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ON THE CONNECTION BETWEEN THE ELLIPTIC EQUATIONS OF THE NAVIER-STOKES TYPE AND THE THEORY OF HARMONIC FUNCTIONALS*

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Abstract. It is established in this paper, under conditions more general than those used by Millikan, that the two-dimensional incompressible viscous flows past finite bodies cannot be equivalent to a variational problem of the Euler-Lagrange type. It has thereby been possible to obtain two harmonic functionals with a close relation to the Navier-Stokes equations.

1. Introduction. Variational formulations of physical problems are well-known for their simplicity and elegance. Mostly problems governed by linear differential or integral equations have been successfully attacked by variational methods. In this paper we explore the possibility of applying variational methods to the Navier-Stokes equations. Millikan [1] has shown that the Navier-Stokes equations cannot be equivalent (except in some exceptional cases) to a variational problem, but under conditions more restrictive than those used in this paper. We find, under more general conditions, that the structure of the Navier-Stokes equations is such that we cannot get a single functional whose extremization yields the equations; instead, we obtain a system of functional differential equations which are equivalent to the Navier-Stokes equations.

2. Analysis for two-dimensional flows. The governing partial differential equations for two-dimensional, incompressible viscous flow past a body are

$$u_x + v_y = 0, \tag{2.1}$$

$$uu_x + vu_y = -(1/\rho)p_x + \nu(u_{xx} + u_{yy}), \tag{2.2a}$$

$$uv_x + vu_y = -(1/\rho)p_y + \nu(v_{xx} + v_{yy}), \tag{2.2b}$$

where u and v are the components of the velocity at (x, y) in a rectangular Cartesian frame, p the hydrostatic pressure, ν the kinematic viscosity, and the suffixes x, y on u or v indicate partial differentiation.

We consider flows past finite bodies kept in an infinite mass of fluid having a free-stream velocity U_∞ (see Fig. 1). The boundary conditions will therefore be

$$u = 0, \quad v = 0 \quad \text{on} \quad S; \tag{2.3}$$

$$\text{for } |x| \to \infty \text{ or } |y| \to \infty, \ u \to U_\infty, \ v \to 0, \ p \to p_\infty; \tag{2.4}$$

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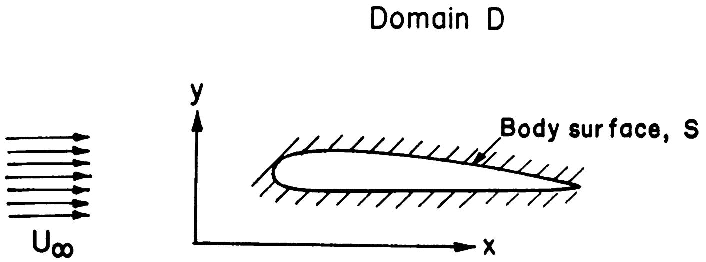

FIG. 1.

where p_∞ is the free-stream hydrostatic pressure. We now pose the following problem: does there exist a functional $A[u, v, p]$ depending on the functions $u(x, y)$, $v(x, y)$ and $p(x, y)$ such that its functional derivatives $\delta A/\delta p$, $\delta A/\delta u$ and $\delta A/\delta v$ when equated to zero yield Eqs. (2.1), (2.2)? Here A is a class of functionals which we call the Euler-Lagrange class, defined by

$$A[u, v, p] = \iint_D L(u, v, p, u_x, v_x, p_x, u_y, v_y, p_y, x, y)\, dx\, dy, \tag{2.5}$$

where L is a function (of the variables listed in parentheses in Eq. (2.5)) satisfying certain mathematical conditions regarding continuity, differentiability, integrability, and D is the domain indicated in Fig. 1. (Millikan [1] took the integrand L in Eq. (2.5) to be independent of p, p_x and p_y and further assumed that L can be expanded in a Taylor series in all the variables.)

