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ON THE CONSTANT SCALAR CURVATURE KÄHLER METRICS (II)—EXISTENCE RESULTS

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Contents

1. Introduction

This is the second paper discussing constant scalar curvature Kähler metrics. We prove Donaldson's conjecture (mentioned in the abstract) as well as the existence part of properness conjecture. For simplicity, we will first consider the case $Aut_0(M, J) = 0$ and then follow with the general case. Here $Aut_0(M, J)$ denotes the identity component of the automorphism group and $Aut_0(M, J) = 0$ means the group is discrete. In the general setting, we will need to study the twisted cscK equation with more subtle constraints. Our main method is to adopt the continuity path introduced in [21] and we need to prove that the set of parameter $t \in [0, 1]$ in the continuity path is both open (c.f. [21]) and closed under suitable geometric constraints. The a priori estimates obtained in [22] and their modifications in Section 3 (where the scalar curvature takes twisted form as in the twisted path introduced in [21]) are the crucial technical ingredients needed in this paper.

We will begin with a brief review of the history of this problem. In 1982 and 1985, E. Calabi published two seminal papers [12, 13] on extremal Kähler metrics where he proved some fundamental theorems on extremal Kähler metrics. His initial vision is that there should be a unique canonical metric in each Kähler class. Levine (c.f. [60]) constructed examples on which there is no extremal metric in any Kähler class. More examples and obstructions are found over the last few decades and huge efforts are devoted to formulate the right conditions (in particular the

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algebraic conditions) under which we can "realize" Calabi's original dream in a suitable format. The well known Yau-Tian-Donaldson conjecture is one of the important formulations which states that on projective manifolds, the cscK metrics exist in a polarized Kähler class if and only if this class is K-stable. It is widely expected among experts that the stability condition needs to be strengthened to a stronger notion such as uniform stability or stability through filtrations, in order to imply the existence of cscK metrics.

In the seminal paper [46], S. K. Donaldson proposed a beautiful program in Kähler geometry, aiming in particular to attack Calabi's renowned problem of existence of cscK metrics. In this celebrated program, Donaldson took the point of view that the space of Kähler metrics is formally a symmetric space of non-compact type and the scalar curvature function is the moment map from the space of almost complex structures compatible with a fixed symplectic form to the Lie algebra of a certain infinite dimensional symplectic structure group, where the said Lie algebra is exactly the space of all real valued smooth functions on the manifold. With this in mind, Calabi's problem of finding a cscK metric is reduced to finding a zero of this moment map in the infinite dimensional space setting. From this beautiful new point of view, S. K. Donaldson proposed a network of problems in Kähler geometry which have inspired many exciting developments over the last two decades, culminating in the recent resolution of Yau's stability conjecture on Kähler-Einstein metrics [24–26].

Let $\mathcal{H}$ denote the space of Kähler potentials in a given Kähler class $(M,[\omega])$. T. Mabuchi [62], S. Semmes [65] and S. K. Donaldson [46] set up an L^2 metric in the space of Kähler potentials:

$$\|\delta\varphi\|_\varphi^2=\int_M(\delta\varphi)^2\omega_\varphi^n,\qquad\forall\,\delta\varphi\in T_\varphi\mathcal{H}.$$

Donaldson [46] conjectured that $\mathcal{H}$ is a genuine metric space with the pathwise distance defined by this L^2 inner product. In [18], the first named author established the existence of $C^{1,1}$ geodesic segment between any two smooth Kähler potentials and proved this conjecture of S.K. Donaldson. He went on to prove (together with E. Calabi) that such a space is necessarily non-positively curved in the sense of Alexandrov [14]. More importantly, S. K. Donaldson proposed Conjecture/Question 1.1 to attack the existence problem:

Conjecture/Question 1.1 ([46]). *Assume $Aut_0(M,J)=0$. Then the following statements are equivalent:*

1. *There is no constant scalar curvature Kähler metric in $\mathcal{H}$;*
2. *There is a potential $\varphi_0\in\mathcal{H}_0$ and there exists a geodesic ray $\rho(t)(t\in[0,\infty))$ in $\mathcal{H}_0$, initiating from φ_0 such that the K-energy is non-increasing;*
3. *For any Kähler potential $\psi\in\mathcal{H}_0$, there exists a geodesic ray $\rho(t)(t\in[0,\infty))$ in $\mathcal{H}_0$, initiating from ψ such that the K-energy is non-increasing.*

In the above, $\mathcal{H}_0=\mathcal{H}\cap\{\phi:I(\phi)=0\}$, where the functional I is defined by (2.7). The reason we need to use $\mathcal{H}_0$ is to preclude the trivial geodesic $\rho(t)=\varphi_0+ct$ where c is a constant.

In the original writing of S. K. Donaldson, he didn't specify the regularity of these geodesic rays in this conjecture. In this paper, we avoid this issue by working in the space $\mathcal{E}^1$ (see Section 2 for definition) in which the potentials have only very weak regularity but the notion of geodesic still makes sense. By Theorem 4.7 of [8], we can extend the notion of K-energy to the space $\mathcal{E}^1$. The precise version of the result we prove is the following which amounts to a weak version of Donaldson's Geodesic stability conjecture:

Theorem 1.1. *The following statements are equivalent.*

(1) *There exists no constant scalar curvature Kähler metrics in* $(M,[\omega_0])$*;*
(2) *Either the Calabi-Futaki invariant of* $(M,[\omega_0])$ *is nonzero or there exists a Kähler potential* $\varphi_0 \in \mathcal{E}_0^1$ *with* $K(\varphi_0)<\infty$*, and a locally finite energy geodesic ray in* $\mathcal{E}_0^1$ *initiating from* φ *where the* K*-energy is non-increasing but it is not parallel to a holomorphic line;*
(3) *Either the Calabi-Futaki invariant of* $(M,[\omega_0])$ *is nonzero or for any Kähler potential* $\varphi_0 \in \mathcal{E}_0^1$ *with* $K(\varphi_0)<\infty$*, there exists a locally finite energy geodesic ray initiated from* φ_0 *where* K*-energy is non-increasing but it is not parallel to a holomorphic line.*

In the above, *holomorphic line* means a continuous curve $h:[0,\infty)\to\mathcal{E}_0^1$, such that for any $t>0$, the $(1,1)$ current $\omega_{h(t)} := \omega_0+\sqrt{-1}\partial\bar\partial h(t)=\sigma_t^*\omega_{h(0)}$ for a one-parameter family $\sigma_t\in G$ and "parallelism" is defined as in Definition 1.4. $\mathcal{E}^1$ is the metric completion of $\mathcal{H}$ under L^1 geodesic distance, and $\mathcal{E}_0^1=\mathcal{E}^1\cap\{\phi: I(\phi)=0\}$, where the functional I is defined as in (2.7). We learned about the idea of using locally finite energy geodesic ray from the recent beautiful work of Darvas-He [39] on Donaldson conjecture in Fano manifolds where they use Ding functional instead of the K-energy functional. From our point of view, both the restriction to canonical Kähler class and the adoption of Ding functional are more of analytical nature.

We will introduce the notion of geodesic stability (c.f. Definition 1.5) and Theorem 1.1 can be reformulated as an equivalence between existence of cscK and geodesic stability. Let us first introduce ¥ invariant associated with geodesic ray and the notion of "parallelism" between two locally finite energy geodesic rays. This invariant characterizes the growth of K-energy along a geodesic ray.

Definition 1.2 (c.f. Definition (3.10) in [20])**.** Let $\phi\in\mathcal{E}_0^1$ with $K(\phi)<\infty$. Let $\rho:[0,\infty)\to\mathcal{E}_0^1$ be a locally finite energy geodesic ray with unit speed such that $K(\rho(t))<\infty$ for any $t\geq 0$. We define:

$$¥[\rho]=\lim\inf_{k\to\infty}\frac{K(\rho(k))}{k}.$$

Remark 1.3. From the convexity of K-energy along locally finite energy geodesic ray (c.f. [8, Theorem 4.7]), we see that actually the above limit exists, namely

$$¥[\rho]=\lim_{k\to\infty}\frac{K(\rho(k))}{k}.$$

Moreover

$$¥[\rho]=\lim_{k\to\infty}\big(K(\rho(k+1))-K(\rho(k))\big).$$

Definition 1.4. Let $\rho_i:[0,\infty)\to\mathcal{E}_0^1$ be two continuous curves, $i=1,2$. We say that ρ_1 and ρ_2 are parallel, if $\sup_{t>0}d_1(\rho_1(t),\rho_2(t))<\infty$.

Obviously, one can modify this according to d_p topology for any $p\geq 1$. We can define a notion of geodesic stability/semi-stability in terms of ¥ invariant as follows:

Definition 1.5. Let $\phi_0 \in \mathcal{E}_0^1$ be such that $K(\phi_0) < \infty$. We say $(M, [\omega_0])$ is geodesic stable at ϕ_0 if for any locally finite energy geodesic ray $\rho : [0, \infty) \to \mathcal{E}_0^1$ with unit speed, exactly one of the following alternative holds:

(1) $\yen[\rho] > 0$,
(2) $\yen[\rho] = 0$, and ρ is parallel to another geodesic ray $\rho' : [0, \infty) \to \mathcal{E}_0^1$, generated from a holomorphic vector field $X \in aut(M, J)$.

We say $(M, [\omega_0])$ is geodesic semistable at ϕ_0 as long as $\yen[\rho] \geq 0$ for all geodesic ray ρ described above.

We say $(M, [\omega_0])$ is geodesic stable(resp. semistable) if it is geodesic stable (resp. semistable) at every $\phi \in \mathcal{E}_0^1$.

We remark that the notion of geodesic stability/semi-stability is independent of the choice of base potential ϕ_0, in virtue of Theorem 1.4 (see below).

Remark 1.6. It is possible to define the ¥ invariant for a locally finite energy geodesic ray in $\mathcal{E}_0^p$ with $p > 1$. Note that a geodesic segment in $\mathcal{E}_0^p$ is automatically a geodesic segment in $\mathcal{E}_0^q$ for any $q \in [1, p]$. Following the preceding definition, one can also define geodesic stability in $\mathcal{E}_0^p (p > 1)$. Note that for a locally given finite energy geodesic ray in $\mathcal{E}_0^p (p > 1)$, the actual value of ¥ invariant in $\mathcal{E}_0^p$ might differ by a positive multiple from the ¥ invariant considered in $\mathcal{E}_0^1$. However, it will not affect the sign of the ¥ invariant for a particular locally finite energy geodesic ray. On the other hand, the collection of locally finite energy geodesic ray in $\mathcal{E}_0^p (p > 1)$ might be strictly contained in the collection of geodesic rays in $\mathcal{E}_0^1$. Therefore, the notion of geodesic stability in the $\mathcal{E}_0^1$ is strongest while the notion of geodesic stability in $\mathcal{E}_0^\infty$ is the weakest. Without going into technicality, we may define geodesic stability in $\mathcal{E}_0^\infty$ as the ¥ invariant being strictly positive for any locally finite energy geodesic ray which lies in $\bigcap_{p \geq 1} \mathcal{E}_0^p$. For interested readers, we refer to the following works and references therein: J. Ross [63], G. Székelyhidi [68], Berman-Boucksom-Jonsson [5], R. Dervan [44].

Using this notion of geodesic stability, we can re-formulate Theorem 1.1 as:

Theorem 1.2. *There exists a cscK metric if and only if* $(M, [\omega_0])$ *is geodesic stable.*

After we prove this theorem, we obtain the following characterization of geodesic semi-stability.

Theorem 1.3. $(M, [\omega_0])$ *is geodesic semistable if and only if the continuity path* $t(R_\varphi - \underline{R}) = (1 - t)(tr_\varphi \omega_0 - n)$ *has a solution for any* $t < 1$.

Consequently, we deduce

Corollary 1.7. *If the* K*-energy is bounded from below in* $(M, [\omega_0])$*, then* $(M, [\omega_0])$ *is geodesic semistable.*

It is an interesting question to ask if the converse is also true. Namely if $(M, [\omega_0])$ is geodesic semistable, does it follow that K-energy is bounded from below? Note even for the corresponding statement in the algebraic case, we don't know how to conclude the existence of a lower bound of K-energy from K-stability or uniform stability except in the Fano manifolds where the authors proved it indirectly in route of CDS's theorem.

We have Theorem 1.4 which is useful to our characterization of borderline case.

Theorem 1.4. *Let $\rho_1(t) : [0,\infty) \to \mathcal{E}_0^1$ be a locally finite energy geodesic ray with unit speed. Then*

(i) *For any $\varphi \in \mathcal{E}_0^1$, there exists at most one unit speed locally finite energy geodesic ray $\rho_2(t) : [0,\infty) \to \mathcal{E}_0^1$ initiating from φ which is parallel to ρ_1. Moreover, $\yen[\rho_1] = \yen[\rho_2]$ for any such geodesic ray ρ_2.*

(ii) *If $\yen[\rho_1] < \infty$ and $K(\varphi) < \infty$, then there exists such a geodesic ray ρ_2 as described in point* (i).

Remark 1.8. It is an interesting question whether such a parallel geodesic ray exists in general, i.e. with no assumption on ¥ invariant.

The uniqueness part and that ¥ invariants for two rays are equal will be proved in Appendix. For the existence part, we first give a proof in the special case with $\rho_1(0), \varphi \in \mathcal{E}^2$. This allows us to use the Calabi-Chen theorem (c.f. [14]) that $(\mathcal{E}^2, d_2)$ is non-positively curved. Note that when $p \neq 2$, the infinite dimensional space $(\mathcal{E}^p, d_p)$ is no longer Riemannian formally. Recall that for φ_0, $\varphi_1 \in \mathcal{H}$, $d_p(\varphi_0, \varphi_1)$ is defined as the infimum of $\int_0^1 \left(\int_M |\partial_t \varphi|^p \omega^n_{\varphi(t,\cdot)} \right)^{\frac{1}{p}} dt$, where the infimum is taken over all smooth curves $\varphi(t,\cdot) : [0,1] \to \mathcal{H}$. The space $\mathcal{E}^p$ is just the metric completion of the space $\mathcal{H}$ under the distance d_p.

Nonetheless, we prove Theorem 1.5, which follows from the NPC (non-positively curved) property when $p = 2$.

Theorem 1.5. *Let $1 \le p < \infty$. Let ϕ_0, ϕ_0', ϕ_1, $\phi_1' \in \mathcal{E}^p$. Denote $\{\phi_{0,t}\}_{t\in[0,1]}$, $\{\phi_{1,t}\}_{t\in[0,1]}$ be the finite energy geodesics connecting ϕ_0 with ϕ_0' and ϕ_1 with ϕ_1' respectively. Then we have*

$$d_p(\phi_{0,t}, \phi_{1,t}) \le (1-t) d_p(\phi_0, \phi_1) + t d_p(\phi_0', \phi_1').$$

Remark 1.9. In [8, Proposition 5.1], the authors obtained Theorem 1.5 for the case $p = 1$, using a representation formula of d_1.

Given the central importance of the notion of K-energy in Donaldson's beautiful program, the first named author proposed the Conjecture/Question 1.10, shortly after [18]:

Conjecture/Question 1.10. *Assume $Aut_0(M,J) = 0$. The existence of constant scalar curvature Kähler metric is equivalent to the properness of K-energy in terms of geodesic distance.*

Here "properness" means that the K-energy tends to $+\infty$ whenever the geodesic distance tends to infinity (c.f. Definition 4.1). The original conjecture naturally chose the distance introduced in [46] which we now call L^2 distance. After a series of fundamental work of T. Darvas on this subject (c.f. [38, ?Darvas1403]), it becomes clear that the L^1 geodesic distance is a natural choice for the properness conjecture. Indeed, the correct formulation appears earlier in Darvas-Rubinstein [40].

Definition 1.11 (c.f. [7, 40])**.** We say K-energy is proper with respect to L^1 geodesic distance modulo $G := Aut_0(M,J)$, if

(1) For any sequence $\{\varphi_i\} \subset \mathcal{H}_0$, $\inf_{\sigma \in G} d_1(\omega_0, \sigma^* \omega_{\varphi_i}) \to \infty$ implies $K(\varphi_i) \to +\infty$,
(2) K-energy is bounded from below.

Henceforth we will denote the group $Aut_0(M,J)$ by G. With this in mind, we will prove that

Theorem 1.6 (Theorem 4.3)**.** *There exists a constant scalar curvature Kähler metric if and only if the K energy functional is proper with respect to the L^1 distance modulo G.*

For properness conjecture, we remark that there is a more well known formulation due to G. Tian where he conjectured that the existence of cscK metrics is equivalent to the properness of K-energy in terms of Aubin functional J (c.f. definition (2.7)). One may say that Tian's conjecture is more of analytical nature while Conjecture/Question 1.10 fits into Donaldson's geometry program in the space of Kähler potentials more naturally. According to T. Darvas (c.f. Theorem 5.5 of [38]), Aubin's J functional and the L^1 distance are equivalent. Therefore, these two properness conjectures are equivalent. Nonetheless, the formulation in Conjecture/Question 1.10 is essential to our proof.

The direction that existence of cscK implies properness has been established by Berman, Darvas and Lu in [7]. For the converse direction, Darvas and Rubinstein in [40] have reduced this problem to a problem of regularity of weak minimizers of K-energy over the space $\mathcal{E}^1$, which we will resolve in Section 5. (In the special case of toric varieties, Zhou-Zhu [77] proved the existence of toric invariant weak minimizers of the modified K-energy under properness assumption and they first proposed the properness definition modulo a group, similar to the one used in Darvas-Rubinstein [40] and also in our paper.) Hence Theorem 1.6 has been established by combining these results. Nonetheless, in this paper we will show how to obtain Theorem 1.6 by solving along the continuity path

$$t(R_\varphi - \underline{R}) = (1-t)(tr_\varphi\omega_0 - n), \ \ t \in [0,1].$$

For this purpose, we develop new estimates for scalar curvature type equations which may be of independent interest.

The existence part of Theorem 1.6 also holds for twisted cscK metric as well (c.f. Theorems 4.1 and 4.2), which is the solution to the equation

$$t(R_\varphi - \underline{R}) = (1-t)(tr_\varphi\chi - \underline{\chi}). \tag{1.1}$$

In the above, $0 < t \leq 1$, χ is a fixed Kähler form, and $\underline{R}$ is the average of scalar curvature, and $\underline{\chi} = \frac{\int_M n\chi\wedge\omega_0^{n-1}}{\int_M \omega_0^n}$. It is well-known that $\underline{R}$ and $\underline{\chi}$ depend only on the Kähler classes $[\omega_0]$ and $[\chi]$.

Now we recall an important notion introduced in [21]:

$$R([\omega_0],\chi) = \sup\{t_0 \in [0,1] : \text{ the above equation can be solved for any } 0 \leq t \leq t_0\} \tag{1.2}$$

In the same paper, the first named author conjectured that this is an invariant of the Kähler class $[\chi]$. In this paper, as a consequence of Theorems 4.1 and 4.2, we will show that if χ_1 and χ_2 are two Kähler forms in the same class, then one has

$$R([\omega_0], \chi_1) = R([\omega_0], \chi_2),$$

so that the quantity $R([\omega_0], [\chi])$ is well-defined and gives rise to an invariant between two Kähler classes $[\omega_0]$, $[\chi]$. Moreover, when the K-energy is bounded from below, the twisted path (1.1) can be solved for any $t < 1$, as long as $t = 0$ can be solved. Thus in this case we have

Theorem 1.7. *Let χ be a Kähler form. If the K-energy is bounded from below on $(M, [\omega_0])$, then $R([\omega_0], [\chi]) = 1$ if and only if one can solve $tr_\varphi\chi = \underline{\chi}$.*

As noted in [21], it is interesting to understand geometrically for what Kähler classes this invariant is 1 but do not admit constant scalar curvature metrics. More broadly, it is interesting to estimate the upper and lower bound of this invariant. It is not hard to see the relation between the invariant introduced in [69] and the invariant introduced above when restricted to the canonical Kähler class in Fano manifolds, where we take $[\chi]$ to be the first Chern class in (1.2). Hopefully, the method used there can be adapted to our setting to get estimate for this new invariant, in particular an upper bound.

T. Darvas and Y. Rubinstein conjectured in [40, Conjecture 2.9] that any minimizer of K-energy over the space $\mathcal{E}^1$ is actually a smooth Kähler potential. This is a bold and imaginative conjecture which might be viewed as a natural generalization of an earlier conjecture by the first named author that any $C^{1,1}$ minimizer of K-energy is smooth (c.f. [18, Conjecture 3]). Under an additional assumption that there exists a smooth cscK metric in the same Kähler class, Darvas-Rubinstein conjecture is verified in [7]. In this paper, we establish this conjecture as an application of properness theorem. Note that Euler-Lagrange equation is not available a priori in our setting, so that the usual approach to the regularity problem in the calculus of variations does not immediately apply. Instead, we need to use the continuity path to overcome this difficulty.

Theorem 1.8 (Theorem 5.1)**.** *Let $\varphi_* \in \mathcal{E}^1$ be such that $K(\varphi_*) = \inf_{\varphi\in\mathcal{E}^1} K(\varphi)$. Then φ_* is smooth and $\omega_{\varphi_*} := \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_*$ is a cscK metric.*

We actually establish a more general result which allows us to consider more general twisted K-energy and we can show the weak minimizers of twisted K-energy are smooth as long as the twisting form is smooth, closed and nonnegative. Previous result due to W. He and Y. Zeng [57] proved Chen's conjecture on the regularity of $C^{1,1}$ minimizers of K-energy with some additional assumption on the positivity of volume form.

E. Calabi believed that every Kähler class should have one canonical representative. E. Calabi's vision has inspired generations of Kähler geometers to work on this exciting problem and without it, this very paper will never exist. To celebrate his vision, we propose to call such a manifold a *Calabi manifold*.

Definition 1.12. A Kähler manifold is called **Calabi manifold** if every Kähler class on it admits an extremal Kähler metric.

Clearly, all compact Riemann surfaces, complex projective spaces $\mathbb{CP}^n$ and all compact Calabi-Yau manifolds [75] are *Calabi manifolds*. Our discussion above asserts

Corollary 1.13. *Any Kähler surface with $C_1 < 0$ and no curve of negative self-intersection is a Calabi surface.*

It is fascinating to understand how large this family of Calabi surfaces is. Following this corollary, one should be able to construct more examples of Calabi manifolds.

To prepare ourselves for the general case, we will need to study a general equation first. As before, we continue our study of the twisted cscK equation

$$t(R_\varphi - \underline{R}) = (1-t)(tr_\varphi\chi - \underline{\chi}), \qquad \text{where } t \in [0,1],$$

but allow in more general form with χ being some fixed smooth real $(1,1)$ form in [22]. In this paper, χ is allowed to vary in a fixed Kähler class with some constraints. More specifically, we consider

$$\chi = \chi_0 + \sqrt{-1}\partial\bar{\partial} f_* \geq 0, \qquad \sup_M f_* = 0, \qquad \int_M e^{-pf_*} < \infty \text{ for some } p > 1. \tag{1.3}$$

We are able to extend many of our previous estimates in [22] to these more general right hand side as (1.3) (some of those will require p to be sufficiently large depending only on dimension n). These new a priori estimates are crucial for us to extend our proof of Donaldson's conjecture on geodesic stability and the Properness conjecture for K-energy to the setting with general automorphism group. For simplicity, we only state and prove the results on constant scalar curvature Kähler metrics in this paper. Analogous results for extremal Kähler metrics can be proved in a similar way using our estimates.

Theorem 1.9 (Theorem 3.3)**.** *Let φ be a smooth solution to* (3.1), (3.2)*, with assumptions in* (1.3) *hold. Suppose additionally that $p \geq \kappa_n$ for some constant κ_n depending only on n. Then for any $p' < p$,*

$$||F + f_*||_{W^{1,2p'}} \leq C_{25.1}, \;\; ||n + \Delta\varphi||_{L^{p'}(\omega_0^n)} \leq C_{25.1}.$$

Here $C_{25.1}$ depends only on an upper bound of entropy $\int_M \log\big(\frac{\omega_\varphi^n}{\omega_0^n}\big)\omega_\varphi^n$, p, p', the bound for $\int_M e^{-pf_*} dvol_g$, $||R||_0$, $\max_M |\beta_0|_g$ and background metric ω_0.

In a series of three fundamental papers [48–50], S. Donaldson proved that on toric Kähler surfaces, the existence of cscK metric is indeed equivalent to the K-stability. This is partially generalized in [15] to extremal Kähler metrics (c.f. [17] and reference therein for interior regularity estimates on Kähler toric varieties). However, in general algebraic Kähler manifolds, one expects that the K-stability might fall short of the existence of cscK metrics; see the evidence provided by [2]. In the special case of Toric Kähler manifold, we prove the YTD conjecture:

Theorem 1.10. *On toric Kähler manifold, the existence of cscK metric is equivalent to uniform stability.*

The above "uniform version" of the YTD conjecture was made by Donaldson [48]. That the existence of cscK implies uniform stability was shown by Chen-Li-Sheng [15] in the toric setting. The general case (algebraic manifolds) follows from combining Corollary B in [11] and Theorem 1.5 in [7]. Note that [43] proved similar implications on algebraic manifolds for twisted cscK when the twisting form is strictly positive. For general Kähler manifolds, Dervan and Ross [45] introduced a notion of uniform stability in non-algebraic settings and proved that uniform stability follows from the properness of the K-energy.

Finally we explain the organization of the paper:

In Section 2, we recall the necessary preliminaries needed for our proof, including the continuity path we will use to solve the cscK equation and the theory of geodesic metric spaces established by Darvas and others.

In Section 3, we derive estimates for scalar curvature type equations with more general right hand side.

In Section 4, we prove the properness conjecture using continuity path. First we handle the special situation when $Aut_0(M,J) = 0$, for which we only need the estimates already established in [22]. Then we move on to the general case, for which the generalized estimates obtained in Section 3 become necessary.

In Section 5, we prove that the weak minimizer of K-energy over the metric space $(\mathcal{E}^1, d_1)$ (c.f. Section 2.2) is given by a smooth cscK potential.

In Sections 6 and 7, we show that the existence of cscK metric is equivalent to geodesic stability. Again we first prove it under the special case when $Aut_0(M, J) = 0$, where the notion of geodesically stability is simpler. Then in Section 7, we move on to the general case. The YTD conjecture for toric setting is treated in Section 7.1.

In the Appendix, we prove some results about the non-positively curved properties of the metric space $(\mathcal{E}^p, d_p)$ which will be useful to us. Such results may be of independent interest.

2. Preliminaries

In this section, we will review some basic concepts in Kähler geometry as well as some fundamental results involving finite energy currents, which will be needed for our proof of Theorems 1.1 and 1.3. In particular, it includes the characterization of the space $(\mathcal{E}^1, d_1)$, a compactness result on bounded subsets of $\mathcal{E}^1$ with finite entropy. We also include results on the convexity of K-energy along $C^{1,1}$ geodesics as well as its extension to the space $\mathcal{E}^1$. For more detailed account on these topics, we refer to a recent survey paper by Demailly [41].

2.1. **K-energy and twisted K-energy.** Let (M, ω_0) be a fixed Kähler class on M. Then we can define the space $\mathcal{H}$ of Kähler metrics cohomologous to ω_0 as:

$$\mathcal{H} = \{\varphi \in C^\infty(M) : \omega_\varphi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}. \tag{2.1}$$

We can introduce the K-energy in terms of its derivative:

$$\frac{dK}{dt}(\varphi) = -\int_M \frac{\partial\varphi}{\partial t}(R_\varphi - \underline{R})\frac{\omega_\varphi^n}{n!}, \ \varphi \in \mathcal{H}. \tag{2.2}$$

Here R_φ is the scalar curvature of ω_φ, and

$$\underline{R} = \frac{[C_1(M)] \cdot [\omega]^{[n-1]}}{[\omega]^{[n]}} = \frac{\int_M R_\varphi \omega_\varphi^n}{\int_M \omega^n}.$$

Following [18], we can write down an explicit formula for $K(\varphi)$:

$$K(\varphi) = \int_M \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)\frac{\omega_{\varphi^n}}{n!} + J_{-Ric}(\varphi), \tag{2.3}$$

where for a $(1,1)$ form χ, we define

(2.4)

$$\begin{aligned} J_\chi(\varphi) &= \int_0^1 \int_M \varphi\left(\chi \wedge \frac{\omega_{\lambda\varphi}^{n-1}}{(n-1)!} - \underline{\chi}\frac{\omega_{\lambda\varphi}^n}{n!}\right)d\lambda \\ &= \frac{1}{n!}\int_M \varphi \sum_{k=0}^{n-1} \chi \wedge \omega_0^k \wedge \omega_\varphi^{n-1-k} - \frac{1}{(n+1)!}\int_M \underline{\chi}\varphi \sum_{k=0}^{n} \omega_0^k \wedge \omega_\varphi^{n-k}. \end{aligned}$$

Here

$$\underline{\chi} = \frac{\int_M \chi \wedge \frac{\omega_0^{n-1}}{(n-1)!}}{\int_M \frac{\omega_0^n}{n!}}.$$

Following formula (2.19), we have

$$\frac{dJ_\chi}{dt} = \int_M \partial_t\varphi(tr_\varphi\chi - \underline{\chi})\frac{\omega_\varphi^n}{n!}.$$

It is well-known that K-energy is convex along smooth geodesics in the space of Kähler potentials.

Let $\beta \geq 0$ be a smooth closed $(1,1)$ form, we define a "*twisted K-energy with respect to β*" by

$$K_\beta(\varphi) = K(\varphi) + J_\beta(\varphi). \tag{2.5}$$

The critical points of $K_\beta(\varphi)$ satisfy the following equations:

$$R_\varphi - \underline{R} = tr_\varphi\beta - \underline{\beta}, \quad \text{where } \underline{\beta} = \frac{\int_M \beta\wedge\frac{\omega_0^{n-1}}{(n-1)!}}{\int_M \frac{\omega_0^n}{n!}}. \tag{2.6}$$

For later use, we also define the functionals $I(\varphi), J(\varphi)$, given by

$$I(\varphi) = \frac{1}{(n+1)!}\int_M \varphi\sum_{k=0}^n \omega_0^k\wedge\omega_\varphi^{n-k}, \qquad J(\varphi) = \int_M \varphi(\omega_0^n - \omega_\varphi^n). \tag{2.7}$$

We also need to consider the more general twisted K-energy, which is defined to be

$$K_{\chi,t} = tK + (1-t)J_\chi. \tag{2.8}$$

Following [18], we can write down Euler-Lagrange equation for twisted K-energy:

$$t(R_\varphi - \underline{R}) = (1-t)(tr_\varphi\chi - \underline{\chi}),\ t\in[0,1]. \tag{2.9}$$

Following [21], for $t>0$, we can rewrite this into two coupled equations:

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F\det g_{i\bar{j}}, \tag{2.10}$$

$$\Delta_\varphi F = -(\underline{R} - \frac{1-t}{t}\underline{\chi}) + tr_\varphi(Ric - \frac{1-t}{t}\chi). \tag{2.11}$$

In the following, we will assume $\chi > 0$, that is, χ is a Kähler form. The equation (2.9) with $t\in[0,1]$ is the continuity path proposed in [21] to solve the cscK equation. More generally, one can consider similar twisted paths in order to solve (2.6). Namely we consider

$$t(R_\varphi - \underline{R}) = t(tr_\varphi\beta - \underline{\beta}) + (1-t)(tr_\varphi\chi - \underline{\chi}). \tag{2.12}$$

The solution to (2.12) is a critical point of $tK_\beta + (1-t)J_\chi$. We will see later that it is actually a minimizer. For $t>0$, this again can be equivalently put as

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F\det g_{i\bar{j}}, \tag{2.13}$$

$$\Delta_\varphi F = -(\underline{R} - \underline{\beta} - \frac{1-t}{t}\underline{\chi}) + tr_\varphi\big(Ric - \beta - \frac{1-t}{t}\chi\big). \tag{2.14}$$

An important question is whether the set of t for which (2.12) can be solved is open. The cited result is only for (2.9), but the same argument would work for (2.12).

Lemma 2.1 ([21,56,76])**.** *Let $\beta\geq 0$ be nonnegative closed smooth $(1,1)$ form and χ be a Kähler form. Suppose that for some $0\leq t_0 < 1$, (2.12) has a solution $\varphi\in C^{4,\alpha}(M)$ with $t=t_0$, then for some $\delta>0$, (2.12) has a solution in $C^{4,\alpha}$ for any $t\in(t_0-\delta, t_0+\delta)\bigcap[0,1)$.*

We observe that we can always make sure (2.9) or (2.12) can be solved for $t = 0$ by choosing $\chi = \omega_0$ or any Kähler form in $[\omega_0]$.

Remark 2.2. Clearly if χ is smooth, it is easy to see by bootstrap that a $C^{4,\alpha}$ solution to (2.9) is actually smooth.

Hence Lemma 2.1 shows the set of t for which (2.9) has a smooth solution is relatively open in $[0, 1)$.

From the Theorem 5.3 of [22], we can conclude that

Proposition 2.3. *Let φ be a smooth solution to* (2.9) *or* (2.12) *with $t > \delta_0 > 0$, normalized so that $\sup_M \varphi = 0$. Then the higher derivatives of φ can be estimated in terms of an upper bound of entropy, defined as $\int_M \log(\frac{\omega_\varphi^n}{\omega_0^n})\omega_\varphi^n$, as well as δ_0.*

Proof. This follows directly from Theorem 5.3 of [22], by taking $f = \underline{R} - \underline{\beta} - \frac{1-t}{t}\underline{\chi}$, and $\eta = Ric(\omega_0) - \beta - \frac{1-t}{t}\chi$. Note that the assumption t being bounded below by δ_0 guarantees f and η is bounded. ☐

2.2. **The complete geodesic metric space $(\mathcal{E}^p, d_p)$.** In Section 3.3 of [54] introduced the following space for any $p \geq 1$:

$$\mathcal{E}^p = \{\varphi \in PSH(M, \omega_0) : \int_M \omega_\varphi^n = \int_M \omega_0^n,\ \int_M |\varphi|^p \omega_\varphi^n < \infty\}. \tag{2.15}$$

In the above, $\varphi \in PSH(M, \omega_0)$ means that $\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi \geq 0$ in the sense of currents. A fundamental conjecture of V. Guedj [53] stated that the completion of the space $\mathcal{H}$ of smooth potentials equipped with the L^2 metric is precisely the space $\mathcal{E}^2(M, \omega_0)$ of potentials of finite energy. This has been shown by Darvas [37]. In [38], he has shown similar characterization holds for general L^p metric. Note that the extension to the L^1 metric is essential and fundamental to our work.

Following Mabuchi, T. Darvas [38] introduced the notion of d_1 on $\mathcal{H}$.

$$||\xi||_\varphi = \int_M |\xi| \frac{\omega_\varphi^n}{n!}, \forall\ \ \xi \in T_\varphi\mathcal{H} = C^\infty(M). \tag{2.16}$$

Using this, we can define the path-length distance d_1 on the space $\mathcal{H}$, i.e. $d_1(u_0, u_1)$ equals the infimum of length of all smooth curves in $\mathcal{H}$, with $\alpha(0) = u_0$, $\alpha(1) = u_1$. T. Darvas [38, Theorem 2] proved, following Chen [18] in the case of d_2, that $(\mathcal{H}, d_1)$ is a metric space.

