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On the construction of designs with three-way blocking

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Abstract. Experimental designs under two-way blocking structure are used to separate out two cross-classified non-interacting sources of variation in the experimental material. However, there may exist situations wherein the experimental area contain some stony patches or other features that tend to clump in compact areas where these designs cannot capture the variability due to these compact areas. For such situations, experimental designs that are capable of removing three sources of variability can be more advantageously used. In this paper, some methods of constructing designs under three-way blocking structure have been developed. Joint information matrix has been derived and the efficiency factor of these designs has been computed as compared to an orthogonal design. List of parameters of the designs obtained has been prepared for number of treatments < 25 along with the efficiency factor for each design.

Keywords: Gerechte designs, partially variance balanced, rectangular association scheme, structurally incomplete, Sudoku designs, variance balanced

1. Introduction

Experimental designs under two-way blocking structure are used to separate out two non-interacting sources of variation in the experimental material. These designs may not be able to mark out stony patches or other features that tend to clump in compact areas. In these situations, where there are three sources of variability, experimental designs under three-way blocking can be advantageously used. Gerechte designs introduced by Behrens [3] are a popular class of designs that are used to separate three sources of variation from the experimental material. In these designs, an $n \times n$ square grid is partitioned into n regions $S_1, S_2, \ldots, S_n$, each containing n cells of the grid. The symbols $1, \ldots, n$ are to be placed in the cells of the grid in such a way that each symbol occurs once in each row, once in each column and once in each region. Stewart and Bradley [7] discussed some adjusted orthogonal, variance balanced, multi-dimensional block designs. Design plans and efficiencies of seven three-dimensional designs, each nearly optimal in defined classes of designs, are listed in this study. Bailey et al. [1,2] have shown that the regions of a Gerechte design are always taken to be spatially compact areas. In these designs, rows, columns and additionally, compact regions are treated as blocks. Gerechte designs were originated in statistical design of agricultural experiments where it is ensured that treatments are fairly exposed to localized variations in the field containing experimental plot. These designs are capable of marking out stony patches, lumpsy soil, or other features that tend to clump in compact areas. They defined orthogonal and multiple Gerechte design and also discussed the randomization procedure of this design. Vaughan [9] discussed the problem of existence of a Gerechte design for a given partition of the cells and explored the relationship of Gerechte design with graph theory and also discussed the situations when a Gerechte design has completion or not. Courtiel and Vaughan [4] showed that for all positive

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integers s and t, any Gerechte framework where each region is either an $s \times t$ rectangle or a $t \times s$ rectangle is realizable.

Sudoku designs form a special case of Gerechte designs that are available for composite numbers. Sudoku design is constructed in square shape and regions are also square in shape and hence came the name Sudoku square design. Sudoku (9 × 9) design with 3 × 3 regions is the most common Sudoku design. Hui-Dong and Ru-Gen [5] discussed the basic properties of $k \times k$ Sudoku squares with construction procedure and then the Sudoku square is made into a new design applied to field experiment. This design can make a layout of k treatments with k replications and control the three-way soil-environmental variation. They have also presented the construction procedure of Sudoku squares, the mathematical model and statistical method of analyzing data from a Sudoku square design. Subramani and Ponnuswamy [8] discussed the construction of a class of Sudoku designs of order m^2. They presented four different models for analyzing the data obtained from such design. They also presented an illustration of analysis with a hypothetical data set. Also, the application potential of these designs in various fields of agriculture has been discussed. Xu et al. [12] studied the Sudoku based space filling designs.

In this paper, some methods of constructing designs under three-way blocking structure have been developed. The information matrix for estimating the contrasts pertaining to treatment effects has been derived and the efficiency factor of these designs has been computed. List of parameters of the designs obtained has been prepared for number of treatments < 25 along with the efficiency factor for each design.

2. Experimental setup and model

Let the v treatments are arranged in an experimental set up under three-way blocking structure in p rows, q columns and m regions (patches). The following four-way classified model with treatments, rows, columns and regions as the four classifications, is considered:

$$\begin{aligned} &y_{ijkh} = \mu + \tau_i + \alpha_j + \beta_k + \gamma_h + e_{ijkh}; \\ &\quad i = 1, 2, \ldots, v; j = 1, 2, \ldots, p; k = 1, 2, \ldots, q; h = 1, 2, \ldots, m \end{aligned} \tag{1}$$

where, y_{ijkh} is the response from the j^{th} row, k^{th} column and h^{th} region receiving i^{th} treatment. μ is the general mean, τ_i is the i^{th} treatment effect, α_j is the j^{th} row effect, β_k is the k^{th} column effect and γ_h is the h^{th} region effect. e_{ijkh} is the error term identically and independently distributed and following normal distribution with mean zero and constant variance. Under this model, the information matrix for estimating the contrasts pertaining to the treatment effects has been obtained. A detailed derivation has been given in the Appendix.

3. Some definitions

Some terms that are used in the successive sections are defined below.

3.1. Regular/Irregular regions

If all the regions belonging to a design under three-way blocking structure are of same size and shape, i.e., same number of treatments are appearing in each region and the shape of each region is uniform, then such designs are said to be regular. Otherwise, they are called irregular.