Giving variations δu, δv and δp to u, v and p respectively we get, from Eq. (2.5),

$$\delta A = \iint_D \left[\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial v}\delta v + \frac{\partial L}{\partial p}\delta p + \frac{\partial L}{\partial u_x}\delta u_x + \frac{\partial L}{\partial v_x}\delta v_x + \frac{\partial L}{\partial p_x}\delta p_x + \frac{\partial L}{\partial u_y}\delta u_y + \frac{\partial L}{\partial v_y}\delta v_y + \frac{\partial L}{\partial p_y}\delta p_y\right] dx\, dy. \tag{2.6}$$

Integrating by parts the relevant terms in Eq. (2.6), and grouping the terms involving δu, δv and δp separately, we get

$$\frac{\delta A}{\delta p} = \frac{\partial L}{\partial p} - \frac{\partial}{\partial x}\frac{\partial L}{\partial p_x} - \frac{\partial}{\partial y}\frac{\partial L}{\partial p_y}, \tag{2.7}$$

$$\frac{\delta A}{\delta u} = \frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial L}{\partial u_y}, \tag{2.8}$$

$$\frac{\delta A}{\delta v} = \frac{\partial L}{\partial v} - \frac{\partial}{\partial x}\frac{\partial L}{\partial v_x} - \frac{\partial}{\partial y}\frac{\partial L}{\partial v_y}, \tag{2.9}$$

provided that

$$\left(\frac{\partial L}{\partial p_x}\right)_S = \left(\frac{\partial L}{\partial p_y}\right)_S = 0. \tag{2.10}$$

Keeping in mind that L is a function of $u, v, p, \cdots$ we get from Eqs. (2.7), (2.8) and (2.9)

$$\frac{\delta A}{\delta p} = \frac{\partial L}{\partial p} - \frac{\partial^2 L}{\partial u\, \partial p_x} u_x - \frac{\partial^2 L}{\partial v\, \partial p_x} v_x - \frac{\partial^2 L}{\partial p\, \partial p_x} p_x - \frac{\partial^2 L}{\partial u\, \partial p_y} u_y - \frac{\partial^2 L}{\partial v\, \partial p_y} v_y - \frac{\partial^2 L}{\partial p\, \partial p_y} p_y$$
$$- \left(\frac{\partial^2 L}{\partial u_y\, \partial p_x} + \frac{\partial^2 L}{\partial u_x\, \partial p_y}\right) u_{xy} - \left(\frac{\partial^2 L}{\partial v_y\, \partial p_x} + \frac{\partial^2 L}{\partial v_x\, \partial p_y}\right) v_{xy} - 2\frac{\partial^2 L}{\partial p_x\, \partial p_y} p_{xy}$$
$$- \frac{\partial^2 L}{\partial u_x\, \partial p_x} u_{xx} - \frac{\partial^2 L}{\partial v_x\, \partial p_x} v_{xx} - \frac{\partial^2 L}{\partial p_x^2} p_{xx} - \frac{\partial^2 L}{\partial p_y\, \partial u_y} u_{yy} - \frac{\partial^2 L}{\partial v_y\, \partial p_y} v_{yy} - \frac{\partial^2 L}{\partial p_y^2} p_{yy}\,, \quad (2.11)$$

$$\frac{\delta A}{\delta u} = \frac{\partial L}{\partial u} - \frac{\partial^2 L}{\partial u\, \partial u_x} u_x - \frac{\partial^2 L}{\partial v\, \partial u_x} v_x - \frac{\partial^2 L}{\partial p\, \partial u_x} p_x - \frac{\partial^2 L}{\partial u\, \partial u_y} u_y - \frac{\partial^2 L}{\partial v\, \partial u_y} v_y - \frac{\partial^2 L}{\partial p\, \partial u_y} p_y$$
$$- 2\frac{\partial^2 L}{\partial u_x\, \partial u_y} u_{xy} - \left(\frac{\partial^2 L}{\partial v_x\, \partial u_y} + \frac{\partial^2 L}{\partial v_y\, \partial u_x}\right) v_{xy} - \left(\frac{\partial^2 L}{\partial p_y\, \partial u_x} + \frac{\partial^2 L}{\partial p_x\, \partial u_y}\right) p_{xy} - \frac{\partial^2 L}{\partial u_x^2} u_{xx}$$
$$- \frac{\partial^2 L}{\partial v_x\, \partial u_x} v_{xx} - \frac{\partial^2 L}{\partial p_x\, \partial u_x} p_{xx} - \frac{\partial^2 L}{\partial u_y^2} u_{yy} - \frac{\partial^2 L}{\partial v_y\, \partial u_y} v_{yy} - \frac{\partial^2 L}{\partial p_y\, \partial u_y} p_{yy}\,, \quad (2.12)$$

