Mon. Not. R. astr. Soc. (1980) **191**, 483–498

On the consequences of the gravothermal catastrophe

D. Lynden-Bell and P. P. Eggleton *Institute of Astronomy, The Observatories, Cambridge CB3 0HA*

Received 1979 October 9; in original form 1979 July 11

Summary. The cores of globular clusters evolve independently of the outer parts and eventually achieve a self-similar evolution with a shrinking almost isothermal portion followed by a power law density profile, $\rho \propto r^{-\alpha}$. It is proved that α is in the range $2 < \alpha < 2.5$ and that both the magnitude of the core energy and the core mass decrease during evolution towards core collapse. In fact $|E_c| \propto M_c^{\zeta}$ where $\zeta = (5 - 2\alpha)/(3 - \alpha)$. The temporal behaviours of the core density and core velocity dispersion are determined in terms of α. The central density always becomes infinite in finite time but both the core mass and the core energy become zero then.

An eigenvalue equation is solved numerically to determine α for a model with energy transport by stellar encounters approximated as heat conduction in a gas. It yields $\alpha = 2.21$, $\zeta \simeq 3/4$ and the detailed density and temperature profiles for the similarity solution of that model.

1 Introduction

The aim of this paper is not to compute accurate models for the evolution of globular clusters but to increase conceptual understanding of the mechanisms that drive that evolution. Because of this aim we shall not shrink from simplifying and idealizing the problems discussed in the belief that a clear understanding may be best achieved thereby. Probably the most drastic idealization that we shall make is to consider only clusters of stars of equal mass. This is a serious matter since a flow of energy from heavy stars to light is one of the driving mechanisms of cluster evolution and significantly speeds it up as Spitzer (1975) emphasizes. However, it is not yet clear that the final evolution of a cluster of stars of different masses is dominated by mechanisms that are not represented in the equal mass case. Further, the unequal mass case is significantly more involved and does not lend itself to any simple and general solution, whereas we show that the equal mass case has such solutions.

Early work on the dynamics of star clusters regarded the escape of stars from a cluster as the primary force driving its dynamical evolution. Encounters continually create stars in the tail of the near-Maxwellian distribution but these leave the cluster so no detailed balance of reverse encounters can establish equilibrium. In the early 1960s Hénon (1960, 1961) showed that this was not the dominant evolutionary process. He pointed out that there was an infinite volume of phase space bound to an isolated cluster and the population of this halo by weak encounters was more important for evolution than the escape of stars, which could only occur due to the rarer hard encounters. At about the same time Antonov (1962) discovered the surprising fact that even the artificial problem of N stars in a perfectly reflecting sphere endowed with a total energy E and mass M does not possess an equilibrium if the sphere has a radius greater than his critical value $r_e = 0.335\, GM^2/(-E)$. Lynden-Bell & Wood (1968) gave a physical explanation of Antonov's discovery in terms of the negative specific heat of the self-gravitating core of the cluster. If Antonov's sphere is expanded beyond his critical radius, the outside of the cluster, which was held in by the box, expands and cools. Heat then flows from the core of the cluster which shrinks and gets hotter (higher temperature), while the outer parts also get hotter on receiving the heat. If the outer parts are too extensive then their temperature rise on receipt of a given quantity of heat dQ will be less than the temperature rise of the core due to loss of that heat. Under these circumstances a gravothermal catastrophe arises in that the flow of heat only serves to increase the temperature gradient. Several workers have followed what happens after the gravothermal catastrophe. Larson (1970a, b) developed a computer code for solving Jeans's equations of stellar hydrodynamics giving the first few moments of the Fokker–Planck equation. His results showed that the core of the cluster shrank in size and in mass, and it left behind a 'halo' which took up a power law of about $\rho \propto r^{-2.4}$ out to the confining sphere. More recently Hachisu *et al.* (1978) and Inagaki & Hachisu (1978) have studied the gravothermal collapse of gaseous spheres with various laws of thermal conductivity. Provided the conductivity is sufficiently large in the middle as compared with the outside these develop in a manner closely similar to Larson's stellar dynamical calculation. A small central core develops and loses mass as its density and temperature increase. Lynden-Bell & Wood (1968) argued that clusters not confined by reflecting spheres would also go through a gravothermal catastrophe once they had developed enough of the Maxwellian core. To demonstrate this they calculated the entropies and energies of the truncated Maxwellian star clusters in both the Woolley (1954) and the Michie–King (Michie 1963; Michie & Bodenheimer 1963; King 1966) sequences and showed that those sequences reached models of maximum entropy (for given mass and energy) at values of $\beta(\psi_0 - \psi_e)$ close to that at which the gravothermal catastrophe occurred for the Maxwellian in a reflecting sphere. Here $\beta = 1/kT$ for the Maxwellian, ψ_0 is the central value of the potential and ψ_e is the potential at the edge of the system. They argued that evolution might proceed approximately along such a sequence of clusters with an increasingly more Maxwell-like distribution until the point of maximum true entropy $S = -k\int f \log f d^6\tau$ was reached. Beyond that point the cluster would still go on increasing its entropy, but in doing so it would get further and further from the approximate models on the sequence which do not have the high entropies needed. The trouble is that the models have been constructed around the idea that Maxwell's distribution is the most probable one, whereas this is only true up to Antonov's critical radius. Thus the models are built around the wrong precepts at the high central concentrations – the real clusters 'want' greater temperature differences between core and halo than those models allow – such states have yet higher entropies than any achieved along the model sequences. Since, according to Boltzmann's H theorem, encounters increase S unless f is exactly Maxwellian, further evolution along the model sequences is not possible; and yet further evolution must and does occur as encounters continue to increase S.

