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On the Construction of Monopoles

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Abstract. We show that any self-dual SU(2) monopole may be constructed either by Ward's twistor method, or Nahm's use of the ADHM construction. The common factor in both approaches is an algebraic curve whose Jacobian is used to linearize the non-linear ordinary differential equations which arise in Nahm's method. We derive the non-singularity condition for the monopole in terms of this curve and apply the result to prove the regularity of axially symmetric solutions.

1. Introduction

We shall be concerned in this paper with constructing solutions to the Bogomolny equations $D\Phi = *F$. Here F is the curvature of an SU(2) connection on $\mathbb{R}^3$, Φ (the Higgs field) is a section of the adjoint bundle, and we are seeking solutions for which $\|\Phi\| = 1 - kr^{-1} + O(r^{-2})$ as $r \to \infty$. These are particular solutions to the static, finite energy Yang-Mills-Higgs equations and we shall often refer to them simply as "monopoles".

There exist already two different approaches to constructing monopoles. One is due to R. S. Ward, using the twistor formalism to reduce the problem to one of holomorphic vector bundles on the algebraic surface $T\mathbb{P}_1$, the tangent bundle of the projective line. Ward's method, extended by Corrigan and Goddard [6] and the author [8], shows that the monopole is determined by an algebraic curve in $T\mathbb{P}_1$. Moreover, as shown in [8], every monopole may be obtained in this way. The main problem of this approach is finding the conditions to impose on the curve in order to ensure that the monopole is non-singular.

The alternative approach, due to Nahm [10], incorporates the non-singularity condition directly and has other formal advantages over the twistor viewpoint. Nahm's method is a bold adaptation of the ADHM construction of instantons [3], replacing matrices by differential operators and the quadratic constraint on the matrices by a non-linear ordinary differential equation:

$$\frac{dT_i}{dz} = \frac{1}{2}\sum_{j,k} \varepsilon_{ijk}[T_j, T_k]$$

for matrices T_1, T_2, T_3. The main problem concerning this construction is to actually solve the equations. There remains also the question of whether this approach yields all monopoles.

In this paper we shall present a synthesis of the two methods which enables us to answer the questions raised above. In fact we shall show that Nahm's equations can be solved by considering a linear flow on the Jacobian of an algebraic curve – the same curve that occurs in Ward's construction. To be precise, we prove the equivalence of the following:

A. A solution to the Bogomolny equations $D\Phi = *F$ on $\mathbb{R}^3$ with boundary conditions as $r \to \infty$,

A1. $\|\Phi\| = 1 - \frac{k}{r} + O(r^{-2})$,

A2. $\frac{\partial \|\Phi\|}{\partial \Omega} = O(r^{-2})$,

A3. $\|D\Phi\| = O(r^{-2})$.

B. A compact algebraic curve $S \subset T\mathbb{P}_1$ in the linear system $|O(2k)|$ satisfying the conditions:

B1. S has no multiple components.

B2. S is real with respect to a standard real structure on $T\mathbb{P}_1$.

B3. L^2 is trivial on S and $L(k-1)$ is real.

B4. $H^0(S, L^z(k-2)) = 0$ for $z \in (0, 2)$.

Here L^z is the holomorphic line bundle on $T\mathbb{P}_1$ defined by $\exp(z\omega)$, where $\omega \in H^1(T\mathbb{P}_1, O)$ is the standard $SL(2, \mathbb{C})$-invariant element.

C. A solution to the differential equation

$$\frac{dT_i}{dz} = \frac{1}{2} \sum_{j,k} \varepsilon_{ijk} [T_j, T_k], \qquad z \in (0, 2),$$

for $k \times k$ matrices $T_i(z)$ $(i = 1, 2, 3)$ satisfying the conditions:

C1. $T_i^* = -T_i$,

C2. $T_i(z) = -\bar{T}_i(2-z)$,

C3. T_i has simple poles at $z = 0$ and $z = 2$ but is otherwise analytic,

C4. at each pole the residues of (T_1, T_2, T_3) define an irreducible representation of SU(2).

Section 2 is devoted to the proof of C $\Rightarrow$ A, and consists of a detailed presentation of Nahm's work [10]. The implication A $\Rightarrow$ B is mainly dealt with in the author's paper [8]. We review the results briefly in Sect. 3, but also prove condition B3, a vanishing theorem for the algebraic curve. This turns out to be the condition on the curve for nonsingularity of the monopole and the result and method are similar to the crucial vanishing theorem $H^1(\mathbb{P}_3, E(-2)) = 0$ for instantons [7]. In Sects. 4, 5, and 6 we show how B $\Rightarrow$ C and in Sect. 7 check that the circle of ideas is complete by showing that the monopole one recovers by pursuing A $\Rightarrow$ B $\Rightarrow$ C $\Rightarrow$ A is gauge equivalent to the original one.