We have the following characterization for $(\mathcal{E}^1, d_1)$:

Theorem 2.1 ([38, Theorem 5.5])**.** *Define*

$$I_1(u, v) = \int_M |u - v| \frac{\omega_u^n}{n!} + \int_M |u - v| \frac{\omega_v^n}{n!},\ u, v \in \mathcal{H}.$$

Then there exists a constant $C > 0$ depending only on n, such that

$$\frac{1}{C} I_1(u, v) \leq d_1(u, v) \leq C I_1(u, v),\ \textit{for any } u, v \in \mathcal{H}. \tag{2.17}$$

For later use, here we describe how to obtain "finite energy geodesics" from the $C^{1,1}$ geodesics between smooth potentials.

Theorem 2.2 ([38, Theorem 2])**.** *The metric completion of $(\mathcal{H}, d_1)$ equals $(\mathcal{E}^1, d_1)$ where*

$$d_1(u_0, u_1) =: \lim_{k\to\infty} d_1(u_0^k, u_1^k),$$

for any smooth decreasing sequence $\{u_i^k\}_{k\geq 1} \subset \mathcal{H}$ *converging pointwise to* $u_i \in \mathcal{E}^1$*. Moreover, for each* $t \in (0,1)$*, define*

$$u_t := \lim_{k\to\infty} u_t^k,\ t \in (0,1),$$

where u_t^k *is the* $C^{1,1}$ *geodesic connecting* u_0^k *and* u_1^k *(c.f.* [18]*). We have* $u_t \in \mathcal{E}^1$*, the curve* $[0,1] \ni t \mapsto u_t$ *is independent of the choice of approximating sequences and is a* d_1*-geodesic in the sense that for some* $c > 0$*,* $d_1(u_t, u_s) = c|t-s|$*, for any* s*,* $t \in [0,1]$*.*

The above limit is pointwise decreasing limit. Since the sequence $\{u_i^k\}_{k\geq 1}$ is decreasing sequence for $i = 0$, 1, we know $\{u_t^k\}_{k\geq 1}$ is also decreasing for $t \in (0,1)$, by comparison principle.

We say $u_t : [0,1] \ni t \to \mathcal{E}^1$ connecting u_0, u_1 is a finite energy geodesic if it is given by the procedure described in Theorem 2.2. The following result shows the limit of finite energy geodesics is again a finite energy geodesic.

Proposition 2.4 ([8, Proposition 4.3])**.** *Suppose* $[0,1] \ni t \to u_t^i \in \mathcal{E}^1$ *is a sequence of finite energy geodesic segments such that* $d_1(u_0^i, u_0)$*,* $d_1(u_1^i, u_1) \to 0$*. Then* $d_1(u_t^i, u_t) \to 0$*, for any* $t \in [0,1]$*, where* $[0,1] \ni t \mapsto u_t \in \mathcal{E}^1$ *is the finite energy geodesic connecting* u_0*,* u_1*.*

Finally we record the following compactness result which will be useful later. This result was first established in [6]. The following version is taken from [8], which is the form most convenient to us.

Lemma 2.5 ([6, Theorem 2.17], [8, Corollary 4.8])**.** *Let* $\{u_i\}_i \subset \mathcal{E}^1$ *be a sequence for which the following condition holds:*

$$\sup_i d_1(0, u_i) < \infty,\ \sup_i K(u_i) < \infty.$$

Then $\{u_i\}_i$ *contains a* d_1*-convergent subsequence.*

2.3. **Convexity of K-energy.** In this subsection, we record some known results about the convexity of K-energy and J_χ functional along $C^{1,1}$ geodesics and also finite energy geodesics. In [18], the first named author proved the following result about the convexity of the functional J_χ.

Theorem 2.3 ([18, Proposition 2])**.** *Let* $\chi \geq 0$ *be a closed* $(1,1)$ *form. Let* u_0*,* $u_1 \in \mathcal{H}$*. Let* $\{u_t\}_{t\in[0,1]}$ *be the* $C^{1,1}$ *geodesic connecting* u_0*,* u_1*. Then* $[0,1] \ni t \mapsto J_\chi(u_t)$ *is convex.*

The convexity of K-energy along smooth geodesics was first observed by T. Mabuchi, c.f. [62]. However, such convexity over non-smooth geodesics is more challenging, and is conjectured by the first named author:

Conjecture/Question 2.6 (Chen)**.** *Let* u_0*,* $u_1 \in \mathcal{H}$*. Let* $\{u_t\}_{t\in[0,1]}$ *be the* $C^{1,1}$ *geodesic connecting* u_0*,* u_1*. Then* $[0,1] \ni t \mapsto K(u_t)$ *is convex.*

This conjecture was verified by the fundamental work of Berman and Berndtsson [4] (c.f. Chen-Li-Păun [29] also).

Theorem 2.4. *Conjecture/Question* 2.6 *is true.*

It turns out that the K-energy and also the functional J_χ can be extended to the space $(\mathcal{E}^1, d_1)$ and is convex along finite energy geodesics. More precisely,

Theorem 2.5 ([8, Theorem 4.7]). *The K-energy defined in* (2.3) *can be extended to a functional $K : \mathcal{E}^1 \to \mathbb{R} \cup \{+\infty\}$. Besides, the extended functional $K|_{\mathcal{E}^1}$ is the greatest d_1-lower semi-continuous extension of $K|_{\mathcal{H}}$. Moreover, $K|_{\mathcal{E}^1}$ is convex along finite energy geodesics of $\mathcal{E}_1$.*

Theorem 2.6 ([8, Propositions 4.4 and 4.5]). *The functional J_χ as defined by* (2.4) *can be extended to be a d_1-continuous functional on $\mathcal{E}^1$. Besides, J_χ is convex along finite energy geodesics.*

2.4. **Chen's decomposition formula for K-energy.** In view of Theorem 1.3, it is important to study, under what conditions, the K-energy functional is proper in a given Kähler class. In [18], the first named author proposed a decomposition formula for K-energy:

$$K(\varphi) = \int_M \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)\frac{\omega_\varphi^n}{n!} + J_{-Ric}(\varphi), \tag{2.18}$$

where the functional J_{-Ric} is defined through its derivatives:

$$\frac{d\,J_{-Ric}}{d\,t} = \int_M \frac{\partial\varphi}{\partial t}(-Ric \wedge \frac{\omega_\varphi^{n-1}}{(n-1)!} + \underline{R}\frac{\omega_\varphi^n}{n!}). \tag{2.19}$$

One key observation in [18] (based on this decomposition formula) is that K-energy has a lower bound if the corresponding J_{-Ric} functional has a lower bound. Note that when the first Chern class is negative, one can choose a background metric such that $-Ric > 0$. Then, J_{-Ric} is convex along $C^{1,1}$ geodesics in $\mathcal{H}$ and is bounded from below if it has a critical point. In [73], Song-Weinkove further pointed out that J_{-Ric} functional being bounded from below is sufficient to imply the properness of K-energy. The research in this direction has been very active and intense (c.f. Chen [18], Fang-Lai-Song-Weinkove [52], Song-Weinkove [74], Li-Shi-Yao [61], R. Dervan [42], and references therein). Combining these results with Properness Theorem 4.1, we have Corollary 2.7.

Corollary 2.7. *There exists a cscK metric in $(M, [\omega])$ if any one of the following conditions holds:*

(1) *There exists a constant $\epsilon \geq 0$ such that $\epsilon < \frac{n+1}{n}\alpha_M([\omega])$ and $\pi C_1(M) < \epsilon[\omega]$ such that*

$$\left(-n\frac{C_1(M)\cdot[\omega]^{n-1}}{[\omega]^n} + \epsilon\right)\cdot[\omega] + (n-1)C_1(M) > 0.$$

Here $\alpha_M(\omega)$ denotes the α-invariant of the Kähler class $(M, [\omega])$ (c.f. [70]*).*

(2) *If*

$$\alpha_M([\omega]) > \frac{C_1(M)\cdot[\omega]^{n-1}}{[\omega]^n}\cdot\frac{n}{n+1}$$

and

$$C_1(M) \geq \frac{C_1(M)\cdot[\omega]^{n-1}}{[\omega]^n}\cdot\frac{n}{n+1}\cdot[\omega].$$

Here part (1) of Corollary 2.7 follows from Li-Shi-Yao [61] (c.f. Fang-Lai-Song-Weinkove [52], Song-Weinkove [74]), part (2) of Corollary 2.7 follows from R. Dervan [42].

Following Donaldson's observation in [47], if a Kähler surface M admits no curve of negative self intersections and has $C_1(M) < 0$, then the condition

$$\frac{2[\omega]\cdot[-C_1(M)]}{[\omega]^2}\cdot[\omega]-[-C_1(M)]>0$$

is satisfied automatically for any Kähler class $[\omega]$ (c.f. Song-Weinkove [73]). Consequently, on any Kähler surface M with $C_1(M) < 0$ with no curve of negative self-intersection, the K-energy is proper for any Kähler class (c.f. Song-Weinkove [74]). It follows that on these surfaces, every Kähler class admits a cscK metric.

Corollary 2.8. *Any Kähler surface with $C_1 < 0$ and no curve of negative self-intersection is a Calabi surface.*

It is fascinating to understand how large this family of Calabi surfaces is. It is possible to construct such examples explicitly.

3. Scalar curvature type equations with singular right hand side

Let (M, J, ω_0) be a compact Kähler manifold. We consider the following scalar curvature type equation:

$$\det(g_{i\bar{j}}+\varphi_{i\bar{j}})=e^F\det g_{i\bar{j}}, \tag{3.1}$$

$$\Delta_\varphi F=tr_\varphi(Ric-\beta)-R_0. \tag{3.2}$$

In the above, $\beta=\beta_0+\sqrt{-1}\partial\bar{\partial}f_*\geq 0$. Also we assume that β_0 is a real bounded $(1,1)$ form and f is normalized to be $\sup_M f_*=0$, $e^{-f_*}\in L^{p_0}(M)$ for some $p_0>1$. R_0 is a bounded function. Here β_0 is bounded just means that if we write β_0 in coordinates as $\beta_0=\sqrt{-1}(\beta_0)_{i\bar{j}}dz_i\wedge d\bar{z}_j$, the coefficients $(\beta_0)_{i\bar{j}}$ are bounded.

As before, φ should be such that $g_{i\bar{j}}+\varphi_{i\bar{j}}>0$ on M, so that $\omega_\varphi:=\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi$ defines a new Kähler metric in the same class as ω_0. We note that (4.28), (4.29) can be combined to give the following scalar curvature type equation:

$$R_\varphi=tr_\varphi\beta+R_0. \tag{3.3}$$

Here R_φ denotes the scalar curvature of the metric ω_φ. In the following, we will always assume that the solution φ is smooth and our goal is to derive a priori estimates.

3.1. Boundedness of $F+f_*$.

The estimate in this subsection only requires a bound for $\int_M e^{-p_0f_*}dvol_g$ for some $p_0>1$. In particular, we don't need any positivity assumption on the form β.

Lemma 3.1. *Let ψ be the solution to the following equation:*

$$\det(g_{i\bar{j}}+\psi_{i\bar{j}})=\frac{e^F\sqrt{F^2+1}}{\int_M e^F\sqrt{F^2+1}dvol_g}\det g_{i\bar{j}}, \tag{3.4}$$

$$\sup_M\psi=0. \tag{3.5}$$

Suppose also $\sup_M\varphi=0$. Then for any $0<\varepsilon_0<1$, there exists a constant C_0, such that

$$F+f_*+\varepsilon_0\psi-2(\max_M|Ric-\beta_0|_g+1)\varphi\leq C_0.$$

Here C_0 depends only on ε_0, the upper bound of the entropy $\int_M Fe^F dvol_g$, the bound for $\max_M|\beta_0|_g$, $||R||_0$ and the background metric (M,ω_0).

Proof. Similar to the cscK case, the proof is by Alexandrov maximum principle. Observe from (3.2) that

$$\Delta_\varphi(F+f_*) = tr_\varphi(Ric-\beta_0) - R. \tag{3.6}$$

Denote $C = 2(\max_M |Ric-\beta_0|_g + 1)$ and we compute

$$\Delta_\varphi(F+f_*+\varepsilon_0\psi - C\varphi) = tr_\varphi(Ric-\beta_0) - R + \varepsilon_0\Delta_\varphi\psi - Cn + Ctr_\varphi g. \tag{3.7}$$

Using arithmetic-geometric mean inequality, we have

$$\Delta_\varphi\psi = g_\varphi^{i\bar{j}}(g_{i\bar{j}}+\psi_{i\bar{j}}) - tr_\varphi g \geq A^{-\frac{1}{n}}(F^2+1)^{\frac{1}{2n}} - tr_\varphi g.$$

Here $A = \int_M e^F\sqrt{F^2+1}dvol_g$. Also due to our choice of the constant C, we obtain from (3.7) that

$$\Delta_\varphi(F+f_*+\varepsilon_0\psi - C\varphi) \geq \frac{C}{2}tr_\varphi g + \varepsilon_0 A^{-\frac{1}{n}}(F^2+1)^{\frac{1}{2n}} - C_1. \tag{3.8}$$

Here C_1 has the said dependence as stated in the lemma. By Proposition 2.1 in [70], there exists $\alpha > 0$, and a constant C_2, such that for any ω_0-psh function ϕ, we have

$$\int_M e^{-\alpha(\phi - \sup_M \phi)}dvol_g \leq C_2.$$

Now denote $u = F + f_* + \varepsilon_0\psi - C\varphi$, $\delta = \frac{\alpha}{2nC}$, and let $0<\theta<1$ to be determined. First for any $p\in M$, we can construct a cut-off function η_p, so that $\eta_p(p) = 1$, $\eta_p \equiv 1-\theta$ outside the ball $B_{d_0}(p)$, and $|\nabla\eta_p| \leq \frac{2\theta}{d_0}$, $|\nabla^2\eta| \leq \frac{2\theta}{d_0^2}$. Here d_0 is a sufficiently small constant depending only on the background metric (M,ω_0). Assume that the function u achieves maximum at the point p_0, then we compute

(3.9)

$$\begin{aligned}
&\Delta_\varphi(e^{\delta u}\eta_{p_0}) = e^{\delta u}\delta^2|\nabla_\varphi u|^2_\varphi\eta_{p_0} + e^{\delta u}\eta_{p_0}\delta\Delta_\varphi u + e^{\delta u}\Delta_\varphi\eta_{p_0} + 2e^{\delta u}\delta\nabla_\varphi u\cdot_\varphi\nabla_\varphi\eta_{p_0}\\
&\geq e^{\delta u}\delta^2|\nabla_\varphi u|^2_\varphi\eta_{p_0} + e^{\delta u}\delta\eta_{p_0}\big(\frac{C}{2}tr_\varphi g + \varepsilon_0A^{-\frac{1}{n}}(F^2+1)^{\frac{1}{2n}} - C_1\big)\\
&- e^{\delta u}|\nabla^2\eta_{p_0}|tr_\varphi g - e^{\delta u}\delta^2|\nabla_\varphi u|^2_\varphi\eta_{p_0} - e^{\delta u}\frac{|\nabla_\varphi\eta_{p_0}|^2_\varphi}{\eta_{p_0}}\\
&\geq e^{\delta u}\eta_{p_0}\big(\frac{\delta C}{2} - \frac{2\theta}{d_0^2(1-\theta)} - \frac{4\theta^2}{d_0^2(1-\theta)}\big)tr_\varphi g + e^{\delta u}\delta\eta_{p_0}\big(\varepsilon_0A^{-\frac{1}{n}}(F^2+1)^{\frac{1}{2n}} - C_1\big).
\end{aligned}$$

Choose θ small enough so that (note that $\delta C = \frac{\alpha}{2n}$)

$$\frac{\delta C}{2} - \frac{2\theta}{d_0^2(1-\theta)} - \frac{4\theta^2}{d_0^2(1-\theta)} > 0.$$

With this choice of θ, (3.9) gives

$$\Delta_\varphi(e^{\delta u}\eta_{p_0}) \geq e^{\delta u}\delta\eta_{p_0}(\varepsilon_0A^{-\frac{1}{n}}(F^2+1)^{\frac{1}{2n}} - C_1) \geq -e^{\delta u}\delta\eta_{p_0}C_1\chi_{\{F\leq C_3\}}. \tag{3.10}$$

Here $\chi_{\{F\leq C_3\}}$ is the indicator function of the set $\{F\leq C_3\}$, and C_3 is a constant determined by the inequality

$$\varepsilon_0A^{-\frac{1}{n}}\big(F^2+1\big)^{\frac{1}{2n}} - C_1 \leq 0 \text{ implies } F\leq C_3.$$

Hence C_3 depends only on ε_0, C_1 and A. We wish to apply Alexandrov maximum principle to (3.10) inside $B_{d_0}(p_0)$, and with a similar derivation as (5.16) in the first paper [22], we obtain:

$$
\begin{aligned}
& e^{\delta u}\eta_{p_0}(p_0) \le \sup_{\partial B_{d_0}(p_0)} e^{\delta u}\eta_{p_0} \\
& \quad + C_n d_0 \bigg(\int_{B_{d_0}(p_0)} \delta^{2n} e^{2F} e^{2n\delta u} \big((\varepsilon_0 A^{-\frac{1}{n}}(F^2+1)^{\frac{1}{2n}} - C_1)^- \big)^{2n} dvol_g \bigg)^{\frac{1}{2n}}.
\end{aligned} \tag{3.11}
$$

To estimate the integral appearing above, observe that $f \le 0$, $\psi \le 0$, then we have

$$
\begin{aligned}
& \int_{B_{d_0}(p_0)} e^{2F} e^{2n\delta u} \big((\varepsilon_0 A^{-\frac{1}{n}}(F^2+1)^{\frac{1}{2n}} - C_1)^- \big)^{2n} dvol_g \\
& \qquad \le \int_M e^{2F+2n\delta F} e^{-2nC\delta\varphi} \chi_{\{F \le C_3\}} C_1^{2n} dvol_g \le e^{(2+2n\delta)C_3} C_2 (C_1)^{2n}.
\end{aligned} \tag{3.12}
$$

Since $\eta_{p_0} \le 1-\theta$ on $\partial B_{d_0}(p_0)$, the result follows from (3.11). Indeed, since $e^{\delta u}$ achieves maximum at p_0, we have

$$
e^{\delta u}(p_0) \le (1-\theta) e^{\delta u}(p_0) + C_n d_0 \delta e^{(2+2n\delta)C_3} C_2 (C_1)^{2n}.
$$

The desired estimate then follows. □

Corollary 3.2. *There exists a constant C_4, such that*

$$
F + f_* \le C_4.
$$

In particular, if φ is normalized so that $\sup_M \varphi = 0$, then

$$
||\varphi||_0 \le C_{4.5}.
$$

Here C_4 and $C_{4.5}$ depend only on the upper bound for the entropy $\int_M e^F F dvol_g$, the bound for $\max_M |\beta_0|_g$, $||R||_0$, p_0 (uniform for $p_0 > 1$ as long as $p_0 - 1$ bounded away from 0), the bound $\int_M e^{-p_0 f_*} dvol_g$ and the background metric (M, ω_0).

Proof. First we obtain from Lemma 3.1 that

$$
\frac{\alpha}{\varepsilon_0}\big(F + f_* - 2(\max_M |Ric - \beta_0|_g + 1)\varphi \big) \le -\alpha\psi + \frac{\alpha C_0}{\varepsilon_0}. \tag{3.13}
$$

Hence for any $p > 1$, if we choose ε_0 so that $p = \frac{\alpha}{\varepsilon_0}$, then we obtain

$$
\int_M e^{p(F+f_*)} dvol_g \le C_5. \tag{3.14}
$$

The constant C_5 has the dependence as described in Lemma 3.1 with additional dependence on p, but will be uniform in p as long as p remains bounded. Choose $\varepsilon_1 = \frac{p_0 - 1}{2}$, then we can estimate

$$
\begin{aligned}
& \int_M e^{(1+\varepsilon_1)F} dvol_g = \int_M e^{(1+\varepsilon_1)(F+f_*)} \cdot e^{-(1+\varepsilon_1)f_*} dvol_g \\
& \quad \le \bigg(\int_M e^{-p_0 f_*} dvol_g \bigg)^{\frac{1+\varepsilon_1}{p_0}} \cdot \bigg(\int_M e^{\frac{p_0}{p_0-(1+\varepsilon_1)}(F+f_*)} dvol_g \bigg)^{1-\frac{1+\varepsilon_1}{p_0}} \le C_{5.5}.
\end{aligned} \tag{3.15}
$$

Here $C_{5.5}$ is uniform in p_0 as long as $p_0 - 1$ is bounded away from 0. Then we can conclude from (3.15) and Kolodziej's main result (c.f. [59]) that

$$
||\varphi||_0,\ ||\psi||_0 \le C_6. \tag{3.16}
$$

The result now follows from Lemma 3.1, with choice of ε_0 so that $\frac{\alpha}{\varepsilon_0} = \frac{p_0}{p_0-(1+\varepsilon_1)} = \frac{2p_0}{p_0-1}$. □

Next we would like to estimate the lower bound for $F + f_*$.

Lemma 3.3. *There exists a constant C_7 such that*

$$F + f_* \geq -C_7.$$

Here C_7 depends only on $||\varphi||_0$, $\max_M |\beta_0|_g$, $||R||_0$, the background metric g, the bound for $\int_M e^{-p_0 f_*} dvol_g$, and p_0 (uniform in p_0 as long as $p_0 - 1$ bounded away from 0). In particular

$$F \geq -C_7.$$

Proof. We choose $C = 2(\max_M |Ric - \beta_0|_g + 1)$. Then we have

$$\Delta_\varphi(F+f_*+C\varphi) = tr_\varphi(Ric-\beta_0)-R_0+Cn-Ctr_\varphi g \leq -tr_\varphi g+||R_0||_0+Cn. \tag{3.17}$$

In [22, Proposition 2.1], we estimated $tr_\varphi g$ from below by $ne^{-\frac{F}{n}}$ and the result follows from maximum principle. Here one cannot do a pointwise maximum principle as before and needs to argue differently.

Choose $\varepsilon_2 = \frac{p_0}{2n(p_0-1)}$, and the cut-off function η_p as in the proof of Lemma 3.1 (with a parameter θ to be chosen later), and denote $u_1 = F + f_* + C\varphi$. Assume the function u_1 achieves minimum at $p_1 \in M$. We may compute

$$\begin{aligned}
\Delta_\varphi(e^{-\varepsilon_2 u_1}\eta_{p_1}) &= e^{-\varepsilon_2 u}(\varepsilon_2^2|\nabla_\varphi u_1|^2_\varphi \eta_{p_1} \\
&\quad - \varepsilon_2\Delta_\varphi u_1 \eta_{p_1} + \Delta_\varphi \eta_{p_1} - 2\varepsilon_2 \nabla_\varphi u_1 \cdot_\varphi \nabla_\varphi \eta_{p_1}) \\
&\geq e^{-\varepsilon_2 u}\big(\varepsilon_2^2|\nabla_\varphi u_1|^2_\varphi \eta_{p_1} + \varepsilon_2 tr_\varphi g \eta_{p_1} - \varepsilon_2(||R||_0 + Cn) - |\nabla^2\eta_{p_1}|_g tr_\varphi g \\
&\quad - \varepsilon_2^2|\nabla_\varphi u_1|^2_\varphi \eta_{p_1} - \frac{|\nabla_\varphi \eta_{p_1}|^2_\varphi}{\eta_{p_1}}\big) \geq e^{-\varepsilon_2 u}\big(tr_\varphi g \eta_{p_1}(\varepsilon_2 - \frac{2\theta}{d_0^2(1-\theta)} - \frac{4\theta^2}{d_0^2(1-\theta)}) \\
&\quad - \varepsilon_2(||R_0||_0 + Cn)\big).
\end{aligned} \tag{3.18}$$

Since $\eta_{p_1} \geq 1-\theta$, we may choose θ sufficiently small so that

$$(1-\theta)\varepsilon_2 - \frac{2\theta}{d_0^2(1-\theta)} - \frac{4\theta^2}{d_0^2(1-\theta)} > 0.$$

With this choice, we then have

$$\Delta_\varphi\big(e^{-\varepsilon_2 u_1}\eta_{p_1}\big) \geq -\varepsilon_2 e^{-\varepsilon_2 u}(||R_0||_0 + Cn). \tag{3.19}$$

Hence if we apply the Alexandrov maximum principle in $B_{d_0}(p_0)$, we have

$$\begin{aligned}
e^{-\varepsilon_2 u_1}\eta_{p_1}(p_1) &\leq \sup_{\partial B_{d_0}(p_1)} e^{-\varepsilon_2 u_1}\eta_{p_1} \\
&\quad + C_n d_0\bigg(\int_M e^{2F} e^{-2n\varepsilon_2 u_1}\varepsilon_2^{2n}(||R_0||_0 + Cn)^{2n} dvol_g\bigg)^{\frac{1}{2n}}.
\end{aligned} \tag{3.20}$$

To estimate the integral appearing above, we may calculate:

$$\begin{aligned}&\int_M e^{2F}e^{-2n\varepsilon_2 u_1}\varepsilon_2^{2n}(||R_0||_0+Cn)^{2n}dvol_g \le C_8\int_M e^{(2-2n\varepsilon_2)F-2n\varepsilon_2 f_*}dvol_g\\ &= C_8\int_M e^{\frac{p_0-2}{p_0-1}F}\cdot e^{-\frac{p_0}{p_0-1}f_*}dvol_g\\ &\le C_8\Big(\int_M e^F dvol_g\Big)^{\frac{p_0-2}{p_0-1}}\cdot\Big(\int_M e^{-p_0 f_*}dvol_g\Big)^{\frac{1}{p_0-1}}.\end{aligned}\tag{3.21}$$

Since we have $\eta_{p_1}=1-\theta$ on $\partial B_{d_0}(p_1)$, the desired estimate then follows in the same way as in the last part of the proof for Lemma 3.1. □

3.2. $W^{2,p}$ **estimate.** In this subsection, we will need to assume $\beta\ge 0$ (or more generally a lower bound for β), besides assuming a bound for $\int_M e^{-p_0 f}dvol_g$ for some $p_0>1$.

Theorem 3.1. *Assume $\beta\ge 0$ in* (3.1), (3.2). *For any $p\ge 1$, there exists a constant C_p, depending only on $||F+f_*||_0$, $||R_0||_0$, $\max_M|\beta_0|_g$, the background metric (M,ω_0), a bound for $\int_M e^{-p_0 f_*}dvol_g$, $||\varphi||_0$ and p, such that*

$$\int_M e^{(p-1)f_*}(n+\Delta\varphi)^p dvol_g\le C_p.$$

Proof. Let $\kappa>0$, $C>0$, $0<\delta<1$ be constants to be chosen later, we will compute:

$$\begin{aligned}&\Delta_\varphi\big(e^{-\kappa(F+\delta f_*+C\varphi)}(n+\Delta\varphi)\big)=\Delta_\varphi\big(e^{-\kappa(F+\delta f_*+C\varphi)}\big)(n+\Delta\varphi)+e^{-\kappa(F+\delta f_*+C\varphi)}\Delta_\varphi(\Delta\varphi)\\ &\quad-2\kappa e^{-\kappa(F+\delta f_*+C\varphi)}\nabla_\varphi(F+\delta f_*+C\varphi)\cdot_\varphi\nabla_\varphi(\Delta\varphi).\end{aligned}\tag{3.22}$$

We can compute

$$\begin{aligned}&\Delta_\varphi\big(e^{-\kappa(F+\delta f_*+C\varphi)}\big)=e^{-\kappa(F+\delta f_*+C\varphi)}\big(\kappa^2|\nabla_\varphi(F+\delta f_*+C\varphi)|_\varphi^2\\ &-\kappa\Delta_\varphi(F+f_*+C\varphi)+\kappa(1-\delta)\Delta_\varphi f_*\big)\\ &=e^{-\kappa(F+\delta f_*+C\varphi)}\kappa^2|\nabla_\varphi(F+\delta f_*+C\varphi)|_\varphi^2\\ &\qquad+e^{-\kappa(F+\delta f_*+C\varphi)}\kappa\big(Ctr_\varphi g-tr_\varphi(Ric-\beta_0)\big)\\ &\qquad+e^{-\kappa(F+\delta f_*+C\varphi)}(\kappa R_0-\kappa Cn)+\kappa(1-\delta)e^{-\kappa(F+\delta f_*+C\varphi)}\Delta_\varphi f_*.\end{aligned}\tag{3.23}$$

We choose $C\ge 2(\max_M|Ric-\beta_0|_g+1)$, then we obtain from above:

$$\begin{aligned}&\Delta_\varphi\big(e^{-\kappa(F+\delta f_*+C\varphi)}\big)\ge e^{-\kappa(F+\delta f_*+C\varphi)}\kappa^2|\nabla_\varphi(F+\delta f_*+C\varphi)|_\varphi^2\\ &\quad+e^{-\kappa(F+\delta f_*+C\varphi)}\frac{\kappa C}{2}tr_\varphi g+\kappa e^{-\kappa(F+\delta f_*+C\varphi)}(1-\delta)\Delta_\varphi f_*\\ &\quad-\kappa e^{-\kappa(F+\delta f_*+C\varphi)}C_9.\end{aligned}\tag{3.24}$$

The constant C_9 appearing above will depend on our choice of C. On the other hand, let $p\in M$, we choose normal coordinate in a neighborhood of p so that

$$g_{i\bar j}(p)=\delta_{ij},\ \nabla g_{i\bar j}(p)=0,\ \varphi_{i\bar j}=\varphi_{i\bar i}\delta_{ij}.$$

We have computed in our first paper [22] (following Yau [75]) that

$$
\begin{aligned}
\Delta_\varphi(\Delta\varphi) &= \frac{R_{i\bar{i}\alpha\bar{\alpha}}(1+\varphi_{i\bar{i}})}{1+\varphi_{\alpha\bar{\alpha}}} + \frac{|\varphi_{\alpha\bar{\beta}i}|^2}{(1+\varphi_{\alpha\bar{\alpha}})(1+\varphi_{\beta\bar{\beta}})} + \Delta F - R \\
&\geq -C_{10} tr_\varphi g(n+\Delta\varphi) + \frac{|\varphi_{\alpha\bar{\beta}i}|^2}{(1+\varphi_{\alpha\bar{\alpha}})(1+\varphi_{\beta\bar{\beta}})} + \Delta F - R.
\end{aligned}
\tag{3.25}
$$

Here C_{10} depends only on the curvature bound of g. Also we notice the complete square similar to our calculation in cscK case:

$$
\begin{aligned}
&\kappa^2|\nabla_\varphi(F+\delta f_*+C\varphi)|_\varphi^2(n+\Delta\varphi) + \frac{|\varphi_{\alpha\bar{\beta}i}|^2}{(1+\varphi_{\alpha\bar{\alpha}})(1+\varphi_{\beta\bar{\beta}})} \\
&\qquad - 2\kappa\nabla_\varphi(F+\delta f_*+C\varphi)\cdot_\varphi \nabla_\varphi(\Delta\varphi) \\
&\geq \kappa^2|\nabla_\varphi(F+\delta f_*+C\varphi)|_\varphi^2(n+\Delta\varphi) + \frac{|\nabla_\varphi(\Delta\varphi)|_\varphi^2}{n+\Delta\varphi} \\
&\qquad - 2\kappa\nabla_\varphi(F+\delta f_*+C\varphi)\cdot_\varphi \nabla_\varphi\Delta\varphi \geq 0.
\end{aligned}
$$

Combining (3.22), (3.24) and (3.25), we conclude

$$
\begin{aligned}
&\Delta_\varphi\big(e^{-\kappa(F+\delta f_*+C\varphi)}(n+\Delta\varphi)\big) \geq e^{-\kappa(F+\delta f_*+C\varphi)}\big(\frac{\kappa C}{2} - C_{10}\big)tr_\varphi g(n+\Delta\varphi) \\
&+ \kappa e^{-\kappa(F+\delta f_*+C\varphi)}(1-\delta)\Delta_\varphi f_*(n+\Delta\varphi) + e^{-\kappa(F+\delta f_*+C\varphi)}\Delta F \\
&- e^{-\kappa(F+\delta f_*+C\varphi)}(C_9\kappa + R).
\end{aligned}
\tag{3.26}
$$

In the following, we will always choose $\kappa \geq 1$, hence if we choose $C \geq 4C_{10}$, we obtain for some constant C_{11}, it holds:

$$
\begin{aligned}
\Delta_\varphi\big(e^{-\kappa(F+\delta f_*+C\varphi)}(n+\Delta\varphi)\big)e^{\kappa(F+\delta f_*+C\varphi)} &\geq \frac{\kappa C}{4}tr_\varphi g(n+\Delta\varphi) \\
&+ \kappa(1-\delta)\Delta_\varphi f_*(n+\Delta\varphi) + \Delta F - \kappa C_{11}.
\end{aligned}
\tag{3.27}
$$

The constant C_{11} above will depend on our choice of C. Let $p \geq 1$, denote $v = e^{-\kappa(F+\delta f_*+C\varphi)}(n+\Delta\varphi)$, we have

$$
\begin{aligned}
&\int_M (p-1)v^{p-2}|\nabla_\varphi v|_\varphi^2 dvol_\varphi = \int_M v^{p-1}(-\Delta_\varphi v)dvol_\varphi \\
&\leq -\int_M v^{p-1}\big(\frac{\kappa C}{4}vtr_\varphi g + e^{-\kappa(F+\delta f_*+C\varphi)}\kappa(1-\delta)\Delta_\varphi f_*(n+\Delta\varphi) \\
&+ e^{-\kappa(F+\delta f_*+C\varphi)}\Delta F - \kappa C_{11}e^{-\kappa(F+\delta f_*+C\varphi)}\big)dvol_\varphi.
\end{aligned}
\tag{3.28}
$$

We will handle the term involving ΔF via integrating by parts, but somewhat differently from the calculation for cscK(here we assume $\kappa > 1$):

(3.29)

$$
\begin{aligned}
-\int_M v^{p-1}e^{-\kappa(F+\delta f_*+C\varphi)}\Delta F dvol_\varphi &= -\int_M v^{p-1}e^{(1-\kappa)F-\kappa\delta f_*-\kappa C\varphi}\Delta F dvol_g \\
= -\int_M v^{p-1}e^{(1-\kappa)F-\kappa\delta f_*-\kappa C\varphi}&\frac{1}{1-\kappa}\Delta\big((1-\kappa)F-\kappa\delta f_*-\kappa C\varphi\big)dvol_g \\
-\int_M v^{p-1}e^{(1-\kappa)F-\kappa\delta f_*-\kappa C\varphi}&\frac{\kappa\delta\Delta f_* + \kappa C\Delta\varphi}{1-\kappa}dvol_g.
\end{aligned}
$$