Hénon (1971a, b, 1973, 1975) and Spitzer and collaborators (Spitzer & Hart 1971a, b; Spitzer & Shapiro 1972; Spitzer & Thuan 1972; Spitzer & Chevalier 1973), have developed large computer codes and performed beautiful calculations on the evolution of star clusters of large mass. Aarseth (1971, 1973, 1974; Aarseth & Hills 1972) has pioneered with somewhat smaller clusters where strict N body techniques are needed. All of these investigators have studied clusters of both equal and unequal masses. However, in spite of these detailed studies, the reality of the occurrence of the gravothermal catastrophe is still doubted – a clear separation from the escape mechanism only occurs in large clusters with very high central concentrations and while Spitzer & Thuan (1972) have found what they believe to be the phenomenon at work in their experiments there is not yet universal agreement (Lightman & Shapiro 1978). Furthermore the pretty experiments of Spitzer & Thuan do not answer all questions about the evolution of equal mass clusters because the numerical techniques become difficult when the central densities are very high. For this reason the way that the energy and mass of the core behave at very high concentrations is not determined with certainty. Several authors (Lightman & Shapiro 1978; Miller & Parker 1964) have deduced from these works that the core energy is constant during evolution and we emphasized that 'if true' this 'fact' demanded a physical explanation (Lynden-Bell 1975). We shall see that very general considerations of the consequences of the gravothermal catastrophe enable us to disprove this 'fact'. In reality both core mass and core energy tend to zero as the core radius shrinks. In the next section we give these general arguments and conclude that not only the evolution of the core, but also its surroundings, are self-similar and leave behind a power law debris ($\rho \propto r^{-\alpha}$) of material that was once in the core. The true value of α requires a detailed calculation from which it emerges as a sort of eigenvalue, but general considerations show that $2 < \alpha < 5/2$. A preliminary determination of α is made from consideration of the exactly self-similar solutions for a conducting gas model of appropriate thermal conductivity. These are determined in Section 3. Not surprisingly the value found, 2.21, is close to that originally found in Larson's numerical work which evolved to an almost power law form with $\alpha = 2.4$. Hachisu *et al.* also found 2.4 but in that calculation they used a thermal conductivity inappropriate for gravitational interactions. Our eigenvalue $\alpha = 2.21$ is, however, the correct asymptotic value of the self-similar solution and can be obtained from a graph of ρ against r if that graph is followed for at least five decades.

2 General arguments for, and properties of, self-similar solutions

Once a cluster has developed a core which is Maxwellian up to values of $\beta(\epsilon - \psi_0)$ of about 7 then the role of the halo is to receive the heat generated as the core shrinks. The outer parts of the halo have a long relaxation time so they hardly change on the time-scale of this heat transfer. Further, as the core shrinks it loses mass and thus generates a new higher density inner halo with a shorter relaxation time. It is this inner halo that receives the heat from the core allowing it to contract, grow hotter and to release yet more mass to make a yet higher density inner halo etc. Thus as the core shrinks it is at all times surrounded by an inner halo of old core debris, and an almost static outer halo that takes no interest in the core's activity but merely responds adiabatically to any changes in potential due to changes in core size. Since this process proceeds until the core is very small, it is reasonable to guess that eventually the initial conditions that determined the size of the original core and the structure of its halo will be irrelevant to the structure of the new core and the new inner halo. If we take a snapshot at any time and then set ourselves the problem of the subsequent evolution we would expect that apart from a scaling the problem would be the same as the problem set at another time. Thus, apart from the rescaling, the solution will be the same. Earlier we emphasized (Lynden-Bell 1975) that such a rescaling must apply to the isothermal parts of the cores because isothermal spheres are homologous, but the above argument extends this to those parts of the halo that are old core debris. We deduce that there will be a similarity solution for the core and that part of the halo that once inhabited the core, while the outer halo will essentially be frozen in its initial state for many core relaxation times. Mathematically such a similarity solution has its density and all other variables scaling in the general form

$$\rho = \rho_c(t)\, \rho_*(r_*) \tag{2.1}$$

where

$$r_* = r/r_c(t) \tag{2.2}$$

where $r_c(t)$ is the radius of the core.

Mathematically it is difficult to calculate precisely with an approximate similarity solution that does not hold for the whole cluster but luckily we may escape this difficulty by a ruse. Since the outer parts of the halo play no part in the evolution of the core it does not matter what the outer part of the halo is like. If our aim is the study of the core and what was once core and is now halo, then we can replace the outer halo by anything we like, always provided that it continues to do nothing! This suggests the following ruse. We replace the outer halo by the structure that it would have had, had it once been part of the core of an enormous cluster. In practice we shall find that we have to add an infinite halo in order to do this at all scales, but this imaginary halo indeed does nothing and is so diffuse that it has no effect on the core or inner halo. With the problem doctored by this strange ruse we can now demand that we have an exact self-similar solution not only in the core and the inner halo but everywhere. We therefore seek such solutions. Now by hypothesis the outer halo does nothing while the core shrinks hence in the halo $\rho(r, t) = \rho_c(t)\, \rho_*[r/r_c(t)]$ must be independent of t. Thus differentiating and writing a dash for differentiation of ρ_* wrt its argument r_*

$$\dot{\rho}_c \rho_* - \frac{\dot{r}_c}{r_c} \rho_c r_* \rho_*' = 0$$

and so

$$\frac{r_* \rho_*'}{\rho_*} = \frac{r_c}{\dot{r}_c} \frac{\dot{\rho}_c}{\rho_c} = -\alpha. \tag{2.3}$$

Since the lhs is a function of r_* alone and the rhs is a function of t alone, α must be a constant. We deduce that

$$\rho_* = A r_*^{-\alpha} \quad \text{in the halo} \tag{2.4}$$

and that

$$\rho_c \propto r_c^{-\alpha}. \tag{2.5}$$

Since ρ_c and r_c are functions of t alone this last relationship holds everywhere. Now the core mass is some definite multiple of $\rho_c r_c^3$ (dependent on whose precise definition we use) and so $M_c \propto r_c^{3-\alpha}$. Similarly the core energy is some multiple of $-GM_c^2/r_c \propto -r_c^{5-2\alpha}$. Hence the core energy is related to the core mass by

$$E_c \propto -M_c^{\zeta} \quad \text{where} \quad \zeta = \frac{5-2\alpha}{3-\alpha}. \tag{2.6}$$

Similarly calling v_c^2 the central velocity dispersion we have

$$v_c^2 \propto GM_c/r_c \propto r_c^{2-\alpha}. \tag{2.7}$$

Now the standard formula for the relaxation time in a stellar system is

$$T_r = v^3(8\pi Gm\rho \log N)^{-1}. \tag{2.8}$$

For the evolution of our system we must have

$$\frac{1}{\rho_c}\frac{d\rho_c}{dt} \propto \frac{1}{T_{rc}} \propto \frac{\rho_c}{v_c^3}(8\pi Gm \log N). \tag{2.9}$$

In any similarity solution it is the scales that vary so the dimensionless quantities appearing in the solution are constant (since they are scale independent). $(T_{rc}/\rho_c)(d\rho_c/dt)$ is dimensionless, and is therefore constant in the similarity solution. (In order to use such arguments strictly we should remark that we allow m the mass of a star to enter only in the relaxation time and nowhere in the momentary equilibrium of the star cluster. Encounterless stellar dynamics has equilibria independent of the masses that make them up.)