3.2. Structurally incomplete designs

A design under three-way blocking structure is said to be structurally incomplete if there is at least one row-column intersection which does not receive any treatment.

4. Methods of construction

The general methods for constructing designs under three-way blocking structure are explained in Sections 4.1 and 4.2 along with appropriate examples.

4.1. Designs under three-way blocking structure involving rectangular regions

Let the number of treatments, v be an even number. Write the v numbers in natural order in the first row of a $v \times v$ array and form $\left(\frac{v}{2}\right)$ consecutive pairs of numbers. Start the second row with the second pair of treatments and end with first pair, in a circular manner. Similarly, the third row starts with third pair and so on, the $\left(\frac{v}{2}\right)^{\text{th}}$ row starts with $\left(\frac{v}{2}\right)^{\text{th}}$ pair. Obtain another set of $\left(\frac{v}{2}\right)$ rows by interchanging the order of each pair and juxtapose these rows below the first set of $\left(\frac{v}{2}\right)$ rows. The resulting array has every treatment appears in each row and each column exactly once. Now, v regions are formed by partitioning the v rows into two equal parts and by taking two consecutive columns together such that every treatment appears in each region precisely once. In this type of designs, rows, columns and regions are complete and the shape of the regions are regular (rectangular). These designs are found to be variance balanced with efficiency factor as unity. For this class of designs parameters are v (number of treatments) $= r$ (number of replications of treatments) $= p$ (number of rows) $= q$ (number of columns) $= m$ (number of regions). General form of information matrix for estimating the elementary contrasts pertaining to treatment effects is obtained using the Model (1) as given below:

$$C_\tau = v\left(I_v - \frac{J_v}{v}\right)$$

The canonical efficiency factor of these designs in comparison to an orthogonal design having the same number of treatments and assuming the same variance has been computed and is unity for all the designs in this class. Parameters of the designs along with estimated variance ($V_{ij}; i, j = 1, 2, \ldots, v; i \neq j$) have been listed in Table 1 for $v < 25$.

Example 1. Let $v = 8$ giving rise to 4 pairs of treatments in the first row written in natural order. Start the second row with the second pair of treatments and end with first pair, in a circular manner. Similarly, the third row starts with third pair and fourth row starts with fourth pair. Remaining 4 rows are obtained by interchanging the order of each pair in the first 4 rows. Taking rectangular regions 4×2 size together, eight regions are formed.

1	2	3	4	5	6	7	8
3	4	5	6	7	8	1	2
5	6	7	8	1	2	3	4
7	8	1	2	3	4	5	6
2	1	4	3	6	5	8	7
4	3	6	5	8	7	2	1
6	5	8	7	2	1	4	3
8	7	2	1	4	3	6	5

The parameters of this design are $v = r = p = q = m = 8$. The information matrix for estimating the elementary contrasts pertaining to treatment effects is $C_\tau = 8(I_8 - \frac{J_8}{8})$ and the estimated variance of elementary contrasts pertaining to treatment effects is 0.25.

Particular Case 1. Designs under three-way blocking structure for rectangular regions with one empty node

Consider an experimental situation where one empty node is present in every region. A class of structurally incomplete designs for three-way elimination of heterogeneity having rectangular regions can be constructed using the designs obtained under Method 4.1. These designs exist when number of treatments is a multiple of four. In the designs obtained under Method 4.1, leave one cell (row column intersection) from each region empty. Without loss of generality, the empty nodes are placed in the array in such a way that each row, each column and each region has only one empty node. This reduces one replication for each of the treatments. The resultant designs are found to be partially variance balanced with the treatments following a three associate-class rectangular association scheme

Table 1
List of designs obtained using Method 4.1

S. No.	v	r	p	q	m	$\boldsymbol{V_{ij}}$
1	4	4	4	4	4	0.5000
2	6	6	6	6	6	0.3333
3	8	8	8	8	8	0.2500
4	10	10	10	10	10	0.2000
5	12	12	12	12	12	0.1667
6	14	14	14	14	14	0.1428
7	16	16	16	16	16	0.1250
8	18	18	18	18	18	0.1111
9	20	20	20	20	20	0.1000
10	22	22	22	22	22	0.0909
11	24	24	24	24	24	0.0833

Table 2
List of designs obtained using particular case 1

S. No.	v	r	p	q	m	$\overline{V}_{ij}$	E_{can}
1	4	3	4	4	4	0.6667	0.6667
2	8	7	8	8	8	0.3054	0.9357
3	12	11	12	12	12	0.1862	0.9763
4	16	15	16	16	16	0.1350	0.9877
5	20	19	20	20	20	0.1060	0.9925
6	24	23	24	24	24	0.0874	0.9949

(Vartak [10,11]). Parameters for this class of designs are $v = p = q = m$ and $r = v - 1$. The average of estimated variances of elementary contrasts pertaining to treatment effects ($\overline{V}_{ij}; i, j = 1, 2, \ldots v; i \neq j$) has been computed using the SAS code developed for the purpose. The canonical efficiency factor of these designs in comparison to an orthogonal design having the same number of treatments assuming the same variance for both the designs (E_{can}) has also been computed. Parameters of the designs along with average variance and efficiency factor have been listed in Table 2 for $v < 25$.

