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ON THE CONSTRUCTION OF CONTACT SUBMANIFOLDS WITH PRESCRIBED TOPOLOGY

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Abstract

We prove the existence of contact submanifolds realizing the Poincaré dual of the top Chern class of a complex vector bundle over a closed contact manifold. This result is analogue in the contact category to Donaldson's construction of symplectic submanifolds. The main tool in the construction is to show the existence of sequences of sections which are asymptotically holomorphic in an appropiate sense and that satisfy a transversality with estimates property directly in the contact category. The description of the obtained contact submanifolds allows us to prove an extension of the Lefschetz hyperplane theorem which completes their topological characterization.

1. Introduction and statement of the main results

Recently, S. Donaldson has proved the existence of symplectic submanifolds that realize the Poincaré dual of a large enough integer multiple of the canonical cohomology class defined by the symplectic structure of a given closed symplectic manifold [4]. The main idea in Donaldson's theorem is to construct symplectic submanifolds as zero sets of appropriate sections of powers of the prequantizable line bundle L over the symplectic manifold M. Later on, D. Auroux and R. Paoletti have proved an extension of Donaldson's theorem, where now a wider family of symplectic submanifolds are constructed as the zero sets of asymptotically holomorphic sections of vector bundles. These bundles are obtained by tensoring an arbitrary complex bundle with large powers of the prequantizable line bundle L [2], [3], [14]. In his paper, D. Auroux also shows that, asymptotically, all submanifolds constructed from a given vector bundle E are isotopic for k large enough. Paoletti

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has also shown that the classical special position theorems adapt to the symplectic category [15]. Moreover, Donaldson's techniques can provide a nearly holomorphic embedding of symplectic manifolds in $\mathbb{CP}^N$ [13]. These results open new directions of research on symplectic geometry and have a wide range of applications (see for instance the review paper [5]). The key idea to understand these works is the concept of ampleness of a complex holomorphic bundle. It allows the flexibilization of the bundles in the holomorphic category by means of increasing their curvatures. Ampleness has been exploited extensively in the Kähler setting. Donaldson in his outbreaking work [4] has translated the definition of ampleness to the symplectic category. The most important idea in this work is the definition of asymptotic holomorphicity for sequences of sections of bundles which are more and more twisted.

In this paper we extend the previous ideas to the contact category. Recall that a contact manifold is an odd dimensional manifold C together with a completely nonintegrable hyperplane distribution D on it. Such a distribution will be called a contact distribution. Not very much is known about the topology of contact manifolds (see the reviews by Y. Eliashberg [8, 6], [12] and references therein). One important tool to obtain general results has been the use of holomorphic methods. The theory of pseudoholomorphic curves—and disks—can be developed in the contact setting (see the recent work on this direction by Y. Eliashberg, H. Hofer and D. Salamon [7], [10]). However, there were not more holomorphic techniques in the contact setting. Even though Donaldson's construction in the symplectic category was a triumph of ideas partly inspired by Kodaira's embedding theorem, the basic tools in the construction of asymptotically holomorphic sections actually used in the proof of the theorem were based in simple methods of local conformal geometry of a symplectic manifold, a subtle transversality theorem with estimates and a globalization process. We will show in this paper how these ideas can be refined to extend Donaldson's construction to the contact category.

A contact submanifold of a contact manifold (C, D) is a submanifold N such that $TN \cap D$ is a contact distribution on N. We will show that, analogously to the symplectic situation, we can construct contact submanifolds as zero sets of sections of complex bundles over the contact manifold. These submanifolds will in general be homologically trivial because of the triviality of the cohomology class defined by the contact structure. However we can adapt to the contact setting a refined version of Donaldson's theorem proved by D. Auroux and R. Paoletti that will allow us to construct contact submanifolds realizing the Poincaré dual of the top Chern class of any vector bundle over the contact manifold. More precisely, we will prove:

Theorem 1. *Let C be a compact contact manifold of dimension $2n+1$ and $E \to C$ a rank r complex bundle over C $(r \leq n)$ with top Chern class $c_r(E)$. Then, there exists a contact submanifold W of C realizing the Poincaré dual of $c_r(E)$ on $H_{2n+1-2r}(C,\mathbb{Z})$. Moreover, the inclusion $i\colon W \to C$ induces an isomorphism on homotopy groups π_p for $p \leq n-r-1$ and a surjection on π_{n-r} (resp. on homology groups).*

Up to now the only results similar to Theorem 1 have been obtained in dimension 3 where a special class of contact submanifolds (in fact, curves) have been constructed for some contact structures. These contact curves the closed orbits of the Reeb vector field. The Weinstein conjecture asserts that such closed orbits always exist; it has been proved in some partial cases [10]. The only general tool to construct submanifolds in general dimension is the Gromov's h-principle, but this does not give a general method to decide when is possible to construct a contact submanifold [9]. Moreover his techniques only apply in codimension greater than two.

Theorem 1 imposes two conditions. One purely topological is the transversality of the submanifolds with respect to the contact structure. The second one is geometric and asserts the nonintegrability of the induced hyperplane structure. Even the topological one is not trivial, because it is not an easy problem to build a submanifold transverse to a hyperplane distribution. In fact, it cannot be solved by a local perturbation of a given manifold.

We will leave the discussion of the relationship between the symplectic and the contact construction for a forthcoming paper. This relationship offers important aspects, for instance in dimension three it would connect our construction with the results on pseudoholomorphic curves in symplectizations obtained in [10, 7].

1.1 Strategy of the proof and the contents of the paper

We will reproduce the main results of Donaldson-Auroux theory directly in the context of exact contact manifolds. For that we will use a characterization of contact submanifolds obtained by using the almost complex geometry of the contact distribution, this is, the contact analogous of the ∂ and $\bar{\partial}$ operators, obtained by projecting the $(1,0)$ and

$(0,1)$ components of the exterior differential to the symplectic bundle defined by the contact distribution D. Then, we will show how to construct a sequence of sections of a rank r complex bundle $E \to C$ that satisfy this characterization asymptotically. Notice that, contrary to the situation in the symplectic category, the line bundle $L \to C$ defined by an exact contact structure with the condition $\operatorname{curv}(L) = i\omega$ is trivial because the cohomology class of $\omega = d\theta$ vanishes. However, the symplectic bundle $D \to C$ defined by the contact structure, carries a conformal class of symplectic structures. Then we will consider the family of symplectic structures on D given by $k\omega$ where $k \in \mathbb{Z}^+$ and ω is as always the restriction of the presymplectic 2-form $d\theta$ to D. This family of symplectic bundle structures will replace the line bundles $L^{\otimes k} \to C$ in Donaldson-Auroux theory.

The main results in Donaldson-Auroux theory adapt immediately to this situation provided that we use a parametrized transversality with estimates theorem which constitutes a generalization of the transversality with estimates argument used in Donaldson's and Auroux' papers (see Section 4.2), and that follows the proof of [2]. The key idea is to approximate locally the contact distribution by a 1-parametric family of symplectic submanifolds, and then a generalization of Auroux' works gives us the result. Then we will prove the existence of contact submanifolds realizing the Poincaré dual of $c_r(E)$.

The result also applies to nonexact contact submanifolds. We need only to develope a $\mathbb{Z}_2$-invariant theory, because every contact manifold can be double covered by an exact contact manifold. This will be developed in Subsection 4.4.

From the topological point of view the obtained submanifolds verify a generalization of the Lefschetz hyperplane theorem. This will be proved in Section 5 and constitutes the second half of Theorem 1. The proof is based in the same Morse theory argument that Donaldson and Auroux have used to obtain their results in the symplectic category.

The conclusion of the main theorem, Theorem 1, is quite striking because it shows that there are contact codimension 2 submanifolds which determine strongly the homotopy type of the initial manifold. Also, these submanifolds do exist in a lot of homology classes of the manifold, always including the trivial one.

Finally, Section 2 is devoted to state some notation and preliminary facts on contact manifolds that will be used elsewhere along the paper, and Section 3 is devoted to introduce and discuss the fundamental notion of sequences of asymptotically holomorphic sections, both in the symplectic and in the contact category, and the notion of transversality with estimates, a notion that will play a central role in the proof of the main results of this paper.

2. Preliminary notions on contact manifolds

2.1 Almost complex structures on contact manifolds

As it was indicated in the introduction a contact manifold is a pair (C, D) where D is a completely nonintegrable hyperplane distribution on C. Locally, such distribution is defined by the kernel of a 1-form θ, this is, $D_x = \operatorname{Ker}\theta(x)$ and, the locally defined 2-form, $d\theta$, is nondegenerate when restricted to D. We observe that if we replace the 1-form θ by $\theta' = f\theta$ with f a nonvanishing function, the corresponding 2-form changes as $d\theta' = df \wedge \theta + f d\theta$. Thus restricted to D, $d\theta'$ and $d\theta$ define the same conformal symplectic structure. In this sense a contact manifold (C, D) carries a canonical conformal symplectic bundle $D \to C$ provided by the hyperplane distribution together with its conformal symplectic structure. Conversely, a conformal symplectic structure defined on a hyperplane distribution D, such that there exists a local potential θ for the local symplectic structure with $\operatorname{Ker}\theta = D$, defines a contact structure.

We will say that the contact structure on C is exact if there exists a globally defined 1-form θ defining the contact distribution D. In such case a global symplectic structure $d\theta$ can be fixed in the conformal class, and the bundle $D \to C$ becomes a symplectic bundle.

An exact contact structure will define a nonsingular vector field R, the Reeb field, by means of $i_R\theta = 1$, $i_R d\theta = 0$. Hence, we have a natural splitting of the tangent bundle $TC = D \oplus \langle R \rangle$ (this is also true in the nonexact case, but $\langle R \rangle$ is not trivial in that case.) We will say that a 1-form α is horizontal if $i_R\alpha = 0$ and vertical otherwise. Notice that a contact form θ is vertical. In fact, any 1-form uniquely decomposes as $\alpha = \alpha_D + a\theta$, with α_D horizontal and $a = i_R\alpha$. The canonical splitting of the tangent bundle will induce a natural decomposition of the cotangent bundle $T^*C = D^* \oplus \langle \theta \rangle$. Notice that changing the contact 1-form θ will make R to change, and the corresponding splittings will change too. The splitting of the tangent bundle TC allows to fix a metric g compatible with the contact structure of the form $g = g_D \oplus g_R$ where g_R is a metric on the 1 dimensional real bundle generated by R

and such that $g_R(R, R) = 1$, and g_D is a metric on D compatible with $d\theta_{|D}$ in the sense that there exists an almost complex structure J in D such that $g_D = d\theta(\cdot, J\cdot)$. Thus, $\langle R\rangle^{\perp} = D$ and,

$$g = g_D + \theta \otimes \theta \tag{2.1}$$

We will say that such g is a contact metric.

2.2 Contact submanifolds

Let $i\colon N \to C$ be a submanifold of the contact manifold (C, D). We will say that N is transverse to the contact structure if i is transverse to the contact distribution D, i.e., if $i_*(TN)$ is transverse to D. The submanifold N is transverse to the contact structure if and only if $i_*(TN)$ is not contained in the contact distribution D at any point of $i(N)$.

Definition 1. A contact submanifold of the contact manifold (C, D_C) is a triple (N, D_N, i) where (N, D_N) is a contact manifold and $i\colon N \to C$ is an embedding such that $D_N = i_*^{-1}(D_C)$.

Let $i\colon N \to C$ be a submanifold of the contact manifold (C, D). Then the set

$$D_N = \{\, u \in TN \mid i_*(u) \in D \,\} = i_*^{-1}(D) \tag{2.2}$$

will define a distribution of codimension 1 on N if the map i is transverse to D. Moreover by construction $i_*(D_N) \subset D_C$. Clearly, if θ_C is a local contact form for the distribution D_C, then $\theta_N = i^*\theta_C$ will define D_N, i.e., $\operatorname{Ker}\theta_N = D_N$. However the pair (N, D_N) will not be in general a contact manifold because the restriction to D_N of $d\theta_N$ could be degenerate. So, to obtain a contact submanifold we need two conditions: transversality to the distribution and nondegeneracy of the induced distribution.

If on the other hand, $i\colon N \to C$ is a submanifold and (N, D_N, i) is a contact submanifold with local form θ_N obtained as restriction of a local form θ_C of C, then i must be transverse to D_C because if this were not the case, at some point x, $i_*(T_xN) \subset D_C(x)$ and $\operatorname{Ker}\theta_C(x) = T_xN \neq \operatorname{Ker}\theta_N(x)$.

We will identify in what follows a submanifold N with its image $i(N)$ on C and will omit the map i in the discussions if there is no risk of confussion. Later, we will need the following fact:

Lemma 1. *Let N be a $(2r+1)$-dimensional submanifold of the contact manifold (C, D). Then N will be a contact submanifold of C if and only if $TN \cap D$ is a conformal symplectic subbundle of D of rank $2r$.*

Proof. We have seen already that N must be transverse to the contact distribution D and that if θ is a local contact form for D, then the 1-form $\theta_N = i^*\theta$ generates the distribution $D_N = TN \bigcap D$ on N where $i \colon N \to C$ denotes the submanifold embedding. Thus, the conformal symplectic structure of D_N will be defined locally by the symplectic 2-form $\omega_N = d\theta_N = i^* d\theta = i^*\omega$ and D_N is locally a symplectic subbundle of D, hence a conformal symplectic subbundle of D of rank $2r$.

Conversely, if $D_N = TN \bigcap D$ is a subbundle of rank $2r$, then N is transverse to D. Consider the local 1-form $\theta_N = i^*\theta$ on N, where θ is a local contact form defining locally the contact distribution D. Clearly, $D_N \subset \operatorname{Ker} \theta_N$, but since D_N defines a hyperplane distribution, $D_N = \operatorname{Ker} \theta_N$. Moreover as D_N is a conformal symplectic subbundle, then $d\theta_N$ is nondegenerate on D_N, hence, D_N is a contact distribution. q.e.d.

2.3 Bundles on contact manifolds

As we pointed out in the introduction, the basic tool to construct symplectic submanifolds is the use of sections of an appropriate line bundle over the given symplectic manifold. For symplectic forms of integer class, the bundle we choose is the line bundle whose curvature is given by the symplectic form itself. In the exact contact manifold case, the class defined by the contact structure is trivial. Thus we shall be considering the trivial line bundle $L = C \times \mathbb{C}$ over C for the discussion to follow. For exact contact manifolds (C, θ) the bundle L comes equipped with a connection ∇_L defined by the contact form itself, namely,

$$\nabla_L s = ds - i\theta s,$$

for any section s of the bundle. The tensor powers $L^{\otimes k}$ of L continue to be trivial but the connections $\nabla_{L^{\otimes k}}$, which they are equipped with, are defined by the 1-forms $ik\theta$, that still continue to define the same contact distribution $D = \operatorname{Ker} k\theta \subset TC$. However the corresponding symplectic structures ω_k on D are also rescaled as $\omega_k = k\omega$. The Reeb vector fields R_k are thus given by $R_k = k^{-1}R$ and we define a metric g_k as

$$g_k = k g_D + k\theta \otimes \theta. \tag{2.3}$$

We will call g_k the rescaled contact metric. Notice that $|R_k|_{g_k} = k^{-1/2}$ for this metric, and so it is not a contact metric.

We shall consider in what follows the family of (trivial) line bundles $L^{\otimes k}$ with the hermitian connections $\nabla_{L^{\otimes k}}$ induced on them by the connection ∇_L, which have curvature $-ik\omega$.