In the standard formula (2.8) N is the number of stars in the system. In our infinite system N is the number of stars that interact with the core. It is natural to take N to be a multiple of the number of stars in the core itself but although there are still similarity solutions when $\log N$ is dependent on time – indeed they have the same separation constant and identical spatial form – nevertheless the extra complication of the formulae for the time dependence outweighs the gain in accuracy (Lynden-Bell 1975). We shall hereafter neglect the time variation of $\log N$. Its value is then absorbed into the constant of proportionality in equation (2.9).

From equations (2.9), (2.7) and (2.5)

$$-\alpha\frac{dr_c}{dt} \propto \frac{r_c^{1-\alpha}}{r_c^{3/2(2-\alpha)}} = r_c^{\alpha/2-2}.$$

Hence

$$r_c^{3-\alpha/2} \propto (t_0 - t) \tag{2.10}$$

where t_0 is a constant of integration and we take $\alpha \neq 6$. From equations (2.7) and (2.9) we obtain

$$\left.\begin{aligned} v_c^2 &\propto (t_0-t)^{(4-2\alpha)/(6-\alpha)} \\ \rho_c &\propto (t_0-t)^{-2\alpha/(6-\alpha)} \\ M_c &\propto (t_0-t)^{(6-2\alpha)/(6-\alpha)} \\ E_c &\propto (t_0-t)^{2(5-2\alpha)/(6-\alpha)}. \end{aligned}\right\} \tag{2.11}$$

Notice that $r_c \to 0$ at the finite time $t = t_0$ provided $\alpha < 6$.

We now turn to the proof that $2 < \alpha < 2.5$. To do this we first prove that any self-similar evolving solution must be infinite. We start by assuming that the total mass M and total energy E are finite and show that this leads to contradiction.

In any similarity solution the scales change but the functional forms such as $\rho_*(r_*)$ remain the same. In our problem constants with dimensions are:

$$G = [M^{-1}L^3T^{-2}]$$

$$M = [M]$$

$$E = [ML^2T^{-2}].$$

Since each is constant and $\rho_*(r_*)$ is some definite functional form we deduce that the core radius is some fixed fraction of the natural length and so

$$r_c \propto \frac{GM^2}{-E} = [L].$$

Thus if M and E are finite constants, r_c is fixed, the core does not shrink and no evolution occurs. Now this is false since the only non-evolving systems are isothermal and these are infinite. Now M finite implies $\alpha > 3$, whereas E finite only requires $\alpha > 2.5$ (we remember that $\rho_* = Ar_*^{-\alpha}$ is only the asymptotic form at large r_*). Thus we now try the supposition that $3 \geqslant \alpha > 2.5$ so that M is inifinite but E is finite. Now E_c/E is dimensionless – in a similarity solution it is only the scales that change, so dimensionless numbers are constant. Hence E_c is constant. This looks like a straightforward proof that the core evolves at constant energy, but it actually leads on to contradiction as we now show. Looking at our asymptotic form for ρ_*, and remembering that the outer halo is unchanging on the time-scale of the evolution of the core, we see that A is a constant of dimensions $ML^{\alpha-3}$. Hence A, G and E form three dimensionfull constants and so

$$r_c \propto \left(\frac{GA^2}{-E}\right)^{1/(2\alpha-5)} = [L].$$

Hence r_c is again fixed and again no evolution is possible. However, the careful reader will have noticed that this argument itself fails when $\alpha = 5/2$ for then both GA^2 and E have the same dimensions and so a changing r_c is possible. This suggests a fixed core energy with $\alpha = 5/2$ is the solution to our problem. That is not the case. At the start of our argument we assumed E the total energy to be finite so $\alpha > 5/2$. Thus $\alpha = 5/2$ is outside the range of the argument's validity but the argument does show that there are no solutions with $\alpha > 5/2$ so both E and M must be infinite in any similarity solution.

At this point in the argument D.L.-B. was much tempted to believe that although the argument leading to E_c = constant and $\alpha = 5/2$ is not strictly valid, nevertheless there might be a solution of this form. However, this too is false. Consider for a moment the system with $\rho \propto r^{-5/2}$ everywhere; then $M(r) = A_1 r^{1/2}$ where A_1 is constant. Hence

$$-\int \frac{GM\,dM}{r} = -\int \frac{GM\,dM}{M^2}\,A_1.$$

The integral is logarithmically divergent not only at large M but also at small M. Thus if the core should ever shrink to zero in finite time the inner halo would have to lose an infinite energy. However, the transport would be through a cluster that was of finite density and velocity dispersion away from the central core and the transport processes could not carry an infinite energy in a finite time. As we have already shown that r_c shrinks to zero in a finite time, both $\alpha = 5/2$ and constant E_c are impossible.