$$\frac{\delta A}{\delta v} = \frac{\partial L}{\partial v} - \frac{\partial^2 L}{\partial u\, \partial v_x} u_x - \frac{\partial^2 L}{\partial v\, \partial v_x} v_x - \frac{\partial^2 L}{\partial p\, \partial v_x} p_x - \frac{\partial^2 L}{\partial u\, \partial v_y} u_y - \frac{\partial^2 L}{\partial v\, \partial v_y} v_y - \frac{\partial^2 L}{\partial p\, \partial v_y} p_y$$
$$- \left(\frac{\partial^2 L}{\partial u_y\, \partial v_x} + \frac{\partial^2 L}{\partial u_x\, \partial v_y}\right) u_{xy} - 2\frac{\partial^2 L}{\partial v_y\, \partial v_x} v_{xy} - \left(\frac{\partial^2 L}{\partial p_y\, \partial v_x} + \frac{\partial^2 L}{\partial p_x\, \partial v_y}\right) p_{xy}$$
$$- \frac{\partial^2 L}{\partial u_x\, \partial v_x} u_{xx} - \frac{\partial^2 L}{\partial v_x^2} v_{xx} - \frac{\partial^2 L}{\partial p_x\, \partial v_x} p_{xx} - \frac{\partial^2 L}{\partial u_y\, \partial v_y} u_{yy} - \frac{\partial^2 L}{\partial v_y^2} v_{yy} - \frac{\partial^2 L}{\partial p_y\, \partial v_y} p_{yy}\,. \quad (2.13)$$

If $\delta A/\delta p = 0$, $\delta A/\delta u = 0$ and $\delta A/\delta v = 0$ are to be equivalent to Eqs. (2.1) and (2.2) then we must have

$$\frac{\partial^2 L}{\partial u_x\, \partial p_x} = \frac{\partial^2 L}{\partial v_x\, \partial p_x} = \frac{\partial^2 L}{\partial u_y\, \partial p_x} + \frac{\partial^2 L}{\partial u_x\, \partial p_y} = \frac{\partial^2 L}{\partial v_y\, \partial p_x} + \frac{\partial^2 L}{\partial v_x\, \partial p_y} = \frac{\partial^2 L}{\partial p_x^2} = \frac{\partial^2 L}{\partial p_x\, \partial p_y}$$
$$= \frac{\partial^2 L}{\partial p_y\, \partial u_y} = \frac{\partial^2 L}{\partial p_y\, \partial v_y} = \frac{\partial^2 L}{\partial p_y^2} = 0, \quad (2.14\text{a, b, } \cdots \text{, i})$$

$$\frac{\partial^2 L}{\partial u_x\, \partial v_x} = \frac{\partial^2 L}{\partial u_x\, \partial u_y} = \frac{\partial^2 L}{\partial u_x\, \partial v_y} + \frac{\partial^2 L}{\partial v_x\, \partial u_y} = \frac{\partial^2 L}{\partial u_y\, \partial v_y} = \frac{\partial^2 L}{\partial v_x\, \partial v_y} = 0, \quad (2.15\text{a, b, } \cdots \text{, e})$$

$$\frac{\partial^2 L}{\partial v_x^2} = \frac{\partial^2 L}{\partial v_y^2} = \frac{\partial^2 L}{\partial u_x^2} = \frac{\partial^2 L}{\partial u_y^2} = \nu. \quad (2.16\text{a, b, c, d})$$

It may be noted that Eqs. (2.14), (2.15) and (2.16) are obtained by comparing the coefficients of the second-order derivatives u_{xx}, u_{xy}, $\cdots$ etc. in $\delta A/\delta p = 0$, $\delta A/\delta u = 0$ and $\delta A/\delta v = 0$ with those in Eqs. (2.1), (2.2a) and (2.2b). At this stage we cannot make a similar comparison for first-order derivatives u_x, u_y, $\cdots$ etc. because the $\partial L/\partial p$, $\partial L/\partial u$ and $\partial L/\partial v$ terms in Eqs. (2.11), (2.12) and (2.13) respectively may contain u_x, u_y, $\cdots$ etc. and thus the coefficients of these derivatives are not known. We thus conclude that the satisfaction of Eqs. (2.14), (2.15) and (2.16) is not a sufficient condition but only a necessary one for the desired equivalence. From Eqs. (2.14e), (2.14f) and (2.14i) we get

$$L = Z(u_x, u_y, v_x, v_y) + p_x Z_5(u_x, u_y, v_x, v_y) + p_y Z_6(u_x, u_y, v_x, v_y). \quad (2.17)$$