We shall consider in addition a rank r hermitian vector bundle $E \to C$ with hermitian connection ∇_E and the tensor products $L^{\otimes k} \otimes E$ equipped with the connection induced by $\nabla_{L^{\otimes k}}$ and ∇_E with curvature

$$R = I \otimes R_E - ik\omega \otimes I, \tag{2.4}$$

where R_E denotes the curvature of ∇_E.

2.4 Contact submanifolds as zero sets of sections of complex bundles

In this section we will take profit of the previous discussion to mimic Donaldson theory in the setting of exact contact manifolds.

Let (C, θ) be a closed exact contact manifold and s a smooth section of the trivial complex line bundle L over C transverse to the zero section of L. The level set $W = s^{-1}(\mathbf{0})$ either will be empty or a smooth real codimension 2 submanifold of C. In the latter case, the tangent space of this submanifold at a given point x will be $\operatorname{Ker} \nabla s(x)$. As we noticed in Section 2.1, there is a natural splitting $T^*C = D^* \oplus \langle\theta\rangle$, and any 1-form α decomposes uniquely as $\alpha = \alpha_D + (i_R\alpha)\theta$. Thus if $u \in T_xC$ denotes a tangent vector to C at x, then $u = u_D + u^\perp$, where $u_D \in D_x$ and $u^\perp = (i_u\theta)R$ is the orthogonal vector to u_D, then $\alpha_D(u) = \alpha(u_D)$. We will denote by α_D either the form so defined and also the restriction of α to the subspace D. Finally, choosing an almost complex structure J on D compatible with the symplectic structure, we can further decompose the complex valued 1-forms on D, $\Lambda^1(D, \mathbb{C})$, in its holomorphic and antiholomorphic components,

$$\Lambda^1(D, \mathbb{C}) = \Lambda^{(1,0)}(D) \oplus \Lambda^{(0,1)}(D),$$

as

$$\alpha_D = \alpha^{(1,0)} + \alpha^{(0,1)},$$

where

$$\alpha^{(1,0)} = \frac{1}{2}(\alpha_D - i\alpha_D \circ J), \qquad \alpha^{(0,1)} = \frac{1}{2}(\alpha_D + i\alpha_D \circ J).$$

Similarly, we can proceed to decompose the space of L-valued 1-forms $\Lambda^1(TC, L)$ and the L-valued 1-forms on D,

$$\Lambda^1(D, L) = \Lambda^{(1,0)}(D, L) \oplus \Lambda^{(0,1)}(D, L).$$

These elementary considerations, lead us to decompose the L-valued 1-form ∇s as

$$\nabla s = \nabla_D s + \nabla^{\perp} s, \tag{2.5}$$

where $\nabla_D s = (\nabla s)_D$ is the component of ∇s along D and $\nabla^{\perp} s = (i_R \nabla s)\theta$. Further $\nabla_D s = \partial_{D,J} s + \bar{\partial}_{D,J} s$, with $\partial_{D,J} s = (\nabla_D s)^{(1,0)} \in \Lambda^{(1,0)}(D, L)$, $\bar{\partial}_{D,J} s = (\nabla_D s)^{(0,1)} \in \Lambda^{(0,1)}(D, L)$ the holomorphic and the antiholomorphic parts of $\nabla_D s$ respectively. Hence, we will write ∇s as

$$\nabla s = \partial_{D,J} s + \bar{\partial}_{D,J} s + \nabla^{\perp} s. \tag{2.6}$$

We will use now the following simple linear algebra result:

Lemma 2. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a $\mathbb{R}$-linear application. Then f decomposes as:*

$$f = f^{(1,0)} + f^{(0,1)},$$

where $f^{(1,0)}$ and $f^{(0,1)}$ are the holomorphic and antiholomorphic parts of the application. Then if $|f^{(1,0)}| > |f^{(0,1)}|$, $\mathrm{Ker}\, f$ is a symplectic subspace of $\mathbb{C}^n$ with respect to the standard symplectic structure.

Two diferent proofs of this result can be found in [11, 4]. The following simple generalization will be useful later on:

Lemma 3. *Let $f \colon \mathbb{C}^n \to \mathbb{C}^r$ be a $\mathbb{R}$-linear application. Then f decomposes as:*

$$f = f^{(1,0)} + f^{(0,1)},$$

where $f^{(1,0)}$ and $f^{(0,1)}$ are the holomorphic and antiholomorphic parts of the application. Given $\gamma > 0$, there exists $c > 0$ such that if $|f^{(0,1)}| < c$ and f has a right inverse f^{-1} verifying that $|f^{-1}| < \gamma^{-1}$, then $\mathrm{Ker} f$ is a symplectic subspace of $\mathbb{C}^n$.

Proof. We will choose $c > 0$ along the proof. Let us take a real basis $\{e'_1, \dots, e'_{2r}\}$ in $\mathbb{C}^r$ such that the system of vectors $S = \{e_1, \dots, e_{2r}\}$, $e_i = f^{-1}(e'_i)$, is orthonormal in $\mathbb{C}^n$. Then we extend

S to a basis $\{e_1, \dots, e_{2n}\}$ in $\mathbb{C}^n$, such that $\{e_{2r+1}, \dots, e_{2n}\}$ is an orthonormal basis of $\mathrm{Ker} f$. Thanks to the bounding of the right inverse f^{-1} we can assure that:

$$|e'_j| > \gamma, \text{ for } j = 1, \dots, 2r.$$

Now we have:

$$|f^{(1,0)}(e_i) - f(e_i)| = |f^{(0,1)}(e_i)| < c, \ \forall i = 1, \dots 2n.$$

It is easy to show that $\{f^{(1,0)}(e_1), \dots, f^{(1,0)}(e_{2r})\}$ is a basis of $\mathbb{C}^r$. To do it we just have to compute $|f^{-1}(e'_j) - f^{-1} \cdot f^{(1,0)}(e_j)| = |f^{-1} \cdot f^{(0,1)}(e_j)| < \gamma^{-1} c$. So we obtain that $\{f^{-1} \cdot f^{(1,0)}(e_1), \dots, f^{-1} \cdot f^{(1,0)}(e_{2r})\}$ is a linear independent system because for c small is close to S which is orthonormal. And therefore we obtain that $\{f^{(1,0)}(e_1), \dots, f^{(1,0)}(e_{2r})\}$ is basis for c small enough. However, recall that the choice of c only depends on γ. Now we define the following linear isomorphism:

$$\begin{array}{rcl} p : \mathbb{C}^r & \rightarrow & \mathbb{C}^r \\ f^{(1,0)}(e_j) & \rightarrow & e'_j \end{array}$$

Then $g = f^{-1} \cdot p$ is a right inverse for $f^{(1,0)}$, and it is easy to show that, perhaps shrinking c, $|g| < 2\gamma^{-1}$. Again the chosen c depends only on γ.

Our objective will be to compare $\mathrm{Ker} f = V$ with $\mathrm{Ker} f^{(1,0)} = V^{(1,0)}$. We recall that we have chosen an orthonormal basis $\{e_{2r+1}, \dots, e_{2n}\}$ in V. Now we define $b_j = e_j + g \cdot f^{(0,1)}(e_j)$, $j = 2r+1, \dots, 2n$. It is easy to show that $|b_j - e_j| \leq 2\gamma^{-1} c$ and $b_j \in V^{(1,0)}$. So $\{b_{2r+1}, \dots, b_{2n}\}$ is a basis of $V^{(1,0)}$, which is arbitrarily close to $\{e_{2r+1}, \dots, e_{2n}\}$. We can conclude that V and $V^{(1,0)}$ are at distance $O(c)$ in the grassmanian of $2(n-r)$ subspaces $\mathrm{Gr}_{\mathbb{R}}(2(n-r), 2n)$.

Now following [4] we construct a function to measure the symplecticity of the subspaces of $\mathbb{R}^{2n} = \mathbb{C}^n$. We define the Kähler angle of a real subspace $V \in \mathrm{Gr}_{\mathbb{R}}(2(n-r), 2n)$ as:

$$\begin{array}{rcl} \alpha : \mathrm{Gr}_{\mathbb{R}}(2(n-r), 2n) & \rightarrow & [-1, 1] \\ V & \mapsto & \dfrac{(\omega_{0|V})^{2(n-r)}}{\mathrm{vol}_{g|V}} \end{array}$$

$\mathrm{vol}_{g|V}$ is the volume form in V defined by the standard metric in $\mathbb{R}^{2n}$ and ω_0 is the standard symplectic form in $\mathbb{R}^{2n}$. So the Kähler angle is positive if and only if V is a symplectic subspace (preserving the orientation). Even more if V is complex, $\alpha(V) = 1$. Obviously, α is a continuous function.

Recall now that $V^{(1,0)}$ is a complex subspace. Thus $\alpha(V^{(1,0)}) = 1$ and by the continuity of α and the compactness of $Gr_{\mathbb{R}}(2(n-r), 2n)$ we obtain that for a fixed c small enough $\alpha(V) > 0$. Therefore V is symplectic for that c and the proof is concluded. q.e.d.

Applying Lemma 2 we can prove the following result which will be the key idea for our approach:

Lemma 4. *Let s be a smooth section of $L \to C$. Then if*

$$|\bar{\partial}_{D,J} s| < |\partial_{D,J} s|, \tag{2.7}$$

at the zero set $W = s^{-1}(\mathbf{0})$, then W is a contact submanifold of C.

Proof. First we prove that TW is transverse to D at a point $x \in W$. By (2.7) we have that $d_D s$ is surjective. So we can find a vector $v \in D$ such that $\langle ds(x), v\rangle = -\langle ds(x), R\rangle$ then $\langle ds(x), v - R\rangle = 0$. Therefore $v - R \in TW$ and it is not in D, and then TW is transverse to D.

It remains to check that D_W is completely nonintegrable, but $D_W = \operatorname{Ker} \nabla_D s$, and if $|\bar{\partial}_{D,J} s| < |\partial_{D,J} s|$, then by Lemma 2, D_W is a symplectic subspace of D and, because of Lemma 1, D_W is a contact distribution. q.e.d.

In what follows we will simply write ∂ and $\bar{\partial}$ instead of $\partial_{D,J}$ and $\bar{\partial}_{D,J}$ respectively whenever it causes no confussion.

2.5 Exact coverings of nonexact contact manifolds

Let C be a smooth manifold and D a hyperplane distribution on it not necessarily of contact type. Let D^0 denote the annihilator of D, i.e., for each $x \in C$, $D^0_x = \{\, \alpha_x \in T^*_x C | \alpha_x(u) = 0, \forall u \in D_x \,\}$. The annihilator D^0 defines a rank 1 subbundle of the cotangent bundle T^*C. The restriction of the canonical symplectic form ω_0 on T^*C to D^0 defines an exact 2-form on it. This form degenerates along the zero section of the bundle $D^0 \to C$. We shall denote by $S_D(C)$ the principal $\mathbb{R}^*$-bundle over C obtained by removing the zero section of D^0. Notice that if there exists a globally defined 1-form θ such that $D = \operatorname{Ker} \theta$, then the bundle $\pi \colon S_D(C) \to C$ will be trivial because θ itself defines a nowhere vanishing smooth section of it. However, the 2-form ω induced on $S_D(C)$ by the canonical symplectic structure ω_0 could be degenerate. It is easy to realize that such form will be symplectic if and only if D is a contact distribution on C. Thus we will call $(S_D(C), \omega)$ the symplectization of the contact manifold (C, D).

If we choose a bundle metric η on the bundle $D^0 \to C$, the sphere bundle defined by it, i.e., $\hat{C} = \{\, \alpha \in D^0 | \eta(\alpha, \alpha) = 1 \,\}$, is a double covering of the manifold C. It is obvious that $\hat{C} \to C$ will be trivial if and only if the contact structure on C is exact. If the contact structure on C is not exact, then, the covering $\hat{C}$ carries an exact contact structure that lifts the one in C. More precisely, we define the lifted contact structure as follows. We shall denote the points in $\hat{C}$ as $\hat{x}$ and by $\pi \colon \hat{C} \to C$ the canonical projection. Then a tangent vector $\hat{u} \in T_{\hat{x}}\hat{C}$ belongs to the contact distribution $\hat{D}$ if $\pi_*(\hat{u}) \in D_x$, with $x = \pi(\hat{x})$. Now it is clear that $S_{\hat{D}}(\hat{C}) = \pi^* S_D(C)$ is trivial, hence the lifted contact structure $\hat{D}$ is exact.

The $\mathbb{Z}_2$ action of the structure group of the double covering $\hat{C} \to C$ is clearly defined by anticontactomorphisms. In other words, there exists an involutive anticontact diffeomorphism $a \colon S_D(C) \to S_D(C)$ given by $a(\alpha) = -\alpha$, for any $\alpha \in \hat{C}$, we mean by this that $a^*\theta = -\theta$. In fact, it extends to a symplectic $\mathbb{Z}_2$ action on $S_D(C)$.

The symplectic manifold $S_D(C)$ carries a distinguished class of almost complex structures. We first notice that for any compatible almost complex structure J_D on the symplectic bundle $D \to C$ we can construct a lifted compatible almost complex structure $\hat{J}$ on $S_D(C)$ as follows. The quotient bundle $TS_D(C)/\pi^*(D) = \nu_C(D)$ is a symplectic bundle and we can split $TS_D(C) = \pi^* D \oplus \nu_C(D)$ by fixing the bundle metric η. We shall choose then a compatible almost complex structure J' on $\nu_C(D)$ and define $\hat{J} = J_D \oplus J'$. More concretely in the exact case, choose a trivialization of $S_D(C) = C \times \mathbb{R}_0^*$, then denoting by λ the scaling coordinate on $\mathbb{R}_0^*$, and by $\pi_D \colon TS_D(C) \to \pi^* D$ the orthogonal projection, we have

$$\hat{J}_{(x,\lambda)}(\dot{x}, \dot{\lambda}) = ((J_D)_x \pi_D(\dot{x}) + \dot{\lambda} R_x, -\theta_x(\dot{x})),$$

where R denotes the Reeb field of the contact structure. This compatible almost complex structure verifies that $a_* \hat{J} = \hat{J}$, i.e, it is invariant under the $\mathbb{Z}_2$ action on $S_D(C)$. Moreover, the lifted distribution D is $\hat{J}$-complex. We shall call it compatible almost complex structure of contact type.

3. Asymptotically holomorphic sections and transversality

Once we have characterized codimension 2 contact submanifolds of C arising as zeroes of sections satisfying the basic inequality (2.7), we will follow [4], [2], [3], to show that such sections do indeed exist. In fact we will do more and show that sections satisfying a weaker holomorphicity condition do exist for large values of an integer parameter k. This property combined with a transversality condition will imply (2.7). Moreover we will extend the discussion to arbitrary hermitian vector bundles, using the transversality condition given by Lemma 3. Let us make these comments precise.

3.1 Definitions

Let X be a riemannian manifold with metric g and $D \subset TX$ be a subbundle carrying an almost complex structure J. Let $E_k \to X$, $k \in \mathbb{N}$, be a family of hermitian complex bundles of rank r equipped with hermitian connections ∇_k.