Example 2. Let $v = 8$. Placing one empty node each in each rectangular region of a design obtained under Method 1, in such a way that each row, each column and each region have only one empty node the following design for 8 treatments in 8 rows, 8 columns and 8 regions each having one empty node in 7 replications can be obtained:

*	2	3	4	5	6	7	8
3	4	*	6	7	8	1	2
5	6	7	8	1	*	3	4
7	8	1	2	3	4	5	*
2	1	4	3	6	5	*	7
4	*	6	5	8	7	2	1
6	5	8	*	2	1	4	3
8	7	2	1	*	3	6	5

For the above design, the information matrix has been computed using SAS code and is given below:

$$\boldsymbol{C}_\tau = \begin{bmatrix} 5.733 & -0.800 & -0.733 & -0.867 & -0.800 & -0.800 & -0.867 & -0.867 \\ -0.800 & 5.733 & -0.867 & -0.733 & -0.800 & -0.800 & -0.867 & -0.867 \\ -0.733 & -0.867 & 5.733 & -0.800 & -0.867 & -0.867 & -0.800 & -0.800 \\ -0.867 & -0.733 & -0.800 & 5.733 & -0.867 & -0.867 & -0.800 & -0.800 \\ -0.800 & -0.800 & -0.867 & -0.867 & 5.733 & -0.800 & -0.733 & -0.867 \\ -0.800 & -0.800 & -0.867 & -0.867 & -0.800 & 5.733 & -0.867 & -0.733 \\ -0.867 & -0.867 & -0.800 & -0.800 & -0.733 & -0.867 & 5.733 & -0.800 \\ -0.867 & -0.867 & -0.800 & -0.800 & -0.867 & -0.733 & -0.800 & 5.733 \end{bmatrix}$$

Particular Case 2. Designs under three-way blocking structure for rectangular regions with one or $\left(\frac{v}{2} - 1\right)$ empty region(s)

Once again consider the designs with rectangular regions obtained under Method 4.1. If any one of the rectangular regions is emptied, i.e., the entire region is not receiving any treatments, the resulting design for three-way blocking structure with an empty region will still hold good all the properties of the original design, except for some reduction in efficiency. This result can be extended to designs having $\left(\frac{v}{2} - 1\right)$ consecutive empty regions and the method yields designs for all even number of treatments. Both types of designs are found to be partially variance balanced with the treatments following a three associate-class rectangular association scheme. Parameters for the class of designs where only one empty rectangular region are $v = p = q = m$ and $r = v - 1$ whereas the parameters for the class of designs with $\left(\frac{v}{2} - 1\right)$ neighbouring empty rectangular regions are $v = p = q = m$ and $r = \left(\frac{v}{2} + 1\right)$.

The average of estimated variances of elementary contrasts pertaining to treatment effects has been computed using the SAS code. The canonical efficiency factor of these designs in comparison to an orthogonal design having the same number of treatments and assuming the same variance has also been computed. Parameters of the designs along with average variance and efficiency factor have been listed in Table 3 for $v < 25$.

Example 3. Let $v = 8$. A design for one region is completely empty is given below:

1	2	3	4	5	6	7	8
3	4	5	6	7	8	1	2
5	6	7	8	1	2	3	4
7	8	1	2	3	4	5	6
*	*	4	3	6	5	8	7
*	*	6	5	8	7	2	1
*	*	8	7	2	1	4	3
*	*	2	1	4	3	6	5

Parameters of this design are $v = p = q = m = 8$ and $r = 7$. For the above design the C-matrix is as follows:

$$\boldsymbol{C}_\tau = \begin{bmatrix} 5.875 & -0.875 & -0.958 & -0.708 & -0.958 & -0.708 & -0.958 & -0.708 \\ -0.875 & 5.875 & -0.708 & -0.958 & -0.708 & -0.958 & -0.708 & -0.958 \\ -0.958 & -0.708 & 5.875 & -0.875 & -0.958 & -0.708 & -0.958 & -0.708 \\ -0.708 & -0.958 & -0.875 & 5.875 & -0.708 & -0.958 & -0.708 & -0.958 \\ -0.958 & -0.708 & -0.958 & -0.708 & 5.875 & -0.875 & -0.958 & -0.708 \\ -0.708 & -0.958 & -0.708 & -0.958 & -0.875 & 5.875 & -0.708 & -0.958 \\ -0.958 & -0.708 & -0.958 & -0.708 & -0.958 & -0.708 & 5.875 & -0.875 \\ -0.708 & -0.958 & -0.708 & -0.958 & -0.708 & -0.958 & -0.875 & 5.875 \end{bmatrix}$$

Example 4. Let $v = 8$. A design where $\left(\frac{v}{2} - 1\right) = 3$ regions are completely empty is shown below:

1	2	3	4	5	6	7	8
3	4	5	6	7	8	1	2
5	6	7	8	1	2	3	4
7	8	1	2	3	4	5	6
*	*	*	*	*	*	8	7
*	*	*	*	*	*	2	1
*	*	*	*	*	*	4	3
*	*	*	*	*	*	6	5

