Journal of Algebraic Combinatorics **11** (2000), 269–275

On the Connection Between Macdonald Polynomials and Demazure Characters

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Received March 30, 1998; Revised November 17, 1998; Accepted December 21, 1998

Abstract. We show that the specialization of nonsymmetric Macdonald polynomials at $t = 0$ are, up to multiplication by a simple factor, characters of Demazure modules for $\widehat{sl(n)}$. This connection furnishes Lie-theoretic proofs of the nonnegativity and monotonicity of Kostka polynomials.

Keywords: affine Lie algebras, Macdonald polynomials, Demazure character

1. Introduction

Macdonald defined a special class of polynomials $P_\lambda(z, q, t)$, called *symmetric Macdonald polynomials*, which form a basis of the symmetric polynomials in $\mathbb{C}(q, t)[z_1, \ldots, z_n]$. These polynomials are indexed by partitions $\lambda \in \mathbb{N}^n$, $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq 0$. They interpolate between several classes of classical polynomials: $P_\lambda(z, 0, t)$ are the Hall-Littlewood polynomials, which, in turn are the Schur functions when $t = 0$. By setting $q = t^\alpha$ and letting t go to 1, one obtains Jack polynomials. In [11], Macdonald mentions that there is no similar interpretation of $P_\lambda(z, q, 0)$. By using the theory of nonsymmetric Macdonald polynomials, we show that the $P_\lambda(z, q, 0)$ are the characters (up to factor) of certain Demazure modules of $\widehat{sl(n)}$. This interpretation allows us to obtain Lie-theoretic proofs of the nonnegativity and monotonicity of Kostka polynomials. In addition, it gives us a branching rule for the decomposition of certain integrable highest weight $\widehat{sl(n)}$-modules under the action of $sl(n)$.

The connection between Demazure characters and symmetric functions has already been explored in [8] using a path realization of the crystal basis. The results in this paper intersect somewhat with those in [8]. The main advantage of our approach is its simplicity and its explanation of the connection with Macdonald polynomials. Nonnegativity and positivity of Kostka polynomials have already been proven by Lascoux-Schützenberger [9], Butler [1], Lusztig [10]. The connection between the branching rule and Kostka polynomials was explored in [5]. A different representation-theoretic interpretation of $P_\lambda(z, q, 0)$ is given in [4].

2. Nonsymmetric Macdonald polynomials

These nonsymmetric analogues of the symmetric Macdonald polynomials were first introduced in [12, 14]. Nonsymmetric Macdonald polynomials $E_\lambda(z, q, t)$ are indexed by compositions $\lambda \in \mathbb{N}^n$ and form a basis of $\mathbb{C}(q, t)[z_1, \ldots, z_n]$. (See [2, 5] for their precise definition). In [6], Knop gives a recursive description of the $E_\lambda(z, q, t)$. We describe this recursion for when $t = 0$. In this case, we have $E_\lambda(z, q, 0) \in \mathbb{Z}[q, q^{-1}][z_1, \ldots, z_n]$. For ease of notation, we will denote $E_\lambda(z, q, 0)$ simply by E_λ from now on. For $i \in [1, \ldots, n-1]$ let s_i be the simple reflection that interchanges z_i and z_{i+1}. Consider the following operators on $\mathbb{Z}[q, q^{-1}][z_1, \ldots, z_n]$:

$$\begin{aligned} \bar{H}_i &:= s_i - z_{i+1}\frac{(1-s_i)}{(z_i - z_{i+1})} \qquad \text{for } i \in [1, \ldots, n-1] \\ \Phi f(z_1, \ldots, z_n) &:= z_n f(q^{-1}z_n, z_1, \ldots, z_{n-1}) \\ \bar{H}_0 &:= \Phi \bar{H}_1 \Phi^{-1} = \Phi^{-1}\bar{H}_{n-1}\Phi \end{aligned}$$

Then the recursion relations are given by [6]

Theorem 1 *The E_λ are generated by application of the $\bar{H}_i$ $(0 \le i < n)$ and Φ to* 1. *More precisely, set $E_{(0^n)} := 1$. The action of Φ and the $\bar{H}_i$ on the set of E_λ for $\lambda \in \mathbb{N}^n$ is as follows*:

$$q^{\lambda_1}\Phi E_{(\lambda_1,\ldots,\lambda_n)} = E_{(\lambda_2,\ldots,\lambda_n,\lambda_1+1)}$$

$$\bar{H}_i E_\lambda = \begin{cases} E_{s_i\lambda} & \textit{if } \lambda_i < \lambda_{i+1} \\ E_\lambda & \textit{if not} \end{cases} \quad \textit{for } 1 \le i \le n-1$$

where $s_i\lambda$ is the composition λ with λ_i and λ_{i+1} interchanged.