We have thus shown that $\alpha < 5/2$ but we still need to show that $\alpha > 2$. This follows from the remark that $\alpha = 2$ is the case of an isothermal sphere. We require that our solutions lose heat from the core to the halo so the temperature of the core must be greater. There is only a decrease of temperature outward for $\alpha > 2$ and only then does the central temperature, v_c^2, increase as t increases towards t_0 – see equation (2.11). A further property of complete similarity solutions can be used as a check that their central non-power-law parts are indeed correct. M_c and E_c tend to zero when r_c tends to zero and both the total energy and mass within any given $r = R$ well out in the halo must remain constant. Thus both $M(R)$ and $E(R)$ must equal the values that will be obtained when $r_c = 0$. But those values will be those obtained from the power law density all the way in to $r_* = 0$. Thus the inward extrapolation of the limiting power laws gives a distribution with exactly the same mass and the same energy as the true similarity solution. Hence

$$M(R) - \frac{4\pi R^3 \rho(R)}{3-\alpha} \to 0 \quad \text{as} \quad R \to \infty.$$

Similarly

$$M_*(r_*) - \frac{r_*^3 \rho_*}{3-\alpha} \to 0.$$

Using Jeans's equation of stellar hydrodynamics

$$-\frac{\partial}{\partial r}(\rho v^2) = \frac{GM\rho}{r^2}$$

and the asymptotic forms at large r we have

$$v_*^2 \to \frac{1}{2(\alpha-1)} \frac{M_*}{r_*} \propto r_*^{2-\alpha}.$$

Since this form of Jeans's equation assumes isotropic velocities, the precise result only follows in that case. None of the other arguments of this section has assumed isotropy of the velocities. *They* are still true in the anisotropic case. By discussing the radii at which the core debris is left Lightman & Shapiro (1978) reached similar conclusions on α's range.

3 Self-similar solutions for a gas sphere in gravothermal evolution

Hachisu *et al.* have studied the evolution of a gaseous system larger than Antonov's critical radius. They point out that it is possible to approximate a stellar dynamical problem by using appropriate dependence of the thermal conductivity on density and temperature. They find by numerical computation that their solution does eventually achieve an approximately self-similar form with $\rho \to r^{-\alpha}$ at large distances and $\alpha \simeq 2.4$. Now 2.4 is very close to the critical value 2.5 and since this small difference is vital to the way the core energy depends on core mass it is important to determine α precisely. It is also interesting to demonstrate that exactly self-similar solutions exist. To determine α accurately we set up the exact equations to be satisfied by a self-similar solution from which it emerges as a separation constant determined from the boundary conditions. On computation we find $\alpha = 2.21$; earlier workers estimated $\alpha \simeq 2.4$.

During the evolution the time dependence is due to the slow transport of heat between different parts of the system. Thus, although the system moves, the equations of hydrostatics are a good approximation at all times. We may therefore write

$$\frac{\partial M}{\partial r} = 4\pi r^2 \rho \tag{3.1}$$

$$\frac{\partial p}{\partial r} = \frac{\partial}{\partial r}(\rho v^2) = -\frac{GM\rho}{r^2} \tag{3.2}$$

where $M(r)$ is the mass within radius r, and where

$$\frac{p}{\rho} = \frac{kT}{m} = v^2 \tag{3.3}$$

if we take v^2 as the rms velocity in one dimension (so that $3v^2$ is the total rms velocity). We need an equation governing 'heat' flow between different parts of the system. In thermal problems one writes

$$\frac{L}{4\pi r^2} = -K\frac{\partial T}{\partial r} \tag{3.4}$$

where K is the thermal conductivity. In stellar interiors K is written as $4acT^3/3\kappa\rho$, where κ is the opacity. In order to approximate a stellar dynamical situation we need to know how K (or κ) depends on ρ, T (or ρ, v). In the elementary theory of thermal conductivity of gases the heat flux is an average of the one-way flux of particles across an imaginary surface times the excess energy per particle between starting point and end point, i.e.

$$\frac{L}{4\pi r^2} = -\rho\cdot\frac{\lambda^2}{\tau}\frac{\partial}{\partial r}\left(\frac{3}{2}\frac{kT}{m}\right), \tag{3.5}$$

where λ is the mean free path between collisions and τ is the time between collision, so that

$$\tau = \frac{\lambda}{v} = \lambda\sqrt{\frac{m}{kT}}. \tag{3.6}$$

However, these concepts are not directly applicable to stellar dynamics since the stars do not move with constant velocity but make several orbits in and out and around the cluster between encounters. In such a stellar system a typical star at any radius moves in and out by about a local Jeans length $\lambda = 1/k_J$ where $k_J^2 v^2 = 4\pi G\rho$. This gives a typical radial distance between encounters. However, the time between those encounters is not λ/v but rather a relaxation time T_r. Hence the energy flux is given by equation (3.5) with $\lambda = k_J^{-1}$ and with τ replaced by T_r

$$\frac{L}{4\pi r^2} = -\frac{C\rho}{k_J^2 T_r}\frac{\partial}{\partial r}\left(\frac{3}{2}v^2\right) \tag{3.7}$$

$$= -3GmC(\log N)\frac{\rho}{v}\frac{\partial v^2}{\partial r} \tag{3.8}$$

where C is a dimensionless constant of order unity. As Lecar pointed out, this implies a conductivity K proportional to $\rho T^{-1/2}$ or an opacity κ proportional to $T^{3.5}\rho^{-2}$. Notice that the details of the argument are unimportant, for the use of dimensions, coupled with the fact that the heat flux must be proportional to the relaxation rate (T_r^{-1}) gives the same dependence of conductivity on ρ, v. The detailed computation of Hachisu *et al.* (1978) was done for a different conductivity in which the heat flows faster when the relaxation time is longer. This is not the physical case. However, Hachisu *et al.* do show that their result is rather insensitive to the details of conduction.

To complete our equations we need the analogue of the first law of thermodynamics, viz.

$$\frac{\partial L}{\partial r} = -4\pi r^2\rho\left\{\left(\frac{\partial}{\partial t}\right)_M\frac{3v^2}{2} + p\left(\frac{\partial}{\partial t}\right)_M\frac{1}{\rho}\right\} \tag{3.9}$$

$$= -4\pi r^2\rho v^2\left(\frac{\partial}{\partial t}\right)_M\log\left(\frac{v^3}{\rho}\right) \tag{3.10}$$

since by equation (3.3) $p = \rho v^2$. The operator $(\partial/\partial t)_M$ follows the mass shell concerned. In what follows, we look for a similarity solution to our equations (3.1), (3.2), (3.8) and (3.10) in terms of a time-dependent scaled radius $r_* = r/r_c(t)$.