The metric g allows to decompose $TX = D \oplus D^\perp$, hence using this decomposition the metric g can be written as $g = g_D \oplus g_D^\perp$. We will consider the family of rescaled metrics $g_k = kg$. The bundle of E_k-valued forms on X decomposes as $\Lambda^1(X, E_k) = \Lambda^1(D, E_k) \oplus \Lambda^1(D^\perp, E_k)$. Moreover, the almost complex structure J on D allows to decompose $\Lambda^1(D, E_k)$ into the holomorphic and the antiholomorphic part

$$\Lambda^1(D, E_k) = \Lambda^{(1,0)}(D, E_k) \oplus \Lambda^{(0,1)}(D, E_k).$$

The covariant differential $\nabla_{L^{\otimes k}} s_k$ of a section s_k of E_k decomposes, in analogy with Equations (2.5-2.6), as

$$\nabla_k s_k = (\nabla_k)_D s_k + \nabla_k^\perp s_k,$$

and

$$(\nabla_k)_D s_k = \partial_{k,D,J} s_k + \bar{\partial}_{k,D,J} s_k.$$

As usual the k, D, J subindex in the $(1,0)$ and $(0,1)$ components of $\nabla_k s_k$ will be omitted if there is no risk of confussion.

Definition 2. A sequence s_k of sections of the bundles E_k is said to be asymptotically (D, J)-holomorphic if there exist constants c_p, $p \in \mathbb{N}$, such that for all k and at every point $x \in X$ the following bounds are satisfied:

$$(3.8) \qquad |s_k| \leq c_0, \quad |\nabla^p s_k|_{g_k} \leq c_p, \quad |\nabla^{p-1} \bar{\partial} s_k|_{g_k} \leq c_p k^{-1/2}, \quad p \geq 1,$$

where the norms of the derivatives are evaluated with respect to the metrics g_k.

The previous definition embraces the notion of asympotically J-holomorphic sequence of sections used in Auroux's work [2]-[3]. In fact, if X is a symplectic manifold (M, ω) of integer class, we shall consider $D = TM$ and an almost complex structure J on M compatible with ω. The metric g will be the metric defined by the almost complex and the symplectic structures by $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. The rescaled metrics g_k will be now kg. Donaldson considered the family of complex line bundles $E_k = L^{\otimes k} \to M$ where $L \to M$ is the prequantizable line bundle whose first chern class is $[\omega]/2\pi$ carrying the connection ∇ whose curvature is $-i\omega$ and Auroux extended the discussion to the family of complex bundles $E_k = E \otimes L^{\otimes k}$, where E is an hermitian rank r bundle over M. Then a sequence of asymptotically (D, J)-holomorphic sections with D and J as above will be simply called asymptotically J-holomorphic.

Moreover, in this paper we will also be concerned with the situation where X is an exact contact manifold (C, θ). Now, the subbundle D will be the contact distribution $\operatorname{Ker}\theta$ and g will be a contact metric on C. Hence the rescaled metrics g_k will have the form $g_k = kg = kg_D + k\theta \otimes \theta$. We shall consider the prequantizable line bundle L over C with connection ∇ whose curvature is $-id\theta$ and the family of bundles $E_k = L^{\otimes k}$, or $E_k = E \otimes L^{\otimes k}$. In this situation a sequence of (D, J)-holomorphic sections will be called asymptotically contact-holomorphic. More precisely:

Definition 3. Let E be a complex hermitian bundle over the exact closed contact manifold (C, θ). Let ∇_E be an hermitian connection on E and ∇_k the sequence of connections on $E_k = E \otimes L^{\otimes k}$ with curvature form $R_E - ikd\theta$. Let J be a compatible almost complex structure on the symplectic bundle $D = \operatorname{Ker}\theta \subset TC$. A sequence of sections s_k of E_k is called asymptotically contact-holomorphic if they verify

$$|s_k| \leq c_0, \quad |\nabla^p s_k|_{g_k} \leq c_p, \quad |\nabla^{p-1}\bar{\partial} s_k|_{g_k} \leq c_p k^{-1/2}, \quad p \geq 1, \tag{3.9}$$

for some family of constants c_p, where the norms of the derivatives are evaluated with respect to the rescaled contact metrics g_k.

If we construct a sequence of asymptotically contact-holomorphic sections (or of asymptotically J-holomorphic sections) such that $|\nabla_D s_k|$ is uniformly bounded below when $s_k = 0$, it is obvious from Lemma 4 that the zero sets W_k of sections s_k for large k will be smooth contact submanifolds (resp. symplectic submanifolds). Thus to guarantee that the zero sets of the sections s_k are contact submanifolds (or symplectic submanifolds in the symplectic category), we need to require a transversality property.

Definition 4. Let s be a section of a complex vector bundle E over the Riemannian manifold X with distribution D, and $\eta > 0$. The section s is said to be η-transverse to $\mathbf{0}$ along D if, at any point $x \in X$ where $|s(x)| < \eta$, the covariant derivative restricted to D, $\nabla_D s \colon D_x \subset T_x X \to E_x$, is surjective and has a right inverse of norm less than η^{-1}.

We will say that a sequence s_k of sections of a family of bundles E_k is asymptotically η-transverse to zero along D, if s_k is η-transverse to $\mathbf{0}$ along D for k large enough.

We will often say simply that s_k is transverse to zero along D if such η does exist. In the symplectic case, we will say for short that s_k is transverse to zero and in the contact case we will say that s_k is transverse to zero along the contact distribution or we will omit the later if there is no risk of confusion.

We will obtain immediately the following:

Lemma 5. *Let s_k be a sequence of asymptotically J-holomorphic sections transverse to zero of the family of vector bundles E_k over the symplectic manifold M. Then, for k large enough, the zero set W_k of s_k is a symplectic submanifold of M.*

Analogously, in the contact case, let s_k be a sequence of asymptotically contact-holomorphic sections of the vector bundles E_k over the contact manifold (C, θ) such that they are transverse to zero. Then, for large enough k, the zero set W_k of s_k is a contact submanifold of C.

Proof. The first statement is the starting point in the work of Donaldson and Auroux to construct symplectic submanifolds. The second statement is obvious from Lemma 2 and Lemma 3. q.e.d.

3.2 Results

The theorem we will prove can thus be stated as follows:

Theorem 2. *Let (C, θ) be a closed exact contact manifold and E a rank r complex bundle over C. There exists a sequence s_k of asymptotically contact-holomorphic sections transverse to $\mathbf{0}$ of the bundles $E_k = E \otimes L^{\otimes k}$ where L is the trivial line bundle over C.*

The similar result in the symplectic category, that constitutes the main result in [2], can be stated as follows.

Theorem 3. *Let (M, ω) be a closed symplectic manifold with symplectic form of integer class and E a rank r complex bundle over M. There exists a sequence of asymptotically J-holomorphic sections transverse to $\mathbf{0}$ of the bundles $E_k = E \otimes L^{\otimes k}$ where L is the prequantizable line bundle over M.*

The strategy to prove Theorems 2 and 3 will be to start with a given family of asymptotically contact-holomorphic sections s_k, (or J-holomorphic in the symplectic case) for instance $s_k = 0$, and perturb them in order to obtain transversality to $\mathbf{0}$. That this can be done in the symplectic category is the content of the following result.

Theorem 4 ([2]). *Let E be a rank r complex bundle over the closed symplectic manifold M. Let J_t, $t \in [0,1]$, be a continuous family of compatible almost complex structures. Let $\epsilon > 0$ and $s_{k,t}$ be a family of sequences of asymptotically J_t-holomorphic sections on $E \otimes L^{\otimes k}$ such that they and their derivatives depend continuously on t. Then, there exist a real number $\eta > 0$ (depending on ϵ, M, $s_{k,t}$ and their derivatives), and a family of asymptotically J_t-holomorpic sequences $\sigma_{k,t}$, such that:*

i) *$\sigma_{k,t}$ and their derivatives depend continuously on t.*

ii) *For k large enough we have, $|\sigma_{k,t} - s_{k,t}|_{C^1, g_k} < \epsilon$.*

iii) *$\sigma_{k,t}$ is η-transverse to $\mathbf{0}$.*

The proof of the theorem is based in three main ingredients: existence of localized sections, local transversality with estimates and a globalization process. This discussion is carried out in full detail in [2] and [3] and we will not repeat it here.

Remark. Auroux' techniques can be easily extended to symplectic manifolds with contact boundary. However, we cannot keep the isotopy results if the manifold has boundary. The main obstacle, roughly speaking, is that we cannot assure that the constructed isotopy does not cross the border. This phenomenom will be also understood locally afterwards.

In the contact category, we can proceed similarly as the following theorem states.

Theorem 5. *Let $\epsilon > 0$ and let s_k be a sequence of asymptotically contact-holomorphic sections of the bundles $E \otimes L^{\otimes k}$ over the closed contact manifold C. Then there exists a real number $\eta > 0$ (depending on ϵ, s_k and their derivatives), and a sequence of asymptotically contact-holomorphic sections σ_k of $E \otimes L^{\otimes k}$ such that*

$$|s_k - \sigma_k|_{C^0, g_k} \leq \epsilon,$$

and σ_k is η-transverse to $\mathbf{0}$.

It is clear that Theorem 5 implies Theorem 2 and the first half of the proof of the main Theorem 1 will be finished.

This perturbation theorem is the analogue of Theorem 2 in [2], which implies Theorem 3 above, for symplectic manifolds. Notice that in such case, the theorem was proved for families of sequences parametrized by $t \in [0, 1]$. This extension does not seem possible in the contact category as we will see later. This is not the only important difference between the symplectic and the contact case. In the symplectic category the perturbed sequence σ_k can be chosen arbitrarily close to the initial in C^1-norm but this will not be the case in the contact case. We only obtain small C^0 perturbations, however we will show that we can construct perturbed sequences satisfying the required transversality property C^1-close to the unperturbed one s_k, in the D directions. This will be made more precise later on.

There is not a straightforward relationship between the contact and symplectic situations. It is possible to define asymptotically contact-holomorphic sequences with different bounds in the Reeb directions, however we define it in this way to be able to compare the contact and symplectic sequences in forthcoming papers.

Theorem 5 will be proved in detail in Section 4. We will see that, in spite of the deep resemblance between the symplectic and the contact category, the proof of Theorem 5 will require a thorough reelaboration of the arguments in [4] and [2].

4. An extension of Donaldson-Auroux theory to exact contact manifolds

As we mentioned already, there are three main ingredients to prove the central Theorems 5 and 2. The first one is the existence of localized asymptotically contact-holomorphic sections. The second one is a transversality theorem with estimates and the third and last one, is a globalization process. We will devote the next three sections to discuss them. We adapt the discussion in [2] to the contact category.

We say that a real number appearing in the statement or along the proof of the results to follow is a universal constant if it depends exclusively on the geometry of the manifold. Also we will say that a polynomial is a universal polynomial if depends only on the geometry of the manifold.

4.1 The local theory

As we have noticed before, to put up Donaldson's construction we need to have localized sections whose behaviour under scaling we control adequately. The precise notion of localization we will need is given by the following definition.

Definition 5. A section s_k of a bundle $E_k \to X$ has Gaussian decay in C^r-norm away from a point $x \in X$ if there exist a polynomial P and a constant $\lambda > 0$ such that for all $y \in X$, $|s_k(y)|$, $|\nabla_k s_k(y)|_{g_k}$, ... $|\nabla_k^r s_k(y)|_{g_k}$, are all bounded by $P(d_k(x,y)) \exp(-\lambda d_k(x,y)^2)$.

The Gaussian decay of a family of sections s_k are said to be uniform if there exists P, λ such that the bounds hold for all sections of the family independent of k and of the point x at which the decay occurs for a given section, (i.e., if P and λ are universal).

The first result we will prove is the existence of sequences of asymptotically contact-holomorphic sections with uniform Gaussian decay in C^2-norm for the bundles $L^{\otimes k}$ over a contact manifold C. To set up this result we use Donaldson's construction as summarized in the following lemma which is a particular instance of Lemma 3 in [2].

Lemma 6. *Let (M, ω) be a symplectic manifold with symplectic form of integer class and L the prequantizable line bundle with first Chern class $c_1(L) = [\omega]/2\pi$. Let J be a compatible almost complex structure in M, which determines a family of metrics $g_k = k\omega(\cdot, J\cdot)$. Then there exists a constant $c_s > 0$ such that given any $x \in M$, there exist sections $s_{k,x}$ of $L^{\otimes k}$ over X with the following properties: the sections $s_{k,x}$ are asymptotically J-holomorphic; the bound $|s_{k,x}| \geq c_s$ holds over the ball of g_k-radius 10 around x; and finally, $s_{k,x}$ has unifrom Gaussian decay in C^2-norm away from x.*

The proof in the contact case is now a direct corollary:

Lemma 7. *Let (C, θ) be a closed contact manifold. There exists* *a universal constant $c_s > 0$, such that given any point $x \in C$, there exists a sequence of asymptotically contact-holomorphic sections $\sigma_{k,x}$ of $L^{\otimes k} \to C$ satisfying $|\sigma_{k,x}| \geq c_s$ at every y in a ball of g_k-radius 10 centered at x and the sections $\sigma_{k,x}$ have uniform Gaussian decay away from x in C^2-norm.*

Proof. We consider (C, θ) embedded in the symplectized manifold $S_D(C)$ as the graph of θ, which we denote as above by $\hat{C}$. Thus the prequantizable bundle of the symplectization restricts to C as the prequantizable bundle L. Moreover the almost complex structure on D can be extended to a contact type almost complex structure $\hat{J}$. The sequence of metrics g_k on C are the restriction of the metrics $k\hat{\omega}(\cdot, \hat{J}\cdot)$ to $\hat{C} \simeq C$. So given a point $x \in C$ we obtain a sequence of sections $\hat{s}_{k,x}$ in $S_D(C)$ with uniform Gaussian decay in C^2-norm away from x. Obviously the restriction $\sigma_{k,x}$ of $\hat{s}_{k,x}$ to $\hat{C}$ satisfies all the required properties. q.e.d.

Now we can construct the contact analogue of the J-holomorphic global sections built by Donaldson. We simply globalize the construction to arrive to the construction of a section $s_k = \sum_j w_j \sigma_j$, $|w_j| \leq 1$, where $\sigma_j = \sigma_{k,x_j}$ denotes a localized section around the point x_j and x_j are the centers of a finite covering by Darboux charts of C, where the distance between centers is bounded below by a fixed constant.

Lemma 8. *For any choice of coefficients w_j such that $|w_j| \leq 1$, the section s_k satisfies:*

$$|s_k| \leq c, \qquad |\bar{\partial} s_k|_{g_k} \leq c k^{-1/2}, \qquad |\nabla \bar{\partial} s_k|_{g_k} \leq c k^{-1/2},$$
$$|\nabla s_k|_{g_k} < c, \qquad |\nabla \nabla s_k|_{g_k} < c,$$

everywhere on C.

In Subsection 4.3 we will precise this result to achieve transversality to $\mathbf{0}$ in the sequence.

We shall describe now the local model we are going to work with. Let x be a point in the contact manifold C. Using the contact Darboux theorem, cfr. [1], we can find a local chart $\varphi \colon U \to U_x$, where $U = B_{2n+1}(r)$ is an open ball of $\mathbb{R}^{2n+1}$ of radius r, U_x is an open neighborhood of x, $\varphi(\mathbf{0}) = x$, and such that $\varphi^*\theta = \theta_0 = ds + x_i dy_i$ with (x_i, y_i, s) natural coordinates on $\mathbb{R}^{2n+1}$. It is obvious that scaling the chart φ by a factor ρ^{-1} we obtain a new chart φ_ρ on $\mathbb{R}^{2n+1}$ that transforms the contact form $k\theta$ in $k(\rho^{-1} ds + \rho^{-2} x_i dy_i)$. Hence, if $\rho = k^{1/2}$, then the contact form $k\theta$ becomes $\theta_\rho = \varphi_\rho^*(k\theta) = k^{1/2}ds + x_i dy_i$. We will call such trivialization a ρ-Darboux chart. It is important to notice that the distribution $\operatorname{Ker}\theta_{k^{1/2}}$ tends to the trivial horizontal distribution $\mathbb{R}^{2n} \times \{0\}$ when k goes to ∞ in a ball of fixed radius in $\mathbb{R}^{2n+1}$. We will formalize this idea with the folowing definitions:

Definition 6. The maximum angle $\angle_M$ between $2m$-dimensional subspaces U, V in $\mathbb{R}^n$ is defined as:

$$\angle_M(U,V) = \max_{u\in U}\{\angle(u,V)\}.$$

Definition 7. A sequence of contact distributions D_k in a set $V \subset \mathbb{R}^{2n+1}$, $r > 0$, is c-asymptotically flat if

$$\angle_M(D_k(0), D_k(y)) < ck^{-1/2}, \quad \forall y \in V.$$

The sequence D_k is said to be asymptotically flat if it is c-asymptotically flat for some $c > 0$.

