohomology of  $\mathbb{G} - Y$ . Specifically, letting  $r = n + k(e - k) = \dim \mathbb{G}$ , we will prove

**PROPOSITION 1.2.** *If  $i \geq r + k'f = n + (f + k)(e - k)$ , then*

$$H^i(\mathbb{G} - Y; \mathbf{Z}) = 0.$$

Observe that the proposition indeed implies the theorem. For if  $n \geq (e - k)(f - k)$ , then

$$r = n + k(e - k) \geq (e - k)f,$$

and so  $H^{2r}(\mathbb{G} - Y; \mathbf{Z}) = 0$ . In particular,  $\mathbb{G} - Y$  is not compact. Therefore  $Y$ , and hence also  $D_k(\sigma)$ , must be non-empty. Similarly, if  $n > (e - k)(f - k)$ , then the proposition implies that  $H_{2r-1}(\mathbb{G} - Y; \mathbf{Q}) = 0$ . Since  $X$ —and thus  $\mathbb{G}$ —is normal, the following lemma yields the vanishing of  $H^1(\mathbb{G}, Y; \mathbf{Q})$ , and hence the connectivity of  $Y$ .

**LEMMA 1.3.** *Let  $G$  be a normal projective variety of dimension  $r$ , and let  $Y \subseteq G$  be a closed algebraic subset. Then there is an injection*

$$H^1(G, Y; \mathbf{Q}) \hookrightarrow H_{2r-1}(G - Y; \mathbf{Q}).$$

*Proof.* This follows from the exact sequence of low degree terms of the Zeeman spectral sequence (cf. [13]). Q.E.D.

Lemma 1.3 was pointed out to us by K. Vilonen. When  $X$  is smooth, one can use Lefschetz duality on  $\mathbb{G}$  in place of Lemma 1.3.

The proof of Proposition 1.2 depends on a simple construction, which we now describe. Consider a homomorphism

$$h: A \rightarrow B$$

of vector bundles of ranks  $a$  and  $b$  on a variety  $W$ . Let  $Y \subseteq W$  be the zero-locus of  $h$ ,  $h$  being

considered as a section of the vector bundle  $A^* \otimes B = \text{Hom}(A, B)$ . Now on the projective bundle  $p: \mathbf{P} = \mathbf{P}(\text{Hom}(B, A)) \rightarrow W$ , there is a “tautological” map

$$p^*B \rightarrow p^*A \otimes \mathcal{O}_{\mathbf{P}}(1).$$

Composing this with  $p^*h: p^*A \rightarrow p^*B$ , one obtains a homomorphism  $h^*: p^*A \rightarrow p^*A \otimes \mathcal{O}_{\mathbf{P}}(1)$ . The trace of  $h^*$  then defines a section

$$\text{tr}(h^*) \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1)).$$

Denote by  $Y^* \subseteq \mathbf{P}$  the zero-locus of  $\text{tr}(h^*)$ . Concretely, one may think of a point in  $\mathbf{P}(\text{Hom}(B, A))$  as a homomorphism  $\varphi: B(x) \rightarrow A(x)$  (defined modulo scalars), where  $x = p(\varphi) \in W$ , and then

$$Y^* = \{\varphi \in \mathbf{P}(\text{Hom}(B, A)) \mid \text{tr}(\varphi \circ h(p(\varphi))) = 0\}.$$

The point to observe is that the projection  $p$  gives rise to a map

$$\mathbf{P} - Y^* \rightarrow W - Y$$

which is locally trivial, with fibres  $\mathbb{C}^{ab-1}$ . In particular,

$$H^*(W - Y) \cong H^*(\mathbf{P} - Y^*) \tag{1.4}$$

is an isomorphism. (Compare [15].)