For the first term in (3.29), we have

$$
\begin{aligned}
&-\int_M v^{p-1}e^{(1-\kappa)F-\kappa\delta f_*-\kappa C\varphi}\frac{1}{1-\kappa}\Delta\big((1-\kappa)F-\kappa\delta f_*-\kappa C\varphi\big)dvol_g\\
&=-\int_M \frac{v^{p-1}e^{(1-\kappa)F-\kappa\delta f_*-\kappa C\varphi}}{\kappa-1}|\nabla\big((1-\kappa)F-\kappa\delta f_*-\kappa C\varphi\big)|^2 dvol_g\\
&-\int_M \frac{p-1}{\kappa-1}v^{p-2}e^{(1-\kappa)F-\kappa\delta f_*-\kappa C\varphi}\nabla v\cdot\nabla\big((1-\kappa)F-\kappa\delta f_*-\kappa C\varphi\big)dvol_g\\
&\le\int_M \frac{(p-1)^2}{2(\kappa-1)}v^{p-3}e^{(1-\kappa)F-\kappa\delta f_*-\kappa C\varphi}|\nabla v|^2 dvol_g\\
&\le\int_M \frac{(p-1)^2}{2(\kappa-1)}v^{p-3}e^{-\kappa(F+\delta f_*+C\varphi)}|\nabla_\varphi v|^2_\varphi(n+\Delta\varphi)dvol_\varphi\\
&=\int_M \frac{(p-1)^2}{2(\kappa-1)}v^{p-2}|\nabla_\varphi v|^2_\varphi dvol_\varphi.
\end{aligned}
\tag{3.30}
$$

From 3rd line to 4th line above, we observed that

$$
\begin{aligned}
&-\frac{p-1}{\kappa-1}v^{p-2}e^{(1-\kappa)F-\kappa\delta f_*-\kappa C\varphi}\nabla v\cdot\nabla\big((1-\kappa)F-\kappa\delta f_*-\kappa C\varphi\big)\\
&\le\frac{v^{p-1}e^{(1-\kappa)F-\kappa\delta f_*-\kappa C\varphi}}{2(\kappa-1)}|\nabla\big((1-\kappa)F-\kappa\delta f_*-\kappa C\varphi\big)|^2\\
&+\frac{(p-1)^2}{2(\kappa-1)}v^{p-3}e^{(1-\kappa)F-\kappa\delta f_*-\kappa C\varphi}|\nabla v|^2.
\end{aligned}
$$

Combining (3.29), (3.30), we see

$$
\begin{aligned}
-\int_M v^{p-1}e^{-\kappa(F+\delta f_*+C\varphi)}\Delta F dvol_\varphi &\le\int_M \frac{(p-1)^2}{2(\kappa-1)}v^{p-2}|\nabla_\varphi v|^2_\varphi dvol_\varphi\\
&-\int_M v^{p-1}e^{-\kappa(F+\delta f_*+C\varphi)}\frac{\kappa\delta\Delta f_*+\kappa C\Delta\varphi}{1-\kappa}dvol_\varphi.
\end{aligned}
\tag{3.31}
$$

Plugging (3.31) back to (3.28), we obtain

$$
\begin{aligned}
&\int_M \big(p-1-\frac{(p-1)^2}{2(\kappa-1)}\big)v^{p-2}|\nabla_\varphi v|^2_\varphi dvol_\varphi\le-\int_M\frac{\kappa C}{4}tr_\varphi g v^p dvol_\varphi\\
&+\int_M v^{p-1}e^{-\kappa(F+\delta f_*+C\varphi)}\big(-\kappa(1-\delta)\Delta_\varphi f_*(n+\Delta\varphi)-\frac{\kappa\delta\Delta f_*}{1-\kappa}\big)dvol_\varphi\\
&+\int_M v^{p-1}e^{-\kappa(F+\delta f_*+C\varphi)}\big(\kappa C_{11}-\frac{\kappa C\Delta\varphi}{1-\kappa}\big)dvol_\varphi.
\end{aligned}
\tag{3.32}
$$

Now we choose $\delta=\frac{\kappa-1}{\kappa}$, then we have

$$
\begin{aligned}
&-\kappa(1-\delta)\Delta_\varphi f_*(n+\Delta\varphi)-\frac{\kappa\delta\Delta f_*}{1-\kappa}=-\Delta_\varphi f_*(n+\Delta\varphi)+\Delta f_*\\
&=-\sum_{i\neq j}\frac{(f_*)_{i\bar i}(1+\varphi_{j\bar j})}{1+\varphi_{i\bar i}}\le\sum_{i\neq j}\frac{(\beta_0)_{i\bar i}(1+\varphi_{j\bar j})}{1+\varphi_{i\bar i}}\le\max_M|\beta_0|_g tr_\varphi g(n+\Delta\varphi).
\end{aligned}
\tag{3.33}
$$

We also have for $\kappa\ge 2$,

$$
\kappa C_{11}-\frac{\kappa C\Delta\varphi}{1-\kappa}\le\kappa(C_{12}+C)(n+\Delta\varphi).
\tag{3.34}
$$

Here we used the fact that $n+\Delta\varphi \ge e^{\frac{F}{n}}$, which is bounded from below in terms of $||f_*+F||_0$. Indeed, $F \ge -f_* - ||f_*+F||_0 \ge -||f_*+F||_0$. Hence if we plug (3.33), (3.34) back to (3.32), we conclude that for $p \ge 1$, $\kappa \ge 2$, C chosen sufficiently large depending only on the curvature bound of the background metric and $\max_M |\beta_0|_g$, we have

(3.35)

$$\int_M (p-1-\frac{(p-1)^2}{\kappa-1})v^{p-2}|\nabla_\varphi v|^2_\varphi dvol_\varphi + \int_M (\frac{\kappa C}{4} - \max_M |\beta_0|_g) tr_\varphi g v^p dvol_\varphi \le \int_M \kappa(C_{12}+C)v^p dvol_\varphi.$$

Next we choose κ so that $\kappa \ge 2$ and $\kappa \ge p$, with this choice, we have

$$p-1-\frac{(p-1)^2}{\kappa-1} \ge 0.$$

Choose C sufficiently so as to satisfy $C \ge 8(\max_M |\beta_0|_g + 1)$, with this choice, we can guarantee

$$\frac{\kappa C}{4} - \max_M |\beta_0|_g \ge \frac{\kappa C}{8} \ge \kappa.$$

Hence we obtain from (3.35) that for some constant C_{13}

$$\int_M e^{-\frac{F}{n-1}}(n+\Delta\varphi)^{\frac{1}{n-1}} v^p dvol_\varphi \le \int_M tr_\varphi g v^p dvol_\varphi \le \int_M C_{13} v^p dvol_\varphi. \tag{3.36}$$

Recall our definition for v, this means:

(3.37)

$$\int_M e^{(\frac{n-2}{n-1}-p\kappa)F - p(\kappa-1)f_* - p\kappa C\varphi}(n+\Delta\varphi)^{p+\frac{1}{n-1}} dvol_g \le C_{13}\int_M e^{(1-p\kappa)F - p(\kappa-1)f_* - p\kappa C\varphi}(n+\Delta\varphi)^p dvol_g.$$

From the boundedness of $F+f$ and φ proved in Corollary 3.2 and Lemma 3.3, we obtain for $p \ge 1$:

$$\int_M e^{(p-\frac{n-2}{n-1})f_*}(n+\Delta\varphi)^{p+\frac{1}{n-1}} dvol_g \le C_{14}\int_M e^{(p-1)f_*}(n+\Delta\varphi)^p dvol_g. \tag{3.38}$$

Take $p = 1 + k\frac{1}{n-1}$ in (3.38) with $k \ge 0$, the result follows from induction on k. □

As a consequence of above calculation, we obtain:

Corollary 3.4. *For any $1 < q < p_0$, there exists a constant $\tilde{C}_q$, depending only on the bound $\int_M e^{-p_0 f_*} dvol_g$, $||R||_0$, $\max_M |\beta|_g$, the bound $||F+f_*||_0$, $||\varphi||_0$, the background metric (M,ω_0), and q, such that*

$$\int_M (n+\Delta\varphi)^q dvol_g \le \tilde{C}_q.$$

Besides, $\tilde{C}_q$ is uniform in q as long as q is bounded away from p_0 and remains bounded.

Proof. Choose $s = \frac{(q-1)p_0}{p_0-1}$, then we can calculate

$$\begin{aligned}\int_M (n+\Delta\varphi)^q dvol_g &= \int_M e^{-sf_*}\cdot e^{sf_*}(n+\Delta\varphi)^q dvol_g\\ &\le \Big(\int_M e^{-p_0 f_*}dvol_g\Big)^{\frac{s}{p_0}}\cdot\Big(\int_M e^{\frac{sp_0}{p_0-s}f_*}(n+\Delta\varphi)^{\frac{p_0 q}{p_0-s}}dvol_g\Big)^{1-\frac{s}{p_0}}.\end{aligned}$$

Notice our choice of s makes $\frac{sp_0}{p_0-s} = \frac{p_0 q}{p_0-s} - 1$, so the result follows from Theorem 3.1. □

3.3. **Estimate on $\nabla(F+f_*)$.** In this section, we continue to assume $\beta \ge 0$. Moreover, we also need p_0 to be sufficiently large depending only on n. Our goal is to obtain the following estimate.

Theorem 3.2. *There exists κ_n, depending only on n, such that as long as $p_0 > \kappa_n$, we have*

$$|\nabla_\varphi(F+f_*)|_\varphi \le C_{14}.$$

Here C_{14} is a constant with the same dependence as in Theorem 3.1.

Proof. Denote $w = F+f_*$, we need to calculate:

$$\begin{aligned}\Delta_\varphi\big(e^{\frac12 w}|\nabla_\varphi w|_\varphi^2\big) &= \Delta_\varphi(e^{\frac12 w})|\nabla_\varphi w|_\varphi^2\\ &\quad + e^{\frac12 w}\Delta_\varphi(|\nabla_\varphi w|_\varphi^2) + e^{\frac12 w}\nabla_\varphi w\cdot_\varphi \nabla_\varphi(|\nabla_\varphi w|_\varphi^2)\\ &= \frac14 e^{\frac12 w}|\nabla_\varphi w|_\varphi^4 + \frac12 e^{\frac12 w}\Delta_\varphi w|\nabla_\varphi w|_\varphi^2\\ &\quad + e^{\frac12 w}\Delta_\varphi(|\nabla_\varphi w|_\varphi^2) + e^{\frac12 w}\nabla_\varphi w\cdot_\varphi\nabla_\varphi(|\nabla_\varphi w|_\varphi^2).\end{aligned} \tag{3.39}$$

Now we have

$$\Delta_\varphi w = tr_\varphi(Ric-\beta_0) - R_0. \tag{3.40}$$

Also

$$\begin{aligned}\Delta_\varphi(|\nabla_\varphi w|_\varphi^2) &= g_\varphi^{i\bar j}g_\varphi^{\kappa\bar\beta}w_{,\kappa i}w_{,\bar\beta\bar j} + g_\varphi^{i\bar j}g_\varphi^{\kappa\bar\beta}w_{,\alpha\bar j}w_{,\bar\beta i} + 2\nabla_\varphi w\cdot_\varphi\nabla_\varphi\Delta_\varphi w\\ &\quad + g_\varphi^{i\bar j}g_\varphi^{\alpha\bar\beta}Ric_{\varphi,i\bar\beta}w_\alpha w_{\bar j}.\end{aligned} \tag{3.41}$$

Besides,

$$\nabla_\varphi w\cdot_\varphi\nabla_\varphi(|\nabla_\varphi w|^2) = Re\big(g_\varphi^{i\bar j}g_\varphi^{\alpha\bar\beta}w_i(w_{,\alpha\bar j}w_{\bar\beta} + w_\alpha w_{,\bar\beta\bar j})\big). \tag{3.42}$$

In the above, $w_{,i\alpha}$ denotes the covariant derivative under the metric g_φ. Again observe the complete square:

$$\begin{aligned}&\frac14|\nabla_\varphi w|_\varphi^4 + g_\varphi^{i\bar j}g_\varphi^{\alpha\bar\beta}w_{,\alpha i}w_{,\bar\beta\bar j} + Re\big(g_\varphi^{i\bar j}g_\varphi^{\alpha\bar\beta}w_i w_\alpha w_{,\bar\beta\bar j}\big)\\ &\quad = g_\varphi^{i\bar j}g_\varphi^{\alpha\bar\beta}(w_{,i\alpha}+\frac12 w_i w_\alpha)(w_{,\bar j\bar\beta}+\frac12 w_{\bar j}w_{\bar\beta}).\end{aligned}$$

Hence we obtain from (3.39):

$$\begin{aligned}\Delta_\varphi(e^{\frac12 w}|\nabla_\varphi w|_\varphi^2) &\ge \frac12 e^{\frac12 w}|\nabla_\varphi w|_\varphi^2\big(tr_\varphi(Ric-\beta_0)-R\big) + e^{\frac12 w}g_\varphi^{i\bar j}g_\varphi^{\alpha\bar\beta}w_{,\alpha\bar j}w_{,\bar\beta i}\\ &\quad + e^{\frac12 w}2\nabla_\varphi w\cdot_\varphi\nabla_\varphi\Delta_\varphi w + g_\varphi^{i\bar j}g_\varphi^{\alpha\bar\beta}Ric_{\varphi,i\bar\beta}w_\alpha w_{\bar j} + Re\big(g_\varphi^{i\bar j}g_\varphi^{\alpha\bar\beta}w_i w_{\bar\beta}w_{,\alpha\bar j}\big).\end{aligned} \tag{3.43}$$

Note that

$$Ric_{\varphi,i\bar\beta} = Ric_{i\bar\beta} - F_{i\bar\beta}.$$

Hence

$$
(3.44)\qquad \begin{aligned}
&g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} Ric_{\varphi,i\bar{\beta}} w_\alpha w_{\bar{j}} + Re\big(g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} w_i w_{\bar{\beta}} w_{,\alpha\bar{j}}\big)\\
&= g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} Ric_{i\bar{\beta}} w_\alpha w_{\bar{j}} + Re\big(g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} w_i w_{\bar{\beta}} (w_{,\alpha\bar{j}} - F_{\alpha\bar{j}})\big)\\
&= g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} Ric_{i\bar{\beta}} w_\alpha w_{\bar{j}} + Re\big(g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} w_i w_{\bar{\beta}} f_{\alpha\bar{j}}\big)\\
&\geq g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} Ric_{i\bar{\beta}} w_\alpha w_{\bar{j}} - Re\big(g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} w_i w_{\bar{\beta}} (\beta_0)_{\alpha\bar{j}}\big).
\end{aligned}
$$

In the last line, we use the fact that $\sqrt{-1}\partial\bar{\partial} f_* = \beta - \beta_0 \geq -\beta_0$, hence $(f_*)_{i\bar{j}} \geq -(\beta_0)_{i\bar{j}}$. Hence we obtain from (3.43):

$$
(3.45)\qquad \begin{aligned}
&\Delta_\varphi(e^{\frac{1}{2}w}|\nabla_\varphi w|_\varphi^2) \geq \frac{1}{2} e^{\frac{1}{2}w} |\nabla_\varphi w|_\varphi^2 \big(tr_\varphi(Ric - \beta_0) - R\big) + e^{\frac{1}{2}w} 2\nabla_\varphi w \cdot_\varphi \nabla_\varphi \Delta_\varphi w\\
&\quad + e^{\frac{1}{2}w} g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} Ric_{i\bar{\beta}} w_\alpha w_{\bar{j}} - e^{\frac{1}{2}w} Re\big(g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} w_i w_{\bar{\beta}} (\beta_0)_{\alpha\bar{j}}\big).
\end{aligned}
$$

Next we estimate:

$$
(3.46)\qquad tr_\varphi\big(Ric - \beta_0\big) - R \geq -C_{15}(tr_\varphi \omega_0 + 1) \geq -C_{15}(e^{-F}(n + \Delta\varphi)^{n-1} + 1).
$$

Also

$$
(3.47)\qquad \begin{aligned}
&g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} Ric_{i\bar{\beta}} w_\alpha w_{\bar{j}} \geq -C_{14.5}(tr_\varphi \omega_0)^2 |\nabla w|^2 \geq -C_{15}(tr_\varphi \omega_0)^2 (n + \Delta\varphi) |\nabla_\varphi w|_\varphi^2\\
&\quad \geq -C_{15} e^{-2F} (n + \Delta\varphi)^{2n-1} |\nabla_\varphi w|_\varphi^2.
\end{aligned}
$$

We can also estimate

$$
(3.48)\qquad \begin{aligned}
-Re\big(g_\varphi^{i\bar{j}} g_\varphi^{\alpha\bar{\beta}} w_i w_{\bar{\beta}} (\beta_0)_{\alpha\bar{j}}\big) &\geq -C_{14.5}(tr_\varphi \omega_0)^2 |\nabla w|^2\\
&\geq -C_{15} e^{-2F} (n + \Delta\varphi)^{2n-1} |\nabla_\varphi w|_\varphi^2.
\end{aligned}
$$

Hence we may conclude from (3.45) that

$$
(3.49)\qquad \begin{aligned}
\Delta_\varphi(e^{\frac{1}{2}w}|\nabla_\varphi w|_\varphi^2) &\geq 2e^{\frac{1}{2}w} \nabla_\varphi w \cdot_\varphi \nabla_\varphi \Delta_\varphi w - e^{\frac{1}{2}w} |\nabla_\varphi w|_\varphi^2\\
&\quad \times C_{15}\big(2e^{-2F}(n + \Delta\varphi)^{2n-1} + e^{-F}(n + \Delta\varphi)^{n-1} + 1\big).
\end{aligned}
$$

Denote $u = e^{\frac{1}{2}w}|\nabla_\varphi w|_\varphi^2 + 1$, $\tilde{G} = C_{15}\big(2e^{-2F}(n + \Delta\varphi)^{2n-1} + e^{-F}(n + \Delta\varphi)^{n-1} + 1\big)$. Then we have

$$
(3.50)\qquad \Delta_\varphi u \geq 2e^{\frac{1}{2}w} \nabla_\varphi w \cdot_\varphi \nabla_\varphi \Delta_\varphi w - u\tilde{G}.
$$

Now let $p \geq 1$, then we have

$$
(3.51)\qquad \begin{aligned}
\int_M (p-1) u^{p-2} |\nabla_\varphi u|_\varphi^2 dvol_\varphi &= \int_M u^{p-1}(-\Delta_\varphi u) dvol_\varphi\\
&\leq \int_M u^p \tilde{G} dvol_\varphi - \int_M 2u^{p-1} e^{\frac{1}{2}w} \nabla_\varphi w \cdot_\varphi \nabla_\varphi \Delta_\varphi w dvol_\varphi.
\end{aligned}
$$

We need to integrate by parts to handle the last term above. We have

(3.52)
$$\begin{aligned}
&-\int_M 2u^{p-1}e^{\frac{1}{2}w}\nabla_\varphi w\cdot_\varphi\nabla_\varphi\Delta_\varphi w dvol_\varphi=\int_M 2u^{p-1}e^{\frac{1}{2}w}(\Delta_\varphi w)^2dvol_\varphi\\
&+\int_M u^{p-1}e^{\frac{1}{2}w}|\nabla_\varphi w|^2_\varphi\Delta_\varphi w dvol_\varphi+\int_M 2(p-1)u^{p-2}e^{\frac{1}{2}w}\nabla_\varphi u\cdot_\varphi\nabla_\varphi w\Delta_\varphi w dvol_\varphi\\
&\le\int_M 2u^{p-1}e^{\frac{1}{2}w}(\Delta_\varphi w)^2dvol_\varphi+\int_M u^p\Delta_\varphi w dvol_\varphi-\int_M u^{p-1}\Delta_\varphi w dvol_\varphi\\
&+\int_M\frac{p-1}{2}u^{p-2}|\nabla_\varphi u|^2_\varphi dvol_\varphi+\int_M 2(p-1)u^{p-2}e^w|\nabla_\varphi w|^2_\varphi(\Delta_\varphi w)^2dvol_\varphi\\
&\le\int_M 2pu^{p-1}e^{\frac{1}{2}w}(\Delta_\varphi w)^2dvol_\varphi+\int_M u^p\big((\Delta_\varphi w)^2+1\big)dvol_\varphi\\
&+\int_M\frac{p-1}{2}u^{p-2}|\nabla_\varphi u|^2_\varphi dvol_\varphi.
\end{aligned}$$

Some explanations of above calculations are in order.

In the first inequality, we observed that

$$u^{p-1}e^{\frac{1}{2}w}|\nabla_\varphi w|^2_\varphi\Delta_\varphi w=u^p\Delta_\varphi w-u^{p-1}\Delta_\varphi w,$$

from our definition of u. Also we observed that

$$\begin{aligned}
&2(p-1)u^{p-2}e^{\frac{1}{2}w}\nabla_\varphi u\cdot_\varphi\nabla_\varphi w\Delta_\varphi w\\
&\quad\le\frac{p-1}{2}u^{p-2}|\nabla_\varphi u|^2_\varphi+2(p-1)u^{p-2}e^w|\nabla_\varphi w|^2_\varphi(\Delta_\varphi w)^2.
\end{aligned}$$

In the second inequality, we noticed that

$$u^p\Delta_\varphi w-u^{p-1}\Delta_\varphi w\le\frac{1}{2}(u^p+u^{p-1})\big(1+(\Delta_\varphi w)^2\big)\le u^p\big(1+(\Delta_\varphi w)^2\big).$$

Hence we conclude from (3.51):

$$\begin{aligned}
&\int_M\frac{p-1}{2}u^{p-2}|\nabla_\varphi u|^2_\varphi dvol_\varphi\le\int_M u^p\big(\tilde G+(\Delta_\varphi w)^2+1\big)dvol_\varphi\\
&+\int_M 2pu^{p-1}e^{\frac{1}{2}w}(\Delta_\varphi w)^2dvol_\varphi\\
&\le\int_M u^p\big(\tilde G+(\Delta_\varphi w)^2+1\big)dvol_\varphi+\int_M 2pu^pe^{\frac{1}{2}w}(\Delta_\varphi w)^2dvol_\varphi.
\end{aligned}\tag{3.53}$$

From 1st line to 2nd line above, we noticed $u\ge 1$. Now denote $G=\tilde G+(\Delta_\varphi w)^2+1+2e^{\frac{1}{2}w}(\Delta_\varphi w)^2$, we have

$$\int_M\frac{p-1}{2}u^{p-2}|\nabla_\varphi u|^2_\varphi dvol_\varphi\le\int_M pu^pGe^F dvol_g.\tag{3.54}$$

For the left hand side, we have

$$\int_M\frac{p-1}{2}u^{p-2}|\nabla_\varphi u|^2_\varphi dvol_\varphi\ge\frac{1}{C_{16}}\int_M\frac{2(p-1)}{p^2}|\nabla_\varphi(u^{\frac{p}{2}})|^2_\varphi dvol_g.\tag{3.55}$$

Let $\varepsilon > 0$ to be determined, we can also estimate

$$\begin{aligned}\int_M |\nabla(u^{\frac{p}{2}})|^{2-\varepsilon} dvol_g &\le \int_M |\nabla_\varphi(u^{\frac{p}{2}})|_\varphi^{2-\varepsilon}(n+\Delta\varphi)^{1-\varepsilon/2} dvol_g \\ &\le \Big(\int_M |\nabla_\varphi(u^{\frac{p}{2}})|^2_\varphi dvol_g\Big)^{\frac{2-\varepsilon}{2}} \times \Big(\int_M (n+\Delta\varphi)^{\frac{2}{\varepsilon}-1} dvol_g\Big)^{\frac{\varepsilon}{2}}.\end{aligned} \tag{3.56}$$

Denote

$$K_\varepsilon = \Big(\int_M (n+\Delta\varphi)^{\frac{2}{\varepsilon}-1} dvol_g\Big)^{\frac{\varepsilon}{2-\varepsilon}}. \tag{3.57}$$

Then we have

$$\begin{aligned}||\nabla(u^{\frac{p}{2}})||^2_{L^{2-\varepsilon}(\omega_0^n)} &\le K_\varepsilon \| |\nabla_\varphi(u^{\frac{p}{2}})|_\varphi\|^2_{L^2(\omega_0^n)} \le \frac{K_\varepsilon C_{16} p^2}{4}\int_M u^{p-2}|\nabla_\varphi u|^2_\varphi dvol_\varphi \\ &\le \frac{K_\varepsilon C_{16} p^3}{2(p-1)}\int_M u^p G e^F dvol_g.\end{aligned} \tag{3.58}$$

In the above, the first inequality follows from (3.56). The second inequality follows from (3.55), and the last inequality uses (3.54).

Apply the Sobolev inequality with exponent $2-\varepsilon$ to conclude

$$\begin{aligned}||u^{\frac{p}{2}}||^2_{L^{\frac{2n(2-\varepsilon)}{2n-2+\varepsilon}}} &\le C_\varepsilon\big(||\nabla(u^{\frac{p}{2}})||^2_{L^{2-\varepsilon}} + ||u^{\frac{p}{2}}||^2_{L^{2-\varepsilon}}\big) \\ &\le C_\varepsilon\big(\frac{K_\varepsilon C_{16} p^3}{2(p-1)}\int_M u^p G e^F dvol_g + ||u^{\frac{p}{2}}||^2_{L^{2-\varepsilon}}\big) \\ &\le D_\varepsilon\big(\frac{K_\varepsilon C_{16} p^3}{2(p-1)}\Big(\int_M u^{\frac{2p}{2-\varepsilon}} dvol_g\Big)^{\frac{2-\varepsilon}{2}} \times \Big(\int_M G^{\frac{2}{\varepsilon}} e^{\frac{2F}{\varepsilon}} dvol_g\Big)^{\frac{\varepsilon}{2}} + ||u^{\frac{p}{2}}||^2_{L^{\frac{4}{2-\varepsilon}}}\big).\end{aligned} \tag{3.59}$$

In the last line above, we use Hölder's inequality to estimate $||u^{\frac{p}{2}}||_{L^{2-\varepsilon}}$, and D_ε depends on C_ε and $vol(M)$. Denote

$$L_\varepsilon = \Big(\int_M G^{\frac{2}{\varepsilon}} e^{\frac{2F}{\varepsilon}} dvol_g\Big)^{\frac{\varepsilon}{2}}. \tag{3.60}$$

Also we choose ε to be sufficiently small so that the following holds:

$$\frac{2n(2-\varepsilon)}{2n-2+\varepsilon} > \frac{4}{2-\varepsilon}. \tag{3.61}$$

Hence we may conclude from (3.59) that

$$||u^{\frac{p}{2}}||^2_{L^{\frac{2n(2-\varepsilon)}{2n-2+\varepsilon}}} \le C_{18}\frac{p^3}{p-1}(K_\varepsilon L_\varepsilon + 1)||u^{\frac{p}{2}}||^2_{L^{\frac{4}{2-\varepsilon}}}. \tag{3.62}$$

We need to have a bound for K_ε, L_ε. Choose $\varepsilon = \frac{1}{2n}$, it is clear that this ε verifies (3.61) since $n \ge 2$. With this choice, from the expressions of K_ε and L_ε in (3.57) and (3.60), we need $\int_M (n+\Delta\varphi)^{4n-1} dvol_g$, $\int_M G^{4n} e^{4nF} dvol_g$ is bounded. First from Corollary 3.4, if $p_0 \ge 4n$, then $\int_M (n+\Delta\varphi)^{4n-1} dvol_g$ is bounded.

While for $\int_M G^{4n}e^{4nF}dvol_g$, first we have

(3.63)
$$\begin{aligned} &G = C_{15}\big(2e^{-2F}(n+\Delta\varphi)^{2n-1} + e^{-F}(n+\Delta\varphi)^{n-1}+1\big) + \big(\Delta_\varphi(F+f_*)\big)^2+1 \\ &+ 2e^{\frac{1}{2}(F+f_*)}(\Delta_\varphi(F+f_*))^2 \le C_{21}(n+\Delta\varphi)^{2n-1} + C_{21}(tr_\varphi g) + C_{21}(tr_\varphi g)^2 \\ &\le C_{21}(n+\Delta\varphi)^{2n-1} + C_{21}e^{-F}(n+\Delta\varphi)^{n-1} + C_{21}e^{-2F}(n+\Delta\varphi)^{2n-2} \\ &\le C_{22}(n+\Delta\varphi)^{2n-1}. \end{aligned}$$

Here we used that F has a lower bound, thanks to Lemma 3.3. Hence

$$\begin{aligned} \int_M G^{\frac{2}{\varepsilon}}e^{\frac{2F}{\varepsilon}}dvol_g &\le \Big(\int_M G^{8n}dvol_g\Big)^{\frac12} \times \Big(\int_M e^{8nF}dvol_g\Big)^{\frac12} \\ &\le \Big(\int_M C_{22}^{8n}(n+\Delta\varphi)^{8n(2n-1)}dvol_g\Big)^{\frac12} \times \Big(\int_M e^{8nF}dvol_g\Big)^{\frac12}. \end{aligned}$$

By Corollaries 3.2 and 3.4, it's enough to assume that $p_0 \ge 8n(2n-1)+1$. With this choice, we know that K_ε and L_ε given by (3.57), (3.60) are bounded with the said dependence in the theorem. Then we can iterate (3.62) as in cscK case to deduce $||u||_{L^\infty}$ is bounded in terms of $||u||_{L^1(\omega_0^n)}$.

To see that we have an estimate for $||u||_{L^1}$, we can compute

$$\Delta_\varphi(e^{\frac12 w}) = \frac14 e^{\frac12 w}|\nabla_\varphi w|_\varphi^2 + \frac12 e^{\frac12 w}\Delta_\varphi w. \tag{3.64}$$

Hence

$$\begin{aligned} \int_M e^{\frac12 w}|\nabla_\varphi w|_\varphi^2 dvol_g &\le C_{23}\int_M e^{\frac12 w}|\nabla_\varphi w|_\varphi^2 dvol_\varphi \le C_{23}\int_M 2e^{\frac12 w}(-\Delta_\varphi w)dvol_\varphi \\ &\le \int_M C_{24}(tr_\varphi\omega_0+1)dvol_\varphi = (n+1)C_{24}vol(M). \end{aligned}$$ □

As an immediate consequence, we observe

Corollary 3.5. *Assume $\beta \ge 0$ in (3.1), (3.2). Suppose $p_0 \ge \kappa_n$, where κ_n is as in Theorem 3.2, then for any $p < p_0$, we have*

$$||\nabla(F+f_*)||_{L^{2p}(\omega_0^n)} \le C_{25}.$$

Here C_{25} has the same dependence as in Theorem 3.1, but additionally on p. Besides, the bound is uniform in p as long as p is bounded away from p_0.

Proof. We know from Theorem 3.2 that $|\nabla_\varphi(F+f_*)|_\varphi \le C_{14}$. On the other hand, we have

$$|\nabla(F+f_*)|^2 \le |\nabla_\varphi(F+f_*)|_\varphi^2(n+\Delta\varphi) \le C_{14}^2(n+\Delta\varphi). \tag{3.65}$$

Hence the result follows from Corollary 3.4. □

Combining the estimates in this section, we can formulate Theorem 3.3.

Theorem 3.3. *Assume $\beta \ge 0$ in (3.1), (3.2). Let φ be a smooth solution to (3.1), (3.2). Suppose $p_0 \ge \kappa_n$ for some constant κ_n depending only on n. Then for any $p < p_0$,*

$$||F+f_*||_{W^{1,2p}} \le C_{25.1}, \ \ ||n+\Delta\varphi||_{L^p(\omega_0^n)} \le C_{25.1}.$$

Here $C_{25.1}$ depends only on an upper bound of entropy $\int_M \log\big(\frac{\omega_\varphi^n}{\omega_0^n}\big)\omega_\varphi^n$, $p_0 > 1$, $p < p_0$, the bound for $\int_M e^{-p_0 f_*}dvol_g$, $||R||_0$, $\max_M|\beta_0|_g$ and background metric

ω_0. Besides, the bound is uniform in p_0 as long as p_0 is bounded away from 1 and p bounded away from p_0.

4. K-energy proper implies existence of cscK

4.1. **The case $Aut_0(M,J)=0$.**

Let the functional I be as given by (2.7), we define

$$\mathcal{H}_0 = \{\varphi \in \mathcal{H} : I(\varphi) = 0\}.$$

Following [40, 72] , we introduce the following notion of properness:

Definition 4.1. We say the K-energy is proper with respect to L^1 geodesic distance if for any sequence $\{\varphi_i\}_{i\geq 1} \subset \mathcal{H}_0$, $\lim_{i\to\infty} d_1(0,\varphi_i) = \infty$ implies $\lim_{i\to\infty} K(\varphi_i) = \infty$.

The goal of this section is to prove the following existence result of cscK metrics.

Theorem 4.1. *Let $\beta \geq 0$ be a smooth closed $(1,1)$ form. Let K_β be defined as in (2.5). Suppose K_β is proper with respect to geodesic distance d_1, then there exists a twisted cscK metric with respect to β (i.e., solves (2.6)).*

For the converse direction, we have

Theorem 4.2 (Main theorem of [7] and Theorem 4.13 of [8])**.** *Let β be as in the previous theorem. Suppose that either*

(1) *$\beta > 0$; or*
(2) *$\beta = 0$ and $Aut_0(M,J) = 0$. Suppose there exists a twisted cscK metric with respect to β (i.e., solves (2.6)), then the functional K_β is proper with respect to geodesic distance d_1.*

In this theorem, the case $\beta = 0$ and $Aut_0(M,J) = 0$ is the main result of [7], and the case with $\beta > 0$ follows from the uniqueness of minimizers of twisted K-energy when the twisting form is Kähler (c.f. [8, Theorem 4.13]). For completeness, we will reproduce the proof in this paper.

First we prove Theorem 4.1. For this we will use the continuous path (2.12) to solve (2.6). Put $\chi = \omega_0$ in (2.12), define

$$S = \{t_0 \in [0,1] : (2.12) \text{ has a smooth solution for any } t \in [0,t_0]\}. \tag{4.1}$$

Remark 4.2. One may also consider the set S', consisting of $t_0 \in [0,1]$ for which (2.12) has a solution with $t = t_0$. In general, $t_0 \in S'$ does not imply $[0,t_0] \subset S'$. For instance, in [?chen-Zeng14], it is shown that if a cscK metric exists (i.e., (2.12) can be solved at $t = 1$), then we can solve this equation for all t sufficiently close to 1, for any $\beta > 0$. However, we can always find a $\chi > 0$ such that (2.12) has no solution with $t = 0$.

By Lemma 2.1, we know the set S is relatively open in $[0,1]$. Also when $t = 0$, (2.12) has a trivial solution, namely $\varphi = 0$. In particular $S \neq \emptyset$. The only remaining issue for the continuity method is the closeness of S. Due to Proposition 2.3, we can conclude the following criterion for closeness:

Lemma 4.3. *Suppose $t_i \in S$, $t_i \nearrow t_* > 0$, and let φ_i be a solution to (2.12) with $t = t_i$. Denote $F_i = \log \frac{\omega_{\varphi_i}^n}{\omega_0^n}$. Suppose that $\sup_i \int_M e^{F_i} F_i dvol_g < \infty$, then $t_* \in S$.*

Proof. We just need to show (2.12), or equivalently the coupled equations (2.13), (2.14) have a smooth solution with $t = t_*$. The assumption implies that we can assume $t_i \geq \delta_0 > 0$ for some $\delta_0 > 0$. Moreover, we can normalize the solution φ_i to (2.12) so that $\sup_M \varphi_i = 0$ and the assumption implies that we have a uniform upper bound of entropy. Then Proposition 2.3 implies that we have a uniform bound for all higher derivative bounds of φ_i. Hence we may take a subsequence of φ_i which converges smoothly. Say $\varphi_i \to \varphi_*$. Then we know that φ_* solves (2.12) with $t = t_*$. □

To connect this criterion with properness, we need some estimates connecting the L^1 geodesic distance d_1 and the I , J_χ functional defined in (2.7), (2.4).