The proof of B $\Rightarrow$ C, relating non-linear differential equations with algebraic curves is analogous to the now familiar method of solving the KdV equation and related equations. There is also a hierarchy of equations of which Nahm's is the first, whose solutions are obtained by taking linear flows in different directions of the Jacobian. We consider these briefly together with other general comments, in Sect. 8, where we also verify the non-singularity of Prasad and Rossi's axially symmetric monopoles.

2. Nahm's Construction

In [10] Nahm succeeded in producing solutions to the Bogomolny equations $D\Phi = *F$ by a striking adaptation of the monad (or ADHM) construction of instantons. We shall review his construction in this section.

First, recall that a solution to the Bogomolny equations in $\mathbb{R}^3$ is equivalent to a solution of the self-duality equations in $\mathbb{R}^4$, which is in addition invariant under the action of the additive group $\mathbb{R}$ of translations in the x_0-direction. To see this, let E be a vector bundle with connection ∇ on $\mathbb{R}^3$ and Φ a section of the adjoint bundle. Let $p: \mathbb{R}^4 \to \mathbb{R}^3$ be the projection. Then, $\nabla' = p^*\nabla - \Phi dx_0$ defines an $\mathbb{R}$-invariant connection on the pulled-back bundle $E' = p^*E$ which has curvature $F' = p^*F - D\Phi \wedge dx_0$. Clearly F' is self-dual with respect to the orientation

$$dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \quad \text{iff} \quad D\Phi = *F .$$

Conversely, suppose E' is an $\mathbb{R}$-invariant vector bundle on $\mathbb{R}^4$, with invariant connection ∇'. Restricting to $\mathbb{R}^3$ we obtain a bundle E, and the group action defines an isomorphism $\alpha: p^*E \to E'$ by

$$\alpha(p^*e)_{(x_0, x)} = x_0 \cdot (e_x), \tag{2.1}$$

where $x_0 \in \mathbb{R}$, $x \in \mathbb{R}^3$, and $x_0 \cdot e$ denotes the action of $\mathbb{R}$ on E'.

Since ∇' is $\mathbb{R}$-invariant, we obtain $\alpha^{-1}\nabla'\alpha = p^*\nabla - \Phi dx_0$ for some connection ∇ on E and section Φ of the adjoint bundle. As the self-duality equations are invariant under the gauge transformation α, it follows by the argument above that (∇, Φ) satisfy the Bogomolny equations.

Now the ADHM construction [2, 3] produces self-dual SU(2) connections on $\mathbb{R}^4$ by considering a $(k+1) \times k$ quaternionic matrix of the form $\Delta(x) = Cx + D$, where C, D are constant matrices and $x \in \mathbb{H}$ is a quaternionic variable. This is to be viewed more invariantly as a map $\Delta(x): W \to V$, where V is a $(k+1)$-dimensional quaternionic vector space with a hermitian inner product compatible with the quaternionic structure, and W is a k-dimensional real vector space. If $\Delta(x)$ is of maximal rank for all x, then the kernel of $\Delta^*(x)$ is a 1-dimensional quaternionic subspace E_x of V. As x varies in $\mathbb{H} \cong \mathbb{R}^4$, E_x describes a vector bundle E over $\mathbb{R}^4$, and the orthogonal projection in V defines a connection on E, from the trivial flat connection on $\mathbb{R}^4 \times V$. The curvature of this connection may be expressed as $F = PCdx\varrho^{-2}d\bar{x}C^*P$, where $\varrho^2 = \Delta^*\Delta$ and P is the orthogonal projection onto E in V. If ϱ^2 is real, this will involve components of the quaternionic 2-form

$$dx \wedge d\bar{x} = (dx_0 + idx_1 + jdx_2 + kdx_3) \wedge (dx_0 - idx_1 - jdx_2 - kdx_3),$$

which are all self-dual. Hence the constraint on Δ necessary to produce self-duality of the connection is that $\Delta^*\Delta$ should be real for all $x \in \mathbb{H}$.