Obviously the distributions associated to the sequence of contact forms $\theta_{k^{1/2}}$ are asymptotically flat in any ball in $\mathbb{R}^{2n+1}$.

Restricting the symplectic bundle D to U_x and using the natural identifications provided by the chart φ, the chosen compatible almost complex structure J define a map:

$$\widetilde{J}\colon U \subset \mathbb{R}^{2n+1} \to \operatorname{End}(D_0),$$

with $D_0 = \operatorname{Ker}\theta_0$, and $\widetilde{J}^2 = -I$. But there is a natural complex structure $\widetilde{J}_0$ in the contact distribution on $\mathbb{R}^{2n+1}$ which is obtained from the standard complex structure J_0 in $\mathbb{R}^{2n} = \mathbb{C}^n$ splitting $\mathbb{R}^{2n+1} = \mathbb{R}^{2n} \times \mathbb{R}$, identifying the factor $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, and lifting J_0 to the canonical contact distribution D_0 by the vertical projection $\mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ along the s axis in $\mathbb{R}^{2n+1}$. Notice that $\widetilde{J}_0$ is well defined because the contact distribution D_0 is not perpendicular to the horizontal hyperplane $\mathbb{R}^{2n} \times \{0\}$ (in fact is tangent at $\mathbf{0}$). We shall denote the Cauchy-Riemann operators defined with respect to the almost complex structures $\widetilde{J}$ and $\widetilde{J}_0$ by $\bar\partial$ and $\bar\partial_0$ respectively.

All the precedent considerations can be applied also to the contact manifold $(B_{2n+1}(r), D_k)$, where D_k is asymptotically flat. We can similarly define a canonical complex structure $\widetilde{J}_k$ in D_k by vertical projection of the canonical one in $\mathbb{R}^{2n} \times \{0\}$, we can define $\bar\partial$ and ∂ operators, etc.

The best holomorphic approximation that we can obtain is expressed in the following lemma which is similar to Lemma 2 in [2]:

Lemma 9. *Near any point x of a contact manifold C, there exists a contact Darboux chart $\varphi\colon U \cong B_{g_k}(x,r) \to V \subset \mathbb{R}^{2n+1}$, $r > 0$, such that φ satisfies $|\nabla\varphi| = O(1)$, $|\nabla\nabla\varphi| = O(1)$, on a ball of universal radius r around $x \in C$, and the restriction of $T\varphi\colon TC \to TV$ to D fails to be (J, J_0)-holomorphic, J_0 the standard complex structure in D_0, by an amount that vanishes at x and grows no faster than the distance to the origin, $|\bar\partial\varphi(y)| = O(|y|)$, and $|\nabla\bar\partial\varphi| = O(1)$. Finally for the inverse map $\varphi^{-1} : V \to B_{g_k}(x,r) \subset C$ the following bounds are verified in a universal way:*

$$|y|^2 = O(d_g(\varphi^{-1}(x), \varphi^{-1}(y))), \quad |\nabla\varphi^{-1}| = O(1), \quad |\nabla\nabla\varphi^{-1}| = O(1).$$

The map φ will be called a nearly contact-holomorphic Darboux chart with respect to the almost complex structure J on D and the canonical complex structure $\widetilde{J}_0$ on $\mathbb{R}^{2n}$.

Proof. We choose a Darboux chart at x, $\hat\varphi : B_{g_k}(x,r) \to V \subset \mathbb{R}^{2n+1}$ verifying $\hat\varphi_*\theta = ds + xdy$. The constant c can be chosen in a universal way because of the compactness of C. We need to assure also that the standard complex structure J_0 in $D_0 \subset \mathbb{R}^{2n+1}$ and $\hat\varphi_*J$ coincide at $\hat\varphi(x) = 0$. We follow the proof of the contact Darboux Theorem in Section H of Appendix 3 in [1]. The proof uses the local immersion in the symplectization through the graph of the local form θ, there we use the symplectic Darboux Theorem to obtain Darboux coordinates $\hat\varphi(y) = (p_0, \dots, p_n, q_0, \dots, q_n)$.

Following [1] we can assure that the contact manifold is locally given by the equation $p_0 = 0$. Notice that in general

$$D_x = \hat\varphi_* D(x) \neq \{p_0 = q_0 = 0\}.$$

But we can choose a standard symplectic basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ in D_x. Also we can choose a standard symplectic basis (e_0, f_0) in $D_x^\perp$, assuring that $p_0(e_0) = 0$. The orthogonal operation is made with respect to the symplectic form in the symplectization. Now, we define the transformation:

$$\begin{array}{rcl} \eta : \mathbb{R}^{2n+2} & \to & \mathbb{R}^{2n+2} \\ \dfrac{\partial}{\partial p_i} & \to & e_i \\ \dfrac{\partial}{\partial q_i} & \to & f_i. \end{array}$$

The map η is symplectic and if we compose $\eta \circ \hat{\varphi}$ we obtain that, in these new Darboux coordinates, C is again locally defined by the equation $p_0 = 0$ and also D_x is complex, in fact $D_x = \{p_0 = q_0 = 0\}$. Finally performing a symplectic transformation in D_x we can assure that $J = (J_0)_{|D}$. From this point we follow the proof of [1] and it is easy to check that the constructed contact Darboux chart verifies that $\varphi^*(J_0)(x) = J(x)$ at the point x. We cannot assure more because the two complex structures are related through a, in general nonvanishing, Nijenhuis type tensor at the origin. By the compactness of C the bounds in the derivatives of φ are easy to check. The last inequalities in the statement of the lemma are assured by the fact that φ is a isometry at x and by the compactness of C. Now following the discussion in Section 2 of [4] it is easy to verify that the bounds in the antiholomorphic parts are correct. q.e.d.

We shall choose the connection ∇ on L defined by $-i\theta$ whose curvature is precisely $-id\theta$. Now we trivialize this line bundle using a local section σ_0 in a neighborhood of a point x described by a nearly contact-holomorphic contact chart as in Lemma 9. We can always choose, perhaps after a gauge transformation, a trivializing section σ_0 in $U \subset \mathbb{R}^{2n+1}$ such that the operator $\bar{\partial}$ takes the following form on sections $\sigma = f\sigma_0$,

$$\bar{\partial}(f\sigma_0) = (\bar{\partial} f + \frac{1}{4}\sum_{\alpha}(z_\alpha (d\bar{z}_\alpha)^{(0,1)} - \bar{z}_\alpha (dz_\alpha)^{(0,1)})f)\sigma_0, \tag{4.10}$$

where we have used the identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$ provided by $z_\alpha = x_\alpha + iy_\alpha$ and $\mathbb{R}^{2n+1} \cong \mathbb{C}^n \times \mathbb{R}$ with coordinates (z, s). The $\bar{\partial}$ operator acting on f in formula (4.10) is defined using the $\widetilde{J}$ complex structure. However using the $\widetilde{J}_0$ structure we obtain the operator $\bar{\partial}_0$ defined as:

$$\bar{\partial}_0(f\sigma_0) = (\bar{\partial}_0 f + \frac{1}{4}\sum_{\alpha} z_\alpha d\bar{z}_\alpha f)\sigma_0.$$

Along the proof of the globalization process we will need detailed information on the local structure of the submanifolds $W_k = s_k^{-1}(0)$. The remaining of this section is devoted to find such description.

Suppose now that we have found a sequence of asymptotically contact-holomorphic sections s_k of the rank r complex vector bundles E_k which are η-transverse to $\mathbf{0}$. By the η-transversality property, their zero sets W_k are contact submanifolds for k large enough. Moreover, we have that $|\nabla\nabla s_k| < c'$, so given $m \in \mathbb{N}$ there exists a constant $c = c(m)$ such that $\forall y \in B_{g_k}(x, c)$,

$$|\nabla s_k(x) - \nabla s_k(y)| < \eta/m. \tag{4.11}$$

This implies that $W_k \bigcap B_{g_k}(x, c)$ is diffeomorphic to a ball if we choose m large enough, and in fact C^1-close to the image of the tangent space at x through a nearly J-holomorphic chart. Namely using a nearly contact-holomorphic chart we interpret W_k as a submanifold in $\mathbb{R}^{2n+1}$. Also, the tangent space of W_k at $\mathbf{0} \in \mathbb{R}^{2n+1}$ can be interpreted as a subpace in $\mathbb{R}^{2n+1}$. We say that, because of Equation (4.11) these submanifolds can be made arbitrarily C^1-close, by taking c small enough.

Notice, now, that the rescaled metric g_k and the rescaled metric g_k^W, defined in the contact manifold W_k using any compatible almost complex structure J_k on $D_k^W = D \bigcap TW_k$, do not coincide along W_k. Moreover, by the bounds (3.9) on the asymptotic contact-holomorphic sequence s_k we can choose an almost holomorphic structure J_k on $D_k^W \subset W_k$ such that $J_{|D_k^W} - J_k : D_k^W \subset TW_k \to D \subset TC$ verifies that $|J_{|D_k^W} - J_k| = O(k^{-1/2})$ with respect to the g_k metric in C. Notice, however, that the Reeb vector field of C does not coincide with the one defined in W_k and so we can not compare g_k and g_k^W, even in this case. However, in the proof of the following proposition we will avoid this comparison, using the η-transversality of the sequence. It is possible to establish such relation, giving an alternative proof of Proposition 1. Let us trivialize W_k in the following form:

Proposition 1. *There exist real numbers $c, \hat{c} > 0$ independent of $k \in \mathbb{N}$ and $x \in C$ such that for any point $x \in W_k$, the set $B_W(c) = B_{g_k}(x, c) \bigcap W_k$, the restriction to W_k of the ball of g_k-radius c around x, is the domain of a contact chart $\psi_k \colon B_W(c) \to \hat{B} = B_{2(n-r)+1}(2)$, where $\hat{B}$ carries an asymptotically flat contact structure θ_k^W. Moreover, ψ_k verifies the following estimates over $\hat{B}$:*

$$|\nabla\psi_k| = O(1), \quad |\nabla\nabla\psi_k| = O(1).$$

Also $B_W(\hat{c}) \subset \psi_k^{-1}(B_{2(n-r)+1}(1))$. Finally ψ_k^{-1} verifies the following bounds:

$$|\nabla\psi_k^{-1}| = O(1), \quad |\nabla\nabla\psi_k^{-1}| = O(1),$$
$$|\bar{\partial}\psi_k^{-1}| = O(k^{-1/2}), \quad |\nabla\bar{\partial}\psi_k^{-1}| = O(k^{-1/2}).$$

All the norms are computed with respect to the g_k metric in C, restricted to W_k, and the standard euclidean metric in $\mathbb{R}^{2(n-r)+1}$. The $\bar{\partial}$ and $\nabla\bar{\partial}$ operators are defined with respect to the J fixed on C and to the standard $\hat{J}_k$ associated to θ_k^W in $\mathbb{R}^{2(n-r)+1}$. Finally, the result also applies when $W_k = C$.

Before starting the proof we pick two simple results from [13]. First we give the following

Definition 8. The minimum angle between two nonzero subspaces U, V of $\mathbb{R}^n$ is defined as follows:

- If $\dim U + \dim V < n$ then $\angle_m(U, V) = 0$.
- If their intersection is not transversal then $\angle_m(U, V) = 0$.
- If their intersection is transversal then let W be their intersection. Define U_c as the orthogonal subspace in U to W, and V_c in the same way. Then $\angle_m(U, V) = \min_{u \in U_c - \{0\}}\{\angle(u, V_c)\} \in [0, \pi/2]$.

The result we will need to use is

Lemma 10 (Proposition 3.7 in [13]). *Given $\epsilon > 0$ and $U \in \mathrm{Gr}(m, n)$, $V \in \mathrm{Gr}(r, n)$ subspaces satisfying that $\angle_m(U, V) > \epsilon$, then there are $\gamma_0 > 0$ and a constant C, depending only on ϵ, such that for any $\gamma < \gamma_0$, if $U' \in Gr(m, n)$ and $V' \in Gr(r, n)$ verify that*

$$\angle_M(U, U') < \gamma, \quad \angle_M(V, V') < \gamma,$$

then U' and V' intersect transversally and $\angle_M(U \bigcap V, U' \bigcap V') < C\gamma$.

Now, let us start the proof.

Proof of Proposition 1. We choose a nearly J-holomorphic chart φ_x on a neighborhood of $x \in C$ as given by Lemma 9. Obviously for k large enough φ_x is well-defined in the ball of g_k-radius c, for any fixed $c > 0$. We scale $\mathbb{R}^{2n+1}$ by a factor $k^{1/2}$ to obtain a new nearly contact-holomorphic chart φ_k, which is now a $k^{1/2}$-Darboux chart. If $W_k = C$ we have finished because by Lemma 9, $\psi_k = \varphi_k$ verifies all the required properties: it is an isometry at x, and the behaviour of its derivatives can be controlled by universal constants in the ball.