*Proof of Proposition 1.2.* We first apply the construction just described to the homomorphism  $\tau: S \rightarrow \pi^*F$  on  $\mathbf{G}$ . Thus we consider the projective bundle

$$p: \mathbf{P} = \mathbf{P}(\text{Hom}(\pi^*F, S)) \rightarrow \mathbf{G},$$

and the section  $\text{tr}(\tau^*) \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1))$ . Let  $Y^* \subseteq \mathbf{P}$  denote the zero locus of  $\text{tr}(\tau^*)$ , and set

$$V^* = \mathbf{P} - Y^*.$$

In view of (1.4), the proposition is equivalent to

$$H^i(V^*; \mathbf{Z}) = 0 \quad \text{for } i \geq \dim V^* + 1. \tag{1.5}$$

On the other hand, consider the projective bundle  $q: \mathbf{P}' = \mathbf{P}(\text{Hom}(F, E)) \rightarrow X$  over  $X$ . Then there is a natural map  $g: \mathbf{P} \rightarrow \mathbf{P}'$ , which has the following concrete description. We may think of a point in  $\mathbf{P}$  as a homomorphism  $\varphi: F(x) \rightarrow S(x)$  (mod scalars), where  $x = \pi \circ p(\varphi) \in X$ ,

and  $S(x) \subseteq E(x)$  is a subspace of dimension  $k'$ . Then  $g(\varphi) \in \mathbf{P}'$  is represented by the composition  $F(x) \rightarrow S(x) \hookrightarrow E(x)$ . Let  $Z^* \subseteq \mathbf{P}'$  denote the divisor of the section  $\text{tr}(\sigma^*) \in \Gamma(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}(1))$  arising from  $\sigma: E \rightarrow F$ . Then  $g^{-1}(Z^*) = Y^*$ , and so  $g$  restricts to a proper morphism

$$h: V^* = \mathbf{P} - Y^* \rightarrow \mathbf{P}' - Z^* = U^*,$$

where  $U^*$  denotes the complement of  $Z^*$  in  $\mathbf{P}'$ . But since  $\text{Hom}(E, F)$  is an ample vector bundle on  $X$ , the line bundle  $\mathcal{O}_{\mathbf{P}'}(1)$  is ample on  $\mathbf{P}' = \mathbf{P}(\text{Hom}(E, F)^*)$ . Thus  $U^*$  is an affine variety. The strategy now is to use the Leray spectral sequence for  $h$  to deduce (1.5) from a theorem on the vanishing of the cohomology of an affine variety.

To this end, we analyze the fibres of  $h$ . For  $0 \leq l \leq k' - 1$ , define a subvariety  $\mathbf{P}'_l \subseteq \mathbf{P}'$  by

$$\mathbf{P}'_l = \{\varphi \in \mathbf{P}(\text{Hom}(F, E)) \mid \text{rank } \varphi \leq k' - l\},$$

and let

$$U^*_l = U^* \cap \mathbf{P}'_l.$$

Then each  $U^*_l$  is an affine variety, with  $U^*_{i+1} \subseteq U^*_i$ , and

$$\text{codim}_{U^*_0} U^*_l = 2kl + l^2 + (f - e)l. \quad (1)$$

If  $\varphi' \in \mathbf{P}'$  is represented by a homomorphism  $\varphi': F(x) \rightarrow E(x)$  ( $x = q(\varphi') \in X$ ), and if  $\varphi \in \mathbf{P}$  is represented by  $\varphi: F(x) \rightarrow S(x)$ , where  $S(x)$  is a  $k'$ -plane in  $E(x)$ , then  $\varphi \in h^{-1}(\varphi')$  if and only if  $\varphi'$  coincides (mod scalars) with the composition  $F(x) \rightarrow S(x) \hookrightarrow E(x)$ . Hence  $h$  maps  $V^*$  birationally onto  $U^*_0$ . Moreover the fibre of  $h$  over a point  $\varphi' \in U^*_i - U^*_{i+1}$  is a Grassmannian  $G(l, l+k)$ , and so has dimension  $lk$ . Since  $\text{codim}_{U^*_0} U^*_i \geq 2lk$ , the following lemma applies to the map  $h: V^* \rightarrow U^*_0$  to yield (1.5).

