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On the Connection between the Boson Transformed Ground State and the Coherent State

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In the "boson method in superconductivity"[1] a certain operator transformation named the "boson transformation"[1),2)] plays a very important role. In this note we show the connection between the boson transformed ground state and the Sudarshan-Glauber coherent state.[3]

In the boson method it is shown that when the electron field $\psi_{\uparrow,\downarrow}(\boldsymbol{x}, t)$ is expressed in terms of quasifermion fields $\phi_{\uparrow,\downarrow}(\boldsymbol{x}, t)$ and the boson field $B(\boldsymbol{x}, t)$ the B.C.S. Hamiltonian takes the free form. The conserved current that is generated by the invariance of the B.C.S. Hamiltonian under the phase transformation is expressible as a sum of contributions from quasifermions $\phi_{\uparrow,\downarrow}(\boldsymbol{x}, t)$ and bosons $B(\boldsymbol{x}, t)$ and each part is conserved separately. Now the charge density $\rho=[\psi_\uparrow^\dagger(\boldsymbol{x}, t)\psi_\uparrow(\boldsymbol{x}, t)+\psi_\downarrow^\dagger(\boldsymbol{x}, t)\psi_\downarrow(\boldsymbol{x}, t)]$ can be expressed as

$$\rho(\boldsymbol{x}, t)=\rho_{\text{q.F.}}(\boldsymbol{x}, t)+\rho_{\text{q.B.}}(\boldsymbol{x}, t). \qquad (1)$$

Further it is shown that

$$Q=\int d^3x\rho(\boldsymbol{x}, t)=\int d^3x\rho_{\text{B}}(\boldsymbol{x}, t).$$

In terms of the boson field $B(\boldsymbol{x}, t)$,

$$\rho_{\text{B}}=-\eta(i\nabla)\pi(\boldsymbol{x}, t), \qquad (2)$$

where $\pi(\boldsymbol{x}, t)=(\partial/\partial t)B(\boldsymbol{x}, t)$ and

$$\eta(\boldsymbol{l})=2\Delta\sqrt{\frac{2R_0}{\lambda}}\frac{1}{v_{\text{B}}(\boldsymbol{l})}\left(1-\frac{2\pi^2}{45}l^2\frac{v_{\text{F}}^2}{\pi^2\Delta^2}\right)$$

when the Coulomb effects are ignored. The boson field satisfies the equation

$$\left[\frac{\partial^2}{\partial t^2}-v_{\text{B}}^2(i\nabla)\nabla^2\right]B(\boldsymbol{x}, t)=0, \qquad (3)$$

where

$$v_{\text{B}}^2(\boldsymbol{l})=\frac{1}{3}V_{\text{F}}^2(1+\lambda N(0))\left\{1-\frac{2V_{\text{F}}^2l^2}{45\Delta^2}\right\}.$$

Here λ is the B.C.S. coupling constant, Δ is the gap, V_{F} is the Fermi velocity and R_0 is a function of Δ and $N(0)$.

The phase transformation[1] $\psi_{\uparrow,\downarrow}\to\psi_{\uparrow,\downarrow}e^{i\theta}$ is generated by the total charge Q as follows:

$$e^{-i\theta Q}\psi_{\uparrow,\downarrow}e^{i\theta Q}=e^{i\theta}\psi_{\uparrow,\downarrow}. \qquad (4)$$

In the quasiparticle picture[1] wherein $\psi_{\uparrow,\downarrow}$ is taken to be expressed in terms of $\Phi_{\uparrow,\downarrow}$, B and π, the above transformation on $\Phi_{\uparrow,\downarrow}$, B and π has to be considered. In view of Eq. (1) it is readily seen that the quasifermions remain a passive spectator under the transformation while the boson field does undergo a change. Writing B as

$$B(\boldsymbol{x}, t)=\sum_l f(\boldsymbol{l})[B_l\exp\{i(\boldsymbol{l}\cdot x-\omega_l t)\}+\text{c.c.})] \qquad (5)$$