Recall that it is easy to assure, perhaps shrinkig c, that:

$$\frac{3}{4}g_k(v, w) \leq g((\varphi_k)_*v, (\varphi_k)_*w) \leq \frac{4}{3}g_k(v, w), \qquad \forall v, w \in T_xB_W(c)), \forall x \in B_W(c). \tag{4.12}$$

Now we take the vector space $L_k = (\varphi_k)_*(x)(W_k)$. We choose an unitary basis $(f_1, \dots, f_{n-r})$ in $(\varphi_k)_*(D \bigcap W_k) \subset D_{k^{1/2}}(\mathbf{0}) = \mathrm{Ker}\theta_{k^{1/2}}(\mathbf{0})$. We complete it to a basis in $D_{k^{1/2}}(\mathbf{0})$ by choosing an unitary basis $(f_{n-r+1}, \dots, f_n)$ in the orthogonal complementary subspace. The basis $(f_1, if_1, \dots, f_n, if_n)$ is orthonormal. We choose $f_{2n+1} \in (\varphi_k)_*(W_k)$, an unitary vector in the orthogonal to $D_k^W(\mathbf{0})$, to complete a basis in $\mathbb{R}^{2n+1}$. Recall now that in $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ the standard basis $(e_1, \dots, e_{2n+1})$ verifies

$$e_{2j} = i \cdot e_{2j-1}, \quad j = 1, \dots, n.$$

So we define an application $\lambda : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ as:

$$\begin{aligned} \lambda(f_j) &= e_{2j-1}, \quad j = 1, \dots, n. \\ \lambda(i \cdot f_j) &= e_{2j}, \quad j = 1, \dots, n. \\ \lambda(f_{2n+1}) &= e_{2n+1}. \end{aligned}$$

By the η-transversality of the sequence and its asymptotic contact-holomorphic bounds we can verify that:

$$d_1 g(v, w) \le g(\lambda v, \lambda w) \le d_2 g(v, w), \quad \forall v, w \in \mathbb{R}^{2n+1}, \tag{4.13}$$

where $d_1, d_2 > 0$. We leave this comprobation to the reader, which basically depends on checking that f_{2n+1} has an angle greater that $\epsilon > 0$ with respect to $D_k(\mathbf{0})$. Now, we call $\Phi_k = \lambda \circ \varphi_k$. We push-forward $\theta_{k^{1/2}}$ through λ to obtain $\theta'_{k^{1/2}}$ in $\mathbb{R}^{2n+1}$. This contact form is, again, asymptotically flat in every ball in $\mathbb{R}^{2n+1}$ by the inequalities (4.13). The map Φ_k, obviously, verifies:

$$\begin{aligned} c_1 g_k(v, w) \le g((\Phi_k)_* v, (\Phi_k)_* w) \le c_2 g_k(v, w), \\ \forall v, w \in TB_{g_k}(x, c), \end{aligned} \tag{4.14}$$

where $c_1, c_2 > 0$ are fixed constants. Also, Φ_k is contact. Moreover, it verifies the same boundings in the antiholomorphic derivatives that φ_k, changing the universal constants. We decompose $\mathbb{R}^{2n+1} = \mathbb{R}^{2(n-r)+1} \oplus \mathbb{R}^{2r}$, where $\mathbb{R}^{2(n-r)+1} = \langle e_1, \dots, e_{2(n-r)}, e_{2n+1} \rangle$ and $\mathbb{R}^{2r} = \langle e_{2(n-r)+1}, \dots, e_{2n} \rangle$

Define $\hat{s_k} = \Phi_k^{-1} \circ s_k$. Now, recall that shrinking c we can make W_k topologically trivial in the ball $B_{g_k}(x, c)$. Even more its image by Φ_k is C^1-close to $\mathbb{R}^{2(n-r)+1}$. So there exists an unique application $\tau_k : \mathbb{R}^{2(n-r)+1} \to \mathbb{R}^{2r}$ defined by:

$$\hat{s_k}(x, \tau_k(x)) = \mathbf{0} \tag{4.15}$$

Notice that $\tau_k(\mathbf{0}) = \mathbf{0}$ and $\nabla\tau_k(\mathbf{0}) = O(k^{-1/2})$. From the former considerations, defining $\psi_k^{-1}(x) = \Phi_k^{-1}(x, \tau_k(x))$ we obtain

$$d_1 g_k(v,w) \le g((\psi_k)_* v, (\psi_k)_* w) \le d_2 g_k(v,w), \qquad \forall v, w \in TB_{g_k}(x,c), \tag{4.16}$$

where $d_1, d_2 > 0$ are fixed constants. From inequalities (4.16) it is easy to check the bounds in the derivatives of ψ_k and ψ_k^{-1} in the statement of the proposition. To do that we define $pr_k^{-1}(x) = (x, \tau_k(x))$. Also define $\theta_k^W = (pr_k^{-1})^* \theta'_{k^{1/2}}$. Obviously ψ_k is a contact chart. Our objective now is to assure that θ_k^W is asymptotically flat in $V = \psi_k(B_{g_k}(x,c) \bigcap W_k)$. We can assure that $V \subset R^{2(n-r)+1}$ is contact, perhaps shrinking c, with respect to the restriction θ'_V of $\theta'_{k^{1/2}}$.

We need more precision to control the behaviour of θ_k^W. Recall that $\Phi_k(W_k)$ is defined by the zero set of $\hat{s_k}$. Define now:

$$Ls_k = \partial_k \hat{s_k}(0) + \nabla_k^{\perp} \hat{s_k}(0).$$

With the usual identifications $Ls_k : \mathbb{R}^{2n+1} \to E \otimes L^{\otimes k}$. We know that Ker $Ls_k = \langle e_1, \dots, e_{2(n-r)}, e_{2n+1} \rangle = \mathbb{R}^{2(n-r)}$. Shrinking c independently of k we can assure that $\hat{s}_k$ is η'-transverse along $D'_k = \mathrm{Ker}\theta'_{k^{1/2}}$ and Ls_k is η'-transverse along D'_k in V, $\eta' > 0$. D'_k is asymptotically flat in V, so given any $0 < t < 1$ we know that $\hat{s}_k$ and Ls_k are $t\eta'$-transverse to the horizontal distribution $D_h = \langle e_1, \dots, e_{2n} \rangle \subset \mathbb{R}^{2n+1}$, when k is large enough. We claim tha the spaces D_h, D'_k and W_k are in the hypothesis of Lemma 10. First, it is clear that $\angle_M(D_h, D'_k) = O(k^{-1/2})$. Second, W_k is the zero set of $\hat{s}_k$ and $\hat{s}_k$ is transverse to D'_k; this inmediately implies that $\angle_m(W_k, D'_k) \ge \epsilon$ (ϵ not depending on k). We leave the check of this property to the careful reader. Therefore, applying Lemma 10, we obtain:

$$\angle_M(D_h \bigcap W_k, D'_k \bigcap W_k) < c' k^{-1/2}. \tag{4.17}$$

The following step is the observation that

$$(pr_k)_*(D_h \bigcap W_k) = D_h \bigcap \mathrm{Ker} Ls_k$$

and

$$(pr_k)_*(D'_k \bigcap W_k) = \mathrm{Ker}\theta_k^W.$$

So, using Equations (4.17) and:

$$d'_1 \langle v, w \rangle \le \langle (pr_k)_* \, v, (pr_k)_* \, w \rangle \le d'_2 \langle v, w \rangle, \qquad \forall v, w \in T\Phi_k(B_{g_k}(x,c)), \tag{4.18}$$

we obtain that θ_k^W is asymptotically flat.

To finish we have only to check the antiholomorphic bounds. A direct argument shows that:

$$|\bar{\partial}\Phi_k| = O(k^{-1/2}), \quad |\nabla\bar{\partial}\Phi_k| = O(k^{-1/2}). \tag{4.19}$$

To bound the derivatives of $\tau_k : \mathbb{R}^{2(n-r)+1} \to \mathbb{R}^{2r}$ we differenciate Equation (4.15). We will compute the derivatives assuming that $\mathbb{R}^{2r}$ is equipped with the standard complex structure. Then the antiholomorphic derivatives are the natural ones. We claim that the obtained bounds are:

$$|\bar{\partial}\tau_k| = O(k^{-1/2}), \quad |\nabla\bar{\partial}\tau_k| = O(k^{-1/2}). \tag{4.20}$$

First, we check that for any vector v in $\mathbb{R}^{2r} = \langle e_{2(n-r)+1}, \dots, e_{2n}\rangle$ we have got $|\nabla\hat{s}_k(v)| > c''$, $c'' > 0$ independent of k (this property is only the η'-transversality of $\hat{s}_k$). With this observation and the asymptotic holomorphic bounds of $\hat{s}_k$, it is easy to set up the inequalities (4.20). Using Equation (4.17) we finally obtain that:

$$|\bar{\partial}pr_k^{-1}| = O(k^{-1/2}), \quad |\nabla\bar{\partial}pr_k^{-1}| = O(k^{-1/2}).$$

Now, the chain rule gives us the desired estimates. q.e.d.

4.2 Transversality with estimates

Now, as in Donaldson's original paper, we have reached the heart of the problem, as far as we have to show that the sections we construct are such that $|\partial s| > \epsilon > 0$ on the zero set of s. The main technical tool to prove it is a finer notion of transversality with estimates than the one used in [4] and which was based on results on the complexity of semialgebraic sets [17]. What we will need is a generalization of the notion of parametrized controlled transversality for families of functions, the one used in [2] and [3].

Theorem 6. *Let $f_k\colon B^+ \times [0,1] \to \mathbb{C}$ be a sequence of functions where B^+ is the ball of radius 11/10 in $\mathbb{C}^n$ and $B^+ \times [0,1]$ is equipped with a sequence of contact forms $\theta(k)$ whose distributions are asymptotically flat. Let $0 < \delta < 1/2$ be a constant and let $\sigma = \delta(\log(\delta^{-1}))^{-p}$, where p is a universal integer. Assume that f_k satisfies the following bounds for k large enough over $B^+ \times [0,1]$*

$$|f_k| \le 1, \quad |\bar{\partial}_0 f_k| \le \sigma, \quad |\nabla\bar{\partial}_0 f_k| \le \sigma,$$

where $\bar{\partial}_0$ is the $(0,1)$ operator defined in $D(k) = \mathrm{Ker}\,\theta(k)$ by the complex structure $\widetilde{J}_0$. Then for k large enough there exists a smooth curve $w_k\colon [0,1] \to \mathbb{C}$ such that $|w_k| < \delta$ and the function $f_k - w_k$ is σ-transverse to zero. Moreover if we have $|\partial f_k/\partial s| < 1$ and $|\partial \nabla f_k/\partial s| < 1$ then we can choose w_k and a fixed function $\Phi : [0,1/2] \to \mathbb{R}$ such that $|d^i w_k/ds^i| < \Phi(\delta)$, $(i = 1,2)$, $d^j w_k/ds^j(0) = 0$ and $d^j w_k/ds^j(1) = 0$, $\forall j \in \mathbb{N}$, where c is a universal constant.

Proof. Recall that the notion of η-transverse to $\mathbf{0}$ involves the restriction of ∇f_k to the contact distribution D_k, Definition 4. We obtain that for k large enough the distribution $D(k) = \mathrm{Ker}\ \theta(k)$ and the horizontal distribution $D_h = \mathbb{C}^n \times \{0\}$ have maximum angle less than $dk^{-1/2}$, $d > 0$ fixed, so we have that the derivatives in the D_k and in the D_h directions are at distance $O(k^{-1/2})$ because $|\partial f_k/\partial s| = O(1)$ (even after perturbing with w_k). But then $\delta(\log(\delta^{-1}))^{-p}$-transversality in D_h direction assures $\delta(\log(\delta^{-1}))^{-p'}$-transversality in D_k direction with $p' > p$ for k large enough. The proof is finished once we prove the following lemma. q.e.d.

Lemma 11. *Let $f\colon B^+ \times [0,1] \to \mathbb{C}$ be a complex valued function, where B^+ is the ball of radius 11/10 in $\mathbb{C}^n$. Let $0 < \delta < 1/2$ be a constant and let $\sigma = \delta(\log(\delta^{-1}))^{-p}$, where p is a suitable fixed universal integer. Assume that f_s satisfies the following bounds over $B^+ \times [0,1]$*

$$|f_s| \leq 1, \qquad |\bar{\partial} f_s| \leq \sigma, \qquad |\nabla \bar{\partial} f_s| \leq \sigma.$$

Then there exists a smooth curve $w\colon [0,1] \to \mathbb{C}$ such that $|w| < \delta$ and the function $f_s - w(s)$ is σ-transverse to zero over the unit ball B. Moreover if we have $|\partial f_s/\partial s| < 1$ and $|\partial \nabla f_s/\partial s| < 1$ then we can choose w and a fixed function $\Phi : [0,1/2] \to \mathbb{R}$ such that $|d^i w/ds^i| < \Phi(\delta)$ $(i = 1,2)$, $d^j w/ds^j(0) = 0$ and $d^j w/ds^j(1) = 0$ for all j, where c is a universal constant.

Proof. The lemma is a generalization of Proposition 3 in [2]. We have to obtain the bounds on the derivatives of the curve w. To be able to show that we will review Auroux's original proof.

The first step in the proof is to approximate f_s by an holomorphic function $\widetilde{f}_s$ such that $|f_s - \widetilde{f}_s|_{B,C^1} < c\sigma$ (see [4]). This process does not hold in B^+ and we need to restrict f to the unit ball. Then we approximate $\widetilde{f}_s$ by a polynomial. We can obtain polynomials g_s such that $|g_s - \hat{f}_s|_{B,C^1} < c\sigma$ and their degree d can be estimated by $O(\log \sigma^{-1})$.

Following the notation in [2] we denote by $Y_{h_s,\epsilon}$ the set of points $x \in B$ such that $|\nabla h_s(x)| < \epsilon$. The ϵ-neighborhood of $h_s(Y_{h_s,\epsilon})$ will be denoted by $Z_{h_s,\epsilon}$. Our objective will be to bound the area of the set $Z_{f_s,\sigma}$. The first observation is that thanks to the C^1 closedness of f_s and g_s, i.e., $|f_s - g_s|_{C^1} < c\sigma$, we can do it for $Z_{g_s,(c+1)\sigma}$. In fact we obtain that $Z_{f_s,\sigma} \subset Z_{g_s,(c+1)\sigma}$. Now we use a result on the complexity of semialgebraic real sets to control $Y_{g_s,(c+1)\sigma}$.

Lemma 12 (Proposition 25 of [4]). *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a polynomial of degree d, and let $S(\theta) \subset \mathbb{R}^m$ be the subset $S(\theta) = \{\, x \in \mathbb{R}^m : |x| \leq 1, |F(x)| \leq 1+\theta \,\}$. Then for arbitrarily small $\theta > 0$ there exist fixed constants C and ν depending only on the dimension m such that $S(0)$ may be decomposed into pieces $S(0) = \bigcup_{j\in J} S_j$, where $card(J) \leq Cd^\nu$, in such way that any pair of points in the same piece S_j can be joined by a path in $S(\theta)$ of length less than Cd^ν.*

Therefore the set $Y_{g_s,(c+1)\sigma}$ can be decomposed in $P(d)$ connected subsets each one of path-length at most $P(d)$ (P is a fixed polynomial). The image of each component is included in a ball B_j of radius $2(c+1)\sigma P(d)$. So $Z_{g_s,(c+1)\sigma}$ is contained in the union of $P(d)$ balls of radius $3(c+1)\sigma P(d)$. This set will be denoted by $Z_s^+(\sigma)$. Also, we define the set $DZ_s^+(\sigma)$ as the union of the same $P(d)$ balls, but with radius $6(c+1)\sigma P(d)$, namely with double radius. The complementary of DZ_s^+ in the ball of radius δ will be denoted by G_s. If we choose δ verifying $\pi\delta^2 > P(d)\pi(3(c+1)P(d))^2\sigma^2$, i.e., $\delta > Q(d)\sigma$ for a fixed polynomial Q, we obtain that G_s is not empty. Therefore we will have shown that there exists $w\colon [0,1] \to \mathbb{C}$ verifying the sought property. We have to put some conditions to assure the continuity and to bound the derivatives of the choice. For this we will study the connected components of G_s. We are going to show that they have their area bounded by $R(d)\sigma^2$ for certain universal polynomial R, except one of them which has area very big.

First we suppose that a given component does not meet the border of the ball. Then its border is defined by the border of $DZ_s^+(\sigma)$. Therefore the length of the border is less than $P(d)2\pi 6(c+1)P(d)\sigma$. So the diameter of the component is less than half of this number and the area is bounded by $Q(d)\sigma^2$ for a certain universal polynomial Q. If the component meets the border of the ball we need to be more careful. Suppose we have two different components verifying this. We can build a curve in $DZ_s^+(\sigma)$ separating the two components. The length of this component can be bounded again by $P'(d)\sigma$ and so, one of the components has area bounded by $Q(d)\sigma^2$ for a certain universal polinomial Q. We have bounded the area of all the components, except perhaps one, by $R'(d)\sigma^2$. Remember that we have bounded the area of $DZ_s^+(\sigma)$ by $Q(d)\sigma^2$, thus we can choose a polynomial R greater than R' and Q and imposing $\pi\delta^2 >> R(d)\sigma^2$ we obtain that the area of the big component can be made arbitrarily large, i.e., bigger than $10R'(d)\sigma^2$ for instance. This is the bound we find in [2] but we are going to demand that the area of this component is greater than at least half of the total area, for instance $3\pi\delta^2/5$. This large component will be denoted by $DV_\sigma^g(s)$. Obviously there will be a big component in the complementary of $Z_s^+(\sigma)$ containing $DV_\sigma^g(s)$ which will be denoted $V_\sigma^g(s)$. Moreover there will be a big component bounded by $Z_{f(t),\sigma}$ which will be denoted by $U_\sigma^f(t)$.