We have suppressed, for convenience in writing, the variables u, v, p, x, y on which the arbitrary functions Z, Z_5 and Z_6 depend. We will follow this notation throughout this paper. Further, using Eqs. (2.14a) and (2.14b), it is clear that Z_5 is independent of u_x and v_x. Similarly, Z_6 does not depend on u_y and v_y, in view of Eqs. (2.14g) and (2.14h). We can therefore write

$$L = Z(u_x, u_y, v_x, v_y) + p_x Z_5(u_y, v_y) + p_y Z_6(u_x, v_x). \tag{2.18}$$

Substituting for L from Eq. (2.18) in Eqs. (2.15b) and (2.15e), we get

$$\frac{\partial^2 Z}{\partial u_x \, \partial u_y} = 0, \qquad \frac{\partial^2 Z}{\partial v_x \, \partial v_y} = 0 \tag{2.19a, b}$$

which yields the result

$$Z = Z_1(u_x, v_y) + Z_2(v_x, u_y) + Z_3(u_x, v_x) + Z_4(u_y, v_y). \tag{2.20}$$

Using Eqs. (2.18), (2.20), (2.15a) and (2.15d) we get

$$\frac{\partial^2 Z_3}{\partial u_x \, \partial v_x} + p_y \frac{\partial^2 Z_6}{\partial u_x \, \partial v_x} = 0, \tag{2.21a}$$

$$\frac{\partial^2 Z_4}{\partial u_y \, \partial v_y} + p_x \frac{\partial^2 Z_5}{\partial u_y \, \partial v_y} = 0, \tag{2.21b}$$

which lead to

$$\frac{\partial^2 Z_3}{\partial u_x \, \partial v_x} = 0, \qquad \frac{\partial^2 Z_4}{\partial u_y \, \partial v_y} = 0, \tag{2.22}$$

$$\frac{\partial^2 Z_6}{\partial u_x \, \partial v_x} = 0, \qquad \frac{\partial^2 Z_5}{\partial u_y \, \partial v_y} = 0. \tag{2.23}$$

Eqs. (2.22) yield

$$Z_3(u_x, v_x) = F_1(u_x) + F_2(v_x),$$
$$Z_4(u_y, v_y) = F_3(u_y) + F_4(v_y),$$

where F_1, F_2, F_3 and F_4 are arbitrary functions and we can absorb the first two of them in Z_1 and Z_2 and thus can drop Z_3 term in Eq. (2.20) without loss of generality. For similar reasons, the Z_4 term in the same equation can also be omitted. In view of this reasoning and Eqs. (2.23) we can write for L the expression

$$L = Z_1(u_x, v_y) + Z_2(v_x, u_y) + p_x[G_1(u_y) + G_2(v_y)] + p_y[G_3(u_x) + G_4(v_x)], \tag{2.24}$$

where G_1, G_2, G_3 and G_4 are arbitrary functions of u_y, v_y, u_x and v_x respectively.

Substituting for L from Eq. (2.24) in Eqs. (2.16), we get

$$\frac{\partial^2 Z_2}{\partial v_x^2} + p_y \frac{\partial^2 G_4}{\partial v_x^2} = \nu,$$

$$\frac{\partial^2 Z_1}{\partial v_y^2} + p_x \frac{\partial^2 G_2}{\partial v_y^2} = \nu,$$

$$\frac{\partial^2 Z_1}{\partial u_x^2} + p_y \frac{\partial^2 G_3}{\partial u_x^2} = \nu,$$

$$\frac{\partial^2 Z_2}{\partial u_y^2} + p_x \frac{\partial^2 G_1}{\partial u_y^2} = \nu.$$

These equations yield

$$\frac{\partial^2 G_4}{\partial v_x^2} = \frac{\partial^2 G_2}{\partial v_y^2} = \frac{\partial^2 G_3}{\partial u_x^2} = \frac{\partial^2 G_1}{\partial u_y^2} = 0,$$

$$\frac{\partial^2 Z_2}{\partial v_x^2} = \frac{\partial^2 Z_1}{\partial v_y^2} = \frac{\partial^2 Z_1}{\partial u_x^2} = \frac{\partial^2 Z_2}{\partial u_y^2} = \nu;$$

solving these, we get

$$L = Y + Y_1 u_x + Y_2 u_y + Y_3 v_x + Y_4 v_y + Y_5 p_x + Y_6 p_y + \tfrac{1}{2}\nu(u_x^2 + u_y^2 + v_x^2 + v_y^2) + Y_7 p_x u_y + Y_8 p_x v_y + Y_9 p_y u_x + Y_{10} p_y v_x + Y_{11} u_x v_y + Y_{12} v_x u_y \,, \tag{2.25}$$