Nahm's approach is to seek analogous vector spaces W, V and a linear map $\Delta(x)=Cx+D$ such that:

(1) $\Delta(x+x_0)=U(x_0)^{-1}\Delta(x)U(x_0)$, where $x_0\to U(x_0)$ is a representation of $\mathbb{R}$ in the group of quaternionic unitary transformations of V. This ensures that the connection produced by the ADHM construction will be $\mathbb{R}$-invariant. Indeed, $U(x_0)$ is essentially the gauge transformation α in (2.1).

(2) $\Delta^*\Delta$ is real.

(3) $\Delta^*\Delta$ is invertible.

(4) The kernel of Δ^* has quaternionic dimension 1.

The major difference between the problem of monopoles and that of instantons is that the spaces W and V are infinite-dimensional and $\Delta(x)$ is a *differential* operator.

More precisely, let $H^0=\mathscr{L}^2[0,2]$ and define a real structure on H^0 (an anti-linear map σ such that $\sigma^2=1$) by $\sigma(f)(z)=\bar{f}(2-z)$. We then set $V=H^0\otimes\mathbb{C}^k\otimes\mathbb{C}^2$, and take $\mathbb{C}^k$ to have a real structure σ' and $\mathbb{C}^2$ to be the quaternions. Thus V has a quaternionic structure and the usual $\mathscr{L}^2$ inner product gives it a compatible hermitian structure.

For the real space W we take

$$W=\{f\in H^1\otimes\mathbb{C}^k \mid f(0)=f(2)=0\},$$

where H^1 is the Sobolev space of functions on $[0,2]$ whose derivatives are in $\mathscr{L}^2$. By the Sobolev embedding theorem such functions are actually continuous, and so have well-defined values at each point. Thus W is well-defined and, with respect to σ and σ', is real.

Now let e_1, e_2, e_3 denote the operation of left multiplication on the quaternions $\mathbb{C}^2$ by i, j, k. Since they are constant in z, and commute with right multiplication on $\mathbb{C}^2$ by quaternions they define quaternionic transformations of V. For the map $\Delta(x): W\to V$ we take (with Nahm), a differential operator of the form

$$\Delta(x)f=\left(x_0+\sum_1^3 x_j e_j\right)f+i\frac{df}{dz}+i\sum_1^3 T_j(z)e_j f, \tag{2.2}$$

where $T_j(z)$ is a $k\times k$ matrix depending analytically on $z\in(0,2)$ and with simple poles at the endpoints. Because of the choice of Sobolev spaces W and V, both $T_j(z)$ and $\frac{d}{dz}$ are bounded operators, so $\Delta(x)$ itself is bounded. It is clearly of the form $Cx+D$ with $C=I$, $D=i\frac{d}{dz}+i\sum T_j e_j$, so we must show first, in order to apply the ADHM construction that D is a quaternionic operator from $W\otimes\mathbb{C}^2$ to V. Now if J denotes the quaternionic structure on $\mathbb{C}^k\otimes\mathbb{C}^2$, the quaternionic structure τ on the functions in V or $W\otimes\mathbb{C}^2$ satisfies $\tau f(z)=J(f(2-z))$. Hence,

$$\begin{aligned} i\frac{d}{dz}(\tau f)&=i\frac{d}{dz}(Jf(2-z))=-Ji\frac{d}{dz}(f(2-z))\\ &=Ji\frac{df}{dz}(2-z)=\tau i\frac{d}{dz}f, \end{aligned}$$

and

$$\begin{aligned} i\sum T_j(z)e_j(\tau f) &= i\sum T_j(z)e_jJf(2-z) \\ &= -Ji\sum \bar{T}_j(z)e_jf(2-z) \\ &= \tau(i\sum T_j(z)e_jf) \quad \text{if} \quad T_j(2-z) = -\bar{T}_j(z). \end{aligned}$$

So let us suppose that

$$T_i(z) = -\bar{T}_i(2-z), \tag{2.3}$$

then $\Delta(x)$ is quaternionic linear.

Next consider the invariance condition (1). We have

$$\begin{aligned} \Delta(x+x_0)f &= \Delta(x)f + x_0 f \\ &= e^{ix_0(z-1)}\Delta(x)e^{-ix_0(z-1)}f, \end{aligned}$$

and

$$\tau(e^{ix_0(z-1)}f) = e^{ix_0(z-1)}\tau f,$$

hence $U(x_0) = e^{ix_0(z-1)}$ is quaternionic and clearly unitary, so condition (1) is satisfied.