**LEMMA 1.6.** *Let  $f: X \rightarrow Y$  be a proper surjective morphism of irreducible varieties, with  $Y$  affine. Assume that for each  $d \geq 0$  the set*

$$Y_d = \{y \in Y \mid \dim f^{-1}(y) \geq d\}$$

*has codimension  $\geq 2d$  in  $Y$  (so that in particular  $f$  is generically finite). Then*

$$H^i(X, \mathbf{Z}) = 0 \quad \text{for } i \geq \dim X + 1.$$

*Proof.* Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \mathbf{Z}) \Rightarrow H^{p+q}(X, \mathbf{Z})$$

(1) Recall that we are assuming that  $f \geq e$ .

for  $f$ . Since  $f$  is proper, the sheaves

$$R^{2d-1}f_*\mathbf{Z}, R^{2d}f_*\mathbf{Z}$$

are supported on  $Y_d$ . But these are constructible sheaves on the affine variety  $Y_d$ , and so their cohomology vanishes in degrees above the dimension of  $Y_d$  (cf. Artin [2, Exp. XIV, § 2, § 3], [16]):

$$H^p(Y, R^{2d-1}f_*\mathbf{Z}) = H^p(Y, R^{2d}f_*\mathbf{Z}) = 0 \quad \text{for } p \geq \dim Y_d + 1.$$

Since  $\dim Y_d + 2d \leq \dim Y = \dim X$ , one thus has  $E_2^{p,q} = 0$  for  $p + q \geq \dim X + 1$ , and the lemma follows. Q.E.D.

This completes the proof of Theorem 1.1.

*Remark 1.7.* At least if  $X$  is non-singular, the same proof works in characteristic  $p$  using étale cohomology in place of singular cohomology. In the part of the argument preceding Lemma 1.3,  $H_c^1(\mathbb{G} - Y)$  is used in place of  $H^1(\mathbb{G}, Y)$ , and Poincaré duality between  $H_c^1(\mathbb{G} - Y)$  and  $H^{2r-1}(\mathbb{G} - Y)$  replaces Lemma 1.3 (cf. Deligne [2, Exp. XVIII]).

*Remark 1.8.* The locus  $D_k(\sigma)$  has a natural scheme structure, given locally by the vanishing of the  $(k + 1) \times (k + 1)$  minors of  $\sigma$ . If  $X$  is non-singular, and  $D_k(\sigma)$  has the expected codimension  $(e - k)(f - k)$ , then  $D_k(\sigma)$  is a Cohen-Macaulay scheme [10]. If, in addition,  $D_k(\sigma)$  is non-singular in codimension one—for example, if  $D_{k-1}(\sigma)$  is the singular locus of  $D_k(\sigma)$ —then the connectivity of  $D_k(\sigma)$  is equivalent to the *irreducibility* of  $D_k(\sigma)$ . Indeed, if  $S$  is the singular locus of  $D_k(\sigma)$ , and  $D_k(\sigma)$  is connected, and if the local ring of  $D_k(\sigma)$  has depth at least two at every point of  $S$ , then a theorem of Hartshorne's [6] asserts that  $D_k(\sigma) - S$  is connected.