Lemma 4.4. *There exists a constant $C > 0$, depending only on n and the background metric ω_0, such that for any $\varphi \in \mathcal{H}_0$, we have*

$$|\sup_M \varphi| \leq C(d_1(0,\varphi) + 1),\ |J_\chi(\varphi)| \leq C \max_M |\chi|_{\omega_0} d_1(0,\varphi). \tag{4.2}$$

Proof. This is well known in the literature and we give a proof for completeness here. We now prove the first estimate. Let $G(x, y)$ be the Green's function defined by the metric ω_0, then we can write:

$$\varphi(x) = \frac{1}{vol(M,\omega_0)}\int_M \varphi(y)\frac{\omega_0^n}{n!}(y) + \frac{1}{vol(M,\omega_0)}\int_M G(x,y)\Delta_{\omega_0}\varphi(y)\frac{\omega_0^n}{n!}(y). \tag{4.3}$$

We know that $\sup_{M\times M} G(x, y) \leq C_{15}$, hence

$$\begin{aligned}
&\int_M G(x,y)\Delta_{\omega_0}\varphi(y)\frac{\omega_0^n}{n!}(y) = \int_M (G(x,y) - C_{15})(\Delta_{\omega_0}\varphi(y) + n)\frac{\omega_0^n}{n!}\\
&- \int_M nG(x,y)\frac{\omega_0^n}{n!} + C_{15}n\int_M \frac{\omega_0^n}{n!} \leq -n\inf_{x\in M}\int_M G(x,y)\frac{\omega_0^n}{n!}\\
&+ C_{15}n\int_M \frac{\omega_0^n}{n!} := C_{16}vol(M,\omega_0).
\end{aligned} \tag{4.4}$$

Take sup in (4.3),

$$\sup_M \varphi \leq \frac{1}{vol(M,\omega_0)}\int_M \varphi\frac{\omega_0^n}{n!} + C_{16} \leq Cd_1(0,\varphi) + C_{16}. \tag{4.5}$$

On the other hand, since $I(\varphi) = 0$, it follows from (2.7) that $\sup_M \varphi \geq 0$, so the first estimate follows. For the second estimate, first we can calculate

$$\begin{aligned}
&\int_M \varphi\sum_{k=0}^{n-1}\chi\wedge\omega_0^k\wedge\omega_\varphi^{n-1-k} - n\int_M \varphi\chi\wedge\omega_0^{n-1}\\
&= \int_M \varphi\sum_{k=0}^{n-2}\chi\wedge\omega_0^k\wedge(\omega_\varphi^{n-1-k} - \omega_0^{n-1-k})\\
&= \int_M -\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\sum_{l=0}^{n-2}(n-1-l)\chi\wedge\omega_0^{n-2-l}\wedge\omega_\varphi^l.
\end{aligned} \tag{4.6}$$

Thus,

$$
\begin{aligned}
&|\int_M \varphi \sum_{k=0}^{n-1} \chi \wedge \omega_0^k \wedge \omega_\varphi^{n-1-k} - \int_M n\varphi\chi\wedge\omega_0^{n-1}| \\
&\leq n \max_M |\chi|_{\omega_0} \int_M \sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\sum_{l=0}^{n-1}\omega_0^{n-1-l}\wedge\omega_\varphi^l \\
&= n \max_M |\chi|_{\omega_0} \int_M \varphi(\omega_\varphi^n - \omega_0^n).
\end{aligned}
$$

Using Theorem 2.1, we conclude

$$
|\int_M \varphi \sum_{k=0}^{n-1} \chi \wedge \omega_0^k \wedge \omega_\varphi^{n-1-k} - \int_M n\varphi\chi\wedge\omega_0^{n-1}| \leq C_n \max_M |\chi|_{\omega_0} d_1(0,\varphi).
$$

Similar calculation shows

$$
|\int_M \underline{\chi}\varphi \sum_{k=0}^{n} \omega_0^k \wedge \omega_\varphi^{n-k} - (n+1)\int_M \underline{\chi}\varphi\omega_0^n| \leq C_n \max_M |\chi|_{\omega_0} d_1(0,\varphi).
$$

On the other hand, the quantities $\int_M n\varphi\chi\wedge\omega_0^{n-1}$ and $\int_M \underline{\chi}\varphi\omega_0^n$ can be bounded in terms of $\max_M |\chi|_{\omega_0} d_1(0,\varphi)$, again due to Theorem 2.1. Now the claimed estimate follows from (2.4). □

From Theorem 2.2, any two elements in $\mathcal{E}^1$ can be connected by a "locally finite energy geodesic" segment. On the other hand, from Theorem 4.7 in [8], we know K_β is convex along locally finite energy geodesic segment. This implies $tK_\beta + (1-t)J_{\omega_0}$ is convex along locally finite energy geodesics. In view of this, we can observe:

Corollary 4.5. *Let φ be a smooth solution to* (2.12) *for some $t \in [0,1]$, then φ minimizes the functional $tK_\beta + (1-t)J_{\omega_0}$ over $\mathcal{E}^1$.*

Proof. Observe that it is sufficient to show that φ minimizes $tK_\beta + (1-t)J_{\omega_0}$ over $\mathcal{H}$, in view of the fact that an element in $\mathcal{E}^1$ can be approximated (under distance d_1) using smooth potentials with convergent entropy, as proved in Theorem 3.2 in [8], while the J_χ functional is continuous under d_1, as shown by Proposition 4.1 and Proposition 4.4 in [8].

Next we can write $tK_\beta + (1-t)J_{\omega_0} = tK + J_{t\beta+(1-t)\omega_0}$. Take $\psi \in \mathcal{H}$. Let $\{u_s\}_{s\in[0,1]}$ be the $C^{1,1}$ geodesic connection φ and ψ, with $u_0 = \varphi$, $u_1 = \psi$. From Lemma 3.5 of [4] and the convexity of K-energy along $C^{1,1}$ geodesics, we conclude:

$$
K(\psi) - K(\varphi) \geq \lim_{s\to 0^+} \frac{K(u_s) - K(u_0)}{s} \geq \int_M (\underline{R} - R_\varphi)\frac{du_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!}. \tag{4.7}
$$

The first inequality used the convexity of K-energy along $C^{1,1}$ geodesics, proved by Berman-Berndtsson [4], and the second inequality is Lemma 3.5 of [4].

On the other hand, let $\{\varphi_s\}_{s\in[0,1]}$ be any smooth curve in $\mathcal{H}$ with $\varphi_0 = \varphi$, $\varphi_1 = \psi$, and let $\chi \geq 0$, we know from the calculation in [18, Proposition 2], that

(4.8)
$$\begin{aligned} J_\chi(\psi) - J_\chi(\varphi) &= \int_M (tr_\varphi\chi - \underline{\chi})\frac{d\varphi_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!} + \int_0^1 (1-s)\frac{d^2}{ds^2}J_\chi(\varphi_s)ds \\ &= \int_M (tr_\varphi\chi - \underline{\chi})\frac{d\varphi_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!} + \int_0^1 (1-s)ds\int_M \Big(\frac{\partial^2\varphi}{\partial s^2} - |\nabla_{\varphi_s}\frac{\partial\varphi_s}{\partial s}|^2_{\varphi_s}\Big)tr_{\varphi_s}\chi\frac{\omega_{\varphi_s}^n}{n!} \\ &+ \int_0^1(1-s)ds\int_M g_{\varphi_s}^{i\bar{j}}g_{\varphi_s}^{k\bar{l}}\chi_{i\bar{l}}\big(\frac{\partial\varphi}{\partial s}\big)_{,k}\big(\frac{\partial\varphi}{\partial s}\big)_{,\bar{j}}\frac{\omega_{\varphi_s}^n}{n!}. \end{aligned}$$

Now we choose $\varphi_s = u_s^\varepsilon$, namely the ε-geodesic(which is smooth by [18]), which means
$$\Big(\frac{\partial^2\varphi_s}{\partial s^2} - |\nabla_{\varphi_s}\frac{\partial\varphi_s}{\partial s}|^2_{\varphi_s}\Big)\det g_{\varphi_s} = \varepsilon\det g_0 \geq 0.$$
Hence we obtain from (4.8) that

$$J_\chi(\psi) - J_\chi(\varphi) \geq \int_M (tr_\varphi\chi - \underline{\chi})\frac{du_s^\varepsilon}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!}. \tag{4.9}$$

Also we know that $u_s^\varepsilon \to u_s$ weakly in $W^{2,p}$ for any $p < \infty$ as $\varepsilon \to 0$. This implies $\frac{du_s^\varepsilon}{ds}|_{s=0}$, as a function on M, is uniformly bounded with its first derivatives. Hence we may conclude $\frac{du_s^\varepsilon}{ds}|_{s=0} \to \frac{du_s}{ds}|_{s=0}$ uniformly. This convergence is sufficient to imply
$$\int_M (tr_\varphi\chi - \underline{\chi})\frac{du_s^\varepsilon}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!} \to \int_M (tr_\varphi\chi - \underline{\chi})\frac{du_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!},\ \text{as } \varepsilon \to 0.$$
Therefore,

$$J_\chi(\psi) - J_\chi(\varphi) \geq \int_M (tr_\varphi\chi - \underline{\chi})\frac{du_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!}. \tag{4.10}$$

Take $\chi = t\beta + (1-t)\omega_0$ in (4.10). Then multiply (4.7) by t, add to (4.10), we conclude

$$\begin{aligned} &\big(tK_\beta + (1-t)J_{\omega_0}\big)(\psi) - \big(tK_\beta + (1-t)J_{\omega_0}\big)(\varphi) \\ &\qquad \geq \int_M \Big(t(\underline{R} - R_\varphi) + (tr_\varphi\chi - \underline{\chi})\Big)\frac{du_s}{ds}|_{s=0}\frac{\omega_\varphi^n}{n!} = 0. \end{aligned} \tag{4.11}$$

The last equality using that φ solves (2.13), (2.14). □

Using this fact, we can obtain the following improvement of Lemma 4.3, which asserts that having control over the geodesic distance d_1 along the path of continuity ensures we can pass to limit.

Lemma 4.6. *Suppose $t_i \in S$, $t_i \nearrow t_* > 0$, and let φ_i be the solution to* (2.12) *with $t = t_i$, normalized so that $I(\varphi_i) = 0$. Suppose* $\sup_i d_1(0, \varphi_i) < \infty$*, then $t_* \in S$.*

Proof. As before, we assume $t_i \geq \delta > 0$. First observe that $\sup_i(t_iK_\beta + (1-t_i)J_{\omega_0})(\varphi_i) < \infty$. Indeed, we know from Corollary 4.5 that φ_i are minimizers of $t_iK_\beta + (1-t_i)J_{\omega_0}$, hence

$$\begin{aligned} t_iK_\beta(\varphi_i) + (1-t_i)J_{\omega_0}(\varphi_i) &\leq K_{\chi,t_i}(0) = t_iK_\beta(0) + (1-t_i)J_{\omega_0}(0) \\ &\leq \max(K_\beta(0), J_{\omega_0}(0)). \end{aligned} \tag{4.12}$$

On the other hand, we know

(4.13)
$$t_iK_\beta(\varphi_i)+(1-t_i)J_{\omega_0}(\varphi_i)=t_i\int_M e^{F_i}F_idvol_g+t_iJ_{-Ric+\beta}(\varphi_i)+(1-t_i)J_{\omega_0}(\varphi_i).$$

Since we assumed $\sup_i d_1(0,\varphi_i)<\infty$, Lemma 4.4 then implies that $\sup_i|J_{-Ric+\beta}(\varphi_i)| + |J_{\omega_0}(\varphi_i)|<\infty$. Consequently, $\sup_i\int_M e^{F_i}F_idvol_g<\infty$ since $t_i\geq\delta>0$. The result then follows from Lemma 4.3. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let S be defined as in (4.1), we just need to prove $S=[0,1]$. First we know from Lemma 2.1 that $t_*>0$. We want to show that $t_*=1$ and $1\in S$. Indeed, if $t_*<1$, then we can take a sequence $t_i\in S$, such that $t_i\nearrow t_*$. Let φ_i be the solution to (2.9) so that $I(\varphi_i)=0$.

As observed in (4.12), $\sup_i\big(t_iK_\beta+(1-t_i)J_{\omega_0}\big)(\varphi_i)<\infty$. On the other hand, since $0\in\mathcal{H}$ is a critical point of J_{ω_0}, we know from Corollary 4.5 that $J_{\omega_0}(\varphi_i)\geq J_{\omega_0}(0)$. Therefore we know $\sup_i K_\beta(\varphi_i)<\infty$. By properness, we can then conclude $\sup_i d_1(0,\varphi_i)<\infty$. From Lemma 4.6 we see $t_*\in S$. But then from Lemma 2.1 and Remark 2.2 we know $t_*+\delta'\in S$ for some $\delta'>0$ small. This contradicts $t_*=\sup S$. Hence we must have $t_*=1$. Repeating the argument in this paragraph, we can finally conclude $1\in S$. □

For completeness, we also include here the proof of Theorem 4.2, following [7,8].

Proof of Theorem 4.2. First we assume that $\beta=0$ and $Aut_0(M,J)=0$. Let $\varphi_0\in\mathcal{H}_0$ be such that $\omega_{\varphi_0}:=\omega_0+\sqrt{-1}\partial\bar\partial\varphi_0$ is cscK. We will show that for some $\varepsilon>0$, and for any $\psi\in\mathcal{H}_0$, $d_1(\varphi_0,\psi)\geq 1$, we have $K(\psi)\geq\varepsilon d_1(\psi,\varphi_0)+K(\varphi_0)$.

Indeed, if this were false, we will have a sequence of $\psi_i\in\mathcal{H}_0$, such that $d_1(\varphi_0,\psi_i)\geq 1$, but $\varepsilon_i:=\frac{K(\psi_i)-K(\varphi_0)}{d_1(\psi_i,\varphi_0)}\to 0$. Let $c^i:t\in[0,d_1(\varphi_0,\psi_i)]\to\mathcal{E}^1$ be the unit speed $C^{1,1}$ geodesic segment connecting φ_0 and ψ_i [18]. Let $\phi_i=c^i(1)$, then $d_1(\phi_i,\varphi_0)=1$. On the other hand, from the convexity of K-energy, we have

$$K(\phi_i)\leq\big(1-\frac{1}{d_1(\psi_i,\varphi_0)}\big)K(\varphi_0)+\frac{1}{d_1(\psi_i,\varphi_0)}K(\psi_i)=K(\varphi_0)+\varepsilon_i. \tag{4.14}$$

By the compactness result Lemma 2.5, there exists a subsequence of $\{\phi_i\}_{i\geq1}\subset\mathcal{E}^1$, denoted by ϕ_{i_j}, such that $\phi_{i_j}\overset{d_1}{\to}\phi_\infty$. Hence $d_1(\varphi_0,\phi_\infty)=1$. From the lower semi-continuity of K-energy(Theorem 4.7 of [8]), we obtain:

$$K(\phi_\infty)\leq\lim_{j\to\infty}\inf K(\phi_{i_j})\leq K(\varphi_0). \tag{4.15}$$

But since φ_0 is a minimizer of K-energy over $\mathcal{E}^1$, it follows that ϕ_∞ is also a minimizer. From Theorem 1.4 of [7], we know ϕ_∞ is also a smooth solution to cscK equation, and there exists $g\in Aut_0(M,J)$, such that $g^*\omega_{\phi_\infty}=\omega_{\varphi_0}$. But we assumed $Aut_0(M,J)=0$, hence $\omega_{\phi_\infty}=\omega_{\varphi_0}$. Therefore $\phi_\infty-\varphi_0$ is constant. But from the normalization $I(\phi_\infty)=I(\varphi_0)=0$, we know $\varphi_0-\phi_\infty=0$, this contradicts $d_1(\varphi_0,\phi_\infty)=1$.

Next we assume $\beta>0$. Let φ^β solves (2.12), normalized so that $I(\varphi^\beta)=0$. We show that for some $\varepsilon>0$, one has $K_\beta(\psi)\geq\varepsilon d_1(\varphi^\beta,\psi)+K_\beta(\varphi^\beta)$ for any $\psi\in\mathcal{H}_0$ with $d_1(\varphi^\beta,\psi)\geq 1$.

Indeed, if this were false, then there exists a sequence of $\psi_i \in \mathcal{H}_0$, such that $d_1(\varphi^\beta, \psi_i) \geq 1$, but $\varepsilon_i' := \frac{K_\beta(\psi_i) - K_\beta(\varphi^\beta)}{d_1(\psi_i, \varphi^\beta)} \to 0$. Note that K-energy is lower semicontinuous with respect to d_1 convergence and J_β is continuous ([8, Proposition 4.4]). Hence K_β is lower semicontinuous as well. So the same argument as last paragraph applies and we get a minimizer of K_β, denoted as $\psi_\infty \in \mathcal{H}_0$, such that $d_1(\psi_\infty, \varphi^\beta) = 1$. But by [8, Theorem 4.13], we know ψ_∞ and φ^β should differ by a constant. Because of the normalization $I(\psi_\infty) = I(\varphi^\beta) = 0$, we know that actually $\psi_\infty = \varphi^\beta$. This contradicts $d_1(\psi_\infty, \varphi^\beta) = 1$. □

As a corollary to this theorem, we show that the supreme of t for which (2.9) can be solved depends only on cohomology class of χ. More precisely,

Corollary 4.7. *Let χ_1, χ_2 be two Kähler forms in the same cohomology class. We define*

$$S_i = \{t_0 \in [0,1] : (2.9) \textit{ with } \chi = \chi_i \textit{ has a smooth solution for any } t \in [0, t_0]\}.$$

Then $S_1 = S_2$. In particular, if we define $R([\omega_0], \chi_i) = \sup S_i$, then $R([\omega_0], \chi_1) = R([\omega_0], \chi_2)$.

Proof. First we know from [36, Proposition 21 and Proposition 22], that existence of smooth solutions to $tr_\varphi \chi_i = \underline{\chi_i}$, $i = 1, 2$ are equivalent. So we may assume both equations are solvable. Then it follows from Lemma 2.1 that $R([\omega_0], \chi_i) > 0$. In virtue of Theorem 4.1 and Theorem 4.2, we just need to show for any $0 < t_0 \leq 1$:

$$K_{\chi_1, t_0} \text{ is proper} \Leftrightarrow K_{\chi_2, t_0} \text{ is proper.} \tag{4.16}$$

Here K_{χ_i, t_0} is defined as in (2.8).

Indeed, suppose $t_0 \in S_1$ and $t_0 < 1$, then for any $0 < t \leq t_0$, (2.9) with $\chi = \chi_1$ has a solution. From Theorem 4.2 applied to $\beta = \frac{1-t}{t}\chi_1$, we know this implies $K_{\chi_1, t}$ is proper, for any $0 < t \leq t_0$. If (4.16) were true, then $K_{\chi_2, t}$ is proper for any $0 < t \leq t_0$. Using Theorem 4.1 again, we know (2.9) with $\chi = \chi_2$ is solvable for any $t \in [0, t_0]$. This means $t_0 \in S_2$.

If $t_0 \in S_1$ and $t_0 = 1$, then it means K-energy is bounded from below, hence $K_{\chi_2, t}$ will be proper for $0 \leq t < 1$ ([36, Proposition 21]). Then Theorem 4.1 implies (2.9) will be solvable for $\chi = \chi_2$ and any $0 \leq t < 1$. While for $t = 1$, the solvability follows from the assumption that $t_0 = 1$, since equation (2.9) for $t = 1$ does not involve χ_1 or χ_2. Therefore $1 \in S_2$.

Now we turn to the proof of (4.16), which is an elementary calculation (c.f. [69]). Since χ_1 and χ_2 are in the same Kähler class, we can write

$$\chi_1 - \chi_2 = \sqrt{-1}\partial\bar{\partial}\nu, \text{ for some smooth function } \nu.$$

From (2.4), we can compute for $\varphi \in \mathcal{H}_0$:

$$\begin{aligned} J_{\chi_1}(\varphi) - J_{\chi_2}(\varphi) &= \frac{1}{n!}\sum_{p=0}^{n-1}\int_M (-\varphi)\sqrt{-1}\partial\bar{\partial}\nu \wedge \omega_0^{n-p-1} \wedge \omega_\varphi^p \\ &= \frac{1}{n!}\sum_{p=0}^{n-1}\int_M -\nu\sqrt{-1}\partial\bar{\partial}\varphi \wedge \omega_0^{n-p-1} \wedge \omega_\varphi^p \\ &= \frac{-1}{n!}\int_M \nu\omega_\varphi^n + \int_M \frac{1}{n!}\nu\omega_0^n. \end{aligned} \tag{4.17}$$

From this it is clear that

$$|J_{\chi_1}(\varphi) - J_{\chi_2}(\varphi)| \le c_n \sup_M |\nu|. \tag{4.18}$$

On the other hand,

$$|K_{\chi_1,t_0}(\varphi) - K_{\chi_2,t_0}(\varphi)| \le (1-t_0)|J_{\chi_1}(\varphi) - J_{\chi_2}(\varphi)| \le c_n \sup_M |\nu|. \tag{4.19}$$

From this (4.16) immediately follows. □

4.2. **General case when** $Aut_0(M,J) \neq 0$**.** In this subsection, we will denote $Aut_0(M,J)$ by G for convenience. Define

$$\mathcal{H}_0 = \{\varphi \in C^\infty(M) : \omega_\varphi := \omega_0 + \sqrt{-1}\partial\bar\partial\varphi \ge 0, \;\; I(\varphi) = 0\}.$$

Here the functional I is defined as

$$I(\varphi) = \frac{1}{(n+1)!}\int_M \varphi \sum_{k=0}^n \omega_0^k \wedge \omega_\varphi^{n-k}.$$

The set $\mathcal{H}_0$ can be identified as the set of Kähler metrics cohomologous to ω_0. We also know that for any $\varphi \in \mathcal{H}_0$, any $\sigma \in G$, one has that $\sigma^*\omega_\varphi$ is still in the Kähler class $[\omega_0]$. Hence there exists a unique element $\psi \in \mathcal{H}_0$, such that $\sigma^*\omega_\varphi = \omega_\psi$. We will write in short as $\sigma.\varphi = \psi$. It is clear that this defines an action of G on $\mathcal{H}_0$.

Let d_1 be the L^1 geodesic distance defined in Section 2.2. Now we try to explain how to extend the notion of properness to the general case. For any given metric ω_0, we may consider its G orbit

$$\mathcal{O}_{\omega_0} = \{\varphi \in \mathcal{H} \mid \sigma^*\omega_0 = \omega_\varphi, \text{ for some } \sigma \in G\}.$$

Note that if ω_0 is a cscK metric, then it is symmetric with respect to a maximal compact subgroup [12, 13]. Moreover, one can check directly that $\mathcal{O}_{\omega_0} \subset \mathcal{H}$ is a totally geodesic submanifold (c.f. Proposition 2.1 in [30]). Therefore, it is natural to define a notion of distance to this submanifold $\mathcal{O}_{\omega_0}$ from any Kähler potential φ by

$$\begin{aligned} d_p(\varphi, \mathcal{O}_{\omega_0}) &= \inf_{\psi \in \mathcal{O}_{\omega_0}} \; d_p(\varphi,\psi) \\ &= \inf_{\sigma\in G, \omega_\psi = \sigma^*\omega_0} \; d_p(\varphi,\psi) \\ &= \inf_{\sigma\in G, \omega_\psi = \sigma^*\omega_\varphi} \; d_p(0,\psi). \end{aligned}$$

More importantly, this infimum can be realized (c.f. Proposition 6.8 and Theorem 7.1 in [40]), i.e., there exists a $\sigma_0 \in G$ such that

$$d_p(\omega_\varphi, \sigma_0^*\omega_0) = d_p(\varphi, \mathcal{O}_{\omega_0}).$$

It means that this distance is positive unless φ lies in this orbit. Motivated by this observation, we extend the properness definition to the general case, following [40]. First, as in [40], one can define

$$d_{1,G}(\varphi,\psi) = \inf_{\sigma_1,\sigma_2 \in G} d_1(\sigma_1.\varphi, \sigma_2.\psi), \text{ for any } \varphi, \psi \in \mathcal{H}_0. \tag{4.20}$$

The group G acts on $\mathcal{H}_0$ by isometry, in the sense that

$$d_1(\sigma.\varphi, \sigma.\psi) = d_1(\varphi,\psi), \text{ for any } \sigma \in G, \text{ any } \varphi, \psi \in \mathcal{H}_0.$$

As a result of this, we see that

$$d_{1,G}(\varphi,\psi) = \inf_{\sigma\in G} d_1(\varphi, \sigma.\psi) = \inf_{\sigma \in G} d_1(\sigma.\varphi, \psi). \tag{4.21}$$

Also it is immediate to check that $d_{1,G}$ satisfies triangle inequality: for any $\varphi_i \in \mathcal{H}_0$, $i = 1, 2, 3$, we have

$$d_{1,G}(\varphi_1, \varphi_3) \leq d_{1,G}(\varphi_1, \varphi_2) + d_{1,G}(\varphi_2, \varphi_3). \tag{4.22}$$

The cscK metrics in the class $[\omega_0]$ are critical points of the K-energy, which is implicitly defined by

$$\frac{dK(\varphi)}{dt} = \int_M \frac{\partial \varphi}{\partial t}(\underline{R} - R_\varphi)\frac{\omega_\varphi^n}{n!}. \tag{4.23}$$

In the above, $\underline{R}$ is the average scalar curvature, R_φ is the scalar curvature of the metric ω_φ. The K-energy has the following explicit formula:

$$K(\varphi) = \int_M \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)\frac{\omega_\varphi^n}{n!} + J_{-Ric}(\varphi), \tag{4.24}$$

where for any $(1, 1)$ form χ, we define J_χ as

(4.25)

$$J_\chi(\varphi) = \int_0^1 \int_M \varphi\left(\chi \wedge \frac{\omega_{\lambda\varphi}^{n-1}}{(n-1)!} - \underline{\chi}\frac{\omega_{\lambda\varphi}^n}{n!}\right)d\lambda$$
$$= \frac{1}{n!}\int_M \varphi \sum_{k=0}^{n-1} \chi \wedge \omega_0^k \wedge \omega_\varphi^{n-1-k} - \frac{1}{(n+1)!}\int_M \underline{\chi}\varphi \sum_{k=0}^{n} \omega_0^k \wedge \omega_\varphi^{n-k}.$$

$$\frac{dJ_\chi(\varphi)}{dt} = \int_M \frac{\partial \varphi}{\partial t}(tr_\varphi \chi - \underline{\chi})\frac{\omega_\varphi^n}{n!}. \tag{4.26}$$

The readers may look up Section 2 for more details. First we make precise the notion of properness of K-energy with respect to $d_{1,G}$, in a similar vein as properness with respect to d_1 introduced in the second paper. The following definition of properness modulo G is due to Zhou-Zhu [77, Definition 0.1].

Definition 4.8. We say K-energy is proper with respect to $d_{1,G}$, if

(1) for any sequence $\{\varphi_i\} \subset \mathcal{H}_0$, $d_{1,G}(0, \varphi_i) \to \infty$ implies $K(\varphi_i) \to +\infty$.
(2) K-energy is bounded from below on $\mathcal{H}$.

Remark 4.9. The first point in the above definition can be replaced with: for any sequence $\{\varphi_i\} \subset \mathcal{H}_0$, $\inf_{\sigma\in G} J(\sigma.\varphi_i) \to \infty$ implies $K(\varphi_i) \to +\infty$, where $J(\varphi) = \int_M \varphi(\omega_0^n - \omega_\varphi^n)$. This follows from the fact that $\frac{1}{C}\inf_{\sigma\in G} J(\sigma.\varphi_i) - C \leq d_{1,G}(0, \varphi) \leq C\inf_{\sigma\in G} J(\sigma.\varphi_i) + C$, for some $C > 0$ and any $\varphi \in \mathcal{H}_0$, which can be found in [40, Lemma 5.11].

In this section, we will prove the following result:

Theorem 4.3. *Suppose that K-energy functional is proper with respect to $d_{1,G}$ as defined in* (4.8)*, then the class* $[\omega_0]$ *admits a cscK metric.*

Remark 4.10. The converse direction has been established by [7] and [40].

As a preliminary step, we observe that the assumption K-energy being bounded from below implies it is invariant under the action of G.

Lemma 4.11. *Suppose that the K-energy is bounded from below, then the K-energy is invariant under the action of G, i.e. $K(\sigma.\varphi) = K(\varphi)$ for any $\varphi \in \mathcal{H}$ and $\sigma \in G$.*

Proof. We will prove this by showing the Calabi-Futaki invariant vanishes. Let $\sigma \in G$, then there exists a holomorphic vector field X which generates a one-parameter path $\{\sigma(t)\}_{t\in\mathbb{R}}$, with $\sigma(0) = id$ and $\sigma(1) = \sigma$.

From the definition of K-energy and Calabi-Futaki invariant, we know that

$$\frac{d}{dt}\big(K(\sigma(t)^*\omega_\varphi)\big) = Re\big(\mathcal{F}(X, [\omega_0])\big) = a.$$

Here a is a constant depending only on the holomorphic vector field X and cohomology class of $[\omega_0]$. Since K-energy is bounded from below on the holomorphic line $\{\sigma(t)^*\omega_\varphi\}_{t\in\mathbb{R}}$, we must have $a = 0$. This implies that $K(\sigma.\varphi) = K(\varphi)$. □

Theorem 4.3 will be proved by solving the following path of continuity:

$$t(R_\varphi - \underline{R}) = (1-t)(tr_\varphi\omega_0 - n),\ t \in [0,1]. \tag{4.27}$$

Let φ solves (4.27), then we call ω_φ to be twisted cscK metric. For $t > 0$, equation (4.27) can be equivalently put as:

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det g_{i\bar{j}}, \tag{4.28}$$

$$\Delta_\varphi F = -\big(\underline{R} - \frac{1-t}{t}n\big) + tr_\varphi\big(Ric(\omega_0) - \frac{1-t}{t}\omega_0\big). \tag{4.29}$$

One important fact about this continuity path is that the set of solvable t is open, more precisely,

Lemma 4.12 ([21, 56, 76])**.** *Suppose for some $0 \le t_0 < 1$,* (4.27) *has a solution $\varphi \in C^{4,\alpha}(M)$ with $t = t_0$, then for some $\delta > 0$,* (4.27) *has a solution in $C^{4,\alpha}(M)$ for any $t \in (t_0 - \delta, t_0 + \delta) \cap [0,1]$.*

Remark 4.13. One can see by bootstrap that the solution φ of (4.27) (or equivalently of (4.28), (4.29) for $t > 0$) is smooth if we know it's in $C^{4,\alpha}$.

Another important fact about twisted path is that solutions to (4.27) are minimizers of the twisted K-energy, defined as

$$K_{\omega_0,t} = tK + (1-t)J_{\omega_0} = t\int_M \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)\frac{\omega_\varphi^n}{n!} + J_{-tRic+(1-t)\omega_0},\ \ t\in[0,1]. \tag{4.30}$$

First we observe that if the K-energy satisfies the assumptions of Definition 4.8, the twisted path (4.27) is solvable for any $0 \le t < 1$. Indeed, we have

Lemma 4.14. *Suppose the K-energy is bounded from below, then* (4.27) *is solvable for $0 \le t < 1$.*

Proof. In view of Theorem 4.1, we just need to verify for $0 < t_0 < 1$, K_{ω_0,t_0} is proper with respect to d_1. More specifically, since we know K-energy is bounded from below, we just need to observe J_{ω_0} is proper with respect to d_1.

To see that J_{ω_0} is proper, this follows from Proposition 22 in [36], which says that for some $\delta > 0$ and some $C > 0$, one has

$$J_{\omega_0}(\varphi) \ge \delta J(\varphi) - C,\ \text{for any}\ \varphi \in \mathcal{H}_0.$$

Here J is Aubin's J-functional, defined as

$$J(\varphi) = \int_M \varphi(\omega_0^n - \omega_\varphi^n).$$

It is elementary to show that $J(\varphi) \ge \frac{1}{C'}d_1(0,\varphi) - C'$ for $\varphi \in \mathcal{H}_0$ (c.f. [40, Proposition 5.5]). Hence we see that J_{ω_0} is proper with respect to d_1. □

Hence to get existence of cscK, the only remaining issue is to understand what happens as $t \to 1$. We will handle this difficulty now. Throughout the rest of this section, we assume the K-energy is proper with respect to $d_{1,G}$, in the sense defined by Definition 4.8.

Let $t_i < 1$, and t_i monotonically increase to 1. Denote $\tilde{\varphi}_i \in \mathcal{H}_0$ to be solutions to (4.27) with $t = t_i$. They exist due to Lemma 4.14. First we show that for the sequence $\tilde{\varphi}_i$, the K-energy is uniformly bounded from above.

Lemma 4.15. *Let $\tilde{\varphi}_i$ be as in previous paragraph, then we have*

$$K_{\omega_0,t_i}(\tilde{\varphi}_i) = \inf_{\mathcal{H}} K_{\omega_0,t_i}(\varphi) \to \inf_{\mathcal{H}} K(\varphi), \text{ as } t_i \to 1. \tag{4.31}$$

Also

$$K(\tilde{\varphi}_i) \to \inf_{\mathcal{H}} K(\varphi), \text{ as } t_i \to 1. \tag{4.32}$$

Proof. That $K_{\omega_0,t_i}(\tilde{\varphi}_i) = \inf_{\mathcal{H}} K_{\omega_0,t_i}(\varphi)$ follows from the convexity of the twisted K-energy and has been proved in Corollary 4.5. By the second part of Definition 4.8, we know that $\inf_{\mathcal{H}} K(\varphi) > -\infty$. On the other hand, let $\varphi^\varepsilon \in \mathcal{H}$ be such that $K(\varphi^\varepsilon) \leq \inf_{\mathcal{H}} K(\varphi) + \varepsilon$, and we know that

$$\limsup_{i\to\infty} K_{\omega_0,t_i}(\tilde{\varphi}_i) \leq \limsup_{i\to\infty} K_{\omega_0,t_i}(\varphi^\varepsilon) = K(\varphi^\varepsilon) \leq \inf_{\mathcal{H}} K(\varphi) + \varepsilon. \tag{4.33}$$

On the other hand, we also know that

$$K_{\omega_0,t_i}(\tilde{\varphi}_i) = t_i K(\tilde{\varphi}_i) + (1-t_i)J_{\omega_0}(\tilde{\varphi}_i) \geq t_i \inf_{\mathcal{H}} K(\varphi) + (1-t_i)J_{\omega_0}(0). \tag{4.34}$$

In the last inequality above, we used the fact that 0 is the solution to $tr_\varphi \omega_0 = n$, therefore a minimizer of J_{ω_0}. Hence we have

$$\liminf_{t_i\to 1} K_{\chi,t_i}(\tilde{\varphi}_i) \geq \inf_{\mathcal{H}} K(\varphi). \tag{4.35}$$

From (4.33) and (4.35), (4.31) follows. To see (4.32), we observe for t_i sufficiently close to 1, we have

$$\inf_{\mathcal{H}} K(\varphi) + \varepsilon \geq t_i K(\tilde{\varphi}_i) + (1-t_i)J_{\omega_0}(\tilde{\varphi}_i) \geq t_i K(\tilde{\varphi}_i) + (1-t_i)J_{\omega_0}(0).$$

The first inequality follows from (4.31). Hence we have

$$\limsup_{t_i\to 1} K(\tilde{\varphi}_i) \leq \lim_{t_i\to 1} \big(\frac{1}{t_i}(\inf_{\mathcal{H}} K(\varphi) + \varepsilon) - \frac{1-t_i}{t_i} J_{\omega_0}(0)\big) \leq \inf_{\mathcal{H}} K(\varphi) + \varepsilon. \tag{4.36}$$

From this (4.32) follows. □

As an immediate consequence of Lemma 4.15 and the properness assumption of K-energy, we deduce

Corollary 4.16. *Let $\tilde{\varphi}_i$ be as in previous lemma, we have*

$$\sup_i d_{1,G}(0, \tilde{\varphi}_i) < \infty.$$

Proposition 4.17 is the key technical result from which Theorem 4.3 immediately follows.