Secondly consider the reality condition (2). We require that $\Delta^*\Delta : W \to W^*$ be real. Now this operator may be written

$$\begin{aligned} \Delta^*\Delta &= \left(\bar{x} + i\frac{d}{dz} + i\sum T_j^* e_j\right)\left(x + i\frac{d}{dz} + i\sum T_k e_k\right) \\ &= -\frac{d^2}{dz^2} + (2ix_0 - \sum (T_j + T_j^*)e_j)\frac{d}{dz} \\ &\quad - \sum \frac{dT_k}{dz}e_k + (\bar{x} + i\sum T_j^* e_j)(x + i\sum T_k e_k). \end{aligned}$$

From the $\frac{d}{dz}$ term we require $T_j + T_j^* = 0$ for reality, and from the zero order term we obtain

$$\sum_i \frac{dT_i}{dz}e_i = \sum_{j,k} T_j T_k e_j e_k.$$

Hence $T_i = -T_i^*$ and must satisfy *Nahm's equations*

$$\left.\begin{aligned} \frac{dT_1}{dz} &= [T_2, T_3] \\ \frac{dT_2}{dz} &= [T_3, T_1] \\ \frac{dT_3}{dz} &= [T_1, T_2]. \end{aligned}\right\} \tag{2.4}$$

If these equations are satisfied and $T_i = -T_i^*$, then we may write

$$\Delta^*\Delta = -\frac{d^2}{dz^2} + 2ix_0\frac{d}{dz} + \|x\|^2 - \sum T_j^2 . \tag{2.5}$$

Suppose there exists $f \in W$ such that $\Delta^*\Delta f = 0$. Then since f vanishes at the endpoints of $[0,2]$, we have from (2.5)

$$0 = \langle \Delta^*\Delta f, f\rangle = \left\langle \frac{df}{dz}, \frac{df}{dz}\right\rangle + \langle (\|x\|^2 + \sum T_j^* T_j) f, f\rangle . \tag{2.6}$$

However, the right hand side is positive unless f is identically zero, hence we have condition (3) that $\Delta^*\Delta$ is invertible.

For the final condition on Δ, that $\dim_{\mathbb{H}} \ker \Delta^* = 1$, we must consider the behaviour of $T_i(z)$ at the boundary of $[0,2]$. Suppose that each T_i has a simple pole at $z=0$, then we may write

$$T_i(z) = \frac{a_i}{z} + b_i(z),$$

where $b_i(z)$ is analytic in a neighbourhood of $z=0$. Thus

$$\frac{dT_i}{dz} = -\frac{a_i}{z^2} + \frac{db_i}{dz},$$

and from Nahm's equations (2.4) we have

$$-a_i = \tfrac{1}{2}\sum_{j,k} \varepsilon_{ijk}[a_j, a_k] .$$

Thus

$$(x_1e_1 + x_2e_2 + x_3e_3) \to -2(x_1a_1 + x_2a_2 + x_3a_3)$$

defines a k-dimensional representation of the Lie algebra of imaginary quaternions and hence a representation of SU(2). We suppose, following Nahm, that this is the unique irreducible representation S^{k-1} on homogeneous polynomials in (z_1, z_2) of degree $(k-1)$, and similarly at the other pole $z=2$.

Now in a neighbourhood of $z=0$, we may write

$$\begin{aligned} i\sum T_j e_j &= \frac{i}{z}\sum a_j \otimes e_j + b(z) \\ &= -\frac{i}{2z}\sum \varrho(e_j)\otimes e_j + b(z), \end{aligned}$$

where $b(z)$ is analytic and ϱ is the representation homomorphism of Lie algebras. If we consider the Casimir operator $C(S) = \sum_j \varrho(e_j)^2$ of a representation S, then

$$C(S^{k-1}\otimes S^1) = C(S^{k-1})\otimes 1 + 2\sum_j \varrho(e_j)\otimes e_j + 1\otimes C(S^1),$$

since the ordinary multiplication of quaternions on $\mathbb{H}\cong\mathbb{C}^2$ is the representation S^1. Thus the residue $T=-i/2\sum\varrho(e_j)\otimes e_j$ of $i\sum T_j e_j$ at $z=0$ may be expressed in terms of Casimir operators. Now the Casimir operator on the irreducible representation S^k is the scalar $-k(k+2)$. Furthermore the tensor product $S^{k-1}\otimes S^1$ is isomorphic to $S^k\oplus S^{k-2}$, decomposing into irreducibles. Hence,

$$\begin{aligned} T&=\frac{i}{4}(-(k-1)(k+1)-3+k(k+2))\\ &=\frac{i}{2}(k-1)\text{ on }S^k,\\ T&=\frac{i}{4}(-(k-1)(k+1)-3+(k-2)k)\\ &=-\frac{i}{2}(k+1)\text{ on }S^{k-2}. \end{aligned} \tag{2.7}$$