Parameters of this design are $v = p = q = m = 8$ and $r = 5$ and the C-matrix is:

$$\boldsymbol{C}_\tau = \begin{bmatrix} 3.625 & -0.625 & -0.875 & -0.125 & -0.875 & -0.125 & -0.875 & -0.125 \\ -0.625 & 3.625 & -0.125 & -0.875 & -0.125 & -0.875 & -0.125 & -0.875 \\ -0.875 & -0.125 & 3.625 & -0.625 & -0.875 & -0.125 & 0.875 & -0.125 \\ -0.125 & -0.875 & -0.625 & 3.625 & -0.125 & -0.875 & -0.125 & -0.875 \\ -0.875 & -0.125 & -0.875 & -0.125 & 3.625 & -0.625 & -0.875 & -0.125 \\ -0.125 & -0.875 & -0.125 & -0.875 & -0.625 & 3.625 & -0.125 & -0.875 \\ -0.875 & -0.125 & -0.875 & -0.125 & -0.875 & -0.125 & 3.625 & -0.625 \\ -0.125 & -0.875 & -0.125 & -0.875 & -0.125 & -0.875 & -0.625 & 3.625 \end{bmatrix}$$

The average of estimated variances of elementary contrasts pertaining to treatment effects is 0.2986 and 0.5286 and the canonical efficiency factor is 0.9567 and 0.7568 for design having one empty region and 3 empty regions, respectively.

Table 3
List of designs obtained using particular case 2

S. No.	v	r	p	q	m	$\overline{V}_{ij}$	E_{can}
1	6	5	6	6	6	0.4378	0.9137
2	6	4	6	6	6	0.6667	0.7500
3	8	7	8	8	8	0.2986	0.9567
4	8	5	8	8	8	0.5286	0.7568
5	10	9	10	10	10	0.2281	0.9741
6	10	6	10	10	10	0.4370	0.7627
7	12	11	12	12	12	0.1850	0.9828
8	12	7	12	12	12	0.3723	0.7674
9	14	13	14	14	14	0.1557	0.9877
10	14	8	14	14	14	0.3242	0.7712
11	16	15	16	16	16	0.1345	0.9908
12	16	9	16	16	16	0.2870	0.7742
13	18	17	18	18	18	0.1185	0.9929
14	18	10	18	18	18	0.2575	0.7766
15	20	19	20	20	20	0.1059	0.9943
16	20	11	20	20	20	0.2335	0.7787
17	22	21	22	22	22	0.0957	0.9954
18	22	12	22	22	22	0.2135	0.7804
19	24	23	24	24	24	0.0873	0.9961
20	24	13	24	24	24	0.1967	0.7819

Table 4
List of designs obtained using particular case 3

S. No.	v	$r_{\mathbf{t}}$	$r_{\mathbf{c}}$	p	q	m	$\overline{V}_{t_it_j}$	$V_{t_ic_0}$	E_{can}
1	8	7	8	8	8	8	0.2987	0.2735	0.9748
2	12	11	12	12	12	12	0.1853	0.1758	0.9891
3	16	15	16	16	16	16	0.1347	0.1298	0.9939
4	20	19	20	20	20	20	0.1059	0.1029	0.9962
5	24	23	24	24	24	24	0.0873	0.0853	0.9973

Table 5
List of designs obtained using Method 4.2

S. No.	v	r	p	q	m	V_{ij}	$E_{\mathbf{can}}$
1	7	6	6	7	14	0.4285	0.7778
2	11	10	10	11	22	0.2272	0.8800
3	19	18	18	19	38	0.1184	0.9383
4	23	22	22	23	46	0.0956	0.9504

Table 6
List of designs obtained using particular case 4

S. No.	v	$r_{\mathbf{t}}$	$r_{\mathbf{c}}$	p	q	m	$V_{t_it_j}$	$V_{t_ic_0}$	E_{can}
1	7	6	12	6	7	7	0.4286	0.3214	0.8484
2	11	10	20	10	11	11	0.2272	0.1705	0.9263
3	19	18	36	18	19	19	0.1184	0.0888	0.9651
4	23	22	44	22	23	23	0.0956	0.0717	0.9725

Particular Case 3. Designs under three-way blocking structure having rectangular regions for comparing test treatments with one control

Construct a class of designs under three-way blocking using Method 4.1. Now, replace v distinct treatments, one each from each region by a control treatment 0 in such a way that each row and each column has only one position replaced by control treatment. This procedure results in a design for comparing v (even) test treatments with a control treatment where the test treatments follow three associate-class rectangular association scheme. Parameters of this class of design are $v = r_c$ (number of times control treatment is replicated in the design) $= p = q = m$, r_t (number of replications of test treatments) $= (v - 1)$.

