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ON THE CONSISTENCY AND INDEPENDENCE OF SOME SET-THEORETICAL AXIOMS

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In this paper by means of simple models it is shown that the five set-theoretical axioms of Extensionality, Replacement, Power-Set, Sum-Set, and Choice are consistent and that each of the axioms of Extensionality, Replacement, and Power-Set is independent from the remaining four axioms. Although the above results are known and can be found in part in [1], it is believed that this paper has some expository merits.

The abovementioned axioms are five of the six axioms of the Zermelo-Fraenkel Theory of Sets, the sixth being Axiom of Infinity. We accept that every element of a set is a set and (without borrowing "=" from Logic) we define two sets u and v to be equal if and only if they possess the same elements, in which case we write $u = v$. With this in mind, the six axioms of the Zermelo-Fraenkel Theory of Sets can be stated as follows [2]:

(1) Axiom of Extensionality *Equal sets are elements of the same sets.*
(2) Axiom of Replacement *If the domain of a functional binary predicate is a set (its values, if any, also being sets) then its range is a set.*
(3) Axiom of Power-Set *The set of the subsets of any set exists.*
(4) Axiom of Sum-Set *The set of the elements of the elements of any set exists.*
(5) Axiom of Choice *There exists a choice function for any set none of whose elements is an empty set.*
(6) Axiom of Infinity *There exists a set with infinitely many elements.*

We observe that Axiom of Replacement is an Axiom Scheme. Now we prove.

Theorem 1 *The axioms of Extensionality, Replacement, Power-Set, Sum-Set, and Choice form a consistent system of axioms.*

Proof: Consider the model whose sets are the symbols

(7) $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \ldots$

and whose membership relation "ϵ" is defined by:

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(8) $\overline{x} \in \overline{y}$ if and only if $x = n_i$ for some $i \leqslant k$, where $y = \sum_{i=1}^{k} 2^{n_i}$ with $n_i \geqslant 0$ and $n_i \neq n_j$ for $i \neq j$.

In other words, $\overline{x}$ *is an element of* $\overline{y}$ *if and only if* x *appears as an exponent in the unique representation of* y *as a sum of distinct powers of* 2.

In what follows, an equality of the form $x = \sum 2^n$ shall always indicate the unique representation of x as a sum of distinct powers of 2. Thus, in the model given by (7) and (8) it is the case that $\overline{0} \in \overline{3}$, $\overline{2} \in \overline{5}$, $\overline{5} \in \overline{38}$, etc., etc. Moreover, $\overline{1} = \{\overline{0}\}$, $\overline{2} = \{\overline{1}\}$, $\overline{3} = \{\overline{0}, \overline{1}\}$, $\overline{4} = \{\overline{2}\}$, $\overline{38} = \{\overline{1}, \overline{2}, \overline{5}\}$, $\overline{128} = \{\overline{7}\}$, etc., etc. Clearly, in the above $\overline{0}$ is the *empty set* because $2^m \neq 0$ for $m = 0, 1, 2, \ldots$.

Now let us observe that, since two distinct natural numbers have distinct abovementioned representations, no two sets listed in (7) are equal. Therefore, in the model given by (7) and (8) the Axiom of Extensionality is valid.

Next, let $\overline{x}$ be a set and $f(u, v)$ be a functional (in u) binary predicate defined on $\overline{x}$ with $x = \sum_{i=1}^{k} 2^{n_i}$. Let $\{\overline{m}_e \mid e = 1, \ldots, h\}$ be the set such that $f(\overline{n}_i, \overline{m}_e)$ for some $i \leqslant k$. Consider $\overline{y}$ with $y = \sum_{e=1}^{h} 2^{m_e}$. Then, obviously $\overline{y}$ is a set which is the range of $f(u, v)$. Thus, in the model under consideration the Axiom of Replacement is valid.

Again, let $\overline{x}$ be a set such that $x = \sum_{i=1}^{k} 2^{n_i}$. Consider $\overline{y}$ with $y = \prod_{i=1}^{k} (1 + 2^{2^{n_i}})$. Then it can be verified that $\overline{y}$ is the power set of $\overline{x}$. For example, since $\overline{5} = \{\overline{0}, \overline{2}\}$, the power-set of $\overline{5}$ is the set

$$\{\overline{0}, \overline{\{\overline{0}\}}, \overline{\{\overline{2}\}}, \overline{\{\overline{0}, \overline{2}\}}\} = \{\overline{0}, \overline{1}, \overline{4}, \overline{5}\} = \overline{51}$$

On the other hand, clearly,

$$(1 + 2^{2^2})(1 + 2^{2^0}) = 2^0 + 2^{2^0} + 2^{2^2} + 2^{2^0 + 2^2} = 51$$

Similarly, the power-set of $\overline{6}$ is the set $\overline{85}$. Consequently, in the model given by (7) and (8) the Axiom of Power-Set is valid.