Let us put

$$\begin{aligned} r &= r_c(t)\, r_* \\ M &= M_c(t)\, M_*(r_*) \\ \rho &= \rho_c(t)\, \rho_*(r_*) \\ v &= v_c(t)\, v_*(r_*) \\ L &= L_c(t)\, L_*(r_*). \end{aligned} \tag{3.11}$$

Generally, if M_* is any function of M, t,

$$\left(\frac{\partial}{\partial t}\right)_M = \left(\frac{\partial}{\partial t}\right)_{M_*} - \left(\frac{\partial M}{\partial t}\right)_{M_*} \left(\frac{\partial}{\partial M}\right)_t \tag{3.12}$$

so that if we separate variables as in equations (3.11), equation (3.10) gives

$$\frac{L_c}{r_c}\frac{dL_*}{dr_*} = -4\pi r_c^2 \rho_c v_c^2 \cdot r_*^2 \rho_* v_*^2 \left\{\frac{d}{dt}\log\frac{v_c^3}{\rho_c} - \frac{d\log M_c}{dt}\frac{d\log v_*^3/\rho_*}{d\log M_*}\right\}. \tag{3.13}$$

Since some portions of this depend only on t and others on M_*, there is consistency only in one or other of the two circumstances:

$$\text{(i)}\quad \frac{d\log(v_c^3/\rho_c)}{dt} = -c_1 \frac{L_c}{4\pi\rho_c v_c^2 r_c^3} \tag{3.14}$$

$$\frac{d\log M_c}{dt} = -c_2 \frac{L_c}{4\pi\rho_c v_c^2 r_c^3} \tag{3.15}$$

$$\frac{dL_*}{dr_*} = r_*^2 \rho_* v_*^2 \left\{c_1 - c_2 \frac{d\log(v_*^3/\rho_*)}{d\log M_*}\right\} \tag{3.16}$$

or alternatively

$$\text{(ii)}\quad \frac{d\log(v_*^3/\rho_*)}{d\log M_*} = c_1 \tag{3.17}$$

$$\frac{dL_*}{dr_*} = c_2 \cdot r_*^2 \rho_* v_*^2 \tag{3.18}$$

$$c_2 L_c = -4\pi r_c^3 \rho_c v_c^2 \left\{\frac{d}{dt}\log\frac{v_c^3}{\rho_c} - c_1 \frac{d\log M_c}{dt}\right\}. \tag{3.19}$$

In both cases c_1, c_2 are separation constants. For $c_1 \neq 0$, equation (3.17) cannot give ρ_*, v_* finite and non-zero at $r_* = 0$, as we require; and for $c_1 = 0$, equation (3.17) implies an adiabatic model (an $n = 3/2$ polytrope) which does not extend to $M_* = \infty$ as we require. Therefore we require alternative (i).

The remaining equations (3.1), (3.2) and (3.8) separate more easily: taking,

$$M_c = 4\pi r_c^2 \rho_c \tag{3.20}$$

$$L_c = 12\pi GmC \log N r_c \rho_c v_c \tag{3.21}$$

$$v_c^2 = GM_c/r_c \tag{3.22}$$

we deduce

$$\frac{dM_*}{dr_*} = r_*^2 \rho_* \tag{3.23}$$

$$\frac{dv_*^2}{dr_*} = -\frac{v_* L_*}{\rho_* r_*^2} \tag{3.24}$$

$$-\frac{d(\rho_* v_*^2)}{dr_*} = \frac{\rho_* M_*}{r_*^2}. \tag{3.25}$$

The ratio $a \equiv c_2/c_1$ is readily related to the parameter α of Section 2. Equations (3.14), (3.15), (3.20) and (3.22) unite to give

$$\rho_c \propto r_c^{-[6(1-a)/(2-a)]} \equiv r_c^{-\alpha} \tag{3.26}$$

so that $a = (3-\alpha)/(3-\alpha/2)$. We then recover equations (2.10) and (2.11), as expected. The discussion of Section 2 also shows that for a self-similar solution to be static at large r_* we must have

$$\frac{d \log M_*}{d \log r_*} \to 3 - \alpha \tag{3.27}$$

$$\frac{d \log v_*}{d \log r_*} \to 1 - \tfrac{1}{2}\alpha \tag{3.28}$$

as $r_* \to \infty$. Using equations (3.24) and (3.25) these can be written as

$$L_* \to (\alpha - 2)\, \rho_* r_* v_* \tag{3.29}$$

$$(2\alpha - 2)\, v_*^2 \to M_*/r_* \tag{3.30}$$

as $r_* \to \infty$. Relations (3.27) and (3.28) will ensure, via equations (3.24) and (3.25), that L_*, ρ_* also tend asymptotically to power laws.

To find α, we have therefore to solve a straightforward eigenvalue problem, viz., the four differential equations (3.16), (3.23), (3.24) and (3.25), with *six* boundary conditions (because c_1, α are also unknown) which are equations (3.29), (3.30) and

$$L_* = M_* = 0, \quad \rho_* = v_* = 1 \quad \text{at} \quad r_* = 0. \tag{3.31}$$

We have taken ρ_c, v_c to be the central values of ρ, v, without loss of generality. For numerical purposes, using a standard package (Eggleton 1971) to solve two-point boundary value problems by relaxation on a finite mesh, it is convenient

(i) to impose the surface boundary conditions at a large but finite value of r_*,

(ii) initially to *specify* c_1, and remove the condition $\rho_* = 1$ at $r_* = 0$ ($c_1 = 1/200$ was found to give $\rho_*(0) \approx 2$). We subsequently rescaled so that $\rho_* = 1$ and found $c_1 = 1/430$,

(iii) to introduce the trivial equation $d\alpha/dr_* = 0$,

(iv) to use as variables $\log(1 + r_*^2)$, $\log(1 + M_*^{2/3})$, $\log(1 + L_*/r_*)$, $\log \rho_*$, $\log v_*$ rather than r_*, m_*, L_*, ρ_*, v_*, since this ensures nearly linear behaviour of each variable with respect to every other over all the mesh.

The value of α was found to be very insensitive to variations in (a) the surface value of r_* (b) the step size; although v_* varies so slowly in relation to ρ_* ($\rho_* \sim v_*^{20}$ in the outer layers) that the surface value of v_* was not very small. Different distributions, and numbers, of mesh points gave α in the range 2.206–2.214, with a preferred value of 2.208 which we think is right to ± 0.001.