*Remark 1.9.* One expects a connectedness theorem such as Theorem 1.1 to extend to a Lefschetz-type result on the vanishing of higher relative homology or homotopy groups (cf. [3, § 9]). If  $X$  is non-singular, Proposition 1.2 and duality give (with notation as before)

$$H_i(\mathbb{G}, Y; \mathbf{Z}) = 0 \quad \text{for } i \leq n - (e - k)(f - k). \tag{*}$$

The corresponding groups  $H^i(X, D_k(\sigma))$  need not vanish, however. For example, the Segre variety  $\mathbf{P}^n \times \mathbf{P}^1$  in  $X = \mathbf{P}^{2n+1}$  is the degeneracy locus  $D_1(\sigma)$ , where  $\sigma$  is a  $2 \times (n + 1)$  matrix of linear forms, but  $H_3(X, D_1(\sigma)) \neq 0$ , even for large  $n$ . Nonetheless there is a partial result, namely that the canonical map

$$H_i(D_k(\sigma); \mathbf{Z}) \rightarrow H_i(X; \mathbf{Z})$$

is *surjective* for  $i \leq n - (e - k)(f - k)$ . This follows from (\*) and the observation that in the diagram

$$\begin{array}{ccc} H_i(Y; \mathbf{Z}) & \longrightarrow & H_i(G; \mathbf{Z}) \\ \downarrow & & \downarrow \\ H_i(D_k(\sigma); \mathbf{Z}) & \longrightarrow & H_i(X; \mathbf{Z}), \end{array}$$

the right-hand vertical map is always surjective.

We close this section with a simple application of Theorem 1.1. Given a morphism  $f: X^n \rightarrow Y^m$  of non-singular varieties, with  $m \geq n$ , let  $df_x$  denote the induced map of tangent spaces at  $x \in X$ . Set

$$S_i(f) = \{x \in X \mid \text{rank } df_x \leq n - i\}.$$

Theorem 1.1 may be applied to the vector bundle homomorphism

$$df: T_X \rightarrow f^*T_Y.$$

If  $\Omega_X^1 \otimes f^*T_Y$  is ample, then  $S_i(f)$  will be non-empty if  $i(m - n + i) \leq n$ , and connected if  $i(m - n + i) < n$ . These hypotheses are satisfied, for example, if  $Y = \mathbf{P}^m$ , if  $\Omega_X^1$  is generated by its global sections, and if  $f: X \rightarrow \mathbf{P}^m$  is *any* finite morphism. The following result applies to more general varieties  $X$ .

**PROPOSITION 1.10.** *Let  $L$  be a very ample line bundle on a smooth  $n$ -dimensional variety  $X$ . Let  $f: X \rightarrow \mathbf{P}^m$  be the morphism defined by a base-point free linear system in  $|L^{\otimes k}|$ , for some  $k \geq 2$ . Then  $S_i(f)$  is non-empty if  $i(m - n + i) \leq n$ , and connected if  $i(m - n + i) < n$ .*

*Proof.* Let  $X \hookrightarrow \mathbf{P}^N$  be the embedding defined by  $L$ , so that  $f: X \rightarrow \mathbf{P}^m$  is given by homogeneous forms  $F_0, \dots, F_m$  of degree  $k$  in the coordinates  $X_0, \dots, X_N$  of  $\mathbf{P}^N$ . Consider the Euler sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^N} \rightarrow \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus(N+1)} \rightarrow T_{\mathbf{P}^N} \rightarrow 0.$$

Define  $E$  to be the kernel of the composition  $L^{\oplus(N+1)} \rightarrow T_{\mathbf{P}^N}|_X \rightarrow N_{X|\mathbf{P}^N}$ , and let  $F = (L^{\otimes k})^{\oplus(m+1)}$ . The Jacobian matrix  $(\partial F_i / \partial X_j)$  determines a morphism  $J: E \rightarrow F$  so that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & E & \longrightarrow & T_X \longrightarrow 0 \\ & & \downarrow \cdot k & & \downarrow J & & \downarrow df \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & F & \longrightarrow & f^*T_{\mathbf{P}^m} \longrightarrow 0 \end{array}$$

commutes. Then  $S_i(f) = D_{n+1-i}(J)$ , and  $E^* \otimes F$  is ample, since it is a quotient of a direct sum of copies of the ample line bundle  $L^* \otimes L^{\otimes k} = L^{\otimes(k-1)}$ . Q.E.D.