The average of estimated variances of elementary contrasts pertaining to test vs. test treatments ($\overline{V}_{t_it_j}; i, j = 1, 2, \ldots, v; i \neq j$) and the estimated variance of elementary contrasts pertaining to test vs. control treatment ($V_{t_ic_0}; i = 1, 2, \ldots v$) have been computed using the SAS code developed for the purpose. The canonical efficiency factor of these designs in comparison to an orthogonal design having the same number of treatments assuming the same variance has also been computed. Parameters of the designs along with estimated average variance of test vs. test as well as test vs. control and canonical efficiency factor have been listed in Table 4 for $v < 25$.

Example 5. Let $v = 8$. By replacing distinct treatments from each region with a control treatment represented by '0' we get the following design:

0	2	3	4	5	6	7	8
3	4	**0**	6	7	8	1	2
5	6	7	8	1	**0**	3	4
7	8	1	2	3	4	5	**0**
2	1	4	3	6	5	**0**	7
4	**0**	6	5	8	7	2	1
6	5	8	**0**	2	1	4	3
8	7	2	1	**0**	3	6	5

Parameters of this design are $v = r_c = p = q = m = 8$, $r_t = 7$. Using the SAS program, the C-matrix for the above design is computed as follows:

$$
\boldsymbol{C}_\tau = \begin{bmatrix}
5.969 & -0.719 & -0.688 & -0.750 & -0.719 & -0.719 & -0.750 & -0.750 & -0.875 \\
-0.719 & 5.969 & -0.750 & -0.688 & -0.719 & -0.719 & -0.750 & -0.750 & -0.875 \\
-0.688 & -0.750 & 5.969 & -0.719 & -0.750 & -0.750 & -0.719 & -0.719 & -0.875 \\
-0.750 & -0.688 & -0.719 & 5.969 & -0.750 & -0.750 & -0.719 & -0.719 & -0.875 \\
-0.719 & -0.719 & -0.750 & -0.750 & 5.969 & -0.719 & -0.688 & -0.750 & -0.875 \\
-0.719 & -0.719 & -0.750 & -0.750 & -0.719 & 5.969 & -0.750 & -0.688 & -0.875 \\
-0.750 & -0.750 & -0.719 & -0.719 & -0.688 & -0.750 & 5.969 & -0.719 & -0.875 \\
-0.750 & -0.750 & -0.719 & -0.719 & -0.750 & -0.688 & -0.719 & 5.969 & -0.875 \\
-0.875 & -0.875 & -0.875 & -0.875 & -0.875 & -0.875 & -0.875 & -0.875 & 7.000
\end{bmatrix}
$$

The average of estimated variances of elementary contrasts pertaining to test vs. test treatments and test vs. control treatment have been computed as 0.2987 and 0.2735, respectively. The efficiency factor of the design is computed as 0.9748. It may be noted that the estimated variance for test vs. control is less than that of test vs. test comparison, indicating the suitability of this type of designs for comparing a set of test treatments with a control under three-way blocking set up.

4.2. Designs under three-way blocking structure using balanced incomplete block designs

This series of designs is constructed by making use of the popular balanced incomplete block designs of $v = 4t+3$ series, where v is a prime number and t is any natural number $\geqslant 1$. Obtain two initial blocks by taking even and odd powers of primitive element mod v. Append these initial blocks of one below the other in a vertical manner retaining the identity of the blocks. Develop these initial blocks cyclically mod v in a horizontal direction resulting in $2v$ blocks. Treat the blocks of the BIB designs as regions, a series of designs for three-way blocking structure in $(v-1)$ rows, v columns and $2v$ regions having $(v-1)$ replications can be obtained.

These designs are seen to be variance balanced. In these designs, rows are complete but columns and regions are incomplete. General form of information matrix for estimating the elementary contrasts pertaining to treatment effects is obtained using the Model (1) as given below:

$$\boldsymbol{C}_\tau = \frac{v(v-3)}{(v-1)}\left(\boldsymbol{I}_v - \frac{\boldsymbol{J}_v}{v}\right)$$

The estimated variance of elementary contrasts pertaining to treatment effects and the canonical efficiency factor of these designs in comparison to an orthogonal design for the same number of treatments and assuming equal variance have been computed using the SAS code developed for the purpose. List of parameters of the class of designs along with estimated variance and canonical efficiency factor have been prepared and given in Table 5 for $v < 25$.

Example 6. For $v = 7$, primitive element is 3. By developing initial blocks 1, 2, 4 and 3, 5, 6 obtained using even and odd powers of primitive element mod 7, two BIB designs can be obtained. Append the blocks of one BIB design below the other and take each block of the design as region, 14 regions each of size 3 can be obtained.

1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3
3	4	5	6	7	1	2
5	6	7	1	2	3	4
6	7	1	2	3	4	5

For this design, the parameters are $v = q = 7$, $p = r = 6$ and $m = 14$ and the information matrix for treatment effects is:

$$\boldsymbol{C}_\tau = 4.667\boldsymbol{I}_7 - 0.667\boldsymbol{J}_7$$

This design is variance balanced with the estimated variance of elementary contrasts pertaining to treatment effects as 0.4285 and the canonical efficiency factor as 0.7778.