Let us now consider the operator

$$\tilde{\Delta}=i\frac{d}{dz}+\left(\frac{1}{z}+\frac{1}{(z-2)}\right)T.$$

It is clear that $\tilde{\Delta}f=0$ has a space of solutions of dimension $\dim S^k=(k+1)$ of the form $c(z^2-2z)^{-(k-1)/2}$, and a $\dim S^{k-2}=(k-1)$ dimensional space of the form $c(z^2-2z)^{(k+1)/2}$.

Thus, as an operator on the Sobolev spaces we are considering, $\dim_{\mathbb{C}}\ker\tilde{\Delta}=(k-1)$ (if $k\geqq 1$), and similarly $\dim_{\mathbb{C}}\ker\tilde{\Delta}^*=(k+1)$. Thus $\tilde{\Delta}:W\otimes\mathbb{C}^2\to V$ is a Fredholm operator with index $(k-1)-(k+1)=-2$.

At $z=0$ and $z=2$, the residues of T_i define an irreducible representation of SU(2) on $\mathbb{C}^k$. By Schur's lemma [and since $T_i^*(z)=-T_i(z)$], there exists $P\in U(k)$ unique modulo scalars such that $\operatorname{Res}_{z=0}T_i=\operatorname{Res}_{z=2}P^{-1}T_iP$. Let Q be a skew-adjoint matrix such that $\exp 2Q=P$. Then $\Delta=e^{-zQ}\tilde{\Delta}e^{zQ}+K$, where K is a matrix valued function which is analytic in a neighbourhood of $[0,2]$. Its regularity implies that it is a compact operator on the Sobolev spaces, and hence by the invariance of index, $\operatorname{index}\Delta=\operatorname{index}\tilde{\Delta}=-2$. However, since $\Delta^*\Delta$ is invertible, $\ker\Delta=0$, so $\dim_{\mathbb{C}}\ker\Delta^*=2$. Since Δ and hence Δ^* is quaternionic, it follows that our final condition (4), that $\dim_{\mathbb{H}}\ker\Delta^*=1$, is satisfied. Thus, we have proved

Theorem (2.8) (Nahm). *Let $T_i(z)$ $(1\leqq i\leqq 3)$ be $k\times k$ matrix-valued functions of $z\in(0,2)$ which satisfy*

(1) $\dfrac{dT_i}{dz}=\dfrac{1}{2}\sum_{j,k}\varepsilon_{ijk}[T_j,T_k]$,

(2) *T_i is analytic for $z\in(0,2)$ with simple poles at $z=0$ and 2,*

(3) *the representation of* SU(2) *defined by the residues of T_i at their poles is irreducible,*

(4) $T_i(z)=-\bar{T}_i(2-z)$,

(5) $T_i^*(z)=-T_i(z)$.

Then the ADHM *construction applied to the quaternionic operator*

$$\Delta(x)=(x_0+\sum x_j e_j)+i\frac{d}{dz}+i\sum T_j e_j$$

gives a non-singular solution to the SU(2) *Bogomolny equations on* $\mathbb{R}^3$.

We shall now investigate the boundary behaviour as $r\to\infty$ of Nahm's solution and show that it satisfies the conditions A1–A3 of Sect. 1.

We have already used the approximation $\tilde{\Delta}$ to Δ. We now put in the x dependence and define $\Delta_0(x)=e^{-zQ}\tilde{\Delta}e^{zQ}+(x_0+\sum x_j e_j)$. Now, although Δ_0 is not quaternionic, it will provide a good approximation for the asymptotic behaviour of Δ as $r\to\infty$. Indeed, it is related to Δ by $\Delta(x)=\Delta_0(x)+A$, where A is smooth in a neighbourhood of $[0,2]$ and independent of x. In particular, index $\Delta_0=\operatorname{index}\Delta=-2$. We also have, from (2.6), $\langle\Delta^*\Delta f,f\rangle\geqq r^2\|f\|^2$ ($r^2=\|x\|^2$). Hence, for sufficiently large r, we also have a positive constant c^2 such that

$$\langle\Delta_0^*\Delta_0 f,f\rangle=\langle(\Delta-A)f,(\Delta-A)f\rangle\geqq c^2r^2\|f\|^2.$$

Hence $\Delta_0^*\Delta_0$ is positive for large r and $\dim_{\mathbb{C}}\ker\Delta_0^*=2$. We shall use these two solutions to approximate $\ker\Delta^*$.