$$q^{\lambda_1-\lambda_n+1}\bar{H}_0 E_\lambda = \begin{cases} E_{(\lambda_n-1,\lambda_2,\ldots,\lambda_{n-1},\lambda_1+1)} & \textit{if } \lambda_1 > \lambda_n - 1 \\ E_\lambda & \textit{if not} \end{cases}$$

To ease notation, we define the operators $\tilde{H}_0$ and $\tilde{\Phi}$ on the set of nonsymmetric Macdonald polynomials:

$$\begin{aligned} \tilde{H}_0 E_{(\lambda_1,\ldots,\lambda_n)} &:= q^{\lambda_1-\lambda_n+1}\bar{H}_0 E_{(\lambda_1,\ldots,\lambda_n)} = E_{(\lambda_n-1,\lambda_2,\ldots,\lambda_{n-1},\lambda_1+1)} \\ \tilde{\Phi} E_{(\lambda_1,\ldots,\lambda_n)} &:= q^{\lambda_1}\Phi E_{(\lambda_1,\ldots,\lambda_n)} = E_{(\lambda_2,\ldots,\lambda_n,\lambda_1+1)} \end{aligned}$$

Although this definition of nonsymmetric Macdonald polynomials is given for only $\lambda \in \mathbb{N}^n$, we can easily extend it to compositions $\lambda \in \mathbb{Z}^n$ by defining

$$E_\lambda := \tilde{\Phi}^{-mn} E_{\lambda+(m^n)} = q^{-(m|\lambda|+nm(m+1)/2)}\Phi^{-mn} E_{\lambda+(m^n)}$$

where m is chosen large enough so that $\lambda + (m^n) = (\lambda_1 + m, \ldots, \lambda_n + m)$ is in $\mathbb{N}^n$. The E_λ are well-defined (don't depend on the choice of m). In fact, let m_1 and m_2, with $m_1 \le m_2$, be two such choices. Then,

$$\tilde{\Phi}^{-m_2n} E_{\lambda+(m_2^n)} = \tilde{\Phi}^{-m_1n}\tilde{\Phi}^{-(m_2-m_1)n} E_{\lambda+(m_2^n)} = \tilde{\Phi}^{-m_1n} E_{\lambda+(m_1^n)},$$

the last equality following from the well-definedness of the E_μ for $\mu \in \mathbb{N}^n$. Note that the E_λ are elements of $\mathbb{Z}[q, q^{-1}][z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$.

We now check that, for $\lambda \in \mathbb{Z}^n \backslash \mathbb{N}^n$, the E_λ satisfy the recursion relations. For $i \neq 0$,

$$E_{s_i \cdot \lambda} = \tilde{\Phi}^{-mn} E_{s_i \cdot \lambda + (m^n)} = \tilde{\Phi}^{-mn} \bar{H}_i \tilde{\Phi}^{mn} E_\lambda = \bar{H}_i E_\lambda$$

Now, let $\lambda^* := (\lambda_n - 1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1 + 1)$ and choose m such that $\lambda^* + (m^n) \in \mathbb{N}^n$. Then

$$E_{\lambda^*} = \tilde{\Phi}^{-mn} E_{\lambda^* + (m^n)} = \tilde{\Phi}^{-mn} \tilde{H}_0 \tilde{\Phi}^{mn} E_\lambda = q^{\lambda_1 - \lambda_n + 1} \bar{H}_0 E_\lambda,$$

the last equality following from the commutativity of the $\bar{H}_i$ with $\tilde{\Phi}^n$. This proves that, for all $\lambda \in \mathbb{Z}^n$, the E_λ satisfy the relations of Theorem 1.