4 The solutions

The eigenvalue $\alpha = 2.21$ can be substituted into expressions (2.11) to obtain the time dependences of the central velocity dispersion and density $v_c^2 \propto (t_0 - t)^{-0.11}$ and $\rho_c \propto (t_0 - t)^{-1.16}$. The core radius $r_c \propto (t_0 - t)^{0.53}$ and the core energy $E_c \propto (t_0 - t)^{0.31}$ and $M_c \propto (t_0 - t)^{0.42}$. The density profile of the similarity solution is given in the $\log \rho - \log r$ plot of Fig. 1. Notice the asymptotic gradient of 2.21 steepens to about 2.4 before flattening off as the centre of the core is approached. We remarked earlier that the mass must be exactly the same as that obtained by extrapolating the asymptotic slope right into the origin. The steepening before the final flattening is therefore necessary, so that ρ_* can lie above the power law at some radii and thereby compensate for its deficit near $r = 0$. The whole profile shrinks to zero at $t = t_0$, leaving the power law $\alpha = 2.21$ at that moment. It is clear that the few body problem takes over from the many body problem just before $t = t_0$ because $M_c \to 0$. What happens after that is still one of the most interesting research problems in stellar dynamics. Both Hénon and Aarseth have made interesting attacks on that problem but we shall not consider it in detail here. We remark, however, that although $v_c^2 \to \infty$ and $\rho_c \to \infty$ in our many body solution, nevertheless this is not a spectacular event as $M_c \to 0$ and even $E_c \to 0$. In that sense little or nothing is involved in the final core collapse. In reality it may be that the core is replaced by a well bound central binary star. Now, from expressions (2.11), $v_c^2 \propto M_c^{-0.26}$. Thus a reduction of the mass in the core by a factor of 10^4 will only increase v_c^2 by a factor of 10. Thus a globular cluster core that originally contained 10^4 stars and had $10\,\text{km s}^{-1}$ as its velocity dispersion might be reduced to a binary star with an orbital velocity of about $30\,\text{km s}^{-1}$.

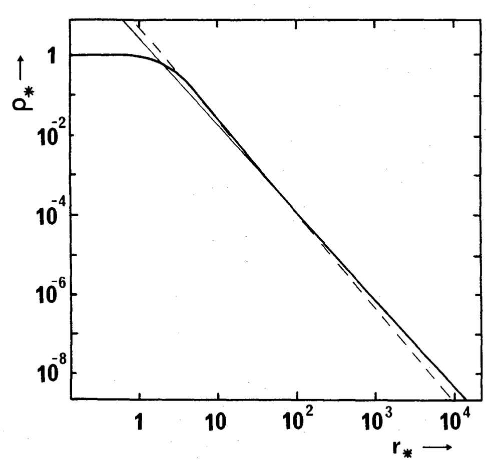

Figure 1. Density profile of the self-similar solution. The broken line has slope 2.4. The continuous straight line has slope 2.21 and the evolution of the unscaled variables can be represented by a sliding of the density profile up this line.

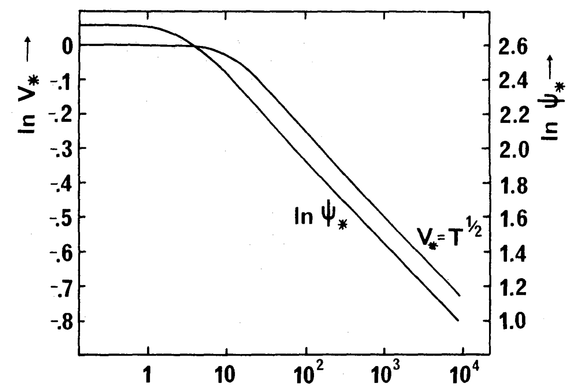

Figure 2. Potential and velocity-dispersion profiles of the self-similar solution.

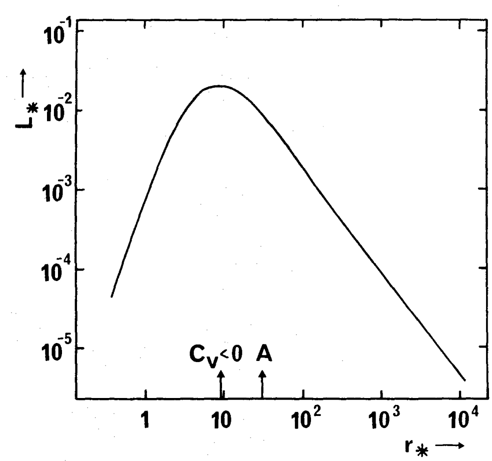

Figure 3. Heat-flux profile of the self-similar solution. At the point A the density has dropped by Antonov's critical factor of 708. The point where the specific heat of the central sphere changes sign is also indicated.

Fig. 2 shows the rather weak dependence of velocity dispersion on radius. Indeed v lies within 5 per cent of being constant out to $14r_c$. Since the central parts are so closely isothermal it is interesting to compare thermodynamic properties with those of isothermal spheres as discussed by Lynden-Bell & Wood. C_V, the specific heat at constant volume, first becomes negative at almost the same radius as that at which L, the heat flux, is a maximum (Fig. 3). This is what the physical argument behind gravothermal runaway suggests ought to be the case. The heat flux has fallen from its maximum by a factor of 2.5 by the time that Antonov's critical density concentration of 708 is reached, and it seems very reasonable that collapse could continue even were the system cut off by an insulating wall at that point as envisaged by Antonov.