Let G_0 be the Green's function for Δ_0 so that $\Delta_0G_0=I-P_0$, $G_0\Delta_0=I$, where P_0 is the orthogonal projection onto $\ker\Delta_0^*$.

Then, since $\langle\Delta_0 f,\Delta_0 f\rangle\geqq c^2r^2\|f\|^2$, we have $\|f\|^2-\|P_0f\|^2\geqq c^2r^2\|G_0f\|^2$. Hence $\|G_0\|\leqq\frac{1}{cr}$ and

$$\|G_0^*\|\leqq\frac{1}{cr}. \tag{2.9}$$

Now since $G_0^*\Delta_0^*=I-P_0$, if $\Delta^*f=0$, then we have

$$f-P_0f=G_0^*(\Delta^*-A^*)f=-G_0^*A^*f,$$

and so, for some constant K,

$$\|f-P_0f\|\leqq\frac{K}{r}\|f\|. \tag{2.10}$$

We can thus approximate, to order r^{-1}, solutions of $\Delta^*f=0$ by solutions of $\Delta_0^*f=0$.

Next choose a direction in $\mathbb{R}^4$ given by a unit quaternion u. Since we shall be interested in connections only in a neighbourhood of $\mathbb{R}^3$, we suppose $u\neq\pm1$, and hence u generates a circle group in SU(2). The representation space $S^{k-1}\otimes S^1$ then splits into 1-dimensional weight spaces with weights

$$k,k-2,k-4,\ldots,-(k-2),-k\quad\text{for } S^k,$$

and

$$k-2,k-4,\ldots,-(k-2)\quad\text{for } S^{k-2}.$$

Consider the action $1\otimes u$ of u on $S^{k-1}\otimes S^1$. This commutes with the action of the representation (i.e., $u\otimes u$) and since the spaces of weight $\pm k$ occur with multiplicity one, they are preserved. Now if u is an imaginary quaternion (i.e., $ru \in \mathbb{R}^3$), then $(1 \otimes u)^2 = -1$, and hence $(1 \otimes u)v_\pm = \pm i v_\pm$, where $v_\pm$ are vectors in the $\pm k$ weight spaces.

Let $g_\pm = g(z)e^{-zQ}v_\pm$, then if $x = ru$

$$\Delta_0^* g_\pm = e^{zQ}\left(i\frac{dg}{dz} - \frac{i}{2}(k-1)\left\{\frac{1}{z} + \frac{1}{(z-2)}\right\}g \pm rig\right)v_\pm$$

from (2.7). Thus $g_\pm = (z^2 - 2z)^{k-1/2}e^{\pm r(z-1) - zQ}v_\pm$ defines an $\mathscr{L}^2$ solution of $\Delta_0^* f = 0$.

Now from (2.10) it follows that a basis for the solutions of $\Delta^* f = 0$ can be found of the form

$$f_\pm = g_\pm / \|g_\pm\| + O(r^{-1}), \tag{2.11}$$

where $g_\pm = (z^2 - 2z)^{k-1/2}e^{\pm r(z-1) - zQ}v_\pm$.

The estimates we require concern the Higgs field Φ. Now from the discussion at the beginning of Sect. 2, Φ is defined in terms of an $\mathbb{R}$-invariant self-dual connection by the formula

$$\Phi(s) = -\nabla'_{\partial/\partial x_0} s|_{x_0 = 0},$$

where s is a section of E' invariant under $\mathbb{R}$. In Nahm's framework, a solution of $\Delta^* f = 0$ of the form $f = e^{ix_0(z-1)}g(x_1, x_2, x_3, z)$ is such an invariant section of $E' = \ker \Delta^*$, and since the self-dual connection ∇' is obtained by orthogonal projection of the ordinary flat derivative, we obtain

$$\Phi(f) = -P\frac{\partial f}{\partial x_0}\bigg|_{x_0 = 0} = P(i(1-z)g),$$

where P is the orthogonal projection onto the kernel of Δ^*. Now, because of (2.10) and (2.11) we can determine Φ to order r^{-1} by considering the operator Δ_0^* and the solutions $g_\pm$.