§ 2. Application to special divisors

Let  $C$  be a non-singular complex projective curve of genus  $g$ , let  $J = \text{Pic}^0(C)$  be the Jacobian of  $C$ , and fix once and for all a base-point  $P_0 \in C$ . Given  $x \in J$ , denote by  $L_x$  the corresponding line bundle of degree zero on  $C$ , and set

$$W_d^r = \{x \in J \mid h^0(C, L_x \otimes \mathcal{O}_C(dP_0)) \geq r + 1\}.$$

Thus  $W_d^r$  parametrizes classes of divisors of degree  $d$  which move in a linear system of (projective) dimension at least  $r$ . By a well-known construction which we review below, if  $m \gg 0$  is a sufficiently large integer, then evaluation at  $t$  distinct points  $P_1, \dots, P_t$  yields a map

$$H^0(C, L_x \otimes \mathcal{O}_C(mP_0)) \rightarrow \bigoplus_{i=1}^t \mathbb{C}(P_i)$$

which globalizes to a homomorphism of vector bundles

$$\sigma_t: E_m \rightarrow F_t \tag{2.1}$$

on  $J$ , of ranks  $m + 1 - g$  and  $t$  respectively. Noting that

$$\ker \sigma(x) = H^0(C, L_x \otimes \mathcal{O}_C(mP_0 - \sum P_i)),$$

and taking  $t = m - d$ , one sees that  $W_d^r \cong D_k(\sigma_{m-d})$ , where  $k = m - g - r$ . In particular, the expected dimension of  $W_d^r$  is given by the Brill-Noether number

$$\rho = g - (r + 1)(g - d + r).$$

For our purposes, the basic fact is

LEMMA 2.2. *For any  $m \geq 2g$  and  $t \geq 1$ ,  $E_m^* \otimes F_t$  is an ample vector bundle on  $J$ .*

Grant the lemma for the moment. Then Theorem 1.1 (a) yields the result of Kempf [9] and Kleiman-Laksov [11, 12] that  $W_d^r$  is non-empty when  $\rho \geq 0$ , while Theorem 1.1 (b) implies

THEOREM 2.3.  *$W_d^r$  is connected if  $\rho > 0$ .*

Furthermore, by Remark 1.9, the homomorphisms

$$H_i(W_d^r; \mathbb{Z}) \rightarrow H_i(J; \mathbb{Z})$$

induced by inclusion are surjective for  $i \leq \rho$ .

On special curves, the loci  $W_d^r$  may well be reducible even when the Brill-Noether number  $\rho$  is positive. For example if  $C$  is trigonal and non-hyperelliptic, of genus 5, then  $W_4^1$  has

two irreducible (but intersecting!) one-dimensional components. In fact, if  $D$  is a trigonal divisor, then  $W_4^1$  is swept out by the two families

$$\{|D+P|\}_{P \in C}, \{|K-D-P|\}_{P \in C}.$$

On a *generic* curve of genus  $g$ , however, this cannot happen. Indeed, Griffiths and Harris [5] have shown that for a general curve,  $W_d^r$  has pure dimension  $\rho$ , and Gieseker [4] has very recently proved that, for general  $C$ ,  $W_d^r$  is singular only along  $W_d^{r+1}$ . Hence from Remark 1.8 one deduces

**COROLLARY 2.4.** *If  $C$  is a generic curve, then  $W_d^r$  is irreducible when  $\rho > 0$ .*

The remainder of this section is devoted to the proof of Lemma 2.2. To fix notation, we start by reviewing the construction of the vector bundle homomorphism (2.1). Let  $\mathcal{L}$  be the universal line bundle on  $J \times C$ , so that  $\mathcal{L}|_{p^{-1}(x)} = L_x$ , where  $p: J \times C \rightarrow J$  and  $q: J \times C \rightarrow C$  are the projections. Letting  $\mathcal{L}(P)$  denote the line bundle  $\mathcal{L}|_{q^{-1}(P)}$  on  $J$ , we normalize  $\mathcal{L}$  (by tensoring with a line bundle from  $J$ ) so that  $\mathcal{L}(P_0) = \mathcal{O}_J$ . Fix  $m \geq 2g - 1$ ; thus  $H^1(C, L_x \otimes \mathcal{O}_C(mP_0)) = 0$  for all  $x \in J$ . Then