Proposition 4.17. *Consider the continuity path* (4.27)*. Suppose for some sequence $t_i \nearrow 1$, there exists a solution $\tilde{\varphi}_i$ to* (4.27) *with $t = t_i$ with $\tilde{\varphi}_i \in \mathcal{H}_0$ and $\sup_i d_{1,G}(0, \tilde{\varphi}_i) < \infty$. Let $\varphi_i \in \mathcal{H}_0$ be in the same G-orbit as $\tilde{\varphi}_i$ such that*

$\sup_i d_1(0,\varphi_i) < \infty$. *Suppose also that K-energy is G-invariant, then* $\{\varphi_i\}_i$ *contains a subsequence which converges in* $C^{1,\alpha}$ *(for any* $0<\alpha<1$*) to a smooth cscK potential.*

Let $\sigma_i \in G$ be such that

$$\sup_i d_1(0,\sigma_i.\tilde{\varphi}_i) < \infty. \tag{4.37}$$

The existence of such a sequence σ_i follows from Corollary 4.16. Denote $\varphi_i = \sigma_i.\tilde{\varphi}_i$. Next we briefly explain how to obtain above proposition.

First we write down the equation satisfied by the sequence φ_i, and they turn out to satisfy an equation in the form studied in Section 2, as shown by Lemma 4.19. Moreover, the integrability exponent p_0 improves to infinity as t_i approaches 1. Hence the estimates in Section 2 allow us to get uniform bounds of φ_i in $W^{2,p}$ for any $p<\infty$. Hence we can use compactness to take limit and we show the limit solves a weak form of cscK equation, as shown in Proposition 4.23. Finally one argues that this weak solution of cscK equation is actually smooth.

As a preliminary step, we show the sequence $\{\varphi_i\}$ has uniformly bounded entropy.

Lemma 4.18. *Denote* $\varphi_i = \sigma_i.\tilde{\varphi}_i$*, then we have*

$$\sup_i \int_M \log\left(\frac{\omega_{\varphi_i}^n}{\omega_0^n}\right)\omega_{\varphi_i}^n < \infty.$$

Proof. First due to the G-invariance of K-energy observed in Lemma 4.11, we have

$$\sup_i K(\varphi_i) = \sup_i K(\tilde{\varphi}_i) < \infty. \tag{4.38}$$

On the other hand, we know from Lemma 4.4 that

$$\sup_i |J_{-Ric}(\varphi_i)| \le \sup_i C_n |Ric|_{\omega_0} d_1(0,\varphi_i) < \infty. \tag{4.39}$$

From (4.38), (4.39), and recall the formula for K-energy in (4.24), the desired conclusion follows. □

Next we derive the equation satisfied by the sequence φ_i. We have the following result:

Lemma 4.19. *Let* θ_i *be such that* $\sigma_i^*\omega_0 = \omega_{\theta_i}$*, with* $\sup_M \theta_i = 0$*. Then* φ_i *satisfies the following equations:*

$$\det(g_{\alpha\bar{\beta}} + (\varphi_i)_{\alpha\bar{\beta}}) = e^{F_i}\det g_{\alpha\bar{\beta}}, \tag{4.40}$$

$$\Delta_{\varphi_i}F_i = -\big(\underline{R} - \frac{1-t_i}{t_i}n\big) + tr_{\varphi_i}\big(Ric(\omega_0) - \frac{1-t_i}{t_i}\omega_{\theta_i}\big). \tag{4.41}$$

Proof. Define $e^{\tilde{F}_i} = \frac{\omega_{\tilde{\varphi}_i}^n}{\omega_0^n}$. We have the following calculations:

$$\sigma_i^*(\omega_{\tilde{\varphi}_i}^n) = (\sigma_i^*\omega_{\tilde{\varphi}_i})^n = \omega_{\varphi_i}^n. \tag{4.42}$$

On the other hand,

$$\sigma_i^*(e^{\tilde{F}_i}\omega_0^n) = e^{\tilde{F}_i\circ\sigma_i}(\sigma_i^*\omega_0)^n. \tag{4.43}$$

So

$$\frac{\omega_{\varphi_i}^n}{(\sigma_i^*\omega_0)^n} = e^{\tilde{F}_i\circ\sigma_i}. \tag{4.44}$$

Hence if we define F_i to be $e^{F_i} = \frac{\omega_{\varphi_i}^n}{\omega_0^n}$, so as to make sure (4.40) always holds, we have

$$F_i = \tilde{F}_i \circ \sigma_i + \log\left(\frac{(\sigma_i^*\omega_0)^n}{\omega_0^n}\right). \tag{4.45}$$

To see (4.41), we go back to (4.29), and note that (4.29) is equivalent to:

$$\sqrt{-1}\partial\bar{\partial}\tilde{F}_i \wedge \frac{\omega_{\tilde{\varphi}_i}^{n-1}}{(n-1)!} = -\big(\underline{R} - \frac{1-t_i}{t_i}n\big)\frac{\omega_{\tilde{\varphi}_i}^n}{n!} + \big(Ric(\omega_0) - \frac{1-t_i}{t_i}\omega_0\big) \wedge \frac{\omega_{\tilde{\varphi}_i}^{n-1}}{(n-1)!}. \tag{4.46}$$

Pulling back using σ_i, we obtain

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}(\tilde{F}_i \circ \sigma_i) \wedge \frac{\omega_{\varphi_i}^{n-1}}{(n-1)!} &= -\big(\underline{R} - \frac{1-t_i}{t_i}n\big)\frac{\omega_{\varphi_i}^n}{n!} \\ &\quad + \big(Ric(\sigma_i^*\omega_0) - \frac{1-t_i}{t_i}\sigma_i^*\omega_0\big) \wedge \frac{\omega_{\varphi_i}^{n-1}}{(n-1)!}. \end{aligned} \tag{4.47}$$

Using (4.45) and recall that

$$\sqrt{-1}\partial\bar{\partial}\log\left(\frac{(\sigma_i^*\omega_0)^n}{\omega_0^n}\right) = Ric(\omega_0) - Ric(\sigma_i^*\omega_0),$$

we conclude

$$\begin{aligned} \big(\sqrt{-1}\partial\bar{\partial}F_i + Ric(\sigma_i^*\omega_0) - Ric(\omega_0)\big) \wedge \frac{\omega_{\varphi_i}^{n-1}}{(n-1)!} &= -\big(\underline{R} - \frac{1-t_i}{t_i}n\big)\frac{\omega_{\varphi_i}^n}{n!} \\ &\quad + \big(Ric(\sigma_i^*\omega_0) - \frac{1-t_i}{t_i}\sigma_i^*\omega_0\big) \wedge \frac{\omega_{\varphi_i}^{n-1}}{(n-1)!}. \end{aligned} \tag{4.48}$$

This is equivalent to (4.41). ☐

Next we would like to use the result obtained in the last section to study the regularity of φ_i. Denote $R_i = \underline{R} - \frac{1-t_i}{t_i}n$, $\beta_i = \frac{1-t_i}{t_i}\omega_{\theta_i}$, $(\beta_0)_i = \frac{1-t_i}{t_i}\omega_0$, and $f_i = \frac{1-t_i}{t_i}\theta_i$. Then we have $\beta_i \geq 0$, and $\beta_i = (\beta_0)_i + \sqrt{-1}\partial\bar{\partial}f_i$. Here we prove a property about the f_i which will be crucial for our proof.

Lemma 4.20. *There exists a constant C_{26}, which depends only on the background metric ω_0, such that for any $p > 1$, there exists $\varepsilon_p > 0$, depending only on p and the background metric ω_0, such that for any $t_i \in (1-\varepsilon_p, 1)$, one has $e^{-f_i} \in L^p(\omega_0^n)$ with $||e^{-f_i}||_{L^p(\omega_0^n)} \leq C_{26}$.*

Proof. Since we know that $\omega_{\theta_i} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\theta_i \geq 0$, with $\sup_M \theta_i = 0$, hence by a result of Tian (c.f. [70, Proposition 2.1]), we know that there exists $\alpha > 0$, $C_{25.5} > 0$, depending only on the background metric ω_0, such that for any $u \in C^2(M)$, $\omega_0 + \sqrt{-1}\partial\bar{\partial}u \geq 0$, one has $\int_M e^{-\alpha(u - \sup_M u)}dvol_g \leq C_{25.5}$.

Given $p > 1$, suppose t_i is sufficiently close to 1 such that $p\frac{1-t_i}{t_i} < \alpha$, then we have

(4.49)

$$\begin{aligned} \int_M e^{-pf_i}dvol_g = \int_M e^{-p\frac{1-t_i}{t_i}\theta_i}dvol_g &\leq \big(\int_M e^{-\alpha\theta_i}dvol_g\big)^{p\frac{1-t_i}{\alpha t_i}} vol(M)^{1-\frac{p(1-t_i)}{\alpha t_i}} \\ &\leq C_{25.5}^{p\frac{1-t_i}{\alpha t_i}} vol(M)^{1-\frac{p(1-t_i)}{\alpha t_i}} \leq \max\big(C_{25.5}, vol(M)\big) := C_{26}. \end{aligned}$$ ☐

As an application of the estimate in Theorem 3.3, we conclude the following uniform estimate for the sequence φ_i.

Proposition 4.21. *For any $p > 1$, there exists a constant C_{27}, and $\varepsilon'_p > 0$, such that for any $t_i \in (1 - \varepsilon'_p, 1)$,*

$$||F_i + f_i||_{W^{1,2p}} \leq C_{27}, \ \ ||n + \Delta\varphi_i||_{L^p(\omega_0^n)} \leq C_{27}.$$

In the above, ε'_p depends only on p and background metric ω_0, and C_{27} depends on p, background metric ω_0 and the uniform entropy bound $\sup_i \int_M \log\big(\frac{\omega_{\varphi_i}^n}{\omega_0^n}\big)\omega_{\varphi_i}^n$.

Proof. We may assume that ε'_p is chosen so small such that for any $t_i \in (1 - \varepsilon'_p, 1)$, $e^{-f_i} \in L^q$ for some $q \geq \kappa_n$. Such smallness depends only on n and the α-invariant of the background metric. The result then follows from Lemma 4.20 and Theorem 3.3. □

With this preparation, we can pass to the limit. Hence we may take a subsequence of φ_i (without relabeled), and a function $\varphi_* \in W^{2,p}$ for any $p < \infty$, and another function $F_* \in W^{1,p}$ for any $p < \infty$, such that

(4.50)
$$\varphi_i \to \varphi_* \text{ in } C^{1,\alpha} \text{ for any } 0<\alpha<1 \text{ and } \sqrt{-1}\partial\bar{\partial}\varphi_i \to \sqrt{-1}\partial\bar{\partial}\varphi_* \text{ weakly in } L^p.$$

(4.51)
$$F_i + f_i \to F_* \text{ in } C^{\alpha} \text{ for any } 0<\alpha<1 \text{ and } \nabla(F_i + f_i) \to \nabla F_* \text{ weakly in } L^p.$$

As a result of (4.50), we have

$$\omega_{\varphi_i}^k \to \omega_{\varphi_*}^k, \text{ weakly in } L^p \text{ for any } 1 \leq k \leq n \text{ and } p < \infty. \tag{4.52}$$

Here we provide an argument(more or less standard) for this weak convergence.

Lemma 4.22. *Suppose the convergence in* (4.50) *holds. Then for any $p < \infty$ and any $1 \leq k \leq n$,*

$$\omega_{\varphi_i}^k \to \omega_{\varphi_*}^k \textit{ weakly in } L^p.$$

Proof. We need to show that, for any ζ, a smooth $(n-k, n-k)$ form, the following convergence holds:

$$\int_M \omega_{\varphi_i}^k \wedge \zeta \to \int_M \omega_{\varphi_*}^k \wedge \zeta, \text{ as } i \to \infty. \tag{4.53}$$

Since $\omega_{\varphi_i}^k$ is uniformly bounded in L^p for any $p < \infty$, (4.53) will imply the same convergence holds for any $\zeta \in L^q$ with $q > 1$. Now we prove (4.53) by induction in k.

First observe that when $k = 1$, (4.53) follows from the weak convergence of $\sqrt{-1}\partial\bar{\partial}\varphi_i$.

Now assume (4.53) holds for $k = l - 1$, we need to show (4.53) holds for $k = l$. Indeed, let ζ be a smooth $(n - l, n - l)$ form, we have

$$\begin{aligned} \int_M \omega_{\varphi_i}^l \wedge \zeta &= \int_M \omega_{\varphi_i}^{l-1} \wedge \omega_0 \wedge \zeta + \int_M \omega_{\varphi_i}^{l-1} \wedge \sqrt{-1}\partial\bar{\partial}\varphi_i \wedge \zeta \\ &= \int_M \omega_{\varphi_i}^{l-1} \wedge \omega_0 \wedge \zeta - \int_M \omega_{\varphi_i}^{l-1} \wedge d^c\varphi_i \wedge d\zeta. \end{aligned} \tag{4.54}$$

Here $d^c = \frac{\sqrt{-1}}{2}(\partial - \bar{\partial})$. From the induction hypothesis, we know that

$$\int_M \omega_{\varphi_i}^{l-1} \wedge \omega_0 \wedge \zeta \to \int_M \omega_{\varphi_*}^{l-1} \wedge \omega_0 \wedge \zeta, \text{ as } i \to \infty. \tag{4.55}$$

On the other hand, we know from (4.50) that $d^c\varphi_i \to d^c\varphi_*$ uniformly, hence $d^c\varphi_i \wedge d\zeta \to d^c\varphi_* \wedge d\zeta$ strongly in L^q for any $q > 1$. This combined with the weak convergence of $\omega_{\varphi_i}^{l-1}$ is sufficient to imply

$$\int_M \omega_{\varphi_i}^{l-1} \wedge d^c\varphi_i \wedge d\zeta \to \int_M \omega_{\varphi_*}^{l-1} \wedge d^c\varphi_* \wedge d\zeta, \text{ as } i \to \infty. \tag{4.56}$$

Combining (4.54), (4.55) and (4.56), we conclude as $i \to \infty$,

$$\int_M \omega_{\varphi_i}^{l} \wedge \zeta \to \int_M \omega_{\varphi_*}^{l-1} \wedge \omega_0 \wedge \zeta - \int_M \omega_{\varphi_*}^{l-1} \wedge d^c\varphi_* \wedge d\zeta = \int_M \omega_{\varphi_*}^{l} \wedge \zeta. \tag{4.57}$$

This proves (4.53) for $k = l$ and finishes the induction. ☐

It is crucial matter to identify the limit. Actually we will show the solution φ_* is a weak solution to cscK in the following sense:

Proposition 4.23. *Let φ_*, F_* be the limit obtained in* (4.50), (4.51). *Then φ_* is a weak solution to cscK in the following sense:*

(1) $\omega_{\varphi_*}^n = e^{F_*}\omega_0^n$,
(2) *For any $\eta \in C^\infty(M)$, we have*

$$-\int_M d^c F_* \wedge d\eta \wedge \frac{\omega_{\varphi_*}^{n-1}}{(n-1)!} = \int_M -\eta \underline{R} \frac{\omega_{\varphi_*}^n}{n!} + \eta Ric \wedge \frac{\omega_{\varphi_*}^{n-1}}{(n-1)!}. \tag{4.58}$$

In the above, $d^c = \frac{\sqrt{-1}}{2}(\partial - \bar\partial)$.

Before we prove this proposition, we need Lemma 4.24, which shows $f_i \to 0$ in L^1. This is needed to justify (1) in the above proposition.

Lemma 4.24. *Recall θ_i is defined as $\sigma_i^*\omega_0 = \omega_{\theta_i}$ with $\sup_M \theta_i = 0$. $f_i = \frac{1-t_i}{t_i}\theta_i$. Then we have*

$$e^{-f_i} \to 1 \text{ in } L^p(\omega_0^n) \text{ as } t_i \to 1 \text{ for any } p < \infty.$$

Proof. First we know from (4.31) that there exists $\varepsilon_i \to 0$, such that

$$\begin{aligned} \inf_{\mathcal{H}} K(\varphi) + \varepsilon_i \geq K_{\omega_0, t_i}(\tilde\varphi_i) &= t_i K(\tilde\varphi_i) + (1-t_i) J_{\omega_0}(\tilde\varphi_i) \\ &\geq t_i \inf_{\mathcal{H}} K(\varphi) + (1-t_i)\delta d_1(0, \tilde\varphi_i) - (1-t_i)C. \end{aligned} \tag{4.59}$$

This implies $(1-t_i)d_1(0,\tilde\varphi_i) \to 0$ as $t_i \to 1$. On the other hand, denote $\tilde\theta_i = \theta_i - \frac{I(\theta_i)}{vol(M)}$, then we have $\tilde\theta_i \in \mathcal{H}_0$ and $\sigma_i.0 = \tilde\theta_i$. Also we know that G acts on $\mathcal{H}_0$ by isometry, hence

$$d_1(0,\tilde\theta_i) - d_1(0,\varphi_i) \leq d_1(\tilde\theta_i, \varphi_i) = d_1(\sigma_i.0, \sigma_i.\tilde\varphi_i) = d_1(0,\tilde\varphi_i). \tag{4.60}$$

Since $\sup_i d_1(0,\varphi_i) < \infty$, we know $(1-t_i)d_1(0,\tilde\theta_i) \to 0$. Therefore from [38, Theorem 5.5], we see that as $t_i \to 1$,

$$(1-t_i)\int_M |\tilde\theta_i|\omega_0^n \leq (1-t_i)d_1(0,\tilde\theta_i) \to 0.$$

Now we claim that

$$I(\theta_i)(1-t_i) \to 0, \text{ as } t_i \to 1. \tag{4.61}$$

If we have shown this claim, then we will have $\int_M |f_i|\omega_0^n \to 0$. Hence at least up to a subsequence, we would have $f_i \to 0$ pointwise outside a measure zero set. This would imply $e^{-pf_i} \to 1$ outside a measure zero set. On the other hand, by taking $p' > p$, we know $\sup_i \int_M e^{-p' f_i}\omega_0^n < \infty$, we can then conclude $\{e^{-pf_i}\}_{i\geq 1}$ is equi-integrable. Then we can conclude $e^{-pf_i} \to 1$ in L^1 using standard results in measure theory.

Hence it only remains to show the claim (4.61). Since we know that $\sup_M \theta_i = 0$, we know that

$$0 \leq \int_M (-\theta_i)\omega_0^n \leq C_{28},\ C_{28} \text{ depends only on background metric } \omega_0.$$

On the other hand,

$$\begin{aligned}
I(\theta_i) + \int_M (-\theta_i)\frac{\omega_0^n}{n!} &= \frac{1}{(n+1)!}\int_M \theta_i \sum_{k=0}^{n}\big(\omega_0^k \wedge \omega_{\theta_i}^{n-k} - \omega_0^n\big)\\
&= \frac{1}{(n+1)!}\int_M \theta_i \sqrt{-1}\partial\bar\partial\theta_i \wedge \sum_{k=0}^{n-1}(n-k)\omega_{\theta_i}^k \wedge \omega_0^{n-k-1}\\
&\geq -\frac{n}{(n+1)!}\int_M \sqrt{-1}\partial\theta_i \wedge \bar\partial\theta_i \wedge \sum_{k=0}^{n-1}\omega_0^k \wedge \omega_{\theta_i}^{n-1-k}\\
&= -\frac{n}{(n+1)!}\int_M \tilde\theta_i(\omega_0^n - \omega_{\tilde\theta_i}^n) \geq -C d_1(0,\tilde\theta_i).
\end{aligned} \tag{4.62}$$

Hence we have

$$0 \geq I(\theta_i) \geq -C'(1 + d_1(0,\tilde\theta_i)).$$

From here the claim (4.61) immediately follows. □

Now we are ready to show Proposition 4.23. We will obtain this as the result of the previous lemma

Proof of Proposition 4.23. First we show the equation (1) holds. First for each fixed i, we have $\omega_{\varphi_i}^n = e^{F_i}\omega_0^n$. (4.52) shows $\omega_{\varphi_i}^n \to \omega_{\varphi_*}^n$ weakly in L^p for any $p < \infty$. For the convergence of the right hand side, we can write $e^{F_i} = e^{F_i + f_i}\cdot e^{-f_i}$. According to (4.51), we see that $F_i + f_i$ is uniformly bounded, and converges to F_* strongly in L^p for $p < \infty$. This implies $e^{F_i+f_i} \to e^{F_*}$ in L^p for any finite p. On the other hand, we have just shown in Lemma 4.24 that $e^{-f_i} \to 1$ in L^p for any $p < \infty$. From here we can conclude $e^{F_i} \to e^{F_*}$ in L^p for $p < \infty$. Hence the equation (1) of Proposition follows.

To see the second equation, first we see from (4.41) that

$$\Delta_{\varphi_i}(F_i + f_i) = -\big(\underline{R} - \frac{1-t_i}{t_i}n\big) + tr_{\varphi_i}\big(Ric - \frac{1-t_i}{t_i}\omega_0\big).$$

This implies for $\eta \in C^\infty(M)$, one has

$$\begin{aligned}
&\int_M (F_i + f_i) d^c d\eta \wedge \frac{\omega_{\varphi_i}^{n-1}}{(n-1)!}\\
&\quad = \int_M -\eta\big(\underline{R} - \frac{1-t_i}{t_i}n\big)\frac{\omega_{\varphi_i}^n}{n!} + \eta\big(Ric - \frac{1-t_i}{t_i}\omega_0\big) \wedge \frac{\omega_{\varphi_i}^{n-1}}{(n-1)!}.
\end{aligned} \tag{4.63}$$

We wish to pass to limit in (4.63) as $t_i \to 1$. First because of (4.52), we can easily conclude:

$$\text{R.H.S. of (4.63)} \to \int_M \eta\Big(-\underline{R}\frac{\omega_{\varphi_*}^n}{n!} + Ric \wedge \frac{\omega_{\varphi_*}^{n-1}}{(n-1)!}\Big). \tag{4.64}$$

For the left hand side, since $F_i + f_i \to F_*$ strongly in L^p, $\omega_{\varphi_i}^{n-1} \to \omega_{\varphi_*}^{n-1}$ weakly in L^p for any $p < \infty$, we can conclude

$$\int_M (F_i + f_i) d^c d\eta \wedge \frac{\omega_{\varphi_i}^{n-1}}{(n-1)!} \to \int_M F_* d^c d\eta \wedge \frac{\omega_{\varphi_*}^{n-1}}{(n-1)!}.$$

Since $F_* \in W^{1,p}$, we have

(4.65)

$$-\int_M d^c F_* \wedge d\eta \wedge \frac{\omega_{\varphi_*}^{n-1}}{(n-1)!} = \int_M F_* d^c d\eta \wedge \frac{\omega_{\varphi_*}^{n-1}}{(n-1)!} = \int_M \eta \Big(-\underline{R} \frac{\omega_{\varphi_*}^n}{n!} + Ric \wedge \frac{\omega_{\varphi_*}^{n-1}}{(n-1)!} \Big).$$

□

Next we argue that ω_{φ_*} is quasi-isometric to ω_0.

Lemma 4.25. *There exists a constant C_{29}, such that $\frac{1}{C_{29}}\omega_0 \le \omega_{\varphi_*} \le C_{29}\omega_0$.*

Proof. We know that $F_* \in W^{1,p}$ for any $p < \infty$, hence we may take $G_k \in C^\infty(M)$, uniformly bounded, and $G_k \to F_*$ in $W^{1,p}$. Let ψ_k be the solution to $\omega_{\psi_k}^n = e^{G_k}\omega_0^n$ with $\sup_M \psi_k = 0$. The result of [28, Theorem 1.1], shows that for any $p < \infty$, one has

$$\sup_k ||\psi_k||_{W^{3,p}} < \infty.$$

Hence up to a subsequence, we can assume that for some $\psi_* \in W^{3,p}$ for any finite p, $\psi_k \to \psi_*$ in $W^{2,p}$ for any finite p. Therefore $\omega_{\psi_*}^n = e^{F_*}\omega_0^n$. Because of uniqueness result of Monge-Ampère equations (c.f. [9, Theorem 1.1]), we can conclude φ_* and ψ_* differ by a constant, hence $\omega_{\varphi_*} = \omega_{\psi_*} \le C_{29}\omega_0$. That $\omega_{\varphi_*} = \omega_{\psi_*} \ge \frac{1}{C_{29}}\omega_0$ follows from F_* is bounded from below. □

As a result of this, we now show that φ_* is actually a smooth cscK.

Corollary 4.26. *φ_* is a smooth solution to cscK.*

Proof. We know from the proof of Lemma 4.25 that $\varphi_* \in W^{3,p}$ for any $p < \infty$, hence we know that $\omega_{\varphi_*} \in C^\alpha$ for any $0 < \alpha < 1$. From (4.58) and Schauder estimate, we conclude $F_* \in C^{2,\alpha}$ for any $0 < \alpha < 1$. Then the higher regularity follows from bootstrap. □

5. Regularity of weak minimizers of K-energy

Our main goal in this section is to show the minimizers of K-energy over $\mathcal{E}^1$ are always smooth. The main ingredients are the continuity path as well as a priori estimates obtained in Section 3. The strategy of the proof is somewhat different from the usual variational problem. Indeed, the usual strategy for variational problem will be first to take some smooth variation of the minimizer, and derive an Euler-Lagrange equation for the minimizer (in weak form). Then one works with the Euler-Lagrange equation to obtain regularity (or partial regularity).

However, the same strategy runs into difficulty here. Indeed, an Euler-Lagrange equation for minimizer is not a priori available, since an arbitrary smooth variation of φ_* does not necessarily preserve the condition that $\omega_\varphi \ge 0$.

To get around this difficulty, we will still use the continuity path and our argument is partly inspired from [7]. The difference here is that the properness theorem (Theorem 4.1) plays a central role. Here we sketch the argument. Take φ_j to be smooth approximations of φ_* (in the space $\mathcal{E}^1$), and we solve continuity path from

φ_j. That K-energy is bounded from below ensures the continuity path is solvable for $t < 1$. We will show the existence of a minimizer ensures that for each fixed j, L^1 geodesic distance remains bounded as $t \to 1$. Hence we can take limit as $t \to 1$ and obtain a cscK potential u_j. Besides, such a sequence of u_j will also be uniformly bounded under L^1 geodesic distance, which follows from the uniform boundedness of φ_j under L^1 geodesic distance. Our a priori estimates allow us to take smooth limit of u_j and conclude that $u_j \to \psi$ smoothly and ψ is a smooth cscK potential. The proof is then finished once we can show ψ and φ_* only differ by an additive constant.

First we show that the existence of minimizers implies existence of smooth cscK metric.

Lemma 5.1. *Suppose that for some $\varphi_* \in \mathcal{E}^1$, we have $K(\varphi_*) = \inf_{\varphi \in \mathcal{E}^1} K(\varphi)$, then there exists a smooth cscK in the class $[\omega_0]$.*

Proof. We consider the continuity path (2.9) with $\chi = \omega_0$. By assumption, K-energy over $\mathcal{E}^1$ is bounded from below. Therefore the twisted K-energy $K_{\omega_0,t}$, defined by (2.8) is proper for any $0 \leq t < 1$. Hence we may invoke Theorem 4.1 with $\beta = \frac{1-t}{t}\omega_0$ to conclude that there exists a solution to (2.9) for any $0 < t < 1$. The only remaining issue is to see what happens in (2.9) as $t \to 1$.

Choose $t_i < 1$ and $t_i \to 1$, and let $\tilde{\varphi}_i$ be solutions to (2.9) with $t = t_i$, normalized up to an additive constant so that $I(\tilde{\varphi}_i) = 0$. Corollary 4.5 implies that $\tilde{\varphi}_i$ is the minimizer to K_{ω_0,t_i}. Therefore we have

(5.1)
$$t_i K(\varphi_*) + (1-t_i) J_{\omega_0}(\tilde{\varphi}_i) \leq t_i K(\tilde{\varphi}_i) + (1-t_i) J_{\omega_0}(\tilde{\varphi}_i) \leq t_i K(\varphi_*) + (1-t_i) J_{\omega_0}(\varphi_*).$$

Hence (5.1) implies that
$$J_{\omega_0}(\tilde{\varphi}_i) \leq J_{\omega_0}(\varphi_*).$$
On the other hand, we know J_{ω_0} is proper, in the sense that $J_{\omega_0}(\varphi) \geq \delta d_1(0,\varphi) - C$, for $\varphi \in \mathcal{H}_0$ (c.f. [36, Proposition 22]). This implies that
$$\sup_i d_1(0, \tilde{\varphi}_i) \leq \frac{1}{\delta}\big(C + J_{\omega_0}(\varphi_*)\big) < \infty.$$
Now from Lemma 4.6 we conclude that (2.9) can be solved up to $t = 1$, and we obtain the existence of a cscK potential. □

The main result of [7] showed the following weak-strong uniqueness property: as long as a smooth cscK exists in the Kähler class $[\omega_0]$, all the minimizers of K-energy over $\mathcal{E}^1$ are smooth cscK. Therefore, we can already conclude the following result:

Theorem 5.1. *Let $\varphi_* \in \mathcal{E}^1$ be such that $K(\varphi_*) = \inf_{\mathcal{E}^1} K(\varphi)$. Then φ_* is smooth, and ω_{φ_*} is a cscK metric.*

Next we will prove a more general version of Theorem 5.1. More precisely, we will prove:

Theorem 5.2. *Let $\chi \geq 0$ be a closed smooth $(1,1)$ form. Define $K_\chi(\varphi) = K(\varphi) + J_\chi(\varphi)$, where $J_\chi(\varphi)$ is defined by (2.4). Let $\varphi_* \in \mathcal{E}^1$ be such that $K_\chi(\varphi_*) = \inf_{\mathcal{E}^1} K_\chi(\varphi)$. Then φ_* is smooth and solves the equation $R_\varphi - \underline{R} = tr_\varphi \chi - \underline{\chi}$.*

Note that one can run the same argument as in Lemma 5.1 to show once there exists a minimizer to K_χ, there exists a smooth solution to

$$R_\varphi - \underline{R} = tr_\varphi \chi - \underline{\chi}. \tag{5.2}$$

However, it is not clear to us whether the argument in [7] can be adapted to this case to show a weak-strong uniqueness result. Namely if there exists a smooth solution to $R_\varphi - \underline{R} = tr_\varphi \chi - \underline{\chi}$, can one conclude all minimizers of K_χ are smooth? Therefore, in the following, we will use a direct argument. This argument is motivated from [7], but now is more straightforward because of the use of properness theorem.

Let φ_* be a minimizer of K_χ. Then by [8, Lemma 1.3], we may take a sequence of $\varphi_j \in \mathcal{H}$, such that $d_1(\varphi_j, \varphi_*) \to 0$, and $K_\chi(\varphi_j) \to K_\chi(\varphi_*)$. Indeed, that lemma asserts the convergence of the entropy part, but the J_{-Ric} and J_χ are continuous under d_1 convergence, by [8, Proposition 4.4].

Since there exists a minimizer to K_χ, the functional K_χ is bounded from below. On the other hand, for each fixed j, by [36, Proposition 22], we know that $J_{\omega_{\varphi_j}}$ is proper. Therefore, for $0 \le t < 1$, the twisted K_χ-energy $K_{\chi, \omega_{\varphi_j}, t} := tK_\chi + (1-t)J_{\omega_{\varphi_j}}$ is proper. Hence we may invoke Theorem 4.1 to conclude there exists a smooth solution to the equation

$$t(R_\varphi - \underline{R}) = (1-t)(tr_\varphi \omega_{\varphi_j} - n) + t(tr_\varphi \chi - \underline{\chi}), \text{ for any } 0 \le t < 1. \tag{5.3}$$

Denote the solution to be φ_j^t, normalized up to an additive constant so that $\varphi_j^t \in \mathcal{H}_0$, namely $I(\varphi_j^t) = 0$.

Since $\chi \ge 0$ and closed, we know that J_χ is convex along $C^{1,1}$ geodesic (though not necessarily strictly convex). Hence the functional K_χ is convex along $C^{1,1}$ geodesic. This again implies the convexity of $tK_\chi + (1-t)J_{\omega_{\varphi_j}}$ along $C^{1,1}$ geodesic. In particular, φ_j^t is a global minimizer of $tK_\chi + (1-t)J_{\omega_{\varphi_j}}$ by Corollary 4.5.