In this case, if we use g_+ and g_- to form a basis for the vector bundle, then the Higgs field is given by

$$\Phi = \begin{pmatrix} \dfrac{\langle i(1-z)g_+, g_+\rangle}{\|g_+\|^2} & \dfrac{\langle i(1-z)g_+, g_-\rangle}{\|g_+\|\,\|g_-\|} \\[2ex] \dfrac{\langle i(1-z)g_-, g_+\rangle}{\|g_+\|\,\|g_-\|} & \dfrac{\langle i(1-z)g_-, g_-\rangle}{\|g_-\|^2} \end{pmatrix}.$$

However,

$$\|g_\pm\|^2 = \int_0^2 (2z - z^2)^{k-1} e^{\pm 2r(z-1)} dz,$$

and so $\|g_\pm\| \to \infty$ as $r \to \infty$. Also

$$\langle i(1-z)g_+, g_-\rangle = \int_0^2 i(1-z)(2z - z^2)^{k-1} dz = 0.$$

Hence the off-diagonal terms in Φ vanish. Furthermore,

$$\lim_{r\to\infty} \frac{\int_0^2 i(1-z)(2z-z^2)^{k-1} e^{\pm 2r(z-1)} dz}{\int_0^2 (2z-z^2)^{k-1} e^{\pm 2r(z-1)} dz} = \mp i.$$

Hence, in this gauge,

$$\Phi \to \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad \text{as} \quad r\to\infty. \tag{2.12}$$

Thus $\|\Phi\|\to 1$ and the eigenspace corresponding to $\mp i$ for large r is isomorphic to the homogeneous bundle $\mathbb{R}^3\backslash 0 \times_{S^1} \mathbb{C}$, where the circle group S^1 acts on $\mathbb{C}$ with weight $\pm k$. The line bundles defined by the eigenspaces of Φ on a large sphere thus have Chern class $\pm k$, so by definition the monopole has charge k.

Consider next the curvature of ∇'. This is given by

$$F = P dx G G^* d\bar{x} P. \tag{2.13}$$

Hence from (2.9) we obtain $\|F\|^2 \leqq 3/c^4 r^4$, and so $\|D\Phi\| = \|F\| = O(r^{-2})$ verifying condition A3.

Now from the Bianchi identity $D^*D\Phi = 0$, hence

$$d^*d\|\Phi\|^2 = -\|D\Phi\|^2 = -\|F\|^2.$$

Thus, from Green's theorem and the estimates

$$\|d\|\Phi\|^2\|^2 = 2\|(D\Phi, \Phi)\|^2 = O(r^{-4}),$$
$$\|\Phi\| \to 1,$$

we find

$$\|\Phi\|^2(x) = 1 - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\|F\|^2(x')}{\|x-x'\|}. \tag{2.14}$$

We now approximate F by F_0. If we set $G = G_0 + B$, then

$$(\Delta_0 + A)(G_0 + B) = I - P,$$

hence $\Delta_0 B = P_0 - P - A G_0$. Thus from (2.9) and (2.10) there is a constant α such that $\|\Delta_0 B\| \leqq \alpha r^{-1}$. Hence, $\|B\| = \|G_0 \Delta_0 B\| \leqq \alpha/cr^2$, i.e. $\|G - G_0\| = O(r^{-2})$, and so from (2.9) $\|GG^* - G_0 G_0^*\| = O(r^{-3})$. Thus from (2.13) we obtain

$$\|F - F_0\| = O(r^{-3}), \tag{2.15}$$

and hence

$$\|F\|^2 - \|F_0\|^2 = O(r^{-5}). \tag{2.16}$$

Now F_0 is the curvature of a homogeneous connection on a direct sum of line bundles over $\mathbb{R}^3\backslash 0$, and so $\|F_0\|^2 = \beta r^{-4}$ for some constant β. Hence from (2.16)

$\|F\|^2 = \beta r^{-4} + O(r^{-5})$, and from (2.14) we may therefore write

$$\|\Phi\|^2(x) = 1 - \frac{1}{4\pi r} \int_{\mathbb{R}^3} \|F\|^2 + O(r^{-2}). \tag{2.17}$$

Now by Stokes' theorem

$$\begin{aligned}\int_{\mathbb{R}^3} \|F\|^2 &= \lim_{R\to\infty} \int_{S_R} *(D\Phi, \Phi) \\ &= \lim_{R\to\infty} \int_{S_R} (F, \Phi) \\ &= \lim_{R\to\infty} \int_{S_R} (F_0, \Phi)\end{aligned}$$

from (2.15). But as $r \to \infty$, the line bundles of which F_0 is the curvature approach the eigenspaces of Φ, so this last integral may be expressed in terms of Chern classes, and we obtain from (2.17)

$$\|\Phi\|^2 = 1 - \frac{2k}{r} + O(r^{-2}),$$

which is the condition A1. It is straightforward to take into account the angular dependence of $\|F\|$ in the estimates, and we obtain all three conditions A1, A2, A3.