$$E_m \stackrel{\text{def}}{=} p_*(\mathcal{L} \otimes q^* \mathcal{O}_C(mP_0))$$

is a vector bundle of rank  $m + 1 - g$ , and pushing forward  $\mathcal{L} \otimes q^* \mathcal{O}_C(mP_0)$  via  $p_*$  commutes with base change. Let  $D_t$  be the divisor  $P_1 + \dots + P_t$  on  $C$ , and set

$$F_t = \bigoplus_{i=1}^t \mathcal{L}(P_i) = p_*(\mathcal{L} \otimes q^* \mathcal{O}_{D_t}).$$

The homomorphism (2.1) arises by taking the direct images of the last two terms in the exact sequence

$$0 \rightarrow \mathcal{L} \otimes q^* \mathcal{O}_C(mP_0 - D_t) \rightarrow \mathcal{L} \otimes q^* \mathcal{O}_C(mP_0) \rightarrow \mathcal{L} \otimes q^* \mathcal{O}_{D_t} \rightarrow 0 \tag{2.5}$$

of sheaves on  $J \times C$ .

Turning to the verification that

$$E_m^* \otimes F_t = \bigoplus_{i=1}^t E_m^* \otimes \mathcal{L}(P_i)$$

is an ample vector bundle, the first point to observe is that it suffices to prove the ampleness of  $E_m^*$ . Indeed, since a direct sum of vector bundles is ample if and only if each summand is [7], it is certainly enough to check the ampleness of  $E^* \otimes \mathcal{L}(P)$  for an arbitrary point  $P \in C$ . Since  $\mathcal{L}(P_0) = \mathcal{O}_J$ , by varying  $P$  over  $C$  one exhibits  $\mathcal{L}(P)$  as a deformation of a trivial

line bundle. But quite generally, if  $X$  is an irreducible projective variety, and  $L$  a line bundle on  $X$  algebraically equivalent to zero, then a vector bundle  $E$  on  $X$  is ample if and only if  $E \otimes L$  is. For if  $\pi: \mathbf{P}(E^*) \rightarrow X$  is the projectivization of  $E^*$ , then we may identify  $\mathbf{P}(E^*)$  with  $\mathbf{P}(E^* \otimes L^*)$ , and the assertion becomes that  $\mathcal{O}_{\mathbf{P}}(1)$  is an ample line bundle on  $\mathbf{P}(E^*)$  if and only if  $\mathcal{O}_{\mathbf{P}}(1) \otimes \pi^*L^*$  is. Observing that  $c_1(\mathcal{O}_{\mathbf{P}}(1) \otimes \pi^*L^*)$  is numerically equivalent to  $c_1(\mathcal{O}_{\mathbf{P}}(1))$ , this follows from Nakai's criterion (cf. [8, Chapter I]).

The ampleness of  $E_m^*$  is equivalent, by definition, to the ampleness of the line bundle  $\mathcal{O}_{\mathbf{P}}(1)$  on  $\mathbf{P} = \mathbf{P}(E_m)$ . Letting  $t=1$  and  $D_t = P_0$  in (2.5) and taking direct images on  $J$ , one obtains (for  $m \geq 2g$ ) an exact sequence

$$0 \rightarrow E_{m-1} \rightarrow E_m \rightarrow \mathcal{O}_J \rightarrow 0.$$

Such a sequence gives rise to a section  $s$  of  $\mathcal{O}_{\mathbf{P}}(1)$  whose divisor is exactly the subvariety  $\mathbf{P}(E_{m-1}) \subseteq \mathbf{P}(E_m)$ . The crucial geometric fact is then the following (cf. [14] or [1]):

$\mathbf{P}(E_m)$  is isomorphic to  $C_m$ , the  $m$ -th symmetric product of  $C$ , and the divisor  $\mathbf{P}(E_{m-1})$  corresponds to the  $(m-1)$ -th symmetric product  $C_{m-1}$ , embedded in  $C_m$  (2.6) via the map  $D \rightarrow D + P_0$ .