Let B_m denote the $\mathbb{Z}[q, q^{-1}]$-vector space generated by all E_λ with $|\lambda| = m$. Then the $\bar{H}_i B_m \subset B_m$ for all i and $\Phi B_m \subset B_{m+1}$. The action of the $\bar{H}_i$ $(i \neq 0)$ and $\tilde{H}_0$ on the E_λ is related to the action of the affine Weyl group on compositions:

$$s_i \cdot (\lambda_1, \ldots, \lambda_n) := (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \ldots, \lambda_n)$$
$$s_o \cdot (\lambda_1, \ldots, \lambda_n) := (\lambda_n - 1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1 + 1)$$

The connection between compositions of a given degree, say m, and elements of the affine Weyl group is as follows. Let $m = kn + i$ where $k \geq 0$ and $0 \leq i < n$. Then the smallest composition is $\eta_m := (k, \ldots k, k+1, \ldots, k+1)$ (i factors of $k+1$ and $n-i$ factors of k). Every composition λ of degree m equals $w \cdot \eta_m$ where w is an affine Weyl group element.

For $w = s_{i_1}, \ldots, s_{i_j}$ a reduced decomposition, we define $\bar{H}_w := \bar{H}_{i_1}, \ldots, \bar{H}_{i_j}$ and $\tilde{H}_w$ the same expression but with $\bar{H}_0$ replaced by $\tilde{H}_0$.

For a composition $\lambda \in \mathbb{Z}^n$, let $\mathrm{u}(\lambda) := \sum_i \frac{\lambda_i(\lambda_i - 1)}{2}$.

Theorem 2 *We can write* $E_\lambda = \tilde{H}_w \tilde{\Phi}^{|\lambda|} \cdot 1 = q^{\mathrm{u}(\lambda)} \bar{H}_w \Phi^{|\lambda|} \cdot 1$ *where w is determined by* $\lambda = w\eta_{|\lambda|}$.

Proof: By the commuting relations of Φ and the $\bar{H}_i$ [6],

$$\begin{cases} \Phi \bar{H}_{i+1} = \bar{H}_i \Phi & i = 1, \ldots, n-2 \\ \Phi^2 \bar{H}_1 = \bar{H}_{n-1} \Phi^2 \end{cases}$$

we need only prove that the power of q is $\mathrm{u}(\lambda)$ by induction. For $i \geq 1$, the actions of the $\bar{H}_i$ do not involve any powers of q. The operator $\bar{H}_0$ equals $\Phi \bar{H}_1 \Phi^{-1}$ by definition. Therefore, we need only check that this holds for Φ. Let $\mu = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + 1)$. We have

$$E_\mu = q^{\lambda_n} \Phi E_\lambda = q^{\lambda_n} q^{\mathrm{u}(\lambda)} \Phi H_w \Phi^{|\lambda|} \cdot 1 = q^{\mathrm{u}(\mu)} H_{w'} \Phi^{|\mu|} \cdot 1$$

where $H_{w'}$ is determined by the above commutation relations. □

Remark We note that $\tilde{\Phi}^{nk+i} \cdot 1 = q^{\mathrm{u}(\eta_{nk+i})} (z_1, \ldots, z_n)^k z_{n-i+1} \cdots z_n$. The $\bar{H}_i$ (all i) commute with multiplication by q and the symmetric function $z_1 \cdots z_n$. Therefore, $E_\lambda = q^{\mathrm{u}(\eta_{nk+i})} (z_1, \ldots, z_n)^k \tilde{H}_w z_{n-i+1} \cdots z_n$. We will use this information in Section 4.

3. Demazure modules of $\widehat{sl(n)}$

Let Λ be a dominant integral weight. Let $V = V(\Lambda)$ be the unique (up to isomorphism) irreducible highest weight $\widehat{sl(n)}$-module with highest weight Λ. Let W be the Weyl group of $\widehat{sl(n)}$. For each $w \in W$, the weight space $V_{w(\Lambda)}$ of weight $w(\Lambda)$ is one-dimensional. We consider $E_w(\Lambda)$, the $\underline{\mathbf{b}}$-module generated by $V_{w(\Lambda)}$, where $\underline{\mathbf{b}}$ is the Borel subalgebra. The $E_w(\Lambda)$, called *Demazure modules*, are finite-dimensional vector spaces which form a filtration of V which is compatible with the Bruhat order on W: $w \le w' \Leftrightarrow E_w(\Lambda) \subseteq E_{w'}(\Lambda)$.

To each Demazure module $E_w(\Lambda)$, we can associate its character $\chi(E_w(\Lambda)$:

$$\chi(E_w(\Lambda)) := \sum_{\mu \text{ weight}} (\dim E_w(\Lambda)_\mu) e^\mu$$

Since the $E_w(\Lambda)$ are finite dimensional, the $\chi(E_w(\Lambda))$ are polynomials in the n simple roots α_i and lie in the group ring for the weight lattice P.