Particular Case 4. Designs under three-way blocking structure using balanced incomplete block designs for comparing test treatments with one control

Consider the class of equi-replicated designs obtained using Method 4.2. Now, treatment substitution method can be used to obtain a class of designs under three-way blocking structure for comparing test treatments with one control. Replace two treatments say, v^{th} and $(v-1)^{\text{th}}$ treatments by the control treatment 0. This doubles the number of replications of the control treatment 0. Treat the remaining $(v-2)$ treatments as test treatments. In the resultant design, control is replicated $r_c = 2(v-1)$ times and $(v-2)$ test treatments replicated $r_t = (v-1)$ times. Other parameters of this class of designs are $q = m = v$ and $p = (v-1)$. The general form of C-matrix is obtained for this class of designs as follows:

$$\boldsymbol{C}_\tau = \begin{bmatrix} \dfrac{v(v-3)}{(v-1)}[\boldsymbol{I}_{v-2} - \dfrac{\boldsymbol{J}_{v-2}}{v}] & \dfrac{2(v-3)}{(v-1)}\mathbf{1}_{v-2} \\ \dfrac{2(v-3)}{(v-1)}\mathbf{1}'_{v-2} & \dfrac{2(v-2)(v-3)}{(v-1)} \end{bmatrix}$$

This design is variance balanced. The estimated variances of elementary contrasts pertaining to test vs. test $(V_{t_i t_j}; i, j = 1, 2, \ldots, v-2; i \neq j)$ and test vs. control treatment $(V_{t_i c_0}; i = 1, 2, \ldots v-2)$ have been computed using the SAS code developed. The canonical efficiency factor of these designs in comparison to an orthogonal design having the same number of treatments assuming the same variance has also been computed. List of parameters of the designs along with estimated variance of to test vs. test as well as test vs. control treatments and efficiency factor have been given in Table 6 for $v < 25$.

Example 7. Let there be $v = 7$ treatments denoted by 1,2, ...,7. By renumbering the treatment numbers 6 and 7 by control treatment 0, the following design for comparing 5 test treatments with a control can be obtained:

1	2	3	4	5	**0**	**0**
2	3	4	5	**0**	**0**	1
4	5	**0**	**0**	1	2	3
3	4	5	**0**	**0**	1	2
5	**0**	**0**	1	2	3	4
0	**0**	1	2	3	4	5

Parameters of this class of design are $q = m = 7$, $p = r_t = 6$ and $r_c = 12$. C-matrix is obtained for this design as follows:

$$\boldsymbol{C}_\tau = \begin{bmatrix} 4.667\boldsymbol{I}_5 - 0.667\boldsymbol{J}_5 & 1.333 \\ 1.333 & 6.667 \end{bmatrix}$$

5. Conclusions

Two methods of construction of designs under three-way blocking structure for regular regions have been developed for different parametric combinations. The first method developed gives variance balanced designs for all even numbers of treatments. Two particular cases of this method yield structurally incomplete, partially variance balanced designs. Second method is developed for designs having number of treatments of the form $4t+3$ and the designs obtained are variance balanced. Both methods can be suitably modified to construct designs for comparing a set of test treatments with a control. These designs give the comparisons between test vs. control treatments more precisely. List of parameters of designs has been given. Canonical efficiency factor for each design has been worked out by writing a SAS code in PROC IML and it is seen that all the designs are having high efficiency factor for sufficient number of treatments.

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Appendix

The model given in Model (1) can be written in matrix notation as follows:

$$\mathbf{Y} = \mu\mathbf{1} + \boldsymbol{\Delta}'\boldsymbol{\tau} + \mathbf{D}_1'\boldsymbol{\alpha} + \mathbf{D}_2'\boldsymbol{\beta} + \mathbf{D}_3'\boldsymbol{\gamma} + \mathbf{e} \tag{2}$$

where, $\mathbf{Y}$ is a $n \times 1$ vector of observations, μ is the grand mean, $\mathbf{1}$ is the $n \times 1$ vector of ones, $\boldsymbol{\Delta}'$ is $n \times v$ incidence matrix of observations versus treatments, $\boldsymbol{\tau}$ is a $v \times 1$ vector of treatment effects, $\mathbf{D'}_1$ is $n \times p$ incidence matrix of observations versus rows, $\boldsymbol{\alpha}$ is $p \times 1$ vector of row effects, $\mathbf{D'}_2$ is $n \times q$ incidence matrix of observations versus columns, $\boldsymbol{\beta}$ is $q \times 1$ vector of column effects, $\mathbf{D'}_3$ is $n \times m$ incidence matrix of observations versus regions, $\boldsymbol{\gamma}$ is $m \times 1$ vector of region effects and **e** is $n \times 1$ vector of random errors with $E(\boldsymbol{e}) = 0$ and $D(\boldsymbol{e}) = \sigma^2\mathbf{I_n}$. This model can be written as

$$\mathbf{Y} = \mathbf{X}_1\boldsymbol{\theta}_1 + \mathbf{X}_2\boldsymbol{\theta}_2 + \mathbf{e} \tag{3}$$

where,

$$\mathbf{X_1} = [\boldsymbol{\Delta}'], \quad \mathbf{X_2} = [\mathbf{1}\ \mathbf{D'}_1\ \mathbf{D'}_2\ \mathbf{D'}_3]',$$

$\boldsymbol{\theta_1} = (\boldsymbol{\tau})$ is the vector of parameter of interest and $\boldsymbol{\theta_2} = (\mathbf{1}\ \boldsymbol{\alpha}\ \boldsymbol{\beta}\ \boldsymbol{\gamma})$ is the vector of nuisance parameters.