Hence we know that

$$tK_\chi(\varphi_j^t) + (1-t)J_{\omega_{\varphi_j}}(\varphi_j) \le tK_\chi(\varphi_j^t) + (1-t)J_{\omega_{\varphi_j}}(\varphi_j^t) \le tK_\chi(\varphi_j) + (1-t)J_{\omega_{\varphi_j}}(\varphi_j). \tag{5.4}$$

The first inequality above uses that φ_j minimizes $J_{\omega_{\varphi_j}}$. Hence

$$\sup_{0<t<1,\, j} K_\chi(\varphi_j^t) \le \sup_j K_\chi(\varphi_j). \tag{5.5}$$

Next we will show that the family of solution φ_j^t are uniformly bounded in d_1. First we have

$$tK_\chi(\varphi_j^t) + (1-t)J_{\omega_{\varphi_j}}(\varphi_j^t) \le tK_\chi(\varphi_*) + (1-t)J_{\omega_{\varphi_j}}(\varphi_*) \le tK_\chi(\varphi_j^t) + (1-t)J_{\omega_{\varphi_j}}(\varphi_*). \tag{5.6}$$

The first inequality follows from that φ_j^t minimizes $tK_\chi + (1-t)J_{\omega_{\varphi_j}}$ and the second inequality follows since φ_* minimizes K_χ. Therefore,

$$J_{\omega_{\varphi_j}}(\varphi_j) \le J_{\omega_{\varphi_j}}(\varphi_j^t) \le J_{\omega_{\varphi_j}}(\varphi_*). \tag{5.7}$$

The first inequality follows from that φ_j is a minimizer of $J_{\omega_{\varphi_j}}$. The second inequality follows from (5.6). As a first observation, we have

Lemma 5.2. *As $j \to \infty$,*

$$J_{\omega_{\varphi_j}}(\varphi_*) - J_{\omega_{\varphi_j}}(\varphi_j) \to 0.$$

Proof. We can compute

(5.8)

$$\begin{aligned} J_{\omega_{\varphi_j}}(\varphi_*) - J_{\omega_{\varphi_j}}(\varphi_j) &= \int_0^1 \frac{d}{d\lambda}\big(J_{\omega_{\varphi_j}}(\lambda\varphi_* + (1-\lambda)\varphi_j)\big)d\lambda \\ &= \int_0^1 d\lambda \int_M (\varphi_* - \varphi_j)\frac{\omega^{n-1}_{\lambda\varphi_*+(1-\lambda)\varphi_j}\wedge\omega_{\varphi_j} - \omega^n_{\lambda\varphi_*+(1-\lambda)\varphi_j}}{(n-1)!} \\ &= \int_0^1 d\lambda \int_M \lambda(\varphi_* - \varphi_j)\wedge\sqrt{-1}\partial\bar\partial(\varphi_j - \varphi_*)\wedge\frac{\omega^{n-1}_{\lambda\varphi_*+(1-\lambda)\varphi_j}}{(n-1)!} \\ &= \int_0^1 d\lambda \int_M \lambda\sqrt{-1}\partial(\varphi_* - \varphi_j)\wedge\bar\partial(\varphi_* - \varphi_j)\wedge\frac{(\lambda\omega_{\varphi_*} + (1-\lambda)\omega_{\varphi_j})^{n-1}}{(n-1)!}. \end{aligned}$$

Define

$$\begin{aligned} I(\varphi_j,\varphi_*) &= \int_M \sqrt{-1}\partial(\varphi_j - \varphi_*)\wedge\bar\partial(\varphi_j - \varphi_*)\wedge\sum_{k=0}^{n-1}\omega^k_{\varphi_j}\wedge\omega^{n-1-k}_{\varphi_*} \\ &= \int_M(\varphi_j - \varphi_*)(\omega^n_{\varphi_*} - \omega^n_{\varphi_j}). \end{aligned} \tag{5.9}$$

Since we know $d_1(\varphi_j,\varphi_*) \geq \frac{1}{C}\int_M|\varphi_j - \varphi_*|(\omega^n_{\varphi_j} + \omega^n_{\varphi_*})$ for some dimensional constant C, by [38, Theorem 5.5], we have $I(\varphi_j,\varphi_*) \leq Cd_1(\varphi_j,\varphi_*) \to 0$. On the other hand, we have $J_{\omega_{\varphi_j}}(\varphi_*) - J_{\omega_{\varphi_j}}(\varphi_j) \leq C'I(\varphi_j,\varphi_*)$ from (5.8) and (5.9). Hence $J_{\omega_{\varphi_j}}(\varphi_*) - J_{\omega_{\varphi_j}}(\varphi_j) \leq C'Cd_1(\varphi_j,\varphi_*) \to 0$. □

Corollary 5.3. *Let $I(\varphi_j,\varphi_j^t)$ be defined similar to* (5.9)*, then we have $\sup_{0<t<1} I(\varphi_j,\varphi_j^t) \to 0$ as $j\to\infty$.*

Proof. From previous lemma and (5.7), we know that as $j\to\infty$,

$$\sup_{0<t<1} J_{\omega_{\varphi_j}}(\varphi_j^t) - J_{\omega_{\varphi_j}}(\varphi_j) \leq J_{\omega_{\varphi_j}}(\varphi_*) - J_{\omega_{\varphi_j}}(\varphi_j) \to 0.$$

On the other hand, we know from (5.8), (5.9) with φ_* replaced by φ_j^t, the following estimate holds:

$$\frac{1}{C_n}(J_{\omega_{\varphi_j}}(\varphi_j^t) - J_{\omega_{\varphi_j}}(\varphi_j)) \leq I(\varphi_j^t,\varphi_j) \leq C_n(J_{\omega_{\varphi_j}}(\varphi_j^t) - J_{\omega_{\varphi_j}}(\varphi_j)).$$

□

Next we would like to show the d_1 distance of φ_j^t remains uniformly bounded. For this we will need the following key lemma:

Lemma 5.4 ([6, Theorem 1.8 and Lemma 1.9])**.** *There exists a dimensional constant C_n, such that for any u, v, $w \in \mathcal{E}^1$, we have*

$$I(u,w) \leq C_n(I(u,v) + I(v,w)).$$

Besides, we have

$$\int_M \sqrt{-1}\partial(u-w)\wedge\bar\partial(u-w)\wedge\omega_v^{n-1} \leq C_n I(u,w)^{\frac{1}{2^{n-1}}}\big(I(u,v)^{1-\frac{1}{2^{n-1}}} + I(w,v)^{1-\frac{1}{2^{n-1}}}\big).$$

As an immediate consequence of this lemma and Corollary 5.3, we see that:

Corollary 5.5. $\sup_{0<t<1} I(\varphi_j^t,\varphi_*) \to 0$ *as* $j\to\infty$*.*

Proof. Indeed,

$$I(\varphi_j^t,\varphi_*)\le C_n(I(\varphi_j^t,\varphi_j)+I(\varphi_j,\varphi_*))\le C_n\big(I(\varphi_j^t,\varphi_j)+Cd_1(\varphi_j,\varphi_*)\big).$$

In the second inequality above, we again used Theorem 5.5 of [38]. □

Using Lemma 5.4, we can show the following:

Lemma 5.6. *There exists a constant C, depending only on $\sup_j d_1(0,\varphi_j)$, n, such that*

$$\sup_{j,0<t<1} d_1(0,\varphi_j^t)\le C.$$

Proof. Denote $d^c=\frac{\sqrt{-1}}{2}(\partial-\bar\partial)$, and let $\varepsilon>0$, we may calculate

(5.10)

$$\begin{aligned}
&J_{\omega_0}(\varphi_j^t)-J_{\omega_{\varphi_j}}(\varphi_j^t)\\
&=\int_0^1\frac{d}{d\lambda}\big(J_{\omega_0}(\lambda\varphi_j^t)-J_{\omega_{\varphi_j}}(\lambda\varphi_j^t)\big)d\lambda\\
&=\int_0^1\int_M\varphi_j^t\Big(\frac{\omega_0\wedge\omega_{\lambda\varphi_j^t}^{n-1}}{(n-1)!}-\frac{\omega_{\varphi_j}\wedge\omega_{\lambda\varphi_j^t}^{n-1}}{(n-1)!}\Big)d\lambda=\int_0^1\int_M d^c\varphi_j^t\wedge d\varphi_j\wedge\frac{\omega_{\lambda\varphi_j^t}^{n-1}}{(n-1)!}d\lambda\\
&\le\varepsilon\int_0^1\int_M d^c\varphi_j^t\wedge d\varphi_j^t\wedge\frac{\omega_{\lambda\varphi_j^t}^{n-1}}{(n-1)!}d\lambda+\frac{1}{\varepsilon}\int_0^1\int_M d^c\varphi_j\wedge d\varphi_j\wedge\frac{\omega_{\lambda\varphi_j^t}^{n-1}}{(n-1)!}d\lambda\\
&\le\varepsilon C_n\int_M d^c\varphi_j^t\wedge d\varphi_j^t\wedge\sum_{k=0}^{n-1}\omega_0^k\wedge\omega_{\varphi_j^t}^{n-1-k}+\frac{C_n}{\varepsilon}\int_M d^c\varphi_j\wedge d\varphi_j\wedge\frac{\omega_{\frac12\varphi_j^t}^{n-1}}{(n-1)!}\\
&\le\varepsilon\tilde C_nd_1(0,\varphi_j^t)+\frac{\tilde C_n}{\varepsilon}I(\varphi_j,0)^{\frac{1}{2^{n-1}}}\Big(I(0,\frac12\varphi_j^t)^{1-\frac{1}{2^{n-1}}}+I(\varphi_j,\frac12\varphi_j^t)^{1-\frac{1}{2^{n-1}}}\Big)\\
&\le\varepsilon\tilde C_nd_1(0,\varphi_j^t)+\frac{\tilde C_n}{\varepsilon}I(0,\varphi_j)^{\frac{1}{2^{n-1}}}\Big(I(0,\frac12\varphi_j^t)^{1-\frac{1}{2^{n-1}}}\\
&\qquad+D_nI(0,\varphi_j)^{1-\frac{1}{2^{n-1}}}+D_nI(0,\frac12\varphi_j^t)^{1-\frac{1}{2^{n-1}}}\Big)\\
&\le\varepsilon\tilde C_nd_1(0,\varphi_j^t)+\varepsilon I(0,\frac12\varphi_j^t)+\varepsilon^{-2^n+1}\big(\tilde C_n(1+D_n)\big)^{2^{n-1}}I(0,\varphi_j).
\end{aligned}$$

In the first line above, we used that $J_{\omega_0}(0)=J_{\omega_{\varphi_j}}(0)=0$, which follows from (2.4). We used the second inequality of Lemma 5.4 in the passage from the 5th line to 6th line, and the first inequality in the passage from 6th line to 7th line. In the passage from 7th line to the last line, we used Young's inequality. Next observe that

$$\begin{aligned}
I(0,\frac12\varphi_j^t)&=\int_M\sqrt{-1}\partial\big(\frac12\varphi_j^t\big)\wedge\bar\partial\big(\frac12\varphi_j^t\big)\wedge\sum_{k=0}^{n-1}\omega_{\frac12\varphi_j^t}^k\wedge\omega_0^{n-1-k}\\
&=\int_M\sqrt{-1}\partial\big(\frac12\varphi_j^t\big)\wedge\bar\partial\big(\frac12\varphi_j^t\big)\wedge\sum_{k=0}^{n-1}\frac{1}{2^k}(\omega_0+\omega_{\varphi_j^t})^k\wedge\omega_0^{n-1-k} \qquad (5.11)\\
&\le C_n\int_M\sqrt{-1}\partial\varphi_j^t\wedge\bar\partial\varphi_j^t\wedge\sum_{k=0}^{n-1}\omega_0^k\wedge\omega_{\varphi_j^t}^{n-1-k}=C_n\int_M\varphi_j^t(\omega_0^n-\omega_{\varphi_j^t}^n)\\
&\le\tilde C_nd_1(0,\varphi_j^t).
\end{aligned}$$

Hence we obtain

$$J_{\omega_0}(\varphi_j^t) \le J_{\omega_{\varphi_j}}(\varphi_j^t) + \varepsilon \tilde{C}_n d_1(0,\varphi_j^t) + 2\varepsilon^{-2^n+1}\big(\tilde{C}_n(1+D_n)\big)^{2^{n-1}} I(0,\varphi_j). \tag{5.12}$$

On the other hand, since we know J_{ω_0} is proper in the following sense:

$$J_{\omega_0}(\varphi) \ge \delta d_1(0,\varphi) - C, \qquad \varphi \in \mathcal{H}_0.$$

Choose ε small enough so that

$$2\varepsilon \tilde{C}_n \le \frac{\delta}{2}.$$

Hence we obtain from (5.12) that

$$d_1(0,\varphi_j^t) \le \frac{2}{\delta}\big(J_{\omega_{\varphi_j}}(\varphi_j^t) + \varepsilon^{-2^n+1}\big(\tilde{C}_n(1+D_n)\big)^{2^{n-1}} I(0,\varphi_j) + C\big). \tag{5.13}$$

Since we know that $I(0,\varphi_j) \le C d_1(0,\varphi_j)$, and $d_1(0,\varphi_j)$ is uniformly bounded, it only remains to find an upper bound for $J_{\omega_{\varphi_j}}(\varphi_j^t)$. In order to bound $J_{\omega_{\varphi_j}}(\varphi_j^t)$ from above, we just need to find an upper bound for $J_{\omega_{\varphi_j}}(\varphi_*)$ thanks to (5.7). For this we can write:

$$\begin{aligned} J_{\omega_{\varphi_j}}(\varphi_*) &= \int_0^1 d\lambda \int_M \varphi_* \Big(\frac{\omega_{\lambda\varphi_*}^{n-1}\wedge\omega_{\varphi_j}}{(n-1)!} - \frac{\omega_{\lambda\varphi_*}^n}{(n-1)!}\Big) \\ &\le \int_0^1 d\lambda \int_M \varphi_* \sqrt{-1}\partial\bar\partial(\varphi_j - \lambda\varphi_*) \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!} \\ &= \int_0^1 d\lambda \int_M \lambda d^c\varphi_* \wedge d\varphi_* \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!} - \int_0^1 d\lambda \int_M d^c\varphi_* \wedge d\varphi_j \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!}. \end{aligned} \tag{5.14}$$

In the above, $d^c = \frac{\sqrt{-1}}{2}(\partial - \bar\partial)$, hence $d^c d = \sqrt{-1}\partial\bar\partial$. For the first term above, it can be bounded in the following way:
(5.15)

$$\int_0^1 d\lambda \int_M \lambda d^c\varphi_* \wedge d\varphi_* \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!} \le \int_M d^c\varphi_* \wedge d\varphi_* \wedge \sum_{k=0}^{n-1} \omega_0^k \wedge \omega_{\varphi_*}^{n-1-k} \le C d_1(0,\varphi_*).$$

For the second term on the right hand side of (5.14),

$$\begin{aligned} -\int_0^1 d\lambda \int_M d^c\varphi_* \wedge d\varphi_j \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!} &\le \frac{1}{2}\int_0^1 d\lambda \int_M d^c\varphi_* \wedge d\varphi_* \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!} \\ &+ \frac{1}{2}\int_0^1 d\lambda \int_M d^c\varphi_j \wedge d\varphi_j \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!}. \end{aligned} \tag{5.16}$$

The first term above can be estimated in the same way as in (5.15). For the second term above, we have

$$
\begin{aligned}
&\int_0^1 d\lambda \int_M \sqrt{-1}\partial\varphi_j \wedge \bar{\partial}\varphi_j \wedge \frac{\omega_{\lambda\varphi_*}^{n-1}}{(n-1)!}\\
&\le C_n \int_M \sqrt{-1}\partial\varphi_j \wedge \bar{\partial}\varphi_j \wedge \frac{\omega_{\frac{1}{2}\varphi_*}^{n-1}}{(n-1)!}\\
&\le C_n I(0,\varphi_j)^{\frac{1}{2^{n-1}}}\left(I(0,\frac{1}{2}\varphi_*)^{1-\frac{1}{2^{n-1}}} + I(\varphi_j,\frac{1}{2}\varphi_*)^{1-\frac{1}{2^{n-1}}}\right)\\
&\le C_n I(0,\varphi_j)^{\frac{1}{2^{n-1}}}\Bigg(I(0,\frac{1}{2}\varphi_*)^{1-\frac{1}{2^{n-1}}} + D_n I(0,\varphi_j)^{1-\frac{1}{2^{n-1}}}\\
&\qquad + D_n I(0,\frac{1}{2}\varphi_*)^{1-\frac{1}{2^{n-1}}}\Bigg).
\end{aligned}
\tag{5.17}
$$

By [38, Theorem 5.5], $I(0,\varphi_j)$ is controlled by $d_1(0,\varphi_j)$ and the calculation in (5.11) shows that $I(0,\frac{1}{2}\varphi_*)$ can be controlled in terms of $d_1(0,\varphi_*)$ respectively. □

Next we are ready to pass to limit. From $\sup_{0<t<1} d_1(0,\varphi_j^t) < \infty$, we may conclude that $\sup_{j,\,0<t<1}|J_{-Ric}(\varphi_j^t)| < \infty$ and $\sup_{j,0<t<1}|J_\chi(\varphi_j^t)| < \infty$ by Lemma 4.4. By (5.5) and our definition of K_χ, we know that $\sup_{j,t}\int_M \log\big(\frac{\omega_{\varphi_j^t}^n}{\omega_0^n}\big)\omega_{\varphi_j^t}^n < \infty$. Hence we may use Lemma 4.3 (the same argument works for K_χ) to conclude that up to a subsequence of t, $\varphi_j^t \to u_j$ as $t \to 1$ and u_j solves (5.2) for each j with $I(u_j) = 0$. This convergence is smooth convergence due to our previous estimates. Again due to the last lemma, we have $\sup_j d_1(0,u_j) \le \sup_{j,t} d_1(0,\varphi_j^t) \le C$ for some fixed constant C depending only on n and $\sup_j d_1(0,\varphi_j)$. Hence we may again assume that up to a subsequence of j, $u_j \to \psi$ smoothly as $j \to \infty$ and ψ is a smooth solution to (5.3). To finish the proof that φ_* is smooth, we just need Lemma 5.7:

Lemma 5.7. *φ_* and ψ differ by an additive constant.*

Proof. By taking limit as $t \to 1$, we can conclude from Corollary 5.5 that $I(u_j,\varphi_*) \to 0$ as $j \to \infty$. On the other hand, since $u_j \to \psi$ smoothly, we have $I(u_j,\psi) \to 0$ as $j \to \infty$. Hence

$$I(\varphi_*,\psi) \le C_n(I(u_j,\varphi_*) + I(u_j,\psi)) \to 0, \text{ as } j \to \infty.$$

That is, $I(\varphi_*,\psi) = 0$. On the other hand, from Lemma 5.8, we know $\varphi_* \in H^1(M)$ and

$$I(\varphi_*,\psi) \ge \int_M |\nabla_\psi(\varphi_* - \psi)|_\psi^2 \omega_\psi^n.$$

Therefore ψ and φ_* differ only up to a constant. □

In the above lemma, we used the following fact.

Lemma 5.8. *Let $\varphi \in \mathcal{E}^1$, then $\varphi \in H^1(M,\omega_0^n)$. Moreover, for any $\psi \in \mathcal{H}$, we have*

$$I(\varphi,\psi) \ge \int_M |\nabla_\psi(\varphi - \psi)|_\psi^2 \omega_\psi^n. \tag{5.18}$$

In the above, $|\nabla_\psi(\varphi-\psi)|_\psi^2 = g_\psi^{i\bar{j}}(\varphi-\psi)_i(\varphi-\psi)_{\bar{j}}$.

Proof. First we assume that both φ, $\psi \in \mathcal{H}$. Then we know that

$$\begin{aligned}I(\varphi,\psi) &= \int_M (\varphi-\psi)(\omega_\psi^n - \omega_\varphi^n)\\ &= \int_M d^c(\varphi-\psi)\wedge d(\varphi-\psi)\wedge\sum_{k=0}^{n-1}\omega_\varphi^k\wedge\omega_\psi^{n-1-k}\\ &\geq \int_M d^c(\varphi-\psi)\wedge d(\varphi-\psi)\wedge\omega_\psi^{n-1} = \int_M |\nabla_\psi(\varphi-\psi)|_\psi^2\omega_\psi^n.\end{aligned}$$

So (5.18) holds as long as $\varphi \in \mathcal{H}$. If $\varphi \in \mathcal{E}^1$, then we can find a sequence $\phi_j \in \mathcal{H}$, such that ϕ_j decreases pointwisely to φ. Such approximation is possible due to the main result of [10]. Also due to Lemma 4.3 of [38], we know that $d_1(\phi_j, \varphi) \to 0$. This implies that $I(\phi_j, \psi) \to I(\phi, \psi)$.

Since (5.18) holds with φ replaced by φ_j, we see that

$$\int_M |\nabla_\psi(\phi_j-\psi)|_\psi^2\omega_\psi^n \leq I(\phi_j,\psi)\to I(\varphi,\psi). \tag{5.19}$$

From $\sup_j d_1(0,\phi_j) < \infty$, we know that $\sup_j \int_M |\phi_j| dvol_g < \infty$. Now (5.19) shows ϕ_j is uniformly bounded in $H^1(M, \omega_\psi^n)$. Hence we can find a subsequence of ϕ_j which converges weakly in $H^1(M, \omega_\psi^n)$, strongly in $L^2(M, \omega_\psi^n)$. Clearly this limit must be φ. This shows $\varphi \in H^1(M, \omega_\psi^n)$, hence also in $H^1(M, \omega_0^n)$. Also we can conclude from (5.19) that

$$\int_M |\nabla_\psi(\varphi-\psi)|_\psi^2\omega_\psi^n \leq \lim\inf_{j\to\infty}\int_M |\nabla_\psi(\phi_j-\psi)|^2\omega_\psi^n \leq \lim\inf_j I(\phi_j,\psi) = I(\varphi,\psi).$$

□

6. Geodesic stability and existence of cscK ($Aut_0(M,J)=0$)

In this section, we prove Theorem 1.2. Similar to the definition of $\mathcal{H}_0$, we define

$$\mathcal{E}_0^1 = \mathcal{E}^1 \cap \{u : I(u) = 0\}.$$

Here $I(u)$ for $u \in \mathcal{E}^1$ is understood as the continuous extension of the functional I from $\mathcal{H}$ to $\mathcal{E}^1$. This is possible because of Proposition 4.1 in [8]. Also we notice that for any u_0, $u_1 \in \mathcal{E}_0^1$, the finite energy geodesic segment (defined by Theorem 2.2) $[0,1] \ni t \to \mathcal{E}^1$ will actually lie in $\mathcal{E}_0^1$. This follows from the fact that the I functional is affine on $C^{1,1}$ geodesics and I can be continuously extended to the space $\mathcal{E}^1$. As before, $\beta \geq 0$ is a smooth closed $(1,1)$ form.

First we note that when $Aut_0(M,J) = 0$ the notion of geodesic stability given by Definition 1.5 simplifies to (since the second alternative in Definition 1.5 does not happen when $Aut_0(M,J) = 0$):

Definition 6.1. Let $\phi_0 \in \mathcal{E}_0^1$ be such that $K(\phi_0) < \infty$. We say $(M, [\omega_0])$ is geodesic stable at ϕ_0 if for any locally finite energy geodesic ray $\rho : [0,\infty) \to \mathcal{E}_0^1$ with unit speed, one has ¥$(\rho) > 0$.

We will first prove the following result in this section, which covers Theorem 1.2 when $Aut_0(M,J) = 0$.

Theorem 6.1. *Suppose that either*

(1) $\beta > 0$ *everywhere; or*
(2) $\beta = 0$ *everywhere and* $Aut_0(M,J) = 0$.

Then the following statements are equivalent:

(1) *There exists no twisted cscK metric with respect to β in $\mathcal{H}_0$.*
(2) *There is an infinite geodesic ray ρ_t with locally finite energy with $K(\rho(0)) < \infty$, $t \in [0,\infty)$ in $\mathcal{E}_0^1$, such that the functional K_β is non-increasing along the ray.*
(3) *For any $\phi \in \mathcal{E}_0^1$ with $K(\phi) < \infty$, there is a locally finite energy geodesic ray starting at ϕ, such that the functional K_β is non-increasing along the ray.*

In the case $\beta > 0$, then from (1) one can additionally conclude K_β is strictly decreasing in (2) and (3) above.

Definition 6.2. Let $[0,\infty) \ni t \to u_t \in \mathcal{E}^1$ be a continuous curve. Then we say u_t is an infinite geodesic ray with locally finite energy, if the following hold:

(1) $d_1(u_t, u_s) = c|t-s|$ for some constant $c > 0$ and any $s, t \in [0,\infty)$.
(2) For any $K > 0$, $[0,K] \ni t \to u_t$ is a finite energy geodesic segment in the sense defined by Theorem 2.2.

Remark 6.3. Observe that the implication (3) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (1) follows from Theorem 4.2, which is already proved in [7, 8]. We will use our a priori estimates and the continuity path (2.9) to resolve the implication (1) $\Rightarrow$ (3). We are partly motivated from arguments in the proof of Theorem 6.5 of [8].

Next we observe Lemma 6.4:

Lemma 6.4. *Consider the continuity path* (2.12). *Suppose there is no twisted cscK metric with respect to β in Kähler class $[\omega_0]$. Denote $t_* = \sup S$, where the set S is defined in* (4.1). *Let $S \ni t_i \nearrow t_*$. Denote φ_i to be the solution to* (2.9) *with $t = t_i$, normalized so that $I(\varphi_i) = 0$. Then we have $\sup_i d_1(0,\varphi_i) = \infty$.*

Proof. Suppose otherwise, then $\sup_i d_1(0,\varphi_i) < \infty$. We can apply Lemma 4.6 to conclude $t_* \in S$. If $t_* < 1$, then we conclude from Lemma 2.1 that $t_* + \delta' \in S$ for some $\delta' > 0$ sufficiently small. This contradicts $t_* = \sup S$. If $t_* = 1$, then $1 \in S$. But this will contradict our assumption that there is no cscK metric in $[\omega_0]$. In either case, the contradiction shows one cannot have $\sup_i d_1(0,\varphi_i) < \infty$. ☐

With the help of above lemma, we are ready to prove (1) $\Rightarrow$ (3) in Theorem 6.1.

Lemma 6.5. *In Theorem* 6.1*, item* (1) *implies item* (3)*.*

Proof. Let φ_i be as in Lemma 6.4, we know that $\sup_i d_1(0,\varphi_i) = \infty$. Hence we may take a subsequence φ_{i_j}, such that $d_1(0,\varphi_{i_j}) \nearrow \infty$. We will construct a geodesic ray as described in Theorem 6.1, point (2) out of this subsequence φ_{i_j}. For simplicity, we will still denote this subsequence by φ_i.

By Theorem 2.2, there exists a unit speed finite energy d_1-geodesic segment connecting ϕ and φ_i, such that the functional I is affine on the segment. Indeed, one can check I is affine on $C^{1,1}$ geodesic and the extension to d_1-geodesic follows from continuity of the functional I (c.f. [8, Proposition 4.1]).

Denote this geodesic by $c^i : [0, d_1(\phi,\varphi_i)] \to \mathcal{E}^1$. Since $I(\phi) = I(\varphi_i) = 0$, we know $I = 0$ on c^i. In other words, $c^i : [0, d_1(\phi,\varphi_i)] \to \mathcal{E}_0^1$. As noted in (4.12), we have

$$\sup_i \big(t_i K_\beta + (1-t_i) J_{\omega_0}\big)(\varphi_i) \leq \max(K_\beta(0), J_{\omega_0}(0)).$$

On the other hand, since the functional J_{ω_0} is convex along $C^{1,1}$ geodesic, and we know 0 is a critical point of J_{ω_0}, we see that

$$J_{\omega_0}(\varphi_i) \geq J_{\omega_0}(0). \tag{6.1}$$

Therefore

$$K_\beta(\varphi_i) \leq \frac{\max(K_\beta(0), J_{\omega_0}(0)) - (1-t_i)J_{\omega_0}(0)}{t_i} \leq C. \tag{6.2}$$

Hence from the convexity of K_β-energy as remarked before, we obtain for any $l \in [0, d_1(\phi, \varphi_i)]$,

$$K_\beta(c^i(l)) \leq (1 - \frac{l}{d_1(\phi,\varphi_i)})K_\beta(\phi) + \frac{l}{d_1(\phi,\varphi_i)}K_\beta(\varphi_i) \leq \max(K_\beta(\phi), C). \tag{6.3}$$

Therefore, for each fixed l, if we consider the sequence $\{c^i(l)\}_{d_1(\phi,\varphi_i)\geq l} \subset \mathcal{E}^1$, it satisfies the assumption in Lemma 2.5. Indeed, $d_1(\phi, c^i(l)) = l$, $\forall i$, which implies $\sup_i |J_\beta(c^i(l))|$ uniformly bounded for fixed l (by Lemma 4.4). Therefore, we have K-energy is uniformly bounded and we may apply Lemma 2.5.

Hence we may take a subsequence $c^{i_j}(l)$, such that $c^{i_j}(l) \to c^\infty(l)$ for some element $c^\infty(l) \in \mathcal{E}^1$ as $j \to \infty$. Since the functional I is continuous under d_1 convergence, we obtain $c^\infty(l) \in \mathcal{E}^1_0$ as well. Clearly we may apply this argument to each $l \in \mathbb{Q}$, then by Cantor's diagonal sequence argument, we can take a subsequence of φ_i, denoted by φ_{i_j}, such that

$$c^{i_j}(l) \to c^\infty(l) \text{ in } d_1, \text{ as } j \to \infty, \text{ for any } l \in \mathbb{Q}. \tag{6.4}$$

Since c^{i_j} are unit speed geodesic segment, we see that for any $r, s \in \mathbb{Q}$, with $0 \leq r, s \leq d_1(\phi, \varphi_{i_j})$, we have $d_1(c^{i_j}(r), c^{i_j}(s)) = |r - s|$. Sending $j \to \infty$ gives

$$d_1(c^\infty(r), c^\infty(s)) = |r-s|, \text{ for any } 0 \leq r, s \in \mathbb{Q}. \tag{6.5}$$

We can then define $c^\infty(r)$ for all $r \in \mathbb{R}$ by requiring $c^\infty(r) = d_1 - \lim_{r_k\in\mathbb{Q}, r_k\to r} c^\infty(r_k)$. From property (6.5) it is easy to see this is well defined, i.e., the said limit exists and does not depend on our choice of sequence r_k. Hence $[0,\infty) \ni r \to c^\infty(r)$ is a unit speed geodesic ray in $\mathcal{E}^1_0$. Besides, if we apply Proposition 2.4 to $[0, r_k]$ for any $r_k > 0$, $r_k \in \mathbb{Q}$, we know $c^{i_j}(r) \to u_k(r)$ for any $r \in [0, r_k]$. Here $[0, r_k] \ni r \to u_k(r)$ is the finite energy geodesic segment connecting ϕ and $c^\infty(r_k)$. Hence we know $c^\infty(r) = u_k(r)$ for any $r \in [0, r_k] \cap \mathbb{Q}$, by (6.4). Therefore $c^\infty(r) = u_k(r)$ for any $r \in [0, r_k]$ by density. Therefore, we have shown $c^\infty|_{[0, d_1(\phi, c^\infty(r))]}$ is the finite energy geodesic segment connecting ϕ and $c^\infty(r)$ for $r \in \mathbb{Q}$. It is easy to extend this to all $r \in \mathbb{R}_+$ by rescaling in time and apply Proposition 2.4 again.

We can now invoke Theorem 4.7, Proposition 4.5 of [8] to conclude $r \longmapsto K(c^\infty(r))$, $r \longmapsto J_\beta(c^\infty(r))$ is convex. Hence $r \longmapsto K_\beta(c^\infty(r))$ is convex as well.

Now from the lower semi-continuity of K_β-energy under d_1-convergence, we obtain from (6.3) that

$$K_\beta(c^\infty(r)) \leq \liminf_{j\to\infty} K_\beta(c^{i_j}(r)) \leq \max(K_\beta(\phi), C), \text{ for all } r \in \mathbb{Q}. \tag{6.6}$$

Using the lower semi-continuity again, we deduce

$$K_\beta(c^\infty(r)) \leq \liminf_{k\to\infty} K_\beta(c^\infty(r_k)) \leq \max(K_\beta(\phi), C). \tag{6.7}$$

Therefore, $(0,\infty) \ni r \longmapsto K_\beta(c^\infty(r))$ is both convex and bounded, this forces K_β-energy must be decreasing along c^∞.

To see the "in addition" part, if K_β is not strictly decreasing, then from the convexity of $r \longmapsto K_\beta(c^\infty(r))$, we can conclude that for some $r_0 > 0$, $K_\beta(c^\infty(r))$ remains a constant for $r \geq r_0$. Since both K and J_β are convex, we know J_β remains linear for $r \geq r_0$. Now [8, Theorem 4.12], shows $c^\infty(r_1) = c^\infty(r_r) + const$ for any r_1, $r_2 \geq r_0$. Because of the normalization $I(c^\infty(r)) = 0$, we know $c^\infty(r_1) = c^\infty(r_2)$ for any r_1, $r_2 \geq r_0$. But this contradicts $d_1(c^\infty(r_1), c^\infty(r_2)) = |r_1 - r_2|$ for any r_1, $r_2 \geq 0$. □

Finally, the implication (2) ⇒ (1) follows immediately from Theorem 4.2.

Proof. Suppose otherwise, namely there exists a twisted cscK metric with respect to β in $\mathcal{H}_0$, denoted by φ^β. Then we can conclude from Theorem 4.2 that the twisted K-energy K_β is proper. In particular, $K_\beta \to +\infty$ along any locally finite energy geodesic ray. This contradicts the assumption in (2). □

We can deduce the following immediate consequence of Theorem 6.1.

Corollary 6.6. *Let $0 < t_0 < 1$, and let χ be a Kähler form. Then the following statements are equivalent:*

(1) *There is no twisted cscK metric with $t = t_0$ in $\mathcal{H}_0$ (i.e. solves* (2.9) *with $t = t_0$).*
(2) *There is an infinite geodesic ray ρ_t of locally finite energy, $t \in [0, \infty)$ in $\mathcal{E}_0^1$, such that the twisted K-energy K_{χ,t_0} (defined by* (2.8)*) is strictly decreasing along the ray.*
(3) *For any $\phi \in \mathcal{E}_0^1$ with $K(\phi) < \infty$, there is a locally finite energy geodesic ray starting at ϕ, such that the twisted K-energy K_{χ,t_0} (defined by* (2.8)*) is strictly decreasing along the ray.*

Also we can show Theorem 1.2 as a consequence (in the special case of $Aut_0(M, J) = 0$, so that geodesic stability reduces to Definition 6.1).

Proof of Theorem 1.2 *when $Aut_0(M, J) = 0$.* First we prove the necessary part. Assume $(M, [\omega_0])$ admits a cscK metric. Let φ_0 be the corresponding cscK potential. Recall we have shown in the proof of Theorem 4.2 (the direction existence implies properness) that for all $\psi \in \mathcal{E}_0^1$, with $d_1(\psi, \varphi_0) \geq 1$, one has $K(\psi) \geq \varepsilon d_1(\psi, \varphi_0) + K(\varphi_0)$. Let $\phi \in \mathcal{E}_0^1$ and $\rho : [0, \infty) \ni t \mapsto \mathcal{E}_0^1$ be a locally finite energy geodesic ray initiating from ϕ. We can assume $\rho(t)$ has unit speed. Then as long as $d_1(\rho(t), \varphi_0) \geq 1$, one has

$$\begin{aligned}
\frac{K(\rho(t)) - K(\phi)}{t} &\geq \frac{\varepsilon d_1(\rho(t), \varphi_0) + K(\varphi_0) - K(\phi)}{t} \\
&\geq \frac{\varepsilon d_1(\rho(t), \phi) - \varepsilon d_1(\phi, \varphi_0) + K(\varphi_0) - K(\phi)}{t} \\
&= \varepsilon - \frac{\varepsilon d_1(\phi, \varphi_0) - K(\varphi_0) + K(\phi)}{t}.
\end{aligned} \tag{6.8}$$

This implies

$$\lim_{t \to \infty} \inf \frac{K(\rho(t)) - K(\phi)}{t} \geq \varepsilon.$$

In particular this means ¥$([\rho]) \geq \varepsilon$. Thus, $(M, [\omega_0])$ is geodesic stable.