So the existence of these large components is assured if we impose $\delta > R(d)\sigma$, for R a universal polynomial depending only on the dimension $2n$. But recall that $d = O(\log\sigma^{-1})$, then we easily obtain that for $0 < \delta < 1/2$ there exists a positive integer p such that $\sigma = \delta(\log(\delta^{-1}))^{-p}$ implies that $\delta > R(d)\sigma$. We select this p for the statement of the lemma.

To finish the proof we would have to assure that $U_\sigma^f(t) = U$ is path connected. Auroux's proof is based in the observation that $\bigcup_s\{\, s \,\} \times Z_{f_s,\sigma}$ is closed. Then it is easy to show that U is semicontinuous and that implies that U is path connected. Once we have got a path on $U_\sigma^f(s)$ is easy to perturb it to get the desired result, but without a precise bound on the derivatives.

We need to follow an alternative way to obtain the required bounds. The important point is the following property:

$$x \in U_\sigma^f(s) \Rightarrow x \in U_{\sigma-\epsilon}^f(s+\epsilon) \tag{4.21}$$

To check it we will only have to prove that:

$$|f_s - f_{s+\epsilon}|_{C^1} < \epsilon \tag{4.22}$$

and as σ-transversality is C^1-stable we have finished. To prove (4.22) we use the following inequalities,

$$\begin{aligned} &|f_s(x) - f_{s+\epsilon}(x)| \leq |f'_{\tau_0}(x)|\epsilon, \\ &|\nabla f_s(x) - \nabla f_{s+\epsilon}(x)| \leq |\partial\nabla f_{\tau_1}/\partial s(x)|\epsilon, \end{aligned}$$

obtained as an application of the mean value theorem. But now using the hypothesis $|\partial f_s/\partial s| < 1$ and $|\partial\nabla f_s/\partial s| < 1$ we obtain the desired result.

Now we use the fact that $DV^g_\sigma(t)$ has area greater than $3\pi\delta^2/5$, then $DV^g_\sigma(s_1)\bigcap DV^g_\sigma(s_2) \neq \emptyset$ for any $s_1, s_2 \in [0,1]$. We divide the segment $[0,1]$ in subsegments of length σ. We choose $w_0 = H_0(0) \in DV^g_\sigma(0)\bigcap DV^g_\sigma(\sigma)$ and we build a line-segment H_0 by choosing $H_0(s) = w_0, \forall s \in [0,\sigma]$. Repeating the process we obtain in each subsegment a function H_i : verifiying that:

1. $H_i : [i\cdot\sigma, (i+1)\cdot\sigma] \to B(0,\delta)$,
2. $H_i(t\sigma) = w_i$, $i < t < i+1$
3. $w_i \in DV^g_\sigma(i\sigma)\bigcap DV^g_\sigma((i+1)\sigma)$.

Now we construct vertical smooth curves V_i connecting w_i and w_{i+1} contained in $DV^g_\sigma((i+1)\sigma)$. We claim that the length of these curves can be bounded by 3δ. To prove it we join w_i and w_{i+1} by using the straight segment $L_i : [0,1] \to B$, $L_i(0) = w_i$ and $L_i(1) = w_{i+1}$. If L cuts $Z^+_{(i+1)\sigma} = S$ then $L_S = L\bigcap S = L[b_1,c_1]\bigcup\cdots\bigcup L[b_l,c_l]$. Then, we change each $L[b_k,c_k]$ by a curve through the border of Z^+_t joining $L(b_l)$ and $L(c_l)$. Obviously the new curve has length less than 2δ plus the total length of the border of Z^+_t, but now recall that along the proof we have bounded this last quantity by δ. Therefore perturbing a little the curve to make it smooth we can assure that its length is less than 3δ. We parametrize the vertical curves V_i by the arc length, i.e., $V_i : [0, \mathrm{length}(V_i)] \to DV^g_\sigma((i+1)\sigma)$ and $|\frac{dV_i}{dt}| = 1$. Moreover, if we construct a curve only lying on $V^g_\sigma((i+1)\sigma)$ we can assure that its curvature is less than $c'(\sigma)^{-1}$, where $c' > 0$ is a fixed constant. We will choose the curves V_i with this additional property. We have found a continuous curve $T = H_0\bigcup V_0\bigcup H_1\cdots$ joining $V^g_\sigma(0)$ with $V^g_\sigma(1)$. By the property (4.21) T is contained in $U^f_{\sigma/2}$. We are going to perturb T to bound the derivatives.

We select a smooth function $\beta\colon [0,1] \to [0,1]$ verifying:

$$\beta(x) = \begin{cases} 0, & x \in [0,1/4], \\ 0 < \beta(x) < 1, & 1/4 < x < 3/4, \\ 1, & x \in [3/4,1]. \end{cases}$$

We compute $|\beta|_{C_2} = c_b$. We denote $\beta_i(x) = \beta(x)\cdot\mathrm{length}(V_i)$, which has norm $|\beta_i|_{C_2} < 3c_b\delta$. Finally we define

$$w((i+1)\sigma + \epsilon) = V_i\left(\beta_i\left(\frac{1}{\sigma}\left(\epsilon + \frac{\sigma}{2}\right)\right)\right), \qquad |\epsilon| \le \sigma/2.$$

The application w is smooth and we obtain that $|\frac{dw}{dt}| < 3c_b\delta/\sigma = c\delta/\sigma$. Also $|\frac{d^2w}{dt^2}| < c'\frac{1}{\sigma^2}$, because of the bounds in the curvature of the curves V_i.

Finally we observe that $|V_i(s) - w_i(s)| < \frac{\sigma}{2}$, then it implies that $w(s)$ is $\sigma/2$ transverse to $\mathbf{0}$. We can find an integer p' such that $\sigma' = \delta(\log(\delta^{-1}))^{-p'} < \sigma/2$ and so the proof is finished. q.e.d.

Remarks. 1. If we change the bounds in the derivatives in s direction by $|\partial f_s/\partial s| < c_0$ and $|\partial \nabla f_s/\partial s| < c_0$ we obtain $|\partial^i w(s)/\partial s^i| < c_0\Phi(\delta)$ $(i = 1, 2)$. We only have to change the application as $\hat{f}(s, x) = f(s/c_0, x)$ which verifies the hypothesis (except by the length of the segment). Then we apply Lemma 11 to this function. The result is reparametrized and is a solution of the problem with the expected bounds.
2. An important observation is in order here. It is not possible in general to obtain a similar result for families of sections and reproduce Auroux's results on isotopy in the contact setting. The main obstruction to this is that the techniques used in proving the parametrized controlled transversality theorem, Lemma 11, do not work for biparametric families of functions as the following elementary example shows. Take the family of functions $f_w = w$, $w \in B_2(1) \subset \mathbb{C}$. To get transversality in this family we have to avoid the zero value what is imposible because it would imply that a cell retracts to its border.

Now we translate the result on transversality to the manifold setting, for this we need to define a concept to control the different asymptotic behaviour of the derivatives in the direction of the Reeb vector field and the directions in the contact distribution D.

Definition 9. A sequence of asymptotically contact-holomorphic sections s_k of the bundles $E \otimes L^{\otimes k}$ has mixed C^2 bounds (c_D, c_R) at a point x if it verifies the following bounds:

$$|s_k(x)| < c_D, \quad |\nabla_D s_k(x)| < c_D, \quad |\nabla_R s_k(x)| < c_R,$$
$$|\bar{\partial} s_k(x)| < c_R k^{-1/2}, |\nabla_D \nabla_D s_k(x)| < c_D,$$
$$|\nabla_R \nabla s_k(x)| < c_R, \quad |\nabla \bar{\partial} s_k(x)| < c_R k^{-1/2}.$$

The sequence has global mixed C^2 bounds (c_D, c_R) if it has these mixed bounds at every point.

An important property is that given two asymptotically contact-holomorphic sequences s_k and s'_k with mixed C^2 bounds (c_D, c_R) and

(c'_D, c'_R) respectively then $s_k+s'_k$ has mixed C^2 bounds $(c_D+c'_D, c_R+c'_R)$. We can now prove the following lemma:

Lemma 13. *Let C be a contact manifold and t_k a sequence of asymptotically contact-holomorphic sections of bundles $E \otimes L^{\otimes k}$ which is η-transverse to $\mathbf{0}$. Also take a sequence of asymptotically contact-holomorphic sections s_k of $L^{\otimes k}$ with mixed C^2 bounds (c_D, c_R). Then given $x \in W_k = Z(t_k)$ and $\delta > 0$ there exists a sequence of sections $\tau_{k,x}$ of $L^{\otimes k}$ and $\sigma = \delta(\log(\delta^{-1}))^{-p}$ (for some integer $p > 0$) verifying that:*

1. *$\tau_{k,x}$ has mixed C^2 bounds*

$$(c_u \delta P(d_k(x,y)) \exp(-\lambda d_k(x,y)^2), c_u c_R \frac{\delta}{\sigma} P(d_k(x,y) \exp(-\lambda d_k(x,y)^2))$$

   at any point y.

2. *$(s_k+\tau_{k,x})_{|W_k}$ is σ-transverse in $B_{g_k}(x, \hat{c}) \bigcap W_k$ for k large enough, where λ and p are universal constants, P is a universal polynomial, $\hat{c}$ and c_u are constants independent of k, x and δ.*

We admit also the case $W_k = C$, i.e., there is no sequence t_k.

Proof. We take a localized asymptotically contact-holomorphic section $\sigma_{k,x}$ as defined in Lemma 7. So in the ball $B_{g_k}(x,c)$, $\sigma_{k,x}$ has norm greater than c_s and we can define in that ball the representative function $f_{k,x}(y) = s_k(y)/\sigma_{k,x}(y)$. We obtain that $f_{k,x}$ has global mixed C^2 bounds $(c_u c_D, c_u c_R)$.

We must notice that c_u is a universal constant associated to $\sigma_{k,x}$ that exists thanks to the bounding of the derivates of s_k and to the inferior bound c_s of $\sigma_{k,x}$. Notice also that even if we have not defined previously the concept of global mixed C^2 bounds for sequences of functions, the definition is obvious.

Using an approximately holomorphic $k^{1/2}$-Darboux trivialization chart ψ_k of W_k provided by Proposition 1 we define a sequence of asymptotically contact-holomorphic functions $\hat{f}_{k,x}$ on the ball of radius 2 in $\mathbb{R}^{2r+1}$. Using the universal constants provided by Proposition 1 we can assure that $\hat{f}_{k,x}$ has global mixed C^2 bounds $(c' c_D, c' c_R)$, where c' is a universal constant, for k large enough. Now we restrict $\hat{f}_{k,x}$ to the set $B_{2r}(1) \times [-1,1]$. The sequence of functions $\hat{f}_{k,x}$ in this set verify the hypothesis of Theorem 6, so we can obtain a sequence of functions $w_k : [-1,1] \to B$ such that $\hat{f}_{k,x}(t,z) + w_k(t)$ is η-transverse for the distribution generated by θ_k^W for k large enough. We know that $|w_k|_{C^2} < c'' c_R \frac{\delta}{\sigma}$ and $|w_k| < \delta$ (c'' universal).

Now we consider an approximately holomorphic chart

$$\varphi\colon B_{g_k}(x, dk^{1/2}) \to \mathbb{R}^{2n+1}$$

at x, using Lemma 9, for the contact manifold (C, θ). The constant d is universal as it is shown in Lemma 9. The ball $\hat{B}$ is mapped by ψ_k^{-1} into the ball $B_{g_k}(x, dk^{-1/2})$. Then we have, making the usual identifications, a function w_k defined on $\varphi(\psi_k^{-1}(B(0,1) \times [-1,1]))$. By the bounds in the derivatives of ψ_k^{-1} obtained in Proposition 1 we know that $|w_k|_{C^2} < c_u c_R \frac{\delta}{\sigma}$ and $|w_k|_{C^0} = c_u \delta$, where c_u is independent of k and x. Also, by Proposition 1, we know that $B_{g_k}(x, \hat{c}) \bigcap W_k \subset \psi_k^{-1}(B_{2(n-r)+1}(1)) \subset \psi_k^{-1}(B_{2(n-r)}(1) \times [-1,1])$. Therefore we have obtained transversality for a radius $\hat{c} > 0$ independent of k and x. To finish we will extend the definition of w_k to all $\mathbb{R}^{2n+1}$.

First, notice that in the case $W_k = \emptyset$, the Darboux chart φ is only an extension of the Darboux chart ψ_k where we have obtained transversality, so we can extend w_k directly by making it constant along the horizontal hyperplanes D_h, and giving it the value $w_k(-1)$ for $t < -1$ and $w_k(1)$ for $t > 1$.

Now consider the case $W_k \neq \emptyset$. The basic idea will be similar as before, but now notice that the horizontal distributions of the two trivializations do not coincide, thus it is not possible to simply extend the functions w_k as we did for the empty case. We can however do it modifying slighty the function w_k as explained in what follows.

The important point to consider here is that the angle[1] between D_h and TW_k at the origin is bounded below in a universal way thanks to the η-transversality of W_k. Moreover, the angle between D_h and TW_k at the points y of W_k is bounded below by a universal constant $\alpha_u > 0$, because of the universal bounds in the first and second derivatives of t_k which guarantees us not only the triviality but also the C^1 proximity of $W_k \bigcap B(x, c)$ with the tangent space of W_k at x (perhaps shrinking c slightly in a universal way).

Now we denote by γ the integral curve defined by the Reeb vector field of W_k passing through the origin of the Darboux chart, and then we extend w_k by making it constant along the horizontal hiperplanes cutting γ. We denote by $\hat{w}_k$ the extension of w_k defined in this way. Now notice that the horizontal integrable distributions D_h in $B(0, dk^{1/2}) \subset \mathbb{R}^{2n+1}$ and $\hat{D}_h$ in $B(0,2) \subset \mathbb{R}^{2r+1}$ are within $O(k^{-1/2})$ of the contact

[1]The angle between an hyperplane and a subspace is defined as the complementary of the angle between the normal vector to the hyperplane and the subspace.

distribution D. Thus in $B_{2r}(0,1) \times [-1,1]$ we obtain that

$$|w_k - \hat{w}_k|_{C^2} = O(k^{-1/2}).$$

Hence, the function $\hat{f}_{k,x} + \hat{w}_k$ will be σ'-transverse where $\sigma' = \delta(\log \delta^{-1})^{-p'}$ with p' a universal constant greater than p, for k large enough.

Finally it is possible to check that the extension of w_k verify the following bounds in the derivatives:

$$|w_k|_{C^2} < c_w c_R \Phi(\delta), \tag{4.23}$$

where c_w is independent of k and x. Now we must extend w_k to the whole $\mathbb{R}^{2n+1}$. The extension can be done simply by taking

$$w_k(z,t) = \begin{cases} w_k(z,-1), & t \leq -1. \\ w_k(z,1), & t \geq 1. \end{cases}$$

We also know by Theorem 6 that all the derivatives of w_k are zero at the border, so the performed extension is C^∞. Also (4.23) is kept for the extended function.