where Y, Y_1, Y_2, $\cdots$, Y_{12} are arbitrary functions of u, v, p and their dependence on these variables is not exhibited in conformity with the notation adopted before. Substituting for L from Eq. (2.25) in Eqs. (2.14c), (2.14d) and (2.15c) we get

$$Y_7 + Y_9 = Y_8 + Y_{10} = Y_{11} + Y_{12} = 0. \tag{2.26}$$

It is now clear that the expression for L given by Eqs. (2.25) and (2.26) satisfies all the Eqs. (2.14), (2.15) and (2.16); hence, the expressions for $\delta A/\delta p$, $\delta A/\delta u$, $\delta A/\delta v$ given by Eqs. (2.11), (2.12) and (2.13) reduce to

$$\frac{\delta A}{\delta p} = \frac{\partial L}{\partial p} - \frac{\partial^2 L}{\partial u\, \partial p_x} u_x - \frac{\partial^2 L}{\partial v\, \partial p_x} v_x - \frac{\partial^2 L}{\partial p\, \partial p_x} p_x - \frac{\partial^2 L}{\partial u\, \partial p_y} u_y - \frac{\partial^2 L}{\partial v\, \partial p_y} v_y - \frac{\partial^2 L}{\partial p\, \partial p_y} p_y \,, \tag{2.27}$$

$$\frac{\delta A}{\delta u} = \frac{\partial L}{\partial u} - \frac{\partial^2 L}{\partial u\, \partial u_x} u_x - \frac{\partial^2 L}{\partial v\, \partial u_x} v_x - \frac{\partial^2 L}{\partial p\, \partial u_x} p_x - \frac{\partial^2 L}{\partial u\, \partial u_y} u_y - \frac{\partial^2 L}{\partial v\, \partial u_y} v_y - \frac{\partial^2 L}{\partial p\, \partial u_y} p_y - \nu(u_{xx} + u_{yy}), \tag{2.28}$$

$$\frac{\delta A}{\delta v} = \frac{\partial L}{\partial v} - \frac{\partial^2 L}{\partial u\, \partial v_x} u_x - \frac{\partial^2 L}{\partial v\, \partial v_x} v_x - \frac{\partial^2 L}{\partial p\, \partial v_x} p_x - \frac{\partial^2 L}{\partial u\, \partial v_y} u_y - \frac{\partial^2 L}{\partial v\, \partial v_y} v_y - \frac{\partial^2 L}{\partial p\, \partial v_y} p_y - \nu(v_{xx} + v_{yy}). \tag{2.29}$$

The viscous terms in Eqs. (2.2a) and (2.2b) appear in the above expressions for $\delta A/\delta u$ and $\delta A/\delta v$, as must be expected. The next task is to explore the possibility of reproducing the inertia and pressure-gradient terms in these expressions, keeping in mind that L must be of the type given by Eq. (2.25). It is obvious that the terms in Eq. (2.25) containing Y, Y_5, Y_6, Y_7, Y_8 and Y_{11} cannot reproduce the inertia terms. We choose

$$Y_5 = u/\rho, \qquad Y_6 = v/\rho, \qquad Y_{11} = \nu, \qquad Y = Y_7 = Y_8 = 0,$$

which means

$$L = Y_1 u_x + Y_2 u_y + Y_3 v_x + Y_4 v_y + (u/\rho)p_x + (v/\rho)p_y + \tfrac{1}{2}\nu(u_x + v_y)^2 + \tfrac{1}{2}\nu(v_x - u_y)^2. \tag{2.30}$$

It may be noted here that this form for L satisfies Eq. (2.10). Substituting for L from Eq. (2.30) into Eqs. (2.27), (2.28) and (2.29), we get

$$\frac{\delta A}{\delta p} = \left(\frac{\partial Y_1}{\partial p} - \frac{1}{\rho}\right)u_x + \left(\frac{\partial Y_4}{\partial p} - \frac{1}{\rho}\right)v_y + \frac{\partial Y_2}{\partial p}u_y + \frac{\partial Y_3}{\partial p}v_x\ , \tag{2.31}$$