Granting (2.6), the ampleness of  $\mathcal{O}_{\mathbf{P}}(1)$ —and hence Lemma 2.2—follows from

LEMMA 2.7. For all  $m \geq 1$ ,  $C_{m-1}$  is an ample divisor on  $C_m$ .

*Proof.* Fix a  $k$ -dimensional subvariety  $V \subseteq C_m$ . By Nakai's criterion, it suffices to show that the intersection number  $\zeta^k \cdot V$  is positive, where  $\zeta$  denotes the numerical equivalence class of  $C_{m-1}$ . For any  $P \in C$ , let  $C_{m-1}(P)$  be the divisor on  $C_m$  obtained by embedding  $C_{m-1}$  in  $C_m$  via the map  $D \rightarrow D + P$ . Then  $C_{m-1} = C_{m-1}(P_0)$  is algebraically—and hence numerically—equivalent to  $C_{m-1}(P)$  for any  $P \in C$ . Now given  $V$  as above, then  $C_{m-1}(P)$  meets  $V$  in a non-empty divisor on  $V$  for almost every  $P \in C$ . (Indeed, this is simply the assertion that given any  $k$ -dimensional family of divisors on  $C$ , almost every  $P \in C$  is contained in some but not all of the divisors in the family.) Therefore, for  $k$  generic points  $P_1, \dots, P_k \in C$ , the intersection

$$C_{m-1}(P_1) \cap \dots \cap C_{m-1}(P_k) \cap V$$

is a finite non-empty set. Since  $\zeta^k \cdot V$  is the degree of the corresponding intersection cycle, the positivity of  $\zeta^k \cdot V$  follows. Q.E.D.

Alternatively, one could prove the lemma by observing that if  $w: C^m \rightarrow C_m$  is the map to  $C_m$  from the Cartesian product of  $C$  with itself  $m$  times, then  $w^* \mathcal{O}(C_{m-1}) = \otimes_{i=1}^m \pi_i^* \mathcal{O}_C(P_0)$ ,

where  $\pi_i: C^m \rightarrow C$  is projection onto the  $i$ th factor. Thus  $w^*O(C_{m-1})$  is ample. But since  $w: C^m \rightarrow C_m$  is finite and surjective, the ampleness of  $O(C_{m-1})$  is equivalent to that of  $w^*O(C_{m-1})$ .

Finally, for the convenience of the reader, we sketch a proof of (2.6). Let

$$u: C_m \rightarrow J$$

be the Abel map, which takes a divisor  $D$  of degree  $m$  to the divisor class of  $D - mP_0$ . We first show that there is a morphism  $v: C_m \rightarrow \mathbf{P}(E_m)$ , compatible with projections to  $J$ , such that

$$C_m \times_{\mathbf{P}(E_m)} \mathbf{P}(E_{m-1}) = C_{m-1}.$$

To this end, let  $\mathcal{D}_m \subseteq C_m \times C$  be the universal divisor of degree  $m$ :  $\mathcal{D}_m = \{(D, P) | P \in D\}$ . Bearing in mind the chosen normalization of  $\mathcal{L}$ , it follows from the universal property of  $J$  that

$$(u \times 1)^*(\mathcal{L} \otimes q^*(mP_0)) = O(\mathcal{D}_m) \otimes p'^*O_{C_m}(-C_{m-1}),$$