Finally we define $h_k = \hat{f}_k + w_k$ and $\tau_{k,x} = w_k \sigma_{k,x}$. $\tau_{k,x}$ is well defined globally because $\sigma_{k,x}(y) = 0$ if $d_k(x,y) > k^{1/4}$. We have to check that $\tau_{k,x}$ has the mixed C^2 bounds required in the statement of the lemma. The following bounds are trivial:

$$\begin{aligned} |\tau_{k,x}(y)| &< C_u \delta P(d_k(x,y)) \exp(-\lambda d_k(x,y)), \\ |\nabla_R \tau_{k,x}(y)| &< c_u c_R \Phi(\delta) P(d_k(x,y)) \exp(-\lambda d_k(x,y)). \\ |\nabla_R \nabla \tau_{k,x}(y)| &< c_u c_R \Phi(\delta) P(d_k(x,y)) \exp(-\lambda d_k(x,y)), \end{aligned}$$

Now we compute the covariant derivative in the D directions of $\tau_{k,x}$. We have:

$$\nabla_D \tau_{k,x} = d_D w_k \cdot s_{k,x} + w_k \nabla_D s_{k,x}. \tag{4.24}$$

We denote by D_h the horizontal distribution in $\mathbb{R}^{2n+1}$, and also we will call D_h to the pull-back of D_h by the application ψ_k which is a distribution defined on $B_{g_k}(x,c)$. We can check that the maximum angle between D and D_h can be bounded by $c_u k^{-1/2} d_k(x,y)(c_u$ universal). And so we obtain that:

$$\begin{aligned} |d_D w_k(x)| &\leq |d_{D_h} w_k(x)| + k^{-1/2} d_k(x,y) |dw_k(x)| \\ &\leq 0 + k^{-1/2} d_k(x,y) c'' c_R \Phi(\delta). \end{aligned}$$

For k large enough we can suppose that:

$$|d_D w_k(y)| \overset{k\to\infty}{\to} 0, \quad d_k(x,y) < ck^{1/4},$$

where $ck^{1/4}$ is the radius of the support of the section $s_{k,x}$ in the Lemma 6, and so of the restricted section $\sigma_{k,x}$ defined in Lemma 7 (c universal). Thus $d_D w_k \cdot \sigma_{k,x}(y) = 0$, if $d_k(x,y) > ck^{1/4}$. Then we have that $d_D w_k \cdot \sigma_{k,x} < \epsilon$ for all $\epsilon > 0$ if we take a large enough k, so we can take it out from the expression (4.24). Now we can bound it:

$$|\nabla_D \tau_{k,x}| < \delta c_u P(d_k(x,y)) \exp(-\lambda d_k(x,y)),$$

as we wanted. To take into account the effect of the first summand we only have to change c_u. To bound $\nabla_D \nabla_D \tau_{k,x}$ we expand as a product again:

$$\nabla_D \nabla_D \tau_{k,x} = \nabla_D d_D w_k s_{k,x} + 2 d w_k \nabla_D s_{k,x} + w_k \nabla_D \nabla_D s_{k,x}.$$

The first two terms can be bounded by an arbitrary small $\epsilon > 0$ following the same ideas that in the precedent case. The third verifies the required bounds by a simple computation.

Let us go to the $\bar\partial$ bounds. We can write:

$$|\bar\partial_D w_k(y)| \leq |\bar\partial_{D_h} w_k(y)| + |(\bar\partial_D - \bar\partial_{D_h}) w_k| \leq c_u c_R \Phi(\delta) k^{-1/2} d_k(x,y).$$

The bound is obtained recalling that D_h and D are at distance $c_u k^{-1/2} d_k(x,y)$. So we finally write:

$$\begin{aligned} |\bar\partial_D \tau_{k,x}(y)| &\leq |\bar\partial_D w_k(y)| |\sigma_{k,x}(y)| + |w_k(y)| |\bar\partial_D \sigma_{k,x}(y)| \\ &\leq c_u c_R \Phi(\delta) k^{-1/2} d_k(x,y) P(d_k(x,y)) \exp(-\lambda d_k(x,y)) \\ &\quad + \delta c_u k^{-1/2} P(d_k(x,y)) \exp(-\lambda d_k(x,y)) \\ &\leq k^{-1/2} c_u c_R \Phi(\delta) P'(d_k(x,y)) \exp(-\lambda d_k(x,y)). \end{aligned} \tag{4.25}$$

The last inequality is based on the asumption $\Phi(\delta) > \delta$, which is obtained if δ is small enough (it will be the usual case), and $P'(t)$ is an universal polynomial greater than $P(t)$ and $t \cdot P(t)$.

Finally we have to bound $\nabla \bar\partial \tau_{k,x}$, and this is made using the same ideas of the former case. So the proof is ended. q.e.d.

4.3 The globalization process

The third ingredient in the proof of Theorem 5 relies on a globalization process invented by Donaldson and refined by Auroux [4, 2]. However we have to generalize the process to adapt it to the situation where we have no uniform control in all the directions.

We will need a technical result which will simplify the proof of Theorem 5:

Proposition 2. *Let t_k be an asymptotically contact-holomorphic sequence of sections of the bundle $E \otimes L^{\otimes k}$ which is η-transverse to $\mathbf{0}$ on an open subset $U \subset C$. If s_k is an asymptotically contact-holomorphic sequence of sections of $L^{\otimes k}$ over U verifying that restricted to $Z(t_k) \bigcap U = W_k \bigcap U$ is η'-transverse to $\mathbf{0}$, we obtain that there exists a constant $\eta'' > 0$ depending only on η, η' and bounds on s_k and t_k such that $t_k \oplus s_k \in \Gamma(U, (E \bigoplus \mathbb{C}) \otimes L^{\otimes k})$ is η''-transverse to $\mathbf{0}$ over U_k^-, where U_k^- is the set of points of U with distance to the border greater than $k^{1/6}$ in g_1 metric.*

Proof. We proceed following step by step [2] Section 3.6. First we compute an inverse for the points $x \in U \bigcap W_k$ such that $|s_k| < \eta'$. We know that ∇t_k is surjective and vanishes in all the directions defined by TW_k. On the other hand s_k has a tangencial component with norm greater than η'. It implies the surjectivity of $t_k \oplus s_k$. We construct a right inverse to check its norm.

We choose $v \in T_x W_k$ with norm $|v| < (\eta')^{-1}$ such that $\nabla s_k(v) = \hat{v}$ has norm equal 1. Also we select $(\hat{v}_1, \dots, \hat{v}_r)$ an ortonormal basis in $E \otimes L^{\otimes k}_{|x}$. And because of the η-transversality of t_k there exists a right inverse R of t_k such that $R(\hat{v}_i) = v_i$ and $|v_i| < \eta - 1$. We have that $\nabla s_k(v_i) = \lambda_i \hat{v}$. The constants λ_i are bounded by a universal constant c thanks to the asymptotic holomorphicity of s_k. Then we can define R' a right inverse for $\nabla(t_k \oplus s_k)$ in the following way:

$$\begin{aligned} R'(\hat{v}) &= v, \\ R'(\hat{v}_i) &= v_i - \lambda_i v, \end{aligned}$$

which obviously has norm bounded by a universal constant γ^{-1}.

Suppose now that t_k and s_k have norm smaller than a certain α at a point x. We will shrink α along the proof. The first bound will be $\alpha < \eta$, then we obtain that t_k is η-transverse to $\mathbf{0}$ at the point. Even more the gradient flow of $|t_k|$ brings x to a zero $y \in W_k$ and because of the η-transversality $d_k(x, y) < \eta^{-1} \cdot \alpha$. For k large enough this distance will be smaller than $k^{1/6}$ and $y \in U \bigcap W_k$ if $x \in U_k^-$. By asymptotic contact-holomorphicity $|\nabla s_k| < c$, then $|s_k(y)| < |s_k(x)| + c\eta^{-1}\alpha$. So we choose α verifying $\alpha < \eta'/2$ and $c\eta^{-1}\alpha < \eta'/2$. Therefore we obtain $|s_k(y)| < \eta'$. And then $t_k \oplus s_k$ is γ-transverse at y, but we know that $|\nabla\nabla t_k \oplus s_k| < c'$, then $|\nabla t_k \oplus s_k(x)| > |\nabla t_k \oplus s_k(y)| - c'\alpha$. Again, we can shrink α to obtain certain η''-transversality to $\mathbf{0}$ with $\eta'' > 0$. q.e.d.

We now show how to derive Theorem 5 from Lemma 13 and Proposition 2. The proof is a generalization of the Donaldson's argument [4] which takes into account the different behaviour along the Reeb vector field and the contact distribution directions.

Proof of Theorem 5. We set a Darboux covering $\{ U_i \}$ of (C, θ) such that $E \otimes L^{\otimes k}$ trivializes on U_i, i.e., we choose contractible neighborhoods. The problem reduces to get certain η-transversality on each U_i perturbing a family s_k of asymptotically contact-holomorphic sections. Because if we obtain η_1-transversality in U_1 by a perturbation, afterwards we can perturb s_k by a perturbation with mixed C^2 bounds $(\eta_1/2, c_R^2)$ to achieve η_2'-transversality in U_2 and then it implies that s_k is $\min\{ \eta_2', \eta_1/2 \} = \eta_2$-transverse to $\mathbf{0}$ in $U_1 \bigcup U_2$. So iterating the process we get σ_k a family of sections contact-holomorphic and η_r-tranverse to $\mathbf{0}$ in C.

So we restrict ourselves to one of these U_i and then $E \otimes L^{\otimes k} = \mathbb{C}^r$. Remark that it is an hermitian vector bundle with connection $\nabla_E \otimes I + I \otimes \nabla_{L^{\otimes k}}$, so for k large enough the contribution of the connection of E in the D directions will be worthless.

We are going to perturb a section $s_k = (s_k^1, \dots, s_k^r)$ adding a section τ_k^j to achieve certain η-transversality on U_i. For this we define $W_k^j = \{ x \in C \mid s_k^1(x) = s_k^2(x) = \cdots = s_k^j(x) = 0 \}$, and $W_k^0 = U_i$. To obtain the transversality we proceed by perturbing in r steps. In each step we will prove that s_k^j can be perturbed to be η_j-transverse to $\mathbf{0}$ when restricted to $W_k^{j-1} \bigcap U_i$. This implies the η-transversality of s_k on $(U_i)_k^-$ by Proposition 2. We make sure that the perturbation can be chosen with mixed C^2 bounds $(\frac{\eta_{i-1}}{2r}, c_R^{ij})$, c_R^{ij} arbitrary but independent of k. It assures that the total perturbation τ_k^j has mixed C^2 bounds $(\eta_{i-1}/2, \sum_j c_R^{ij})$ which are the required bounds to globalize the process.

Now we fix our atention to prove the following result which finishes the proof:

Proposition 3. *Given an asymptotically contact-holomorphic sequence t_k of sections of vector bundles $E \otimes L^{\otimes k}$ η-transverse to $\mathbf{0}$ on an open subset U with zero set W_k, let s_k be an asymptotically contact-holomorphic sequence of sections of $L^{\otimes k}$ with global mixed C^2-bounds (c_D, c_R). Then given $\delta > 0$, there exists an asymptotically contact-holomorphic sequence of sections τ_k of $L^{\otimes k}$ verifying:*

1. *$s_k + \tau_k$ restricted to W_k is η'-transverse to $\mathbf{0}$.*

2. *τ_k has global mixed C^2 bounds (δ, c'_R), where c'_R is independent of k.*

Obviously if we prove Proposition 3, then the proof of Theorem 5 is complete. q.e.d.

Proof of Proposition 3. The key idea is a small generalization of Donaldson's argument to control the interference between local perturbations. We will take a finite set of points $S = \bigcup_m S_m$ of $U \bigcap W_k$ verifying:

- All the points of S_m are at distance greater than a fixed Δ (to be chosen later).

- The number of subsets S_m is of the order of $O(\Delta^{2n+1})$.

- The balls $B(x_p, c)$, $x_p \in S$ form an open covering of $U \bigcap W_k$. c is the universal constant obtained in Lemma 13.

That such set will exist can be easily seen by considering a covering like in [4], §2, pages 680-681, with a distance 2Δ between balls. Then we select only those that cut W_k and we take a point in the corresponding intersection as a center of a new collection of balls of radius c. These new balls obviously cover W_k and the distance between points on the same set S_m is clearly greater than Δ.

We are going to get transversality all over the balls centered at points of S_m at a time using Lemma 13 perturbing $\delta/2$ in C^0 norm. Taking into account the exponential decay of the mixed C^2 bounds obtained in the perturbation, we can assure that at a point $x \in S_m$ the rest of the perturbations due to S_m has mixed C_2 bounds

$$(K(\delta \exp(-\lambda D^2)), K(\frac{\delta}{\sigma} \exp(-\lambda D^2))),$$

K universal constant, in the ball $B(x,c)$. Hence choosing D large enough (but independent of k) we can assure that the perturbed section is $\eta_m = \sigma/2$-transverse $\mathbf{0}$ in $E_m = \bigcup_{x\in S_m} B(x,c) \bigcap W_k$.

We have only to iterate the process assuring that the perturbation in the set E_{m+1} has to have global mixed C_2 bounds $(\eta_m/2, c_R^{m+1})$ not to destroy the achieved transversality in S_m. The important point now is that this condition on D, that assures noninterference among the elements of S_{m+1}, can be written as:

$$\eta_{m+1} > c\eta_m \exp(-\lambda D^2), \tag{4.26}$$

where c is a universal constant. It has to be so because δ has to be a sligthly smaller than η_m. Equation (4.26) has to be verified in all the steps of the process. Recalling the formula for σ obtained in Lemma 13 we write:

$$\exp(\lambda D^2) > K_0(\log(\eta_m^{-1}))^p$$

K_0 is a universal constant. Now recall that the number of subsets S_m is of order $O(D^{2n+1})$. So the process has $N = O(D^{2n+1})$ iterations. We have to study the sequence η_m. In Lemma 24 in [4] is proved that it satisfies $\log(\eta_m^{-1})^p = O(m\log(m))$. Therefore we obtain $(\log\eta_N^{-1})^p = O(D^{2NP}(\log D^{2N})^p)$ which can be bounded by $\exp(\lambda D^2)$ for D large enough.

To conclude the proof we recall that in Lemma 13 the section s_k need to have well defined global mixed C^2 bounds. For this we have to make sure that the numbers c_R^m can be bounded independently of k. This will assure also the asymptotic contact-holomorphicity of the resultant sequence. But this is the case because c_R^1 is a function of δ, the c_R bound of s_k, and some universal constants. In general c_R^{m+1} is a function of c_R^m, δ_m and some universal constants. So if D is independent of k we can assure the universal bounding of c_R^m for all m. The proof is ended. q.e.d.

Remark. In general it is not possible to perturb uniformly a one-parameter family of asymptotically contact-holomorphic sections to make them transverse to $\mathbf{0}$. An immediate consequence of this negative observation is that the families of contact submanifolds constructed using this method could fall in different isotopy classes.