We now define *Demazure operators*. For each α_i, we define an operator Δ_i on P:

$$\Delta_i := \frac{1 - e^{-\alpha_i} s_i}{1 - e^{-\alpha_i}}$$

where s_i is the simple reflection with respect to α_i. Let $w = s_{i_1} s_{i_2} \cdots s_{i_j}$ be a reduced decomposition. Then, we can define $\Delta_w := \Delta_{i_1} \Delta_{i_2} \cdots \Delta_{i_j}$ and Δ_w does not depend on the choice of reduced decomposition. The connection between characters and Demazure operators is given by [3, 7, 13]:

Theorem 3 $\chi(E_w(\Lambda)) = \Delta_w(e^\Lambda)$.

4. Macdonald polynomials and Demazure module characters

Let $\Lambda_0, \ldots, \Lambda_{n-1}$ denote the n fundamental weights of $\widehat{sl(n)}$ defined by $(\Lambda_i, \alpha_j) = \delta_{ij}$. Let $\delta = \sum_{i=0}^{n-1} \alpha_i$. Let π be the ring homomorphism $\pi : \mathbb{Z}[q, q^{-1}][z_1, \ldots, z_n] \to P$ defined by: $\pi(z_i) = e^{\Lambda_i - \Lambda_{i-1}}$ for $i < n$, $\pi(z_n) = e^{\Lambda_0 - \Lambda_{n-1}}$ and $\pi(q) = e^{-\delta}$.

Theorem 4 *The operator $\bar{H}_i$ is equivalent to the Demazure operator Δ_i in the sense that the following diagram commutes*:

$$\begin{array}{ccc} \mathbb{Z}[q, q^{-1}][z_1, \ldots, z_n] & \xrightarrow{\pi} & P \\ \bar{H}_i \downarrow & & \downarrow \Delta_i(e^{\Lambda_0} \cdot \quad) \\ \mathbb{Z}[q, q^{-1}][z_1, \ldots, z_n] & \xrightarrow{\pi} & P \end{array}$$

Proof: We have that $\bar{H}_i$, $i \neq 0$ (resp. $\bar{H}_0$) commutes with multiplication by z_j for $j \neq i$ or $i+1$ (resp. z_1 or z_n). Therefore, one only needs to verify this equivalence on the monomials $z_i^a z_{i+1}^b$ (resp. $z_1^a z_n^b$). This is done by direct computation. $\square$

Let C be the following "change of basis" operator on P: $C(e^{\Lambda_0}) = e^{\Lambda_{n-1}}$ and $C(e^{\Lambda_i}) = e^{\Lambda_{i-1}-\delta}$ for $1 \leq i \leq n-1$.

Theorem 5 *The operator Φ is equivalent to the operator C in the sense that the following diagram commutes*:

$$\begin{array}{ccc} \mathbb{Z}[q,q^{-1}][z_1,\ldots,z_n] & \xrightarrow{\pi} & P \\ \Phi\downarrow & & \downarrow C \\ \mathbb{Z}[q,q^{-1}][z_1,\ldots,z_n] & \xrightarrow{\pi} & P \end{array}$$

Proof: By direct computation. □

Theorems 1, 4, 5 along with the preceding Remark give us our main result:

Theorem 6 *Through the π homomorphism, we can identify $q^{-\mathrm{u}(\lambda)+\mathrm{u}(\eta_{|\lambda|})}E_\lambda$ with $\chi(E_w(\Lambda_i))$ where $i = |\lambda| \bmod n$ and where w is an affine Weyl group element defined by $\lambda = w\eta_{|\lambda|}$.*

Proof: We have that

$$E_\lambda = q^{\mathrm{u}(\lambda)}\bar{H}_w\Phi^{|\lambda|}\cdot 1 = q^{\mathrm{u}(\lambda)-\mathrm{u}(\eta_{|\lambda|})}(z_1\cdots z_n)^k\bar{H}_w z_{n-i+1}\cdots z_n.$$

We have $\pi(z_1z_2\cdots z_n) = 1$ and $\pi(z_{n-i+1}\cdots z_n) = e^{\Lambda_i}$. Therefore,

$$\pi(E_\lambda) = q^{\mathrm{u}(\lambda)-\mathrm{u}(\eta_{|\lambda|})}\Delta_w e^{\Lambda_i}.$$

□

Remark

1. $\pi(E_\lambda)$ having nonnegative coefficients implies that E_λ has nonnegative coefficients.
2. By setting $q = 1$, one obtains the *real character* of a Demazure module (see [15]). For λ a partition, we have the factorization ([11], p. 324)

$$P_\lambda(z,1,0) = e_{\lambda'}(z) = \prod_{i=1}^{n} e_i^{\lambda_i-\lambda_{i+1}}(z)$$

where $e_i(z)$ is the ith elementary symmetric function. This gives us a similar factorization of

$$\chi(E_w(\Lambda)) = q^{-\mathrm{u}(\lambda)+\mathrm{u}(\eta_{|\lambda|})}\prod_{i=1}^{n-1} e_i(\pi(z))^{\lambda_i-\lambda_{i+1}}.$$

Previous examples of this factorization are found in [8, 15].