Fig. 4 demonstrates that over all the system outside the core $\rho_* \propto \psi_*^{10.6}$ so that the system is close to a polytrope of index 10.6 except at the centre. Fig. 5 shows that in the core and on into an overlap region ρ_* is very close to $\exp(\psi_* - \psi_{*0})$ as in an infinite isothermal sphere. We found that the law

$$\rho_* = \left(\frac{\psi_*}{16.74}\right)^{10.4} \left[1 + \left(\frac{\psi_*}{14.55}\right)^{8.7}\right] \tag{4.1}$$

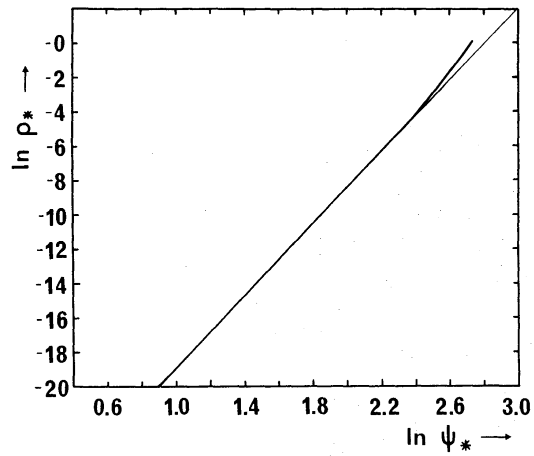

Figure 4. Density against potential in the self-similar solution. A polytrope would yield a straight line; the one indicated has index 10.6.

fitted this graph to 3 per cent or better over the whole range from the centre out to the power law region, although we were surprised that this formula fitted the exponential part with such good accuracy.

Although the gas model has lost many of the details of a proper stellar dynamical treatment, we consider that the same basic process is at work in this gas model as in the stellar dynamics of globular clusters. We therefore believe that when we perform a full stellar dynamical calculation we shall find a self-similar solution with a density profile very close to that of the gas model. It is therefore of interest to calculate a simple stellar-dynamical distribution function that reproduces the density distribution that we have found. In the approximation that the distribution function is isotropic, it is the solution of the integral equation

$$\rho(\psi) = 4\pi \int_{-\psi}^{0} f(\epsilon)[2(\epsilon + \psi)]^{1/2} d\epsilon.$$

It is well known that this can be cast into the form of Abel's integral equation and that the solution for f is

$$f(\epsilon) = -\frac{1}{2\sqrt{2}\pi^2}\frac{\partial}{\partial \epsilon}\int_0^{-\epsilon} \frac{\partial\rho/\partial\psi}{\sqrt{(-\epsilon)-\psi}}\, d\psi.$$

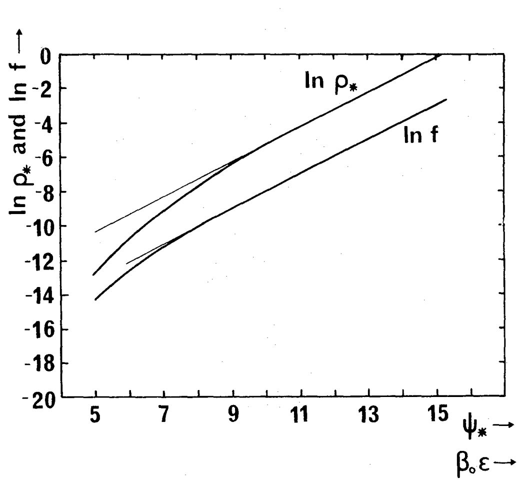

Figure 5. (i) Natural log density versus potential. Notice that in the central region ρ_* is approximately exponential. (ii) Natural log of the distribution function plotted against energy. f is Maxwellian for well bound energies but approaches a power law near the energy of escape.

Performing these operations on our formula (4.1) for $\rho(\psi)$ we find

$$v_c^3 \rho_c^{-1} f(\epsilon) = 0.0134 \left(\frac{-\epsilon_*}{15.3}\right)^{8.9} \left[1 + 3.92 \left(\frac{-\epsilon_*}{15.3}\right)^{8.7}\right] \tag{4.2}$$

where $\epsilon_* = \beta_0 \epsilon = \epsilon / v_c^2$.

In Fig. 5, $\log f$ is plotted against $(-\epsilon)$. It is evident that f is close to Maxwellian for $\beta_0\epsilon$ from 7 into the centre at $\beta_0 \epsilon \simeq 15.3$. However, $f(\epsilon)$ becomes a power law $(-\epsilon)^{8.9}$ for values of ϵ closer to the energy of escape. The form of distribution function is close to those found by Hénon and by Spitzer in their computations of isolated clusters. However, the power law régime is not well developed in their models because their clusters are finite and the outer parts of them were never in the self-similar part of the clusters.

At $k_1 = \beta_0(\epsilon + \psi_0) = 8.7$, f falls below a Maxwellian by a factor 2. Thus significant modification of the Maxwellian occurs near $k_1 = 8.5$ which is the value of k_1 at which maximum true entropy is achieved along Woolley's sequence of truncated Maxwellians (Lynden-Bell & Wood 1968). Here the true entropy is $S = -k \int f \log f d^6\tau$ and the sequence is considered at fixed energy and mass with k_1 being varied.

Possible application of the stellar dynamical catastrophe to the formation of central black holes in stellar systems was considered by Hachisu *et al.* who found only very small black holes in extreme cases. This confirms the idea that the more dissipative collapse of gas is necessary to make such objects (Lynden-Bell 1970, 1978).

Acknowledgments

The basic arguments of Sections 1 and 2 were completed while D.L.-B. was a Fairchild Scholar at the California Institute of Technology where discussions with Dr Goldreich were most helpful. Dr M. Lecar drew the work of Hachisu *et al.* to our attention and stimulated us to solve for the self-similar solution of the gaseous model with his revision of Hachisu's conductivity. Drs Whelan, Papaloizou and Fall helped in delineating the difficulties with the asymptotic form discussed in Appendix 1.