with a suitable normalization factor $f(\boldsymbol{l})$ we find that

$$e^{i\theta Q}=e^{(z^*B_0-zB_0^\dagger)}, \qquad (6)$$

where $z=\eta(0)\theta$. Under this transformation $B(\boldsymbol{x}, t)$ is transformed as

$$B(\boldsymbol{x}, t)\to B(\boldsymbol{x}, t)+\eta(0)\theta \qquad (7)$$

and the equation for the boson is invariant. However this transformation takes the ground state to

$$e^{(z^*B_0-zB_0^\dagger)}|0\rangle=e^{-1/2|z|^2}\sum_n\frac{z^n}{n!}|n\rangle=|z\rangle \qquad (8)$$

with

$$|n\rangle=\frac{(B_0^\dagger)^n}{\sqrt{n!}}|0\rangle.$$

Immediately we recognize that $|z\rangle$ is just the Sudarshan-Glauber[3] coherent state representation. The ground state with condensed boson is a coherent state.

We shall now generalize this to a continuous representation so that θ can be a function of $\boldsymbol{x}$. In doing so we require that this should pass over to a continuous representation of the coherent state.

To do this we define

$$a(\boldsymbol{x},t) = \eta(i\nabla)\sum_\alpha (f_\alpha(\boldsymbol{x},t)B_\alpha + f_\alpha^*(\boldsymbol{x},t)B_\alpha^\dagger) = \sum_\alpha (g_\alpha(\boldsymbol{x},t)a_\alpha + g_\alpha^*(\boldsymbol{x},t)a_\alpha^\dagger), \tag{9}$$

where

$$g_\alpha(\boldsymbol{x},t)a_\alpha = \eta(i\nabla)f_\alpha(\boldsymbol{x},t)B_\alpha.$$

Here $f_\alpha(\boldsymbol{x},t)$ is a solution of Eq. (3). Also requiring

$$\theta(\boldsymbol{x},t) = \sum_\alpha [\theta_\alpha f_\alpha(\boldsymbol{x},t) + \theta_\alpha^* f_\alpha^*(\boldsymbol{x},t)], \tag{10}$$

where θ also satisfies Eq. (3). Now the generalization of (8) to several degrees of freedom is as follows:

$$|\{z_\alpha\}\rangle = \exp\sum_\alpha (\theta_\alpha a_\alpha - \theta_\alpha^* a_\alpha^\dagger)|0\rangle, \tag{11}$$

where

$$a_\alpha = i\int d^3x g_\alpha^*(\boldsymbol{x},t)\overleftrightarrow{\frac{\partial}{\partial t}}a(\boldsymbol{x},t),$$

$$\theta_\alpha = i\int d^3x f_\alpha^*(\boldsymbol{x},t)\overleftrightarrow{\frac{\partial}{\partial t}}\theta(\boldsymbol{x},t). \tag{12}$$

This together with the completeness condition for the function $f_\alpha(\boldsymbol{x},t)$ gives

$$|\{z_\alpha\}\rangle = \exp i\left\{\int d^3x a(\boldsymbol{x},t)\overleftrightarrow{\frac{\partial}{\partial t}}\theta(\boldsymbol{x},t)\right\}|0\rangle = U(\theta,\dot\theta)|0\rangle,$$

where

$$U(\theta,\dot\theta) = \exp i\left\{\int d^3x\left[\dot\theta(\boldsymbol{x},t)\eta(i\nabla)B(\boldsymbol{x},t) - \theta(\boldsymbol{x},t)\eta(i\nabla)\frac{\partial}{\partial t}B(\boldsymbol{x},t)\right]\right\}$$

which is the invariant boson transformation of Umezawa.

The identification of the boson transformation with the coherent state is useful, for now one can use Klauder's "weak correspondence principle"[5] to identify $\theta(\boldsymbol{x},t)$ as the corresponding classical field for the condensed boson field. This is also the content of the statement of Umezawa et al. that $\theta(\boldsymbol{x},t)$ controls the Bose condensation.

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