5. Positivity and monotonicity of Kostka polynomials

Recall that $P_\lambda(z, q, t)$ denotes the symmetric Macdonald polynomial associated to the partition λ.

Theorem 7 *For λ a partition, we have $E_\lambda(z, q, 0) = P_\lambda(z, q, 0)$.*

Proof: Consider $\sum_{w\in W} \bar{H}_w E_\lambda(z, q, t)$. It is symmetric and satisfies the same defining conditions as $P_\lambda(z, q, t)$ (see [6]), therefore is a scalar multiple of it. When $t = 0$, we have $\bar{H}_w E_\lambda(z, q, 0) = E_\lambda(z, q, 0)$. By comparing coefficients of the leading coefficient z^λ in both $E_\lambda(z, q, 0)$ and $P_\lambda(z, q, 0)$, we see that we have equality. □

Recall that one has the following order relation on partitions: two partitions γ and μ such that $|\gamma| = |\mu|$ satisfy $\gamma < \mu$ if $\gamma_1 + \cdots + \gamma_i \leq \mu_1 + \cdots + \mu_i$ for all i with strict inequality for some i.

It is known [[11], VI (8.11)] that $P_\lambda(z, q, 0) = \sum_{\mu\leq\lambda} K_{\mu\lambda}(q, 0)s_\mu(z)$ where K is the Kostka function and the s_μ are the Schur functions. In addition, it is known [[11], p. 355] that $K_{\mu\lambda}(q, 0) = K_{\mu'\lambda'}(q)$ where μ' (resp. λ') is the dual partition of μ (resp. λ). It follows that $P_\lambda(z, q, 0) = \sum_{\mu\leq\lambda} K_{\mu'\lambda'}(q)s_\mu(z)$.

Theorem 8 *The $K_{\mu\lambda}(q)$ have positive coefficients.*

Proof: We have that $P_\lambda(z, q, 0)$ is invariant under the $\bar{H}_i$ (for $i \neq 0$). This is equivalent to saying that the Demazure module $E_w(\Lambda_0)$ decomposes as a direct sum of simple $sl(n)$-modules. In fact, we have the following decomposition:

$$E_w(\Lambda_0) = \oplus_{j\in\mathbb{Z}}(E_w(\Lambda_0))_{j\delta}$$

where $(E_w(\Lambda_0))_{j\delta}$ is just the direct sum of weight spaces whose weights are of the form $\upsilon = \kappa + j\delta$ where κ is some weight for $sl(n)$. (In other words, these are all weights that satisfy $\langle\upsilon, d\rangle = j$ where d is the scaling element.) Since δ is orthogonal to the Cartan subalgebra of $sl(n)$, each $(E_w(\Lambda_0))_{j\delta}$ is a direct sum of irreducible $sl(n)$-modules. Let $\lambda = w\nu_{|\lambda|}$. The $P_\lambda(z, q, 0)$ merely represents the character $\chi(E_w(\Lambda_0))$ as seen in this light; since the $s_\mu(z)$ is a character of an irreducible $sl(n)$-module, the coefficient of q^j in $K_{\mu'\lambda'}(q)$ is the multiplicity of the $sl(n)$-module of highest weight $\mu - j\delta$ in $E_w(\Lambda_0)$. Therefore, the $K_{\mu'\lambda'}(q)$ have positive coefficients. □

Remark A consequence of this theorem is that the Kostka numbers $K_{\mu\lambda}(1)$ are the multiplicities of the (finite-dimensional) $sl(n)$-modules in the Demazure modules $E_w(\Lambda)$.