$$\frac{\delta A}{\delta u} = \left(\frac{\partial Y_3}{\partial u} - \frac{\partial Y_1}{\partial v}\right)v_x + \left(\frac{\partial Y_4}{\partial u} - \frac{\partial Y_2}{\partial v}\right)v_y - \frac{\partial Y_2}{\partial p}p_y + \left(\frac{1}{\rho} - \frac{\partial Y_1}{\partial p}\right)p_x - \nu(u_{xx} + u_{yy}), \tag{2.32}$$

$$\frac{\delta A}{\delta v} = \left(\frac{\partial Y_1}{\partial v} - \frac{\partial Y_3}{\partial u}\right)u_x + \left(\frac{\partial Y_2}{\partial v} - \frac{\partial Y_4}{\partial u}\right)u_y + \left(\frac{1}{\rho} - \frac{\partial Y_4}{\partial p}\right)p_y - \frac{\partial Y_3}{\partial p}p_x - \nu(v_{xx} + v_{yy}). \tag{2.33}$$

We must choose

$$\frac{\partial Y_1}{\partial p} = \frac{\partial Y_2}{\partial p} = \frac{\partial Y_3}{\partial p} = \frac{\partial Y_4}{\partial p} = 0$$

to reproduce correctly the terms $(u_x + v_y)/\rho$ in Eq. (2.31) and the pressure-gradient terms in Eqs. (2.32) and (2.33). From the expressions for $\delta A/\delta u$ and $\delta A/\delta v$ it is obvious that no functions Y_1, Y_2, Y_3 and Y_4 exist which will reproduce the inertia terms in Eqs. (2.32) and (2.33). We therefore conclude that *there does not exist a single functional $A[u, v, p]$ belonging to the Euler-Lagrange class whose functional derivatives with respect to u and v when equated to zero would yield the momentum equations.* Eqs. (2.32) and (2.33), on the other hand, possess an interesting property by virtue of which a close connection can be established between the two-dimensional Navier-Stokes equations and analytic functionals (see the Appendix). To establish this, let

$$D[u, v, p] = \iint_D \left[\tfrac{1}{2}\nu(\operatorname{div}\mathbf{u})^2 + \tfrac{1}{2}\nu(\operatorname{curl}\mathbf{u})^2 + \frac{1}{\rho}\mathbf{u}\cdot \operatorname{grad} p\right] dx\, dy, \tag{2.34}$$

and

$$E[u, v] = -\iint_D (Y_1u_x + Y_2u_y + Y_3v_x + Y_4v_y)\, dx\, dy. \tag{2.35}$$

We then get

$$(\delta D/\delta u) = (1/\rho)p_x - \nu(u_{xx} + v_{yy}), \tag{2.36a}$$

$$(\delta D/\delta v) = (1/\rho)p_y - \nu(v_{xx} + v_{yy}), \tag{2.36b}$$

$$(\delta D/\delta p) = -(1/\rho)(u_x + v_y), \tag{2.36c}$$

and

$$(\delta E/\delta u) = -(uv_x + vv_y), \tag{2.37a}$$

$$(\delta E/\delta v) = uu_x + vv_y\ , \tag{2.37b}$$

if we choose Y_1, Y_2, Y_3 and Y_4 so as to satisfy

$$\frac{\partial Y_3}{\partial u} - \frac{\partial Y_1}{\partial v} = u, \qquad \frac{\partial Y_4}{\partial u} - \frac{\partial Y_2}{\partial v} = v. \tag{2.38}$$

Using Eqs. (2.36), (2.37) and (2.38), we can write the continuity and the momentum equations equivalently as

$$\frac{\delta D}{\delta p} = 0, \tag{2.39}$$

$$\frac{\delta E}{\delta u} = \frac{\delta D}{\delta v}, \tag{2.40}$$

$$\frac{\delta E}{\delta v} = -\frac{\delta D}{\delta u}. \tag{2.41}$$

Eqs. (2.40) and (2.41) are precisely the Cauchy-Riemann conditions for $E + iD$ to be an analytic functional of $u + iv$. In order to establish the existence of E compatible with Eqs. (2.38), we observe that among infinitely many solutions of Eq. (2.38) two simple solutions are

a) $Y_1 = Y_4 = 0,\ Y_2 = -\frac{1}{2}v^2,\ Y = \frac{1}{2}u^2$

giving

$$E = \iint_D \frac{1}{2}(v^2 u_y - u^2 v_x)\,dx\,dy, \tag{2.42}$$

and

b) $Y_1 = Y_4 = 0,\ Y_3 = -Y_2 = (u^2 + v^2)/2$

giving

$$E = \iint_D \frac{1}{2}(u^2 + v^2)(u_y - v_x)\,dx\,dy. \tag{2.43}$$

3. Some concluding remarks.

a) Interpretation of the functionals D and E. We observe that when (u, v, p) represents a fluid dynamical motion, the div **u** term in Eq. (2.34) vanishes and so does the term **u**. grad p, as it gives the work done by the expansion of the incompressible fluid against pressure forces. The integrand for D is then just the square of the vorticity and in such a case Lamb [3] has shown that D is equal to the total dissipation of energy occurring in a unit time. The drag on the body is then equal to $\rho D/U_\infty$.