4.4 Equivariant Donaldson-Auroux theory for free discrete group actions

In order to apply Donaldson-Auroux theory to the nonexact contact case, we will need to lift the problem to a contact manifold with a $\mathbb{Z}_2$ anticontact action, namely its exact covering. The contact submanifolds obtained by a direct application of Donaldson-Auroux theory as in Section 4 are not $\mathbb{Z}_2$-invariant in general, and they will project to a manifold with singularities (of codimension $4r$ generically). Thus we shall need to adapt the construction of contact sections to a $\mathbb{Z}_2$ setting. We shall do that in two steps, first, we shall find an equivariant construction for free finite group actions acting by exact contactomorphisms. An exact contactomorphism is an application $\phi : (C, \theta) \to (C, \theta)$ verifying that $\phi^* \theta = \theta$. Notice that the condition of preserving the contact structure is weaker than this because the application must verify only that $\phi_* D = D$. Second, we will refine the construction of asymptotic J-holomorphic sections to solve the required case, because in our case the $\mathbb{Z}_2$ action is not exact contact. In fact, the condition we have is $a^* \theta = -\theta$, where a is the nonidentity element of $\mathbb{Z}_2$. This kind of $\mathbb{Z}_2$-action will be called anticontact.

Exact contactomorphic actions.

Let G be a finite group acting freely and by exact contactomorphisms on the compact exact contact manifold (C, θ). If we denote by g the elements of the group, this means that $g^* \theta = \theta$ for all $g \in G$. The action induces a symplectic action (in fact, hamiltonian) on the symplectization. It is known that a hamiltonian action on a symplectic manifold M induces a natural bundle action of G on the line bundle $L^{\otimes k} \to M$ when k is large enough, see [16]. So we conclude that G acts on $L^{\otimes k} \to S_D(C)$ when k is large enough. We also assume that E is G-invariant, i.e., there exists a bundle action of G on it. So, the pull-back $\pi^* E$ to the symplectization is acted by G. Thus the group G acts symplectically in $\pi^* E \otimes L^{\otimes k}$ when k is large enough. Interpreting C as a submanifold of $S_D(C)$ by means of the graph of θ, we obtain that G acts in the restriction of $\pi^* E \otimes L^{\otimes k}$ to C. But this restricted bundle is $E \otimes L^{\otimes k}$. Therefore for k large enough G acts on the family of bundles we need.

It is a standard fact the existence of G-invariant complex structures on D compatible with $d\theta$, for any compact group acting on D. We choose one of these compatible almost complex G-invariant J's. So we can reproduce all Donaldson-Auroux preliminar construction in a G-invariant setting, arriving to the construction of an asymptotically contact-holomorphic sequence of sections s_k. Summarizing, Subsection 4.1 can be carried out in a G-invariant setting (in fact for G a compact Lie group).

Now, consider Subsections 4.2 and 4.3 for a finite group G acting on C. The important point is to prove Proposition 3 for G-invariant sequences. We are going to follow the proof making changes whenever necessary. Notice that we can choose a G-invariant set of points $S = \bigcup S_m$ on C, moreover we need to assure that S_m is G-invariant. In fact, starting with any set S which verifies the three conditions required in Proposition 3, we will construct S_G as $S_G = \bigcup_{g\in G} g \cdot S$. Also, $(S_G)_m = \bigcup_{g\in G} g \cdot S_m$. The action is free, C is compact and G is finite so there exists $m_G > 0$ such that:

$$d(x, gx) > m_G, \quad \forall x \in C, g \neq e. \tag{4.27}$$

Suppose that the radius of each S_m is less than $m_G/3$. This can be done simply by subdividing each S_m into a number of subsets (this number is independent of k). Therefore, for k large enough, the set S_G verifies the three properties needed to globalize. Recall that the difficult point is to assure that the g_k distance between 2 points in $(S_G)_m$ is greater than D.

Finally, we are going to obtain transversality in a G-invariant asymptotically J-holomorphic sequence s_k, preserving the G-invariance. The idea is to use the normal argument, but in all the points $\{ gp_i \}_{g\in G}$ at a time. We apply the local transversality result to a point $p \in (S_G)_m$, and we search a complex section $\tau_{k,p}$ which obtains a certain η-transversality in a neighborhood of p of g_k-radius $\hat{c}$.

Up to now all the objects in the construction are G-invariant, except $\tau_{k,p}$. Recall from (4.27) that:

$$\lim_{k\to\infty} d_k(g_1 p, g_2 p) = \infty, \quad g_1 \neq g_2,$$

and so for k large enough the supports of $(g_1)_* \tau_{k,p}$ and $(g_2)_* \tau_{k,p}$ are disjoint. This implies that $\tau_{k,p}^G = \Sigma_{g\in G} g_* \tau_{k,p}$, which is obviously G-invariant, obtains η-transversality in $\bigcup_{g\in G} B_{g_k}(gp, c)$.

So, after adding $\tau_{k,p}^G$, the new section is again G-invariant. Now we follow step by step the proof of Proposition 3 to obtain a G-invariant asymptoticallycontact-holomorphic sections with certain $\eta > 0$ transversality.

Thus, we have proved:

Proposition 4. *Let G be a finite group acting freely and by exact contactomorphisms on an exact contact manifold (C, θ). Suppose also that G acts on a complex vector bundle E. Then there exists a G-invariant sequence of asymptotically J-holomomorphic sections of the vector bundle $E \otimes L^{\otimes k}$ transverse to zero.*

Remark. Following step by step the precedent discussion we can adapt the results to the case of symplectic manifolds acted symplectically and freely by a finite group. We refer the reader to Donaldson's article [4] for details of the standard construction, the reader can check that the adaptation is direct.

Anticontactomorphic $\mathbb{Z}_2$-actions

The precedent computation solves the problem of finding G-invariant sections in a contact manifold acted by a free action of a finite group. But we know that it is almost the situation in the nonexact contact case. We have shown in Section 2.5 that choosing a bundle metric, η, in $S_D(C)$ we construct a double covering $\hat{C}$ of C selecting the vectors v such that $\eta(v, v) = 1$ in the fibres. The $\mathbb{Z}_2$ action given by the application $a(\alpha) = -\alpha$ is anticontactomorphic and free. Then we are going to obtain $\mathbb{Z}_2$-invariant contact submanifolds. The projection to C of these submanifolds will produce smooth contact submanifolds on C which are Poincaré dual of the top Chern class of E. To do it we fix a contact form θ on $\hat{C}$. One can impose that $a^*\theta = -\theta$ with a suitable choice of θ. This implies that $a^*d\theta = -d\theta$. Moreover we can fix an almost complex structure satisfying that $a^*J = -J$. This implies that $a^*g_k = g_k$. Now, the problem is that $L^{\otimes k}$ is not $\mathbb{Z}_2$-invariant. In fact, the aplication a lifts to a bundle morphism $\widetilde{a} : L \to \bar{L}$ defined by the identity in the fibres, recall that $L = \hat{C} \times \mathbb{R}$. Moreover $\widetilde{a}$ preserves the connection in the bundles, because,

$$\begin{aligned} \nabla_L &= d + i\theta. \\ a^*\nabla_L &= d - i\theta. \\ \nabla_{\overline{L}} &= d - i\theta. \end{aligned}$$

Finally check that if s_k is an asymptotically contact-holomorphic sequence of sections s_k defined on the bundles $E \otimes L^{\otimes k}$, being E equivariant, then we have that a^*s_k is an asymptotically contact-holomorphic sequence of sections on the bundles $E\otimes \overline{L}^{\otimes k}$ with respect to the contact form $-\theta$ and the almost-complex structure $-J$. So, with the canonical identification $L\simeq \overline{L}$, we claim that a^*s_k is an asymptotically holomorphic sequence of sections on the bundles $E\otimes L^{\otimes k}$ with respect to the contact form θ and the almost complex structure J. To check it we only need to observe that we are permuting the sign in the almost complex structure in the bundle and in the manifold, so the Cauchy-Riemman condition does not change. At this point we can repeat all the proof given for exact contactomorphic actions, because all the ideas apply. We arrive in this way to the following result.

Corollary 1. *Let C be a (possibly nonexact) contact manifold. Let E a rank r complex vector bundle over C. Then, there exists a sequence of submanifolds W_k of codimension $2r$, such that, for k large enough, they are contact submanifolds of C and they are Poincaré dual of the top Chern class of E.*

5. The topology of contact submanifolds

In this section we give the topological characterization of the constructed submanifolds. The important point is that Lefschetz hyperplane theorem works in the contact case.

5.1 Lefschetz theorem for contact submanifolds

Following Donaldson-Auroux techniques [2] we are going to generalize the classical Lefschetz hyperplane theorem to the contact category. Thus, the idea is to adapt the symplectic argument. For the exact case, we start with any asymptotically contact-holomorphic sequence η-transverse to zero of sections s_k of the bundles $E\otimes L^{\otimes k}\to C$. We define on $C-W_k$, the smooth real function

$$\varphi_k=\log|s_k|^2. \tag{5.27}$$

In the nonexact case we pass to the double covering $\hat{C}$ and there we construct $\hat{\varphi}_k$ as in Equation (5.27). But it is clear that $\hat{\varphi}_k$ is $\mathbb{Z}_2$-invariant, so we can quotient and define a function φ_k on C. Along the proof we make the computations only for the function φ_k defined on the exact case. But, it is an easy exercise to check that these computations work in the nonexact case.

The proof of the isomorphisms and surjections between the homotopy and homology groups of C and W_k for k large enough enunciated in Theorem 1, is equivalent to assure that all the critical points of φ_k have index greater than $n-r$.

First we claim that any critical point x of φ_k verifies $|s_k(x)| \geq \eta$. This is a simple observation because of the η-transversality of the sequence s_k. In fact, if this were not the case, then $\nabla s_k(x)$ will be surjective, and so there exists a vector $v \in T_xM$ such that $\nabla_v s_k(x) = \overline{s_k(x)}$. But,

$$d\varphi_k = \frac{1}{|s_k|^2}(\langle \nabla s_k, s_k\rangle + \langle s_k, \nabla s_k\rangle),$$

hence $\langle \nabla_v s_k, s_k\rangle > 0$ and then $d(\varphi_k)_v(x) > 0$. Therefore x is not a critical point.

We decompose now $d\varphi_k(x) \in T_x^*C$ as in Equation (2.6):

$$d\varphi_k = \partial\varphi_k + \bar{\partial}\varphi_k + d^\perp\varphi_k.$$

Then we have:

$$\partial\varphi_k = \frac{1}{|s_k|^2}(\langle \partial s_k, s_k\rangle + \langle s_k, \bar{\partial} s_k\rangle). \tag{5.28}$$

On the other hand, it is clear that at a critical point x of φ_k, we have

$$\partial\varphi_k(x) = \bar{\partial}\varphi_k(x) = d^\perp\varphi_k(x) = 0.$$

Because of Equation (5.28) we obtain easily the following bounds in a small neighborhood of the critical point x,

$$|\langle \partial s_k, s_k\rangle| < c|s_k|, \qquad |\langle s_k, \bar{\partial} s_k\rangle| < c|s_k|, \tag{5.29}$$

where c is independent of k.

Differenciating again Equation (5.28) and restricting ourselves to the critical point x we obtain:

(5.30)
$$\bar{\partial}\partial\varphi_k = \frac{1}{|s_k|^2}(\langle \bar{\partial}\partial s_k, s_k\rangle - \langle \partial s_k, \partial s_k\rangle + \langle \bar{\partial} s_k, \bar{\partial} s_k\rangle + \langle s_k, \partial\bar{\partial} s_k\rangle).$$

Recall that $\bar{\partial}\partial + \partial\bar{\partial} = R^{1,1}$, where $R^{1,1}$ is the $(1,1)$ component of the curvature of the connection ∇ of the bundle $E \otimes L^{\otimes k}$ restricted to D, which is equal to $-ik\omega \otimes I + R_E^{1,1}$ because of Equation (2.4), where $R_E^{1,1}$ denotes the $(1,1)$ component of the curvature of the connection ∇_E of E restricted to D. Then we can write (5.30) as follows:

$$\begin{aligned} \bar{\partial}\partial\varphi_k = & -ik\omega + \frac{1}{|s_k|^2}(\langle R_E^{1,1} s_k, s_k\rangle - \langle \partial\bar{\partial} s_k, s_k\rangle \\ & + \langle s_k, \partial\bar{\partial} s_k\rangle - \langle \partial s_k, \partial s_k\rangle + \langle \bar{\partial} s_k, \bar{\partial} s_k\rangle). \end{aligned} \tag{5.31}$$

For simplicity in what follows we will use the norms defined in the metric g_1 and the equivalent bounds for the asymptotic holomorphic sequences, as given in [2]. Now we need to restrict our attention to a subspace where the first term in the r.h.s. of Equation (5.31) expression dominates the second for k large enough. A sufficient condition for this is to ensure that $|\partial s_k| = O(1)$ because, thanks to the asymptotic contact-holomorphicity of the sequence s_k, the second term in the r.h.s. of eq. (5.31) will be of the order $O(k^{1/2})$ compared to the order $O(k)$ of the first term.

We define the following complex subspace of D_x:

$$H_x = \{v \in D_x, |\ \partial_v s_k(x) = \lambda s_k(x) \text{ for some } \lambda\ \in \mathbb{C}\},$$

it is clear that if $v \neq 0$ is in H we have:

$$|\partial s_k(v)| = \frac{|\langle \partial s_k(v), s_k\rangle|}{|s_k|} < C\frac{|s_k||v|}{|s_k|} = C|v|.$$

and the last inequality is based in the bound given by Equation (5.29). Then we see that $|\partial s_{k|H}| = O(1)$ and consequently:

$$\bar{\partial}\partial\varphi_k = -ik\omega + O(k^{1/2}),$$

on H.

Moreover we must notice that $-2i\bar{\partial}\partial\varphi_k(u, Ju) = H_{\varphi_k}(u) + H_{\varphi_k}(Ju)$, where H_{φ_k} is the Hessian of φ_k restricted to D. Obviously for k large this number is negative because, $\omega(\cdot, J\cdot)$ is a metric.

Finally, let us suppose that the index of the critical point x were less than $n - r + 1$, then there will exist a subspace $P \subset D_x$ of real dimension at least $n+r$ such that H_{φ_k} is nonnegative on it, and $P \bigcap JP$

will have dimension at least $2r$. On the other hand the subspace H has dimension at least $2n-2r+2$ because of the linearity of ∂s_k on D and the η-transversality of s_k. Hence, the intersection $H\bigcap(P\bigcap JP)$ will be nonvoid which is a contradiction. Then an standard Morse theory argument gives us the isomorphisms and surjections required.

Remark. Notice that in the previous proof it is absolutely essential the bound below for $|s_k|$ at critical points, and this is a direct consequence of the η-transversality of the sequence of asymptotically contact-holomorphic sections s_k.

5.2 The Chern polynomial of contact submanifolds

We can compute the Chern polynomial, $c(D\bigcap TW_k)=c(D_{W_k})$, of the hyperplane bundle distribution D_{W_k} of the contact submanifold W_k in terms of the Chern polynomial of the total bundle $E\otimes L^{\otimes k}$ and the Chern polynomial of the contact distribution D restricted to W_k. For this we recall that W_k is always transverse to D, then we have that

$$D_{|W_k}\cong D_{W_k}\oplus\nu(W_k)$$

where $\nu(W_k)$ denotes the normal bundle D/D_{W_k}. We can identify the normal bundle $\nu(W_k)$ with a complex subbundle of D by choosing a compatible metric such that the Reeb vector field R on C will be orthogonal to D and D_{W_k} a complex subspace. Then the symplectic orthogonal $D^{\perp}_{W_k}$ is complex too.

Now, we have:

$$i^*c(D)=i^*c(E\otimes L^{\otimes k})\cdot c(D_{W_k})$$

and, finally, we obtain the formula:

$$c(D_{W_k})=i^*\left(c(E\otimes L^{\otimes k})^{-1}\cdot c(D)\right).$$

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