Again, let $\overline{x}$ be a set such that $x = \sum_{i=1}^{k} 2^{n_i}$, and let $n_i = \sum_{j=1}^{v_i} 2^{n_{ij}}$. Consider $\overline{y}$ with $y = \sum_{e=1}^{p} 2^{m_e}$ where $m_e \neq m_u$ for $e \neq u$ and $m_e = n_{ij}$ for some i and j. Then, obviously, $\overline{y}$ is the sum-set of $\overline{x}$. For example, the sum-set of $\overline{6}$ is the set $\overline{3}$ since

$$6 = 2^{2^1} + 2^{2^0} \text{ and } \overline{3} = \{\overline{1}, \overline{0}\}$$

Similarly, the sum-set of $\overline{35}$ is the set $\overline{5}$ since

$$35 = 2^{2^2 + 2^0} + 2^{2^0} + 2^0 \text{ and } \overline{5} = \{\overline{2}, \overline{0}\}.$$

Thus, in the above model the Axiom of Sum-Set is valid.

Again, let $\overline{x}$ be a set such that $x = \sum_{i=1}^{k} 2^{n_i}$ with $n_i \neq 0$ and let $n_i = \sum_{j=1}^{v_i} 2^{n_{i_j}}$. Consider $\overline{y}$ given by

$$\overline{y} = \{(\overline{n}_i, \overline{t}_i) \mid . . .\}, \text{ with } t_i = \max_j n_{i_j} \text{ and } i = 1, 2, . . ., k.$$

Then, obviously, $\overline{y}$ is a choice function of $\overline{x}$. For example, according to the above, a choice function of $\overline{0}$ is $\overline{0}$; a choice function of $\overline{2}$ is $\overline{4096}$ because, $\overline{2} = \{\overline{1}\} = \{\{\overline{0}\}\}$ and

$$\{(\overline{1}, \overline{0})\} = \{\{\{\overline{1}\}, \{\overline{1}, \overline{0}\}\}\} = \{\{\overline{2}, \overline{3}\}\} = \{\overline{12}\} = \overline{4096}.$$

Similarly, according to the above, a choice function of $\overline{4}$ is $\overline{1208925819614629174706176}$ because $\overline{4} = \{\overline{2}\} = \{\{\overline{1}\}\}$ and $\{(\overline{2}, \overline{1})\} = \{\{\{\overline{2}\}, \{\overline{2}, \overline{1}\}\}\} = \{\{\overline{4}, \overline{6}\}\} = \{\overline{80}\}$, and $2^{80} = 1208925819614629174706176$. Consequently, in the model given by (7) and (8) the Axiom of Choice is valid.

Thus, Theorem 1 is proved.

Corollary *The Axiom of Infinity is independent of Axioms* (1), (2), (3), (4), *and* (5).

Proof: In view of (8) each of the sets listed in (7) has finitely many elements. Hence in this model the Axiom of Infinity is not valid. Thus, the Corollary follows from the proof of Theorem 1.

Next we prove

Theorem 2 *The Axiom of Extensionality is independent of* (2), (3), (4), *and* (5).

Proof: Consider the model in which each of the symbols

(9) $\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, ...$

is a set and

(10) $\overline{x} \in \overline{y}$ if and only if $x = n_i$ for some $i \leq k$ or $x = m_j$ for some $j \leq h$, where

$$\frac{y}{2} = \sum_{i=1}^{k} 2^{n_i} \quad \text{or} \quad \frac{y-1}{2} = \sum_{j=1}^{h} 2^{m_j}$$

Accordingly, $\overline{0} = \overline{1}$, $\overline{2} = \overline{3} = \{\overline{0}\}$, $\overline{4} = \overline{5} = \{\overline{1}\}$. However, $\overline{0} = \overline{1}$, but, for example, $\overline{1} \in \overline{4}$ and $\overline{0} \notin \overline{4}$. Thus, in the model given by (9) and (10), equal sets are not elements of the same sets. Hence, the Axiom of Extensionality is not valid. On the other hand, by arguments very much the same as those given in the proof of Theorem 1, it can be verified that axioms (2), (3), (4), and (5) are valid in the model given by (9) and (10). Thus, Theorem 2 is proved.

Theorem 3 *The Axiom of Replacement is independent of axioms* (1), (3), (4), *and* (5).

Proof: Consider the model in which the symbol

(11) a

is the one and only set and

(12) $$a \in a$$

Clearly, $f(u, v)$ given by $(u = v) \wedge (v \notin v)$ is a functional (in u) binary predicate which is defined on a. However, its range (the empty set) does not exist in the model given by (11) and (12). Thus, the Axiom of Replacement is not valid. Also, a is its own power-set, sum-set, and choice function. Therefore, in this model axioms (1), (3), (4), and (5) are valid and Theorem 3 is proved.

Theorem 4 *The Axiom of Power-Set is independent of axioms* (1), (2), (4), *and* (5).

Proof: Consider the model in which the symbol

(13) $$e$$

is the one and only set and

(14) $$e \notin e$$

Since the only set e of the model given by (13) and (14) is an empty set and since a power-set of e must contain e as an element, we see that in the above model the Axiom of Power-Set is not valid. On the other hand, in the above model, the axioms of Extensionality, Replacement, Sum-Set, and Choice are trivially valid. Thus, Theorem 4 is proved.

Remark: For obvious reasons, the proof of the independence of the Axiom of Sum-Set from axioms (1), (2), (3), and (5), or, from axioms (1), (2), (3), (5), and (6) requires a model in which there exists a set with infinitely many elements. The same is true for the proof of the independence of the Axiom of Choice from axioms (1), (2), (3), and (4), or, from axioms (1), (2), (3), (4), and (6). These proofs are beyond the scope of this paper.

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