Now, the information matrix for treatment effects is given by

$$\mathbf{C} = \mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{X}_1. \tag{4}$$

where,

$$\mathbf{X}_1'\mathbf{X}_1 = \boldsymbol{\Delta}\boldsymbol{\Delta}' = \mathbf{R}_\tau$$
$$\mathbf{X}_1'\mathbf{X}_2 = \left[\boldsymbol{\Delta}\mathbf{1},\ \boldsymbol{\Delta}\mathbf{D}_1',\ \boldsymbol{\Delta}\mathbf{D}_2',\ \boldsymbol{\Delta}\mathbf{D}_3'\right] = [\mathbf{r}_\tau,\ \mathbf{N}_1,\ \mathbf{N}_2,\ \mathbf{N}_3]$$
$$\mathbf{X}_2'\mathbf{X}_2 = \begin{pmatrix} \mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{D}_1' & \mathbf{1}'\mathbf{D}_2' & \mathbf{1}'\mathbf{D}_3' \\ \mathbf{D}_1\mathbf{1} & \mathbf{D}_1\mathbf{D}_1' & \mathbf{D}_1\mathbf{D}_2' & \mathbf{D}_1\mathbf{D}_3' \\ \mathbf{D}_2\mathbf{1} & \mathbf{D}_2\mathbf{D}_1' & \mathbf{D}_2\mathbf{D}_2' & \mathbf{D}_2\mathbf{D}_3' \\ \mathbf{D}_3\mathbf{1} & \mathbf{D}_3\mathbf{D}_1' & \mathbf{D}_3\mathbf{D}_2' & \mathbf{D}_3\mathbf{D}_3' \end{pmatrix} = \begin{pmatrix} n & \mathbf{k}_\alpha' & \mathbf{k}_\beta' & \mathbf{k}_\gamma' \\ \mathbf{k}_\alpha & \mathbf{K}_\alpha & \mathbf{M}_1 & \mathbf{M}_2 \\ \mathbf{k}_\beta & \mathbf{M}_1' & \mathbf{K}_\beta & \mathbf{M}_3 \\ \mathbf{k}_\gamma & \mathbf{M}_2' & \mathbf{M}_3' & \mathbf{K}_\gamma \end{pmatrix}.$$

Here, $\boldsymbol{r}_\tau = (r_{\tau 1}, r_{\tau 2}, \ldots, r_{\tau v})'$ is the $v \times 1$ replication vector of treatments, $\boldsymbol{k}_\alpha = (k_{\alpha 1}, k_{\alpha 2}, \ldots, k_{\alpha p})'$ is the $p \times 1$ vector of row sizes, $\boldsymbol{k}_\beta = (k_{\boldsymbol{\beta}1}, k_{\boldsymbol{\beta}2}, \ldots, k_{\boldsymbol{\beta}q})'$ is the $q \times 1$ vector of column sizes and $\boldsymbol{k}_\gamma = (k_{\gamma 1}, k_{\gamma 2}, \ldots, k_{\gamma k})'$ is the $m \times 1$ vector of region sizes. Further, $\mathbf{K}_\alpha = Diag(k_{\alpha 1}, k_{\alpha 2}, \ldots, k_{\alpha p})$ is the diagonal matrix of row-sizes, $\mathbf{K}_\beta = Diag(k_{\boldsymbol{\beta}1}, k_{\boldsymbol{\beta}2}, \ldots, k_{\boldsymbol{\beta}q})$ is the diagonal matrix of column-sizes and $\mathbf{K}_\gamma = Diag(k_{\gamma 1}, k_{\gamma 2}, \ldots, k_{\gamma m})$ is the diagonal matrix of region-sizes. $\mathbf{N}_1$ is the incidence matrix of treatments versus rows, $\mathbf{N}_2$ is the incidence matrix of treatments versus columns, $\mathbf{N}_3$ is the incidence matrix of treatments versus regions, $\mathbf{M}_1$ is the incidence matrix of rows versus columns, $\mathbf{M}_2$ is the incidence matrix of rows versus regions and $\mathbf{M}_3$ is the incidence matrix of columns versus regions.

If $\mathbf{X} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{D} \end{pmatrix}$ then $\mathbf{X}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{F}\mathbf{E}^{-1}\mathbf{F}' & -\mathbf{F}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{F}' & \mathbf{E}^{-1} \end{pmatrix}$

where, $\mathbf{F} = \mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{E} = \mathbf{D} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B}$.