where  $p': C_m \times C \rightarrow C_m$  denotes the projection. Since taking the direct image of  $\mathcal{L} \otimes q^*O_c(mP_0)$  on  $J$  commutes with base-change, one then has

$$p'_*O(\mathcal{D}_m) \otimes O_C(-C_{m-1}) = u^*E_m.$$

Now the canonical section of  $O(\mathcal{D}_m)$  gives rise to a nowhere vanishing section of  $p'_*O(\mathcal{D}_m)$ , and so one obtains an inclusion

$$O(-C_{m-1}) \hookrightarrow u^*E_m$$

of vector bundles on  $C_m$ . But this is equivalent to giving a morphism  $v: C_m \rightarrow \mathbf{P}(E_m)$  over  $J$  such that  $v^{-1}(\mathbf{P}(E_{m-1})) = C_{m-1}$ . Note next that  $v$  is bijective. Indeed, it suffices to check this fibre by fibre over  $J$ , where it is clear. Then since  $\mathbf{P}(E_m)$  is smooth, it follows that  $v$  is an isomorphism. (In positive characteristic, one would observe in addition that since  $C_m \times_{\mathbf{P}(E_m)} \mathbf{P}(E_{m-1}) = C_{m-1}$  is smooth,  $v$  must be separable.) Q.E.D.

*Remark 2.8.* The proof of Lemma 2.2 works for curves over an arbitrary algebraically closed field. Granting the results of [4] and [5] in positive characteristic, and making use of Remark 1.7, one deduces that Theorem 2.3 and Corollary 2.4 are also valid in arbitrary characteristic.

### References

- [1] ARBARELLO, E., CORNALBA, M., GRIFFITHS, P. & HARRIS, J., *Geometry of Algebraic Curves*. To appear.
- [2] ARTIN, M., GROTHENDIECK, A. & VERDIER, J. L., *Théorie des topos et cohomologie étale des schémas*. Springer Lecture Notes in Math. 305 (1973). [SGA4].

- [3] FULTON, W. & LAZARSFELD, R., Connectivity and its applications in algebraic geometry. To appear in the *Proceedings of the First Midwest Algebraic Geometry Conference* (Springer Lecture Notes).
- [4] GIESEKER, D., Stable curves and special divisors, I. Preprint.
- [5] GRIFFITHS, P. & HARRIS, J., On the variety of special linear systems on a general algebraic curve. *Duke Math. J.*, 47 (1980), 237–272.
- [6] HARTSHORNE, R., Complete intersections and connectedness. *Amer. J. Math.*, 84 (1962), 497–508.
- [7] ——— Ample vector bundles. *Publ. Math. IHES*, 29 (1966), 63–94.
- [8] ——— *Ample subvarieties of algebraic varieties*. Springer Lecture Notes in Math. 156 (1970).
- [9] KEMPF, G., *Schubert methods with an application to algebraic curves*. Publ. Math. Centrum, Amsterdam, 1971.
- [10] KEMPF, G. & LAKSOV, D., The determinantal formula of Schubert calculus. *Acta Math.*, 132 (1974), 153–162.
- [11] KLEIMAN, S. & LAKSOV, D., On the existence of special divisors. *Amer. J. Math.*, 94 (1972), 431–436.
- [12] ——— Another proof of the existence of special divisors. *Acta Math.*, 132 (1974), 163–176.
- [13] MCCRORY, C., Zeeman's filtration of homology. *Trans. Amer. Math. Soc.*, 250 (1979), 147–166.
- [14] SCHWARTZENBERGER, R. L. E., Jacobians and symmetric products. *Illinois J. Math.*, 7 (1963), 257–268.
- [15] SOMMESE, A., Subvarieties of abelian varieties. *Math. Ann.*, 233 (1978), 229–256.
- [16] VILONEN, K., *The cohomology of affine varieties*. Masters thesis at Brown University, June 1980.

*Received December 19, 1980*