Now we want to show the converse. We assume $(M, [\omega_0])$ is geodesic stable and we want to prove that there is a cscK metric in the Kähler class. Suppose otherwise, then according to Theorem 6.1 with $\beta = 0$, point (3), we know that there exists a locally finite energy geodesic ray $\rho : [0,\infty) \ni t \mapsto \mathcal{E}_0^1$, initiating from $\phi \in \mathcal{E}_0^1$ with $K(\phi) < \infty$, such that the K-energy is non-increasing. It is clear that for this geodesic ray, one has $¥([\rho]) \leq 0$. This contradicts the assumption of geodesic stability at φ. This finishes the proof. □

7. Geodesic stability and existence of cscK (general case)

In this section, we show that geodesic stability in the sense of Definition 1.5 is equivalent to the existence of cscK, when $Aut_0(M,J) \neq 0$. As before, we denote $G = Aut_0(M,J)$. The main result we will prove in this section is:

Theorem 7.1. *The following statements are equivalent:*

(1) *The Kähler class* $[\omega_0]$ *admits a cscK metric.*
(2) *There exists* $\phi_0 \in \mathcal{E}_0^1$ *with* $K(\phi_0) < \infty$*, such that* $(M,[\omega_0])$ *is geodesic stable at* ϕ_0*.*
(3) $(M,[\omega_0])$ *is geodesic stable.*

Here geodesic stability is defined as in Definition 1.5. Observe that the implication (3) ⇒ (2) is trivial. Therefore we will focus on the implications (2) ⇒ (1) and (1) ⇒ (3). First we show the implication (2) ⇒ (1). As a preliminary step, we observe that (2) implies that K-energy is invariant under G.

Lemma 7.1. *If* $(M,[\omega_0])$ *is geodesic semistable at* ϕ_0*, in particular, if* (2) *of Theorem* 7.1 *holds, then the* K*-energy is invariant under* G*.*

Proof. Let $\sigma \in G$, and let $\varphi \in \mathcal{H}_0$, we need to check $K(\varphi) = K(\sigma.\varphi)$. Here $\sigma.\varphi$ is defined as in the beginning of Section 3. We will prove the desired result by showing that the Calabi-Futaki invariant must vanish. To see why this implies our result, let X be a holomorphic vector field and $\{\sigma(t)\}_{t\in\mathbb{R}}$ be the one-parameter family of holomorphic transformation generated by $Re(X)$, such that σ lies inside the one-parameter subgroup $\{\sigma(t)\}_{t\in\mathbb{R}}$. Define $\varphi_t := \sigma(t).\varphi \in \mathcal{H}_0$. Then for any $t \in \mathbb{R}$ we have

(7.1)
$$\frac{dK(\varphi_t)}{dt} = \int_M \partial_t\varphi(\underline{R} - R_\varphi)dvol_\varphi = -\int_M Re(X)(\xi)dvol_\varphi = -Re\big(\mathcal{F}(X,[\omega_0])\big).$$

In the above, ξ is a function chosen so that $\Delta_\varphi\xi = R_\varphi - \underline{R}$. $\mathcal{F}(X,[\omega_0])$ is the Calabi-Futaki invariant which depends only on X and Kähler class $[\omega_0]$. So the right hand side of (7.1) is a constant. Our result immediately follows as long as we can show Claim 7.2:

Claim 7.2.
$$\frac{d}{dt}(K(\varphi_t)) = 0.$$

To see the claim, we can assume that $\frac{d}{dt}(K(\varphi_t)) := a < 0$, and consider the holomorphic ray $\{\varphi_t\}_{t\in[0,\infty)}$. If instead we have $a > 0$, we can consider the holomorphic ray $\{\varphi_t\}_{t\in(-\infty,0]}$, and the same argument below applies.

First we show that $d_1(\varphi,\varphi_t) \to \infty$ as $t \to \infty$. Indeed, we know that $K(\varphi_t) = K(\varphi)+at \leq K(\varphi)$. If there exists a sequence of $t_k \to \infty$, such that $\sup_k d_1(\varphi,\varphi_{t_k}) < \infty$, then we may apply [6, Theorem 2.17], or [8, Corollary 4.8], to conclude that there exists a subsequence t_{k_l}, and $\varphi_0 \in \mathcal{E}^1$, such that $d_1(\varphi_{t_{k_l}},\varphi_0) \to 0$. But then from the lower semicontinuity of K-energy, we know that $K(\varphi_0) \leq \liminf_{l\to\infty} K(\varphi_{t_{k_l}}) = -\infty$. This is a contradiction.

Besides, we also have $d_1(\varphi,\varphi_t)\le Ct$ for some $C>0$. Indeed, if denote $\theta=\partial_t\varphi|_{t=0}$, then $\partial_t\varphi(t)=\theta(\sigma(t))$. To see this, fix $t_0>0$, we can compute

$$\frac{d}{dt}\big(\sigma(t)^*\omega_\varphi\big)|_{t=t_0}=\sqrt{-1}\partial\bar\partial\big(\partial_t\varphi|_{t=t_0}\big)=\frac{d}{dt}\sigma(t_0)^*\big(\sigma(t)^*\omega_\varphi\big)|_{t=0}=\sigma(t_0)^*(\sqrt{-1}\partial\bar\partial\theta)$$
$$=\sqrt{-1}\partial\bar\partial\big(\theta\circ\sigma(t_0)\big).$$

Hence $\partial_t\varphi|_{t=t_0}=\theta\circ\sigma(t_0)+h(t_0)$, for some function h, with $h(0)=0$. Then from the normalization $I(\varphi_t)=0$, we get

$$0=\frac{d}{dt}I(\varphi_t)=\int_M\partial_t\varphi\frac{\omega_{\varphi_t}^n}{n!}=\int_M\big(\theta\circ\sigma(t)+h(t)\big)\sigma(t)^*\left(\frac{\omega_\varphi^n}{n!}\right)=\int_M\theta\frac{\omega_\varphi^n}{n!}+h(t)vol(M).$$

Since $h(0)=0$, we have $\int_M\theta\frac{\omega_\varphi^n}{n!}=0$, which implies $h(t)=0$ for all t. But then

$$d_1(\varphi,\varphi_\tau)\le\int_0^\tau\int_M|\partial_t\varphi(t)|\frac{\omega_{\varphi_t}^n}{n!}dt=\tau\int_M|\theta|\frac{\omega_\varphi^n}{n!}.$$

Let $t_k\nearrow\infty$ and let $\rho_k(s):[0,d_1(\phi_0,\varphi_{t_k})]\to\mathcal{E}_0^1$ be the unit speed finite energy geodesic connecting ϕ_0 and φ_{t_k}. Using the convexity of K-energy along ρ_k (c.f. [4]), we know that for any $s\in[0,d_1(\phi_0,\varphi_{t_k})]$,

$$\begin{aligned}K(\rho_k(s))&\le\big(1-\frac{s}{d_1(\phi_0,\varphi_{t_k})}\big)K(\phi_0)+\frac{s}{d_1(\phi_0,\varphi_{t_k})}K(\varphi_{t_k})\\&=\big(1-\frac{s}{d_1(\phi_0,\varphi_{t_k})}\big)K(\phi_0)+\frac{s(K(\varphi)+at_k)}{d_1(\phi_0,\varphi_{t_k})}\\&\le\max(K(\phi_0),K(\varphi))+\frac{sat_k}{d_1(\phi_0,\varphi_{t_k})}\\&\le\max(K(\phi_0),K(\varphi))+\frac{sat_k}{d_1(\varphi,\varphi_{t_k})+d_1(\phi_0,\varphi)}\\&\le\max(K(\phi_0),K(\varphi))+\frac{sat_k}{Ct_k+d_1(\phi_0,\varphi)}.\end{aligned}\tag{7.2}$$

In the first line of (7.2), we used the convexity of K-energy along ρ_k. From the first to the second line, we used that $K(\varphi_{t_k})=K(\varphi)+t_ka$. From the third to the fourth line, we used triangle inequality for d_1 and also $a<0$. From the fourth line to the last line, we used $d_1(\varphi,\varphi_{t_k})\le Ct_k$.

In particular, for each fixed s, the K-energy is bounded from above, uniform in k. Hence we can use the compactness result [8, Corollary 4.8], to conclude there exists a subsequence $\rho_{k_l}(s)$ which converges under d_1 distance. Then we may apply the same argument as in the proof of (1) $\Rightarrow$ (3) in Theorem 6.1 to conclude there exists a subsequence k_l, such that for all $s\ge0$, $\rho_{k_l}(s)$ converges under d_1 distance. And the limit, denoted as $\rho_\infty(s)$, is a unit speed locally finite energy geodesic ray initiating from ϕ_0. Using the lower semicontinuity of K-energy, we obtain from (7.2):

$$K(\rho_\infty(s))\le\liminf_l K(\rho_{k_l}(s))\le\max(K(\varphi),K(\phi_0))+\frac{sa}{C}.$$

Hence we get

$$¥[\rho_\infty]=\lim_{s\to\infty}\frac{K(\rho(s))}{s}\le\frac{a}{C}<0.$$

This contradicts the geodesic semi-stability. ☐

As a preliminary step, we show that (2) implies K-energy is bounded from below.

Proposition 7.3. *Under the assumption of point* (2) *of Theorem* 7.1*, we have that K-energy is bounded from below.*

Proof. Suppose otherwise, then there exists a sequence of potentials $\tilde{\varphi}_i \in \mathcal{E}_0^1$, such that $K(\tilde{\varphi}_i) \to -\infty$. We can choose $\sigma_i \in G$, such that for $\varphi_i := \sigma_i.\tilde{\varphi}_i \in \mathcal{E}_0^1$, we have $d_{1,G}(\phi_0, \tilde{\varphi}_i) \leq d_1(\phi_0, \varphi_i) \leq d_{1,G}(\phi_0, \tilde{\varphi}_i) + 1$. Because we have shown K-energy is invariant under G, we know $K(\varphi_i) \to -\infty$ as well. Next we distinguish two cases and we show there is contradiction in both cases.

(1) $\sup_i d_1(\phi_0, \varphi_i) < \infty$. We can invoke [6, Theorem 2.17], or [8, Corollary 4.8], to conclude that there exists a subsequence $\varphi_{i_k} \xrightarrow{d_1} \psi \in \mathcal{E}^1$. Because of lower semicontinuity of K-energy (c.f. [8, Theorem 4.7]), we see that $K(\psi) \leq \liminf_{i_k} K(\varphi_{i_k}) = -\infty$. This is not possible.

(2) $\sup_i d_1(\phi_0, \varphi_i) = \infty$. Without loss of generality, we can assume $d_1(\phi_0, \varphi_i) \to \infty$. Let $\rho_i : [0, d_1(\phi_0, \varphi_i)] \to \mathcal{E}_0^1$ be unit speed geodesic segment connecting ϕ_0 with φ_i. Since K-energy is convex along ρ_i (c.f. [8, Theorem 4.7]), we conclude that for any $t \in [0, d_1(\phi_0, \varphi_i)]$,

$$K(\rho_i(t)) \leq (1 - \frac{t}{d_1(\phi_0, \varphi_i)})K(\phi_0) + \frac{t}{d_1(\phi_0, \varphi_i)} K(\varphi_i) \leq \max(K(\phi_0), K(\varphi_i)). \tag{7.3}$$

Hence for each fixed $t > 0$, we may apply [8, Corollary 4.8] to conclude there exists a subsequence, denoted as i_k, such that $\rho_{i_k}(t)$ converges under d_1. Repeating the argument of Lemma 6.5, one can actually conclude it is possible to take a subsequence i_k, such that $\rho_{i_k}(t)$ converges for all $t \in \mathbb{R}$, and the limit $\rho_\infty(t)$ is a unit speed locally finite energy geodesic ray (first use Cantor's process to get a subsequence which converges for all $t \in \mathbb{Q}$, then use geodesic property to extend to $t \in \mathbb{R}$). Also because of lower semicontinuity of K-energy and (7.3), we actually have K-energy is uniformly bounded from above on ρ_∞. Due to convexity, the alternative (1) in Definition 1.5 cannot hold for ρ_∞. Hence ρ_∞ must be in the second alternative, which means ρ_∞ is parallel to a geodesic ray ρ', which is generated from a holomorphic vector field. This implies ρ_∞ is $d_{1,G}$ bounded. Indeed, for any $t > 0$,

$$\begin{aligned} d_{1,G}(\rho_\infty(0), \rho_\infty(t)) &\leq d_{1,G}(\rho_\infty(0), \rho'(0)) + d_{1,G}(\rho'(0), \rho'(t)) + d_{1,G}(\rho'(t), \rho_\infty(t)) \\ &\leq d_1(\rho_\infty(0), \rho'(0)) + \sup_{t>0} d_1(\rho'(t), \rho_\infty(t)). \end{aligned}$$

In the above, note that $d_{1,G}(\rho'(0), \rho'(t)) = 0$ since ρ' is generated from a one-parameter family of holomorphic automorphism. Also we have $\sup_{t>0} d_1(\rho'(t), \rho_\infty(t)) < \infty$ since ρ' and ρ_∞ are parallel.

On the other hand, due to Lemma 7.4, we know that $d_{1,G}(\rho_i(t), \phi_0) \geq t - 1$, for any $t \in [1, d_1(\phi_0, \varphi_i)]$. Therefore,

$$\begin{aligned} d_{1,G}(\rho_\infty(t), \phi_0) &\geq d_{1,G}(\rho_i(t), \phi_0) - d_{1,G}(\rho_i(t), \rho_\infty(t)) \geq t - 1 - d_1(\rho_i(t), \rho_\infty(t)) \\ &\to t - 1, \text{ as } i \to \infty. \end{aligned}$$

This contradicts that ρ_∞ is $d_{1,G}$ bounded. □

Above proof involves the use of Lemma 7.4:

Lemma 7.4. *Let φ, $\psi \in \mathcal{E}_0^1$. Suppose that for some $\varepsilon > 0$, we have $d_1(\varphi, \psi) \leq d_{1,G}(\varphi, \psi) + \varepsilon$. Let $\rho : [0, K] \to \mathcal{E}_0^1$ be a finite energy geodesic connecting φ and ψ, then we have $d_{1,G}(\varphi, \rho(t)) \geq d_1(\varphi, \rho(t)) - \varepsilon$.*

Proof. Let $\sigma \in G$ be arbitrary, we need to show

$$d_1(\varphi, \sigma.\rho(t)) \geq d_1(\varphi, \rho(t)) - \varepsilon. \tag{7.4}$$

Indeed,

$$\begin{aligned} d_{1,G}(\varphi, \psi) &\leq d_1(\varphi, \sigma.\psi) \leq d_1(\varphi, \sigma.\rho(t)) + d_1(\sigma.\rho(t), \sigma.\psi) \\ &= d_1(\varphi, \sigma.\rho(t)) + d_1(\rho(t), \psi) = d_1(\varphi, \sigma.\rho(t)) + d_1(\varphi, \psi) - d_1(\varphi, \rho(t)) \\ &\leq d_1(\varphi, \sigma.\rho(t)) + d_{1,G}(\varphi, \psi) + \varepsilon - d_1(\varphi, \rho(t)). \end{aligned}$$

In the first equality of the second line, we use that G is d_1-isometry. In the second equality, we use that $\rho(t)$ is a geodesic. In the last inequality, we use our assumption. (7.4) immediately follows from this calculation. □

With this preparation, we are ready to prove (2) ⇒ (1).

Proof. Consider the continuity path (4.27). Since we have shown K-energy is bounded from below, we know from Lemma 4.14 to conclude that (4.27) can be solved for any $t < 1$ (This follows from the properness of twisted K-energy $tK + (1-t)J_{\omega_0}$.)

Let $t_i \nearrow 1$, and let $\tilde{\varphi}_i$ be solution to (4.27). We distinguish two cases:

(1) $\sup_i d_{1,G}(\phi_0, \tilde{\varphi}_i) < \infty$. Since we have shown K-energy is invariant under the action of G in Lemma 7.1, Proposition 4.17 applies and we are done.

(2) $\sup_i d_{1,G}(\phi_0, \tilde{\varphi}_i) = \infty$. We will show contradiction occurs in this case. Without loss of generality, we may assume $d_{1,G}(\phi_0, \tilde{\varphi}_i) \to \infty$. We may find $\sigma_i \in G$, such that for $\varphi_i = \sigma_i.\tilde{\varphi}_i$, we have $d_{1,G}(\phi_0, \tilde{\varphi}_i) \leq d_1(\phi_0, \varphi_i) \leq d_{1,G}(\phi_0, \tilde{\varphi}_i) + 1$. From Lemma 4.15, we know that in particular $\sup_i K(\tilde{\varphi}_i) < \infty$. From G-invariance of K-energy, we know that $\sup_i K(\varphi_i) < \infty$. From now on, the argument is very similar to Proposition 7.3. Indeed, let ρ_i be the unit speed finite energy geodesic connecting ϕ_0, φ_i. From the convexity of K-energy, we see that K-energy is uniformly bounded from above on ρ_i(independent of i). Hence we may take limit and get a geodesic ray ρ_∞ initiating from ϕ_0, on which the K-energy is decreasing. Hence the first alternative in Definition 1.5 fails for ρ_∞. On the other hand, the argument of Proposition 7.3 shows that ρ_∞ is $d_{1,G}$ unbounded. Hence the second alternative in Definition 1.5 fails as well. Therefore ρ_∞ violates geodesic stability at ϕ_0. □

Remark 7.5. In the proof for existence, we observe that one can weaken the second alternative in Definition 1.5 to only assume this geodesic ray is $d_{1,G}$ bounded.

Next we will move on to show the implication (1) ⇒ (3).

Proof of (1) ⇒ (3). Without loss of generality, we may assume ω_0 itself is cscK. By the main result of [7] and [40], the existence of cscK metric implies that K-energy is G-invariant and $K(\varphi) \geq C d_{1,G}(0, \varphi) - D$, for some constant $C > 0$, $D > 0$.

Let $\phi \in \mathcal{E}_0^1$ be such that $K(\phi) < \infty$ and let $\rho : [0, \infty) \to \mathcal{E}_0^1$ be a geodesic ray initiating from ϕ. There is no loss of generality to assume it is of unit speed. Namely $d_1(\rho(s), \rho(t)) = |s - t|$, for any s, $t \geq 0$. Again we distinguish two cases:

(1) K-energy is unbounded from above on ρ. Since K-energy is convex on ρ, we see that we are in the first alternative of Definition 1.5.

(2) K-energy is bounded from above on ρ. We need to argue that we are in the second alternative of Definition 1.5. Actually we will show that ρ is parallel to a geodesic ray which initiates from 0 and consists of minimizers of K-energy. From the main result of Section 5, we know the ray consists of cscK potentials. Then the

uniqueness result of [4, Theorem 1.3] applies and shows they differ from each other by a holomorphic transformation.

Let $t_k > 0$ be such that $t_k \to \infty$. Let $r_k : [0, d_1(0, \rho(t_k))] \to \mathcal{E}_0^1$ be the unit speed finite energy geodesic segment connecting 0 and $\rho(t_k)$. Due to the convexity of K-energy along r_k and cscK being minimizers of K-energy, we know for $t \in [0, d_1(0, \rho(t_k))]$,

$$\begin{aligned} K(r_k(t)) &\leq (1 - \frac{t}{d_1(0, \rho(t_k))})K(0) + \frac{t}{d_1(0, \rho(t_k))}K(\rho(t_k)) \\ &\leq (1 - \frac{t}{d_1(0, \rho(t_k))}) \inf_{\mathcal{E}^1} K + \frac{t}{d_1(0, \rho(t_k))} \sup_{t \geq 0} K(\rho(t)). \end{aligned} \tag{7.5}$$

In particular, this shows that K-energy is uniformly bounded from above, independent of k and t. Hence we may repeat the argument of Lemma 6.5 (in particular we use the compactness result [8, Corollary 4.8]), to conclude that one may take a subsequence, denoted as k_l, such that $r_{k_l}(t) \to r_\infty(t)$ for any $t \geq 0$, and $r_\infty(t)$ is a locally finite energy geodesic ray with unit speed. Now one can replace k by k_l in (7.5) and take the limit $k_l \to \infty$, we see that

$$K(r_\infty(t)) \leq \liminf_{k_l} K(r_{k_l}(t)) \leq \inf_{\mathcal{E}^1} K, \text{ for any } t \geq 0. \tag{7.6}$$

This again uses lower semicontinuity of K-energy with respect to d_1-convergence (c.f. [8, Theorem 4.7]). So we get r_∞ is a unit speed geodesic ray consisting of minimizers of K-energy. The only matter left is to show r_∞ and ρ are parallel. We prove this in Lemma 7.6. □

Lemma 7.6. *Let $\rho : [0, \infty) \to \mathcal{E}_0^1$ be a locally finite energy geodesic ray with unit speed. Let $t_k \nearrow \infty$, $\phi \in \mathcal{E}_0^1$, and $r_k : [0, d_1(\phi, \rho(t_k))] \to \mathcal{E}_0^1$ be the finite energy geodesic connecting ϕ and $\rho(t_k)$ with unit speed. Suppose $r_k(t) \to r_\infty(t)$ as $k \to \infty$ in d_1, for any $t \geq 0$. Then r_∞ is a locally finite energy geodesic with unit speed parallel to ρ.*

Proof. That r_∞ is a unit speed locally finite energy geodesic follows the same argument in the proof of (1) ⇒ (3) in Theorem 6.1. It only remains to show that r_∞ and ρ are parallel.

Fix $t > 0$, we may take t_k sufficiently large so that $t_k \geq t + d_1(\phi, \rho(0))$. Define s so as to satisfy

$$\frac{t}{t_k} = \frac{s}{d_1(\phi, \rho(t_k))}.$$

Observe that

$$d_1(\rho(t), r_k(t)) \leq d_1(\rho(t), r_k(s)) + d_1(r_k(s), r_k(t)) = d_1(\rho(t), r_k(s)) + |s - t|. \tag{7.7}$$

Now

$$\begin{aligned} |s - t| &= t\frac{|t_k - d_1(\phi, \rho(t_k))|}{t_k} = t\frac{|d_1(\rho(0), \rho(t_k)) - d_1(\phi, \rho(t_k))|}{t_k} \\ &\leq t\frac{d_1(\rho(0), \phi)}{t_k} \leq d_1(\rho(0), \phi). \end{aligned} \tag{7.8}$$

Hence it only remains to bound $d_1(\rho(t), r_k(s))$. For this we consider the reparametrization: for $\tau \in [0, 1]$, define $\tilde{\rho}(\tau) = \rho\big((1-\tau)t_k\big)$, $\tilde{r}_k(\tau) = r_k\big((1-\tau)d_1(\phi, \rho(t_k))\big)$.

First we consider the case where one has ϕ, $\rho(0) \in \mathcal{E}^2$. The main result of [14] and also the extension in [37] shows that $(\mathcal{E}^2, d_2)$ is non-positively curved. Hence

$$d_1(\tilde{\rho}(\tau), \tilde{r}_k(\tau)) \leq d_2(\tilde{\rho}(\tau), \tilde{r}_k(\tau)) \leq \tau d_2(\tilde{\rho}(1), \tilde{r}_k(1)) = \tau d_2(\rho(0), \phi), \text{ for any } \tau \in [0,1].$$

Now we take $\tau = 1 - \frac{t}{t_k}$ to conclude

$$d_1\big(\rho(t), r_k\big(t t_k^{-1} d_1(\phi, \rho(t_k))\big)\big) = d_1(\rho(t), r_k(s)) \leq \big(1 - \frac{t}{t_k}\big) d_2(\rho(0), \phi) \leq d_2(\rho(0), \phi). \tag{7.9}$$

Combining (7.7), (7.8), (7.9), we conclude that $d_1(\rho(t), r_k(t)) \leq 2d_2(\phi, \rho(0))$ for all t_k sufficiently large. We can send $k \to \infty$ and use that $r_k(t) \to r_\infty(t)$ in d_1 to conclude that

$$d_1(\rho(t), r_\infty(t)) \leq 2d_2(\phi, \rho(0)).$$

In the general case where we don't assume that $\rho(0)$ or $\phi \in \mathcal{E}^2$, we need to use Theorem A.1 to conclude

$$d_1(\tilde{\rho}(\tau), \tilde{r}_k(\tau)) \leq \tau d_1(\tilde{\rho}(1), \tilde{r}_k(1)) = \tau d_1(\rho(0), \phi). \tag{7.10}$$

Then the rest of the above argument goes through but we no longer need to use d_2 distance. □

Next we will prove Theorem 1.1, as an application of equivalence between geodesic stability and existence of cscK metric. Again observe that the implication (3) ⇒ (2) is trivial. It only remains to show the implications (2) ⇒ (1) and (1) ⇒ (3).

Proof of Theorem 1.1. First we show (2) ⇒ (1). If Calabi-Futaki invariant is nonzero, then we know cscK metric cannot exist.

In the other case, let $\rho : [0, \infty) \to \mathcal{E}_0^1$ be such a geodesic ray as described in (2), initiating from φ. We show that this geodesic ray violates the geodesic stability at φ. Indeed, since K-energy is non-increasing on ρ, we have $\yen[\rho] \leq 0$.

If $\yen[\rho] < 0$, then it violates both alternatives in Definition 1.5.

If $\yen[\rho] = 0$, then Definition 1.5 requires ρ to be parallel to a geodesic ray generated from a holomorphic vector field, but we assumed this is not the case.

Next we show (1) ⇒ (3). If Calabi-Futaki invariant is nonzero, then (3) already holds. Now suppose this invariant is zero and there exists $\varphi \in \mathcal{E}_0^1$, such that all geodesic rays either have K-energy unbounded from above or parallel to a holomorphic ray. Observe that Calabi-Futaki invariant being zero means K-energy is G-invariant. Also for all geodesic rays ρ initiating from φ, either $\yen[\rho] > 0$ (when K-energy is unbounded) or ρ is bounded under $d_{1,G}$, when ρ is parallel to a holomorphic ray, following the argument of Proposition 7.3. As observed in Remark 7.5, this is sufficient to imply cscK metric exists. □

Finally we prove Theorem 1.3.

Proof of Theorem 1.3. First we assume that $(M, [\omega_0])$ is geodesic semistable. Fix $0 < t_0 < 1$, if there is no solution to the twisted equation $t_0(R_\varphi - \underline{R}) = (1 - t_0)(tr_\varphi \omega_0 - n)$, then we can apply Corollary 6.6 to conclude there exists a locally finite energy geodesic ray with unit speed $\rho(s) : [0, \infty) \to \mathcal{E}_0^1$, such that $K_{\omega_0, t_0} = t_0 K + (1 - t_0) J_{\omega_0}$ is non-increasing along ρ. On the other hand, from [36, Proposition

21], we know that $J_{\omega_0}(\varphi) \geq Cd_1(0,\varphi) - D$, for some constant C, $D > 0$ and any $\varphi \in \mathcal{H}_0^1$. This implies

$$K_{\omega_0,t_0}(\rho(0)) \geq K_{\omega_0,t_0}(\rho(s)) \geq t_0 K(\rho(s)) + (1-t_0)Cs - (1-t_0)D.$$

This means $¥[\rho] \leq -\frac{C(1-t_0)}{t_0} < 0$, contradicting the geodesic semi-stability.

Then we assume that the twisted equation can be solved for any $0 < t < 1$. Since we know the solutions are minimizers of the twisted K-energy from Corollary 4.5, we see that K_{ω_0,t_0} are bounded from below. From this we can conclude that for any locally finite energy geodesic ray,

$$\begin{aligned}-C_{t_0} &\leq t_0 K(\rho(s)) + (1-t_0)J_{\omega_0}(\rho(s)) \leq t_0 K(\rho(s)) + (1-t_0)C' d_1(0,\rho(s)) \\ &\leq t_0 K(\rho(s)) + (1-t_0)C'(d_1(0,\rho(0)) + s).\end{aligned}$$

In the second inequality above, we used Lemma 4.4. Here C' depends only on the background metric ω_0. In the last inequality, we use that $\rho(s)$ is of unit speed.

Hence

$$¥[\rho] = \lim_{s\to\infty} \frac{K(\rho(s))}{s} \geq -\frac{(1-t_0)C'}{t_0}.$$

Since $t_0 < 1$ is arbitrary, we actually have $¥[\rho] \geq 0$. □

7.1. **Toric Kähler manifolds.** Now we turn our attention to the special case of toric Kähler manifolds in this subsection and present the proof of one version of Yau-Tian-Donaldson conjecture in this setting. There is a general set up of differential geometric framework on toric differential manifolds (c.f. Guillemin [55]). For any polarized toric Kähler manifold $(M, [\omega_0], L)$, there is a corresponding Delzant polytope $P \subset \mathbb{R}^n$ representing it; and any toric invariant Kähler potential in $[\omega_0]$ can be represented by a symplectic potential in $\bar{P}$. The equation for constant scalar curvature metrics becomes a real fourth order equation in terms of the symplectic potential, i.e., Abreu's equation; see Abreu [1]. Working within a general differential-geometric framework developed by Guillemin [55], Abreu [1], Donaldson proved Yau-Tian-Donaldson's conjecture for two dimensional toric Kähler manifolds in [48]. In this subsection, we extend Donaldson's theorem to all dimensional toric Kähler manifolds and prove an analogous theorem that the existence of constant scalar curvature metric is equivalent to the uniform stability of the polarization. Our proof is inspired by ideas in Section 5 of Donaldson [48].

In the following, $d\sigma$ denotes the standard surface measure on the boundary and $d\mu$ is the n-dimensional Lebesgue measure on the polytope. Following [48], we denote u_0 to be the following smooth convex function in P

$$u_0(x) = \frac{1}{2}\sum_k \delta_k(x)\log\delta_k(x). \tag{7.11}$$

Here $\delta_k(x)$ is the linear distance to the kth boundary face. Here u_0 corresponds to the smooth toric invariant background Kähler metric ω_0. Now we denote $\mathcal{S}$ to be the set of continuous convex functions u (we will call them "symplectic potentials" below) on $\bar{P}$ such that:

- $u - u_0$ is smooth on $\bar{P}$,
- The restriction of u on the lower dimensional polytopes on ∂P is smooth and strictly convex.

Due to the work of Abreu [1], Guillemin [55] and Proposition 3.1.7 of Donaldson [48], we now know that the functions in $\mathcal{S}$ have one-to-one correspondence with toric invariant Kähler potentials in $[\omega_0]$ under appropriate normalization of the Kähler potentials and symplectic potentials respectively. We refer the readers to Abreu [1], Guillemin [55] and [48] for more details about this correspondence.

It is important to remark that the geodesic equation in the space of Kähler potentials $\mathcal{H} = \{\varphi : \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}$ takes a simple and elegant form in the toric invariant setting: for any two symplectic potentials $u_1, u_2 \in \mathcal{S}$, the geodesic segment connecting u_1 and u_2 is the linear interpolation (c.f. D. Guan [51]):

$$(1-s)u_1 + su_2, \qquad s \in [0,1],$$

which holds for any L^p Finsler norm with $p \geq 1$. Thus, for any $p \geq 1$, the L^p distance $d_p(u_1, u_2)$ takes a simple formula

$$d_p(u_1, u_2) = \left(\int_P |u_1 - u_2|^p d\,\mu\right)^{\frac{1}{p}}.$$

We can define a linear functional on $\mathcal{S}$:

$$\mathcal{L}_P(f) = \int_{\partial P} f\, d\,\sigma - A\int_P f\, d\,\mu, \qquad \text{where } A = \frac{\int_{\partial P} d\,\sigma}{\int_P d\,\mu}.$$

Since our purpose is to study the existence of cscK metrics, we may assume that the Futaki invariant is zero, which is a necessary condition for the existence of cscK metrics. Under toric setting, it means that this functional $\mathcal{L}_P$ vanishes for all affine-linear functions on P. (In Remark 7.11 we will observe that the vanishing of the Futaki invariant will be entailed by each Definition from 0.1 to 0.4 below.) Therefore, it is natural to normalize an element $f \in \mathcal{S}$ in the following way: Pick a point $p \in P$, we say a function $f \in \mathcal{S}$ is *normalized* if $f \geq 0$ in P and $f(p) = 0$. Without loss of generality and for later convenience, we may choose the point p so that u_0 (given by (7.11)) achieves minimum at p. Note that u_0 must achieve minimum in the interior of P, since one has $\partial_\nu u_0(x_0) = -\infty$, for any vector ν pointing inward of P, at any point $x_0 \in \partial P$. So that $u_0 + c_0$ is normalized, for some constant $c_0 \in \mathbb{R}$.

The K-energy on the symplectic side now takes the following simple form:

$$F_A(u) = -\int_P \log\det(u_{ij})d\,\mu + \mathcal{L}_P(u), \qquad \forall\, u \in \mathcal{S}. \tag{7.12}$$

For the convenience of readers, we list various notions of stability as follows. Note that in the following, we do not require the functions f to be in $\mathcal{S}$.

Definition 7.7. L^1 **stability**: For all convex functions f defined on P^* (the union of P and its codimension 1 faces) whose boundary values lie in $L^1(\partial P, d\,\sigma)$, we have $\mathcal{L}_P(f) \geq 0$. The equality holds only if f is affine.

Definition 7.8. Uniform stability: $\mathcal{L}_P(f) \geq 0$ for all piecewise linear convex functions. Moreover, there is an $\varepsilon > 0$ such that for all piecewise linear convex functions f on P which are *normalized*, we have

$$\mathcal{L}_P(f) \geq \varepsilon \int_{\partial P} f\, d\,\sigma. \tag{7.13}$$

Definition 7.9. Filtrated stable (in the sense of G. Székelyhidi [68]): For all convex, continuous functions f on $\bar{P}$, we have $\mathcal{L}_P(f) \geq 0$. The equality holds only if f is affine.

Definition 7.10. ***K*-stable**: For all piecewise linear convex functions f on P, we have $\mathcal{L}_P(f) \geq 0$. The equality holds only if f is affine.

Remark 7.11.

(1) The definitions above imply that $\mathcal{L}_P(f) = 0$ if f is affine. Indeed, since both f and $-f$ are convex, one has $\mathcal{L}_P(f) \geq 0$ and $\mathcal{L}_P(-f) \geq 0$.

(2) Under uniform stability assumption as Definition 7.8, one can conclude that (7.13) holds for all normalized convex continuous function f on $\bar{P}$ which is smooth on P. This immediately follows from the fact that any such f can be approximated uniformly on $\bar{P}$ by a sequence of normalized piecewise linear functions f_i.

Indeed, for each integer $i \geq 1$, one can find a finite subset $E_i \subset P$ such that $p \in E_i$, $E_i \subset E_{i+1}$ and $\cup_{i\geq 1} E_i$ is dense on P. Then one may define $f_i(x) = \max_{q\in E_i} L_q(x)$, where $L_q(x) = f(q) + (\nabla f(q), x - q)$ is the linear approximation of f at q.