References

Aarseth, S. J., 1971. *Astrophys. Space Sci.*, **14**, 20.
Aarseth, S. J., 1973. *Vistas Astr.*, **15**, 13.
Aarseth, S. J., 1974. *Astr. Astrophys.*, **35**, 237.
Aarseth, S. J. & Hills, J. G., 1972. *Astr. Astrophys.*, **21**, 255.
Antonov, V. A., 1962. *Bull. Leningrad State Univ. N. 7*, 135.
Eggleton, P. P., 1971. *Mon. Not. R. astr. Soc.*, **151**, 351.
Hachisu, I., Nakada, Y., Nomoto, K. & Sugimoto, D., 1978. *Prog. Theor. Phys.*, **60**, 393.
Hénon, M., 1960. *Ann. Astrophys.*, **23**, 668.
Hénon, M., 1961. *Ann. Astrophys.*, **24**, 380.
Hénon, M., 1971a. *Astrophys. Space Sci.*, **13**, 284.
Hénon, M., 1971b. *Astrophys. Space Sci.*, **14**, 751.
Hénon, M., 1973. *Dynamical Structure & Evolution of Stellar Systems*, p. 183, eds Martinet & Mayor, Geneva Observatory.
Hénon, M., 1975. *IAU Symp. 69*, 134.
Inagaki, S. & Hachisu, I., 1978. *Publs astr. Soc. Japan*, **30**, 39.
King, I. R., 1966. *Astr. J.*, **71**, 64.
Larson, R. B., 1970a. *Mon. Not. R. astr. Soc.*, **147**, 323.
Larson, R. B., 1970b. *Mon. Not. R. astr. Soc.*, **150**, 93.
Lightman, A. P. & Shapiro, S. L., 1978. *Rev. Mod. Phys.*, **50**, 437.
Lynden-Bell, D., 1970. *Nuclei of Galaxies*, p. 527, ed. O'Connell, D., *Pont Acad. Sci. Scripta Varia*, **35**.

Lynden-Bell, D., 1975. *IAU Symp. 69*, 27.
Lynden-Bell, D., 1978. *Phys. Scripta*, 17, 185.
Lynden-Bell, D. & Wood, R., 1968. *Mon. Not. R. astr. Soc.*, **138**, 495.
Michie, R. W., 1963. *Mon. Not. R. astr. Soc.*, **125**, 127.
Michie, R. W. & Bodenheimer, P. H., 1963. *Mon. Not. R. astr. Soc.*, **126**, 269.
Miller, R. H. & Parker, E. N., 1964. *Astrophys. J.*, **140**, 50.
Spitzer, L., 1975. *Dynamics of Stellar Systems, IAU Symp. 69*, 3, ed. Hayli, A., Reidel, Dordrecht.
Spitzer, L. & Chevalier, R. A., 1973. *Astrophys. J.*, **183**, 565.
Spitzer, L. & Hart, M. H., 1971a. *Astrophys. J.*, **164**, 399.
Spitzer, L. & Hart, M. H., 1971b. *Astrophys. J.*, **166**, 483.
Spitzer, L. & Shapiro, S. L., 1972. *Astrophys. J.*, **173**, 529.
Spitzer, L. & Thuan, T. X., 1972. *Astrophys. J.*, **175**, 31.
Woolley, R. v. d. R., 1954. *Mon. Not. R. astr. Soc.*, **114**, 191.

Appendix 1: The number of free parameters in the asymptotic solutions

There are two free parameters c_1, a in the solutions to the four equations (3.16) and (3.23)–(3.25) integrated outwards from the centre where equation (3.31) holds as the boundary condition. With a and therefore α known, there is only one new free parameter A in the asymptotic power laws. However, all four of the functions L, M, v, ρ must be fitted at some chosen intermediate point so it looks unlikely that this can be done with only three parameters to vary. In fact there is another free parameter in the asymptotic form as we now demonstrate. We linearize our solutions about their asymptotic forms writing

$$M_* = \frac{Ar_*^{3-\alpha}}{3-\alpha}(1+m_*)$$

$$\rho_* = Ar_*^{-\alpha}[1+m_* + (3-\alpha)^{-1} m_l]$$

where $m_l = dm_*/d(\log r)$.

We also write

$$v_* = \left[\frac{Ar^{-(\alpha-2)}}{2(\alpha-1)(3-\alpha)}\right]^{1/2}(1+u).$$

We keep only first order terms in m and u and substitute these expressions into equations (3.23)–(3.25) and (3.15) to obtain

$$m_{ll} + (3-\alpha)\, m_l + 2(\alpha-1)(3-\alpha)\, m_* = -2(3-\alpha)[u_l - 2(\alpha-1)\, u] \tag{A1}$$

and

$$A_2 \exp[-(3-\alpha/2)\, l] = m_{ll} + 3/2(4-\alpha)\, m_l - 3(3-\alpha)\, u_l \tag{A2}$$

where A_2 is a constant.

The dominant asymptotic form of solution for m_* that obeys the boundary conditions is

$$m_* = A_3 \exp(-\lambda l) \tag{A3}$$

where $\lambda^2 - 2\lambda(5/2 - \alpha) - 6/5(\alpha-1) = 0$ and $\lambda > 0$. Thus

$$\lambda = (5/2-\alpha) + [(5/2-\alpha)^2 + 6/5(\alpha-1)]^{1/2} \tag{A4}$$

A_3 is the missing free parameter. The inhomogeneous solution to equations (A1) and (A2) is not dominant since $3 - \alpha/2 > \lambda$ as given by equation (A4), over the range $2 < \alpha < 2.5$.

Appendix 2: Differential conductivity laws – criterion for the gravothermal catastrophe

Hachisu *et al.* explored a number of conductivity laws by making many different trials. Crucial to the form of solution in which the core shrinks losing mass and energy almost independently of the halo is the condition that the thermal time-scale is smallest in the inner parts. If asymptotically $\rho \propto r^{-\alpha}$ then $T \propto r^{2-\alpha}$ and $K \propto \rho^{a'}T^{b}$, so the energy within r is proportional to $r^{5-2\alpha}$ and the thermal time-scale is $E/KrT \propto r^{2-2b-\alpha(1-a'-b)}$ as given by them. We have shown that the binding energy of the core and inner halo gets greater and greater as $\alpha \to 5/2$. If we slowly change the conductivity law to allow heat to flow through the outer parts more readily, the binding energy of the inner parts will increase and α will approach the value 5/2. We expect our form of gravothermal evolution provided that a' and b are such that the thermal time-scale still increases outward even in the limiting case $\alpha = 5/2$. Thus the condition for the gravothermal catastrophe to occur is that $2 - 2b - \alpha(1 - a' - b) \geqslant 0$ with α set equal to 5/2. This condition becomes

$$b \geqslant 1 - 5a'.$$

This condition agrees with the numerical experiments of Hachisu *et al.* Our b is their β and our a' is their α. They have labelled one figure $\alpha = ½$ but this is a misprint for $\alpha = - ½$ as is clear from the positioning of the diagram and the accompanying text.