Note: If rank($\mathbf{A}$)+rank($\mathbf{D}-\mathbf{B}'\mathbf{A}^{-}\mathbf{B}$) $=$ rank($\mathbf{X}$), then one can use true inverse as well as generalized inverse of matrix $\mathbf{X}$ interchangeably as given in Searle [6].

$$(\mathbf{X}_2'\mathbf{X}_2)^{-} = \begin{pmatrix} 0 & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{0} & \mathbf{A}_{12}' & \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{0} & \mathbf{A}_{13}' & \mathbf{K}_{12}' & \mathbf{K}_{22} \end{pmatrix},$$

where

$$\mathbf{A}_{11} = \mathbf{K}_\alpha^{-1} + [\mathbf{K}_\alpha^{-1}(\mathbf{M}_1\mathbf{K}_{11}\mathbf{M}_1' + \mathbf{M}_1\mathbf{K}_{12}\mathbf{M}_2' + \mathbf{M}_2\mathbf{K}_{12}'\mathbf{M}_1' + \mathbf{M}_2\mathbf{K}_{22}\mathbf{M}_2')\mathbf{K}_\alpha^{-1}],$$
$$\mathbf{A}_{12} = -\mathbf{K}_\alpha^{-1}(\mathbf{M}_1\mathbf{K}_{11} + \mathbf{M}_2\mathbf{K}_{12}'),$$
$$\mathbf{A}_{13} = -\mathbf{K}_\alpha^{-1}(\mathbf{M}_1\mathbf{K}_{12} + \mathbf{M}_2\mathbf{K}_{22}), \mathbf{K}_{11} = \mathbf{K}_\beta^{-1} + \mathbf{K}_\beta^{-1}\mathbf{M}_3(\mathbf{K}_\gamma - \mathbf{M}_3'\mathbf{K}_\beta^{-1}\mathbf{M}_3)^{-}\mathbf{M}_3'\mathbf{K}_\beta^{-1},$$
$$\mathbf{K}_{12} = -\mathbf{K}_\beta^{-1}\mathbf{M}_3(\mathbf{K}_\gamma - \mathbf{M}_3'\mathbf{K}_\beta^{-1}\mathbf{M}_3)^{-} \text{and}$$
$$\mathbf{K}_{22} = (\mathbf{K}_\gamma - \mathbf{M}_3'\mathbf{K}_\beta^{-1}\mathbf{M}_3)^{-}.$$

Thus, the expression for information matrix for treatment effects in designs under three-way blocking structure is simplified as

$$\boldsymbol{C} = \mathbf{R}_\tau - \{(\mathbf{N}_1\mathbf{A}_{11}\mathbf{N}_1' + \mathbf{N}_1\mathbf{A}_{12}\mathbf{N}_2' + \mathbf{N}_1\mathbf{A}_{13}\mathbf{N}_3') + (\mathbf{N}_2\mathbf{A}_{12}'\mathbf{N}_1' + \mathbf{N}_2\mathbf{K}_{11}\mathbf{N}_2' + \mathbf{N}_2\mathbf{K}_{12}\mathbf{N}_3') + (\mathbf{N}_3\mathbf{A}_{13}'\mathbf{N}_1' + \mathbf{N}_3\mathbf{K}_{12}'\mathbf{N}_2' + \mathbf{N}_3\mathbf{K}_{22}\mathbf{N}_3')\}. \quad (5)$$

References

[1] R.A. Bailey, J. Kunert and R.J. Martin, Some comments on Gerechte designs II: Randomization, analysis and other methods that allow for inter-plot dependence, *J Agron Crop Sci* **166** (1991), 101–111.
[2] R.A. Bailey, P.J. Cameron and R. Connelly, Sudoku, Gerechte designs, resolutions, affine space, spreads, reguli and hamming codes, *The American Math Monthly* **115** (2008), 383–404.
[3] W.U. Behrens, Feldversuchsanordnungen mit verbessertem Ausgleich der Bodenunterschiede, *Zeitschrift fur Landwirtschaftliches Versuchsund Untersuchungswesen* **2** (1956), 176–193.
[4] J. Courtiel and E.R. Vaughan, Gerechte designs with rectangular regions, *J Comb Designs* **20**(2) (2012), 112–123.
[5] M.O. Hui-Dong and X.U. Ru-Gen, Sudoku square – A new design in field experiment, *Acta Agronomica Sinica* **34**(9) (2008), 1489–1493.
[6] S.R. Searle, Matrix algebra useful for statistics, John Wiley and Sons, New York, 1982.
[7] F.P. Stewart and R.A. Bradley, Some adjusted orthogonal, variance balanced, multidimensional block designs, *Commun Statist Theory Meth* **19**(6) (1990), 2109–2144.
[8] J. Subramani and K.N. Ponnuswamy, Construction and analysis of Sudoku designs, *Model Assisted Statist Appl* **4** (2009), 287–301.
[9] E.R. Vaughan, The complexity of constructing gerechte designs, *The Electronic J Comb* **16** (2009), 1–8.
[10] M.N. Vartak, On an application of kronecker product of matrices to statistical designs, *Ann Math Statist* **26** (1955), 420–438.
[11] M.N. Vartak, The non-existence of certain PBIB designs, *Ann Math Statist* **30** (1959), 1051–1062.
[12] Xu, Xu, B. Haaland and Z.G.P. Qian, Sudoku-based space-filling designs, *Biometrika Series 3* **98** (2011), 711–720.