Following Sommerfeld [4], we can define the 'mass density' of the vortex distribution equal to the vorticity, and get the result that E is the negative of the net kinetic energy due to this 'mass' distribution.

b) An approximate method of solution of flow problems. In this subsection we exploit the property of analyticity of $E + iD$ for getting an approximate method to obtain flows past finite bodies. Let us presume that

$$u = \sum_{k=1}^{\infty} a_k\phi_k(x, y) \tag{3.1}$$

$$v = \sum_{k=1}^{\infty} b_k\phi_k(x, y), \tag{3.2}$$

$$p = \sum_{k=1}^{\infty} c_k\chi_k(x, y), \tag{3.3}$$

where ϕ_k and χ_k, $k = 1, 2, \cdots, \infty$, are linearly independent sets of functions. One has to be careful in the choice of these sets, as u and p are not L_2 functions. The functionals D and E can then be written as

$$D = g(a_1, a_2, \cdots; b_1, b_2, \cdots; c_1, c_2, \cdots),$$
$$E = f(a_1, a_2, \cdots; b_1, b_2, \cdots),$$

where f and g are functions of a countably infinite number of variables. In fact, f and g are quadratic and cubic functions respectively of these variables. We can then state that $E + iD$ being an analytic functional of $u + iv$ is equivalent to $f + ig$ being analytic (holomorphic) in $a_k + ib_k$, $k = 1, 2, \cdots \infty$. This result can be easily demonstrated by the following approach. Consider the variation in D:

$$\delta D = \sum_{k=1}^{\infty} \left(\frac{\partial g}{\partial a_k} \delta a_k + \frac{\partial g}{\partial b_k} \delta b_k + \frac{\partial g}{\partial c_k} \delta c_k \right), \tag{3.4}$$

which is equal to

$$\iint_D \left(\frac{\delta D}{\delta u} \delta u + \frac{\delta D}{\partial v} \delta v + \frac{\delta D}{\delta p} \delta p \right) dx\, dy. \tag{3.5}$$

Substituting for δu and δv from Eqs. (3.1) and (3.2) into Eq. (3.5) we get, by comparing coefficients of δa_k, δb_k in the resultant expression with those in Eq. (3.4), the expressions

$$\frac{\partial g}{\partial a_k} = \int \frac{\delta D}{\partial u} \phi_k(x, y)\, dx\, dy, \tag{3.6}$$

$$\frac{\partial g}{\partial b_k} = \int \frac{\delta D}{\delta v} \phi_k(x, y)\, dx\, dy. \tag{3.7}$$

Similarly, we can show that

$$\frac{\partial f}{\partial a_k} = \int \frac{\delta E}{\delta u} \phi_k(x, y)\, dx\, dy, \tag{3.8}$$

$$\frac{\partial f}{\partial b_k} = \int \frac{\delta E}{\delta v} \phi_k(x, y)\, dx\, dy. \tag{3.9}$$

The functional Cauchy-Riemann conditions (2.40) and (2.41), in view of Eqs. (3.6) through (3.9), become

$$\partial f/\partial a_k = \partial g/\partial b_k, \tag{3.10}$$

$$\partial f/\partial b_k = -\partial g/\partial a_k, \tag{3.11}$$

thus establishing the result that $f + ig$ is analytic in $a_k + ib_k$, $k = 1, 2, \cdots, \infty$. It may be noted that the continuity equations become

$$\partial g/\partial c_k = 0, \quad k = 1, 2, \cdots, \infty. \tag{3.12}$$

Now Eqs. (3.10), (3.11) and (3.12) are only algebraic equations (in fact, quadratic) in a_k, b_k and c_k. Solution of these equations then yields a solution to the flow problem.