Recall that $V = V(\Lambda_i)$ is the irreducible highest weight $\widehat{sl(n)}$-module of highest weight Λ_i. We have that $\chi(V) = \lim_{\ell(w)\to\infty} \chi(E_w(\Lambda_i))$. We can now describe the branching rule for V in terms of Kostka polynomials (see [5]). Let $\{\lambda^j\}$ be an "increasing" sequence of partitions in the sense that $\lambda^j := w_j\nu_{|\lambda^j|}$ where $\lim_j \ell(w_j) = \infty$ and where $|\lambda^j| = i(\text{mod}\, n)$. We must choose $\nu_{|\lambda^j|}$ such that the resulting λ^j are still partitions.

Corollary 1 *The multiplicity of the $sl(n)$-module of weight μ in V is given by*

$$\lim_{j\to\infty} q^{-\mathrm{u}(\lambda^j)+\mathrm{u}(\nu_{|\lambda^j|})} K_{\mu'\lambda^{j'}}(q)$$

We also have a monotonicity result. Let $\tilde{K}_{\lambda\mu}(q) := q^{-\mathrm{u}(\mu)} K_{\lambda\mu}(q)$. Recall that if $\lambda = w\nu_m$ and $\gamma = w'\nu_m$, $\lambda \neq \gamma$ are partitions, then $\lambda < \gamma$ if and only if $w < w'$ in the Bruhat order, where w and w' are chosen to have smallest length.

Theorem 9 $\tilde{K}_{\lambda\mu}(q) - \tilde{K}_{\lambda\nu}(q)$ *has nonnegative coefficients when* $\nu \geq \mu$.

Proof: Let $\upsilon < \gamma$ be two partitions such that $\upsilon = w'\eta_{|\lambda|}$ and $\gamma = w\eta_{|\lambda|}$. Then $w' < w$ and $E_{w'}(\Lambda) \subset E_w(\Lambda)$. The coefficient of q^j in $\tilde{K}_{\nu'\gamma'}(q) - \tilde{K}_{\nu'\upsilon'}(q)$ is the multiplicity of the $sl(n)$-module of weight $\nu - j\delta$ ($j \in \mathbb{Z}$) in $E_w(\Lambda)/E_{w'}(\Lambda)$. Therefore, it has positive coefficients. □

Acknowledgments

The connection between Demazure characters and Macdonald polynomials was pointed out by I. Cherednik. The author thanks the referees for helpful comments.

References

1. L.M. Butler, "Subgroup lattices and symmetric functions," *Memoirs of the A.M.S* **112**(539), (1994).
2. I. Cherednik, "Double affine Hecke algebras and Macdonald's conjectures," *Ann. Math.* **141** (1995), 191–216.
3. M. Demazure, "Désingularisation des variétés de Schubert généralisées," *Ann. Scient. Ec. Norm. Sup.* **6** (1974), 53–88.
4. A. Garsia and C. Procesi, "On certain graded S_n-modules and the q-Kostka polynomials," *Adv. Math* **94**(1) (1992), 82–138.
5. A.N. Kirillov, "Dilogarithm identities," hep-th/9408113 v2, 25, Aug 1994.
6. F. Knop, "Integrality of two variable Kostka functions," *J. reine angew. Math.* **482** (1997), 177–189.
7. S. Kumar, "Demazure character formula in arbitrary Kac-Moody setting," *Invent. Math.* **89** (1987), 395–423.
8. A. Kuniba, K.C. Misra, M. Okado, T. Takagi, and J. Uchiyama, "Characters of Demazure modules and solvable lattice models," q-alg 9707004, 3 July 1997.
9. A. Lascoux and M.-P. Schützenberger, "Sur une conjecture de H. O. Foulkes," *C.R. Acad. Sci. Paris* **286A** (1978), no. 7, 323–324.
10. G. Lusztig, "Green polynomials and singularities of unipotent classes," *Advances in Math.* **42** (1981), 169–178.
11. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Oxford Math. Mon. Clarendon Press, Oxford, 1995.
12. I.G. Macdonald, "Affine Hecke algebras and orthogonal polynomials," Seminaire Bourbaki, 1996, Vol. 1994/1995, *Astérisque* No. 237, Exp. No. 797, 4, 189–207.
13. O. Mathieu, "Formule de Demazure-Weyl et généralisation du théorème de Borel-Weil-Bott," *C.R. Acad. Sci. Paris, Sér. I Math.* **303**(9) (1986), 391–394.
14. E. Opdam, "Harmonic analysis for certain representations of graded Hecke algebras," *Acta Math.* **175** (1995), 75–121.
15. Y. Sanderson, "Real characters of Demazure modules for rank two affine Lie algebras," *J. Algebra* **184** (1996), 985–1000.