From these definitions, one can easily see the following:

$$\text{Uniform Stability} \Longrightarrow \text{Filtrated Stability} \Longrightarrow L^1 \text{ stability} \Longrightarrow K\text{-stability}.$$

In fact, for the converse direction, the following is true:

Proposition 7.12. *If P is L^1 stable, then it is both uniform stable and filtrated stable.*

Proof. It follows directly from Proposition 5.2.2 in Donaldson [48] that L^1 stable implies uniform stable, then it is clear that it is also filtrated stable. □

Proposition 7.13. *If P is L^1 stable, then the K-energy is proper in the sense of L^1 distance among all toric invariant potentials which correspond to normalized symplectic potentials in $\mathcal{S}$.*

Proof. First from Proposition 7.12, we may assume that P is uniform stable. In other words, there is a positive constant $\varepsilon > 0$ such that for all piecewise linear convex functions f on P which are *normalized*, we have

$$\mathcal{L}_P(f) \geq \varepsilon \int_{\partial P} f \, d\sigma. \tag{7.14}$$

Following Remark 7.11, we know that (7.14) holds for all normalized convex functions which are continuous on $\bar{P}$ and smooth on P. Now we can appeal to Lemma 2.3 in [77] (which is first proved in [48]) that the following holds:

$$F_A(u) \geq \delta \int_P u d\mu - C, \ \forall u \in \mathcal{S}, \ u \text{ is normalized}$$

for some positive constants δ, C which depend only on the constant ε in (7.14) (or alternatively, Definition 7.8), the function u_0 and the polytope P.

On the other hand, we note that $d_1(u, u_0 + c_0) = \int_P |u - u_0 - c_0| d\mu \leq \int_P u d\mu + \int_P (u_0 + c_0) d\mu$. The last inequality is due to the fact that both u and $u_0 + c_0$ are nonnegative since they are normalized. Hence we can conclude that $F_A(u) \geq \delta d_1(u, u_0 + c_0) - C$ for all $u \in \mathcal{S}$. □

Now we can use the properness theorem to deduce the existence of cscK metrics. Indeed, if one assumes the L^1 stability, then the above proposition shows that the K-energy is proper in terms of the L^1-distance, when restricted to the set of Kähler potentials which are normalized in the sense above. However, our previous properness theorem requires that $\varphi \in \mathcal{H}_0$. This is a different normalization than mentioned above on the symplectic side. Recall that

$$\mathcal{H}_0 = \{\varphi : \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0\} \bigcap \{\varphi : I(\varphi) = 0\}, \text{ where } I(\varphi) = \int_M \varphi \sum_{i=0}^{n} \omega_0^{n-i} \wedge \omega_\varphi^n,$$

which is an affine function along any geodesic segment in $\mathcal{H}$. Now we explain in detail how we switch to Kähler side. We denote $\mathcal{H}_{0,T}$ to be the toric invariant elements in $\mathcal{H}_0$, then the following holds:

Lemma 7.14. *There exists $\delta > 0$, $C > 0$, such that for any $\varphi \in \mathcal{H}_{0,T}$, one has*

$$K(\varphi) \geq -C + \delta \inf_{\sigma \in (\mathbb{C}^*)^n} J(\sigma.\varphi).$$

Here $J(\varphi) = \int_M \varphi(\omega_0^n - \omega_\varphi^n)$.

Proof. The constants δ and C appearing below may change from line to line.

Let v be the symplectic potential in $\mathcal{S}$ corresponding to φ. Let u be the normalized convex function from v (namely $u - v$ is an affine function).

Let $\tilde{\varphi}$ be the Kähler potential corresponding to u via the Legendre transform. Following a normalization argument in the proof of Proposition 2.4 [77], one can find $\sigma_0 \in (\mathbb{C}^*)^n$, such that $\tilde{\varphi} = \sigma_0.\varphi + c$, for some $c \in \mathbb{R}$.

From Proposition 7.13, there are two constants δ, C such that $F_A(u) \geq -C + \delta d_1(u_0 + c_0, u)$ for any $u \in \mathcal{S}$ and normalized, where c_0 is chosen so that $u_0 + c_0$ is normalized. This is possible since the point p is chosen to be the minimum point of u_0. Switching to Kähler side, we have $K(\tilde{\varphi}) \geq -C + \delta d_1(0, \tilde{\varphi})$.

According to Darvas [38, Theorem 3], we know that

$$\frac{1}{C_1} \int_M |\tilde{\varphi}|(\omega_0^n + \omega_{\tilde{\varphi}}^n) \leq d_1(0, \tilde{\varphi}) \leq C_1 \int_M |\tilde{\varphi}|(\omega_0^n + \omega_{\tilde{\varphi}}^n),$$

for some uniform constant $C_1 > 0$. Hence we see that

$$K(\tilde{\varphi}) \geq -C + \delta J(\tilde{\varphi}),$$

where $J(\tilde{\varphi}) = \int_M \tilde{\varphi}(\omega_0^n - \omega_{\tilde{\varphi}}^n)$.

On the other hand, since the Futaki invariant is zero, we know that K-energy is invariant under the action of $\sigma \in (\mathbb{C}^*)^n$, hence:

$$K(\tilde{\varphi}) = K(\sigma_0.\varphi) = K(\varphi)$$

and

$$J(\tilde{\varphi}) = J(\sigma_0.\varphi) \geq \inf_{\sigma \in (\mathbb{C}^*)^n} J(\sigma.\varphi).$$

So that we finally get the desired properness of K-energy on $\mathcal{H}_{0,T}$:

$$K(\varphi) \geq -C + \delta \inf_{\sigma \in (\mathbb{C}^*)^n} J(\sigma.\varphi), \ \varphi \in \mathcal{H}_{0,T}.$$

□

Note that our continuity path $(1-t)tr_{\omega_\varphi}\omega_0 = t(R_\varphi - \underline{R})$, $t \in [0,1]$ is invariant under the torus action, and $(\mathbb{C}^*)^n$ acts on $\mathcal{H}_{0,T}$, hence our proof for the Theorem 4.3 carries over (see also Remark 4.9). Thus we can conclude the existence of a cscK potential in $\mathcal{H}_{0,T}$.

For the converse, Theorem 4.6 of Chen-Li-Sheng [16] shows that the existence of cscK metric will imply the uniform stability as defined above. Hence we may conclude

Theorem 7.2. *On toric Kähler manifolds, the existence of a toric invariant cscK metric in the class* $[\omega_0]$ *is equivalent to the* L^1 *stability.*

With Theorem 7.2 in mind, one wonders if we can replace the L^1 stability condition by some algebraic conditions which can be checked relatively easily.

Appendix A

Our goal in this section is to prove the following result, which is used in the proof of Theorem 7.1.

Theorem A.1. *Let* $1 \le p < \infty$. *Let* ϕ_0, ϕ_0', ϕ_1, $\phi_1' \in \mathcal{E}^p$. *Denote* $\{\phi_{0,t}\}_{t\in[0,1]}$, $\{\phi_{1,t}\}_{t\in[0,1]}$ *be two finite energy geodesics, such that* $\phi_{0,t}$ *connects* ϕ_0 *and* ϕ_0', $\phi_{1,t}$ *connects* ϕ_1 *and* ϕ_1'. *Then we have*

$$d_p(\phi_{0,t},\phi_{1,t}) \le (1-t)d_p(\phi_0,\phi_1) + t d_p(\phi_0',\phi_1').$$

When $p=2$, this result follows from that $(\mathcal{E}^2, d_2)$ is NPC, proved in [37] (see also [14]). For general p, we were not able to prove $(\mathcal{E}^p, d_p)$ is NPC in the sense of Alexandrov. Nevertheless, above weaker result still holds.

In the following argument, we will mostly follow the notation in [14]. Let $\varphi(x,s,t) \in C^\infty(M\times[0,1]\times[0,1])$ be such that $\varphi(\cdot,s,t)\in\mathcal{H}$. Denote $X = \partial_t\varphi$, $Y = \partial_s\varphi$. Given $U \in C^\infty(M\times[0,1]\times[0,1])$, consider the connection first introduced by Mabuchi:

$$\nabla_X U = \partial_t U - \nabla_\varphi\partial_t\varphi\cdot_\varphi\nabla_\varphi U,\ \ \nabla_Y U = \partial_s U - \nabla_\varphi\partial_s\varphi\cdot_\varphi\nabla_\varphi U. \tag{A.1}$$

The dot product in the above equation has the following expression in local coordinates:

$$\nabla_\varphi u\cdot_\varphi\nabla_\varphi v = \frac{1}{2}g_\varphi^{i\bar{j}}\big(u_i v_{\bar{j}} + v_i u_{\bar{j}}\big).$$

Given ψ_1, $\psi_2 \in C^\infty(M)$, we denote

$$(\psi_1,\psi_2) = \int_M \psi_1\psi_2 dvol_\varphi.$$

This is the so-called Mabuchi's metric on $\mathcal{H}$.

Given φ_0, $\varphi_1\in\mathcal{H}$, and $\varepsilon>0$, one can consider the so-called ε-geodesic, introduced in [18]:

$$\begin{aligned}&\big(\partial_t^2\varphi - |\nabla_\varphi\partial_t\varphi|_\varphi^2\big)\det g_\varphi = \varepsilon\det g_0 \text{ for } (x,t)\in M\times[0,1]\\ &\varphi|_{t=0}=\varphi_0,\ \varphi|_{t=1}=\varphi_1.\end{aligned} \tag{A.2}$$

It is shown in [18] that (A.2) can be written as a complex Monge-Ampère equation on $M\times[0,1]$ with non-degenerate and smooth right hand side, hence is smooth.

The key to prove Theorem A.1 is the following estimate:

Proposition A.1. *Let $\varphi_i : s \in [0,1] \to \mathcal{H}$, $i = 0,1$ be two smooth curves in $\mathcal{H}$. Let $\chi : \mathbb{R} \to \mathbb{R}_+$ be smooth and convex. Suppose that for each $s \in [0,1]$, $[0,1] \ni t \mapsto \varphi_\varepsilon(s,t)$ is the ε-geodesic connecting $\varphi_0(s)$ and $\varphi_1(s)$. Denote $X = \partial_t\varphi$, $Y = \partial_s\varphi$, then we have*

$$\partial_t^2 \int_M \chi(\partial_s\varphi) dvol_\varphi \geq \int_M \chi''(\partial_s\varphi)(\nabla_X Y)^2 dvol_\varphi.$$

We will postpone the proof of this proposition later, and we will show next how to use this proposition to deduce Theorem A.1.

First we apply Proposition A.1 to obtain

Lemma A.2. *Let ϕ_0, ϕ_0', ϕ_1, $\phi_1' \in \mathcal{H}$. Let $c_1(s) : [0,1] \to \mathcal{H}$ be a smooth curve connecting ϕ_0 and ϕ_1, $c_2(s) : [0,1] \to \mathcal{H}$ be a smooth curve connecting ϕ_0' and ϕ_1'. Let $\{\varphi^\varepsilon(s,t)\}_{(s,t)\in[0,1]^2}$ be such that for each fixed s, $[0,1] \ni t \mapsto \varphi^\varepsilon(s,t)$ is the ε-geodesic connecting $c_1(s)$ with $c_2(s)$. Denote $L_p^\varepsilon(t)$ be the length of the curve $[0,1] \ni s \mapsto \varphi^\varepsilon(s,t) \in \mathcal{H}$ under the distance d_p, then $t \mapsto L_p^\varepsilon(t)$ is convex.*

Proof. In the following, we will write $L_p^\varepsilon(t)$ simply as $L_p(t)$. By definition, we have

$$L_p(t) = \int_0^1 \left(\int_M |\partial_s\varphi|^p dvol_\varphi \right)^{\frac{1}{p}} ds. \tag{A.3}$$

Denote $\chi_\delta(x) = (x^2 + \delta^2)^{\frac{p}{2}}$ and put $L_{p,\delta}(t) = \int_0^1 \left(\int_M \chi_\delta(\partial_s\varphi) dvol_\varphi \right)^{\frac{1}{p}} ds = \int_0^1 |Y|_{\chi_\delta}^{\frac{1}{p}} ds$. Here for simplicity, we use the notation: $|Y|_{\chi_\delta} = \int_M \chi_\delta(\partial_s\varphi) dvol_\varphi$. Then we have

$$\frac{d^2}{dt^2} L_{p,\delta}(t) = \int_0^1 \partial_t^2(|Y|_{\chi_\delta}^{\frac{1}{p}}) ds.$$

We claim that $\partial_t^2(|Y|_{\chi_\delta}^{\frac{1}{p}}) \geq 0$. If this were true, then we know $t \mapsto L_{p,\delta}(t)$ is convex. Also we know that $L_{p,\delta}(t) \to L_p(t)$ for each $t \in [0,1]$ as $\delta \to 0$. This will imply the desired result. Hence it only remains to verify the claim. We can compute

$$\begin{aligned} \partial_t^2\big(|Y|_{\chi_\delta}^{\frac{1}{p}}\big) &= \partial_t\big(\frac{1}{p}|Y|_{\chi_\delta}^{\frac{1}{p}-1}\partial_t(|Y|_{\chi_\delta})\big) \\ &= \frac{1}{p}|Y|_{\chi_\delta}^{\frac{1}{p}-1}\partial_t^2(|Y|_{\chi_\delta}) - \frac{1}{p}\big(1-\frac{1}{p}\big)|Y|_{\chi_\delta}^{\frac{1}{p}-2}|\partial_t(|Y|_{\chi_\delta})|^2 \\ &\geq \frac{1}{p}|Y|_{\chi_\delta}^{\frac{1}{p}-1}\left(\int_M \chi_\delta''(\partial_s\varphi)(\nabla_X Y)^2 dvol_\varphi - \big(1-\frac{1}{p}\big)|Y|_{\chi_\delta}^{-1}|\partial_t(|Y|_{\chi_\delta})|^2\right). \end{aligned} \tag{A.4}$$

In the last inequality, we used Proposition A.1.

On the other hand

(A.5)

$$\begin{aligned} \partial_t(|Y|_{\chi_\delta}) &= \int_M \big(\chi_\delta'(\partial_s\varphi)\partial_{st}\varphi + \chi_\delta(\partial_s\varphi)\Delta_\varphi(\partial_t\varphi)\big) dvol_\varphi \\ &= \int_M \chi_\delta'(\partial_s\varphi)\big(\partial_{st}\varphi - \nabla_\varphi\partial_s\varphi \cdot_\varphi \nabla_\varphi\partial_t\varphi\big) dvol_\varphi = \int_M \chi_\delta'(\partial_s\varphi)\nabla_X Y dvol_\varphi. \end{aligned}$$

Hence we may apply Cauchy-Schwarz inequality to get

$$|\partial_t(|Y|_{\chi_\delta})|^2 \leq \int_M \frac{(\chi_\delta'(\partial_s\varphi))^2}{\chi_\delta''(\partial_s\varphi)} dvol_\varphi \times \int_M \chi_\delta''(\partial_s\varphi)(\nabla_X Y)^2 dvol_\varphi. \tag{A.6}$$

It is straightforward to calculate

$$\chi'_\delta(x) = p(x^2+\delta^2)^{\frac{p}{2}-1}x.$$
$$\chi''_\delta(x) = p(p-2)(x^2+\delta^2)^{\frac{p}{2}-2}x^2 + p(x^2+\delta^2)^{\frac{p}{2}-1}.$$

Therefore

$$\begin{aligned}\chi''_\delta \chi_\delta &= p(p-2)(x^2+\delta^2)^{p-2}x^2 + p(x^2+\delta^2)^{p-1} \\ &\geq p(p-1)(x^2+\delta^2)^{p-2}x^2 = \frac{p-1}{p}(\chi'_\delta)^2.\end{aligned}$$

Hence we obtain from (A.6) that

(A.7)

$$\begin{aligned}(1-\frac{1}{p})|\partial_t(|Y|_{\chi_\delta})|^2 &\leq \int_M \frac{p-1}{p}\frac{(\chi'_\delta(\partial_s\varphi))^2}{\chi''_\delta(\partial_s\varphi)}dvol_\varphi \times \int_M \chi''_\delta(\partial_s\varphi)(\nabla_X Y)^2 dvol_\varphi \\ &\leq \int_M \chi_\delta(\partial_s\varphi)dvol_\varphi \times \int_M \chi''_\delta(\partial_s\varphi)(\nabla_X Y)^2 dvol_\varphi.\end{aligned}$$

Combining (A.4) and (A.7), the result follows. □

As a consequence, we have

Corollary A.3. *Let ϕ_0, ϕ'_0, ϕ_1, $\phi'_1 \in \mathcal{H}$. Let $\{\rho_0(t)\}_{t\in[0,1]}$ be the $C^{1,1}$ geodesic connecting ϕ_0 and ϕ'_0, and $\{\rho_1(t)\}_{t\in[0,1]}$ be the $C^{1,1}$ geodesic connecting ϕ_1 and ϕ'_1. Then we have*

$$d_p(\rho_0(t),\rho_1(t)) \leq (1-t)d_p(\phi_0,\phi_1) + td_p(\phi'_0,\phi'_1), \text{ for any } t\in[0,1].$$

Proof. Let $\varepsilon > 0$. Let $c_1^\varepsilon(s) : [0,1] \to \mathcal{H}$ be the ε-geodesic connecting ϕ_0 and ϕ_1, $c_2^\varepsilon(s) : [0,1] \to \mathcal{H}$ be the ε-geodesic connecting ϕ'_0 and ϕ'_1. Then define $\{\varphi^\varepsilon(s,t)\}_{(s,t)\in[0,1]^2}$ such that for each fixed s, $t \mapsto \varphi^\varepsilon(s,t)$ is the ε-geodesic connecting $c_1^\varepsilon(s)$, $c_2^\varepsilon(s)$.

We can apply the previous lemma to conclude that

$$d_p(\varphi^\varepsilon(0,t),\varphi^\varepsilon(1,t)) \leq L_p^\varepsilon(t) \leq (1-t)L_p^\varepsilon(0) + tL_p^\varepsilon(1). \tag{A.8}$$

Then we let $\varepsilon \to 0$. Since $t \mapsto \varphi^\varepsilon(0,t)$ is the ε-geodesic connecting ϕ_0, ϕ'_0, we have $\varphi^\varepsilon(0,t) \to \rho_0(t)$ uniformly (c.f. [18, Lemma 7, point 3]), hence in d_p distance, for each fixed t, as $\varepsilon \to 0$. Similarly, $\varphi^\varepsilon(1,t) \to \rho_1(t)$ in d_p. Therefore,

$$d_p(\varphi^\varepsilon(0,t),\varphi^\varepsilon(1,t)) \to d_p(\rho_0(t),\rho_1(t)), \text{ as } \varepsilon \to 0.$$

While $L_p^\varepsilon(0)$ is the length of c_1^ε, hence $L_p^\varepsilon(0) \to d_p(\phi_0,\phi_1)$ as $\varepsilon \to 0$. Similarly $L_p^\varepsilon(1) \to d_p(\phi'_0,\phi'_1)$. □

Now we are ready to prove Theorem A.1, via an approximating argument.

Proof of Theorem A.1. We choose smooth approximations of ϕ_0, ϕ'_0, ϕ_1, ϕ'_1. Namely we choose $\phi_{0,k} \to \phi_0$, $\phi'_{0,k} \to \phi'_0$, $\phi_{1,k} \to \phi_1$, $\phi'_{1,k} \to \phi'_1$ as $k \to \infty$ under distance d_p. Then from previous corollary, we know

$$d_p(\phi_{0,k}(t),\phi_{1,k}(t)) \leq (1-t)d_p(\phi_{0,k},\phi_{1,k}) + td_p(\phi'_{0,k},\phi'_{1,k}), \text{ for any } t\in[0,1]. \tag{A.9}$$

In the above, $\{\phi_{0,k}(t)\}_{t\in[0,1]}$ is the $C^{1,1}$ geodesic connecting $\phi_{0,k}$, $\phi'_{0,k}$ and $\{\phi_{1,k}(t)\}_{t\in[0,1]}$ is the $C^{1,1}$ geodesic connecting $\phi_{1,k}$, $\phi'_{1,k}$.

From the end point stability of finite energy geodesic segment (c.f. [8, Proposition 4.3]), we know that $\phi_{0,k}(t) \to \phi_{0,t}$ in d_p as $k \to \infty$, and $\phi_{1,k}(t) \to \phi_{1,t}$ as $k \to \infty$. Taking limit as $k \to \infty$ in (A.9), the result follows. □

It only remains to show Proposition A.1.

Proof. For simplicity, we denote $|Y|_\chi = \int_M \chi(\partial_s\varphi)dvol_\varphi$. Then we may calculate:

$$\begin{aligned}\partial_t(|Y|_\chi) &= \int_M \big(\chi'(\partial_s\varphi)\partial_{st}\varphi + \chi(\partial_s\varphi)\Delta_\varphi(\partial_t\varphi)\big)dvol_\varphi \\ &= \int_M \chi'(\partial_s\varphi)\big(\partial_{st}\varphi - \nabla_\varphi\partial_s\varphi\cdot_\varphi\nabla_\varphi\partial_t\varphi\big)dvol_\varphi = (\chi'(\partial_s\varphi),\nabla_Y X).\end{aligned}\tag{A.10}$$

Differentiate in t once more, we have

$$\begin{aligned}\partial_t^2(|Y|_\chi) &= \int_M \chi''(\partial_s\varphi)\partial_{st}\varphi\nabla_Y X dvol_\varphi + \int_M \chi'(\partial_s\varphi)\partial_t(\nabla_Y X)dvol_\varphi \\ &+ \int_M \chi'(\partial_s\varphi)\nabla_Y X\Delta_\varphi(\partial_t\varphi)dvol_\varphi \\ &= \int_M \chi''(\partial_s\varphi)(\nabla_Y X)^2 dvol_\varphi + (\chi'(\partial_s\varphi),\nabla_X\nabla_Y X) \\ &= \int_M \chi''(\partial_s\varphi)(\nabla_Y X)^2 dvol_\varphi + (\chi'(\partial_s\varphi),\nabla_Y\nabla_X X) \\ &\qquad + (\chi'(\partial_s\varphi),\nabla_X\nabla_Y X - \nabla_Y\nabla_X X).\end{aligned}\tag{A.11}$$

Since $t\mapsto\varphi^\varepsilon(s,t)$ is an ε-geodesic, we have $\nabla_X X = \varepsilon H$, where $H = \frac{\det g_0}{\det g_\varphi}$. Hence

$$\begin{aligned}(\chi'(\partial_s\varphi),\nabla_Y\nabla_X X) &= \int_M \chi'(\partial_s\varphi)\varepsilon\nabla_Y H dvol_\varphi \\ &= \int_M \varepsilon\chi'(\partial_s\varphi)(\partial_s H - \nabla_\varphi\partial_s\varphi\cdot_\varphi\nabla_\varphi H)dvol_\varphi \\ &= \int_M \varepsilon\chi'(\partial_s\varphi)(-H\Delta_\varphi(\partial_s\varphi) - \nabla_\varphi\partial_s\varphi\cdot_\varphi\nabla_\varphi H)dvol_\varphi \\ &= \int_M \varepsilon\chi''(\partial_s\varphi)H|\nabla_\varphi\partial_s\varphi|_\varphi^2 dvol_\varphi \ge 0.\end{aligned}\tag{A.12}$$

From third line to the last line above, we integrated by parts. Hence it only remains to handle the term $(\chi'(\partial_s\varphi),\nabla_X\nabla_Y X - \nabla_Y\nabla_X X)$. Lemma A.13 shows this term is ≥ 0, so we are done. ∎

Lemma A.4.

(A.13)

$$\begin{aligned}(\chi'(\partial_s\varphi),\nabla_Y\nabla_X X - \nabla_X\nabla_Y X) &= \int_M \frac{1}{4}\chi''(\partial_s\varphi)g_\varphi^{i\bar{j}}\Big((\partial_t\varphi)_i(\partial_s\varphi)_{\bar{j}} - (\partial_s\varphi)_i(\partial_t\varphi)_{\bar{j}}\Big) \\ \times g_\varphi^{p\bar{q}}\Big((\partial_t\varphi)_p(\partial_s\varphi)_{\bar{q}} - (\partial_t\varphi)_{\bar{q}}(\partial_s\varphi)_p\Big)dvol_\varphi &= -\int_M \chi''(\partial_s\varphi)\big(\{\partial_t\varphi,\partial_s\varphi\}\big)^2 dvol_\varphi.\end{aligned}$$

In the above, $\{\cdot,\cdot\}$ is the Poisson product, defined as

$$\{f,g\}_\varphi := Im\big(g_\varphi^{i\bar{j}}f_i g_{\bar{j}}\big),\ f,\, g\in C^\infty(M),\ \varphi\in\mathcal{H}.$$

In particular, if $\chi''\ge 0$, the expression in (A.13) ≤ 0.

When $\chi(x) = \frac{1}{2}x^2$, this lemma just expresses the well-known fact that $\mathcal{H}$ has nonpositive sectional curvature under Mabuchi metric.

Proof. We know that the curvature operator can be represented in terms of Poisson product:

$$R_\varphi(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z = \{\{X,Y\},Z\},\ X,\,Y,\,Z\in C^\infty(M),\ \varphi\in\mathcal{H}.$$

Therefore,

$$\begin{aligned}
&\big(\chi'(\partial_s\varphi), \nabla_Y\nabla_X X - \nabla_X\nabla_Y X\big) = -\int_M \chi'(\partial_s\varphi) R_\varphi(\partial_t\varphi,\partial_s\varphi)\partial_t\varphi dvol_\varphi\\
&= -\int_M \chi'(\partial_s\varphi)\{\{\partial_t\varphi,\partial_s\varphi\},\partial_t\varphi\} dvol_\varphi\\
&= -\int_M Im\Big(\chi'(\partial_s\varphi) g_\varphi^{i\bar j}\{\partial_t\varphi,\partial_s\varphi\}_i(\partial_t\varphi)_{\bar j}\Big) dvol_\varphi\\
&= \int_M Im\Big(\chi'(\partial_s\varphi) g_\varphi^{i\bar j}\{\partial_t\varphi,\partial_s\varphi\}(\partial_t\varphi)_{i\bar j}\Big) dvol_\varphi\\
&+ \int_M Im\Big(\chi''(\partial_s\varphi) g_\varphi^{i\bar j}(\partial_s\varphi)_i\{\partial_t\varphi,\partial_s\varphi\}(\partial_t\varphi)_{\bar j}\Big) dvol_\varphi\\
&= -\int_M \chi''(\partial_s\varphi)\Big(\{\partial_t\varphi,\partial_s\varphi\}\Big)^2 dvol_\varphi.
\end{aligned}\tag{A.14}$$

From the third line to fourth line above, we integrated by parts. Also we noticed that $g_\varphi^{i\bar j}(\partial_t\varphi)_{i\bar j} = \Delta_\varphi(\partial_t\varphi)$ is real. □

As an immediate consequence of Theorem A.1, we have

Corollary A.5. *Let $\rho_i : [0,\infty) \to \mathcal{E}_0^p$, $i = 1, 2$ be two locally finite energy geodesic rays, then the function $t \mapsto d_p(\rho_1(t), \rho_2(t))$ is convex on $[0,\infty)$.*

As a consequence of this corollary and elementary properties of convex functions on $[0,\infty)$, we can conclude

Corollary A.6. *Let $\rho_i : [0,\infty) \to \mathcal{E}_0^p$, $i = 1, 2$ be two locally finite energy geodesic rays. Then exactly one of the two alternative holds:*

1. *The limit $\lim_{t\to\infty} \frac{d_p(\rho_1(t),\rho_2(t))}{t}$ exists and is positive (may be $+\infty$);*
2. *$t \mapsto d_p(\rho_1(t), \rho_2(t))$ is decreasing. In particular, $d_p(\rho_1(t), \rho_2(t)) \le d_p(\rho_1(0), \rho_2(0))$ for any $t > 0$.*

The rest of this section is devoted to proving Theorem 1.4. First the uniqueness of such a geodesic ray ρ_2 parallel to ρ_1 initiating from φ follows immediately from Corollary A.6. The existence part is given by Lemma 7.6. Here we need the assumption $¥[\rho_1] < \infty$ to show that for each fixed t, $K(r_k(t))$ is uniformly bounded from above when k is sufficiently large (by convexity of K-energy), and then we can use the compactness result of [8, Corollary 4.8] to conclude the convergence of $\{r_k(t)\}_k$ up to a subsequence.

It only remains to check that ¥ invariants are equal for two parallel locally finite energy geodesic rays.

Proposition A.7. *Suppose $\rho_i : [0,\infty) \to \mathcal{E}_0^p$, $i = 1, 2$ are two parallel geodesic rays with unit speed, then we have $¥[\rho_1] = ¥[\rho_2]$.*

Proof. It is clear that we just need to show $¥[\rho_1] \le ¥[\rho_2]$. The reverse inequality can be obtained by reversing the role of ρ_1 and ρ_2. Also we may assume that $¥[\rho_2] < \infty$, otherwise there is nothing to prove.

Choose $t_k \nearrow \infty$, and let $r_k : [0, d_1(\rho_1(0), \rho_2(t_k))] \to \mathcal{E}_0^1$ be the unit speed geodesic segment connecting $\rho_1(0)$ and $\rho_2(t_k)$ (with $t = 0$ corresponding to $\rho_1(0)$). Let $t \in [0, d_1(\rho_1(0), \rho_2(t_k))]$, we know from the convexity of K-energy:

$$
\begin{aligned}
K(r_k(t)) &\leq \big(1 - \frac{t}{d_1(\rho_1(0), \rho_2(t_k))}\big)K(\rho_1(0)) + \frac{t}{d_1(\rho_1(0), \rho_2(t_k))}K(\rho_2(t_k)) \\
&\leq \big(1 - \frac{t}{d_1(\rho_1(0), \rho_2(t_k))}\big)K(\rho_1(0)) + \frac{t}{t_k - d_1(\rho_1(0), \rho_2(0))}K(\rho_2(t_k)).
\end{aligned}
\tag{A.15}
$$

In the second inequality, we used

$$
d_1(\rho_1(0), \rho_2(t_k)) \geq d_1(\rho_2(0), \rho_2(t_k)) - d_1(\rho_1(0), \rho_2(0)) = t_k - d_1(\rho_1(0), \rho_2(0)).
$$

Hence

$$
\frac{K(r_k(t))}{t} \leq \left(\frac{1}{t} - \frac{1}{d_1(\rho_1(0), \rho_2(t_k))}\right)K(\rho_1(0)) + \frac{t_k}{t_k - d_1(\rho_1(0), \rho_2(0))}a. \tag{A.16}
$$

Next we make the following claim

Claim A.8. $r_k(t) \to \rho_1(t)$, for fixed $t \geq 0$ in d_1 distance as $k \to \infty$.

Assuming this claim for the moment, we can fix t, and take limit in (A.16) as $k \to \infty$, and use lower semicontinuity of K-energy to get:

$$
\frac{K(\rho_1(t))}{t} \leq \liminf_k \frac{K(r_k(t))}{t} \leq \frac{K(\rho_1(0))}{t} + a, \text{ for any } t > 0. \tag{A.17}
$$

Then we take limit as $t \to \infty$, and conclude $¥[\rho_1] \leq a$.

Now it only remains to show the claim. We define the reparametrization: for $\tau \in [0,1]$, $\tilde{r}_k(\tau) = r_k\big(\tau d_1(\rho_1(0), \rho_2(t_k))\big)$, $\tilde{\rho}_1(\tau) = \rho_1\big(\tau d_1(\rho_1(0), \rho_2(t_k))\big)$. Then we may use Theorem A.1 to conclude (here $s_k = d_1(\rho_1(0), \rho_2(t_k))$)

$$
d_1(\tilde{r}_k(\tau), \tilde{\rho}_1(\tau)) \leq \tau d_1(\rho_2(t_k), \rho_1(s_k)), \text{ for any } \tau \in [0,1]. \tag{A.18}
$$

Then choose $\tau = \frac{t}{s_k}$, we have

$$
\begin{aligned}
d_1(r_k(t), \rho_1(t)) &\leq \frac{t}{s_k} d_1(\rho_2(t_k), \rho_1(s_k)) \\
&\leq \frac{t}{s_k}\big(d_1(\rho_2(t_k), \rho_1(t_k)) + d_1(\rho_1(t_k), \rho_1(s_k))\big) \\
&\leq \frac{t}{s_k}\big(\sup_t d_1(\rho_2(t), \rho_1(t)) + |t_k - s_k|\big) \\
&\leq \frac{t}{s_k}\big(\sup_t d_1(\rho_2(t), \rho_1(t)) + d_1(\rho_1(0), \rho_2(0))\big).
\end{aligned}
\tag{A.19}
$$

In the second inequality, we used triangle inequality.

In the third inequality, we used that ρ_1 is a unit speed geodesic ray.

In the last inequality, we used triangle inequality again to conclude

$$
|t_k - s_k| = |d_1(\rho_2(0), \rho_2(t_k)) - d_1(\rho_1(0), \rho_2(t_k))| \leq d_1(\rho_1(0), \rho_2(0)).
$$

Finally we let $k \to \infty$ in (A.19) to see the claim. □

Next we observe that ¥-invariant has the following "lower semicontinuity" property.

Proposition A.9. *Let ρ_k, $\rho : [0,\infty) \to \mathcal{E}_0^p$ be locally finite energy geodesic rays with unit speed. Define $d_k = \lim_{t\to\infty} \frac{d_p(\rho_k(t),\rho(t))}{t}$ (This is well-defined according to Corollary* A.6*). Suppose that $d_k \to 0$ and $d_p(\rho_k(0),\rho(0)) \to 0$ as $k\to\infty$, then $¥[\rho] \leq \liminf_{k\to\infty} ¥[\rho_k]$.*

Proof. Observe that for any $s>0$, we have $d_p(\rho_k(s),\rho(s)) \to 0$. Indeed, from the convexity property of $t \mapsto d_p(\rho_k(t),\rho(t))$ obtained in Corollary A.5, we know that for any $s' > s > 0$, and any k

$$\frac{d_p(\rho_k(s),\rho(s)) - d_p(\rho_k(0),\rho(0))}{s} \leq \frac{d_p(\rho_k(s'),\rho(s')) - d_p(\rho_k(0),\rho(0))}{s'}.$$

Let $s' \to \infty$, we know that

$$\frac{d_p(\rho_k(s),\rho(s))}{s} \leq d_k + \frac{d_p(\rho_k(0),\rho(0))}{s} \to 0, \text{ as } k\to\infty \text{ by assumption.} \tag{A.20}$$

Hence from the lower semicontinuity with respect to d_p convergence, we can conclude that

$$\frac{K(\rho(s))}{s} \leq \liminf_{k\to\infty} \frac{K(\rho_k(s))}{s}, \text{ for any } s>0. \tag{A.21}$$

On the other hand, from the convexity of K-energy along ρ_k, it follows that for any $s'' > s > 0$,

$$\frac{K(\rho_k(s))}{s} \leq \frac{K(\rho_k(s''))}{s''} + \left(\frac{1}{s} - \frac{1}{s''}\right) K(\rho_k(0)). \tag{A.22}$$

Let $s'' \to \infty$ in the above and use the definition of ¥-invariant, we conclude

$$\frac{K(\rho_k(s))}{s} \leq ¥[\rho_k] + \frac{K(\rho_k(0))}{s}, \text{ for any } s>0. \tag{A.23}$$

Finally we let $k\to\infty$ in (A.23) and combine (A.21), we see

$$\frac{K(\rho(s))}{s} \leq \liminf_k \frac{K(\rho_k(s))}{s} \leq \liminf_k ¥[\rho_k] + \frac{K(\rho(0))}{s}, \text{ for any } s>0. \tag{A.24}$$

Finally we let $s\to\infty$ in (A.24) to conclude the proof. ☐

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