c) Functional power series representation of $E + iD$. If $G[w(x, y)]$ is a cubic analytic functional, then (see the Appendix)

$$G[w(x, y)] = K_0 + \int_D K_1(x, y)w(x, y)\,dx\,dy$$
$$+ \int_D K_2(x_1, y_1; x_2, y_2)w(x_1, y_1)w(x_2, y_2)\,dx_1\,dy_1\,dx_2\,dy_2$$
$$+ \int_D K_3(x_1, y_1; x_2, y_2; x_3, y_3)w(x_1, y_1)$$
$$\cdot w(x_2, y_2)w(x_3, y_3)\,dx_1\,dy_1\,dx_2\,dy_2\,dx_3\,dy_3, \tag{3.13}$$

where we have denoted, for convenience in writing, multiple integration by a single integration symbol. In view of the fact that $E + iD$ is a cubic functional, analytic at that w which satisfies the Navier-Stokes equations, it is clear that $E + iD$ must be equal to $G[w(x, y)]$. We may use this property for generating solutions of the Navier-Stokes equations. This can be achieved by assuming some specific kernels K_0, K_1, K_2 and K_3 and then by finding a w which makes $E + iD$ equal to $G[w(x, y)]$. Further, if $w = 0$ gives a closed curve and w tends to a constant at infinity, then that w represents a solution to the flow past the corresponding body.

d) Extension to three-dimensional flows. The extension of the above analysis to three dimensions is conceptually not very difficult but somewhat involved mathematically. We will not go into the analysis here but shall be content with the following observation. Following Deshpande [5] we can write the Navier-Stokes equations for three-dimensional flows as

$$\text{CURL}\,(\mathbf{E}) = \text{GRAD}\,(D)$$

where CURL and GRAD are functional differential operators, $\mathbf{E}$ is a vector functional and D a scalar functional defined by

$$E_n = -\int_V \frac{1}{5}\,\epsilon_{rnt} u_r u_s\, D_{ts}\,dx_1\,dx_2\,dx_3,$$
$$D = \int_V \left[\tfrac{1}{2}\nu(\operatorname{div}\mathbf{u})^2 + \tfrac{1}{2}\nu(\operatorname{curl}\mathbf{u})^2 + \frac{1}{\rho}\,\mathbf{u}\cdot\operatorname{grad} p\right] dx_1\,dx_2\,dx_3.$$

Here ϵ_{rnt} is the alternating Cartesian tensor, u_r is the rth component of the velocity vector $\mathbf{u}$ and the dummy suffix notation is used in the above equations.

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Appendix. The concept of an analytic functional used in this paper is closely related to the analytical functional introduced by Volterra [2]. Following Volterra we define

$$G[w(t)] = K_0 + \int_a^b K_1(\xi)w(\xi)\,d\xi + \int_a^b\int_a^b K_2(\xi_1, \xi_2)w(\xi_1)w(\xi_2)\,d\xi_1\,d\xi_2$$
$$+ \cdots + \int_a^b \cdots \int_a^b K_n(\xi_1, \xi_2, \cdots, \xi_n)w(\xi_1)\cdots w(\xi_n)\,d\xi_1\cdots d\xi_n + \cdots, \tag{A1}$$

where $w(t)$ belongs to a ceratin functional field (e.g. the set of all continuous functions $w(t)$, $t \in [a, b]$), K_n is a symmetric kernel in all its variables $\xi_1, \xi_2, \cdots, \xi_n$ and it is presumed that the above series (A.1) is convergent if $|w(t)| < R$. It may be noted that the series (A.1) is a functional power series.

If we now regard w as a complex function $u(x, y) + iv(x, y)$ of two variables x and y defined over a certain domain D and further make all the kernels also into complex functions of several variables $x_1, y_1; x_2, y_2; \cdots$ we get

$$G[w(x, y)] = K_0 + \iint_D K_1(x, y)w(x, y)\, dx\, dy + \iiiint_D K_2(x_1, y_1; x_2, y_2)w(x_1, y_1)w(x_2, y_2)\, dx_1\, dy_1\, dx_2\, dy_2 + \cdots ; \quad \text{(A.2)}$$

we call this an analytic functional provided the series (A.2) converges for $|w(x, y)| < R$. We may take u and v as continuous functions defined on D and having continuous first-order partial derivatives everywhere in D. We can easily show that $G[w(x, y)]$ defined by Eq. (A.2) satisfies the Cauchy-Riemann conditions

$$\delta H/\delta u = \delta J/\delta v, \qquad \delta H/\delta v = -\delta J/\delta u,$$

where H and J are the real and imaginary parts respectively of $G[w(x, y)]$.

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