In fact Taubes (unpublished) has shown, using the methods of [9], that the conditions A1–A3 are consequences of the equations $D\Phi = *F$ and the single condition $\|\Phi\| \to 1$.

We have thus seen that Nahm's construction produces a monopole with the required asymptotic conditions, passing from the realm of ordinary differential equations to that of partial differential equations. In the next section we pass from partial differential equations to algebraic geometry, using twistor methods.

3. The Spectral Curve

Nahm's construction started by interpreting the Bogomolny equations as x_0-translation invariant solutions of the self-duality equations in $\mathbb{R}^4$. It is well-known that a self-dual SU(2) connection on $\mathbb{R}^4$ corresponds using the Penrose twistor theory to a holomorphic rank 2 vector bundle on the complex 3-manifold $\mathbb{P}_3 \backslash \mathbb{P}_1$, which is trivial on every real line and quaternionic with respect to the real structure on $\mathbb{P}_3$ induced by a quaternionic structure on $\mathbb{C}^4$ (see [2, 4, 5]). The action of translation in the x_0-direction induces a free holomorphic action of the additive group $\mathbb{C}$ of complex numbers on $\mathbb{P}_3 \backslash \mathbb{P}_1$, whose quotient is an algebraic surface. It may be identified with the total space of the tangent bundle $T\mathbb{P}_1$ to the projective line.

Thus, a solution of the SU(2) Bogomolny equations on $\mathbb{R}^3$ corresponds to a holomorphic rank 2 vector bundle $\tilde{E}$ on $T\mathbb{P}_1$ which is quaternionic and trivial on every real section of $\pi: T\mathbb{P}_1 \to \mathbb{P}_1$, these being the projections of real lines in $\mathbb{P}_3$.

This correspondence is dealt with directly, without passing to $\mathbb{R}^4$, in [8], to which we refer for details of the following. The complex surface $T\mathbb{P}_1$ may be thought of as the space of oriented straight lines in $\mathbb{R}^3$, and then the bundle $\tilde{E}$ is defined by:

$$\tilde{E}_z = \{s \in \Gamma(\gamma_z, E) | (\nabla_U - i\Phi)s = 0\}.$$

Here U is the unit tangent vector along the oriented geodesic γ_z corresponding to a point $z \in T\mathbb{P}_1$.

If (∇, Φ) satisfy conditions A1–A3, then there are two distinguished holomorphic line bundles L^+ and L^- in $\tilde{E}$ defined by:

$$L_z^{\pm} = \{s \in \tilde{E}_z | s(t) \to 0 \text{ as } t \to \pm\infty\}.$$

The *spectral curve* S is defined by $S = \{z \in T\mathbb{P}_1 | L_z^+ = L_z^-\}$ and is a compact algebraic curve.

We denote by L the holomorphic line bundle on $T\mathbb{P}_1$ corresponding to the trivial $U(1)$ solution of the Bogomolny equations $\Phi = i$, and by $O(k)$ the pull-back from $\mathbb{P}_1$ of the unique line bundle of degree k. We cover $\mathbb{P}_1$ by two standard affine open sets U_0, U_1 and let ζ be a coordinate on U_0. Then $\frac{d}{d\zeta}$ trivializes the tangent bundle over U_0 and we take local coordinates (η, ζ) on $\pi^{-1}(U_0)$ defined by $(\eta, \zeta) \to \eta \frac{d}{d\zeta}$. With respect to the open covering $\tilde{U}_i = \pi^{-1}(U_i)$ of $T\mathbb{P}_1$, the line bundle L is defined by the transition function $e^{\eta/\zeta}$ on $\tilde{U}_0 \cap \tilde{U}_1$.

It is shown in [8] that $L^+ \cong L(-k)$ and $L^- \cong L^*(-k)$. Hence, since $L^+ = L^-$ on S, we have the fundamental constraint $L \cong L^*$ on S.

The natural real structure on $T\mathbb{P}_1$ is defined by $\tau(\eta, \zeta) = (-\bar{\eta}/\bar{\zeta}^2, -\bar{\zeta}^{-1})$ and corresponds to the operation of altering the orientation on each straight line in $\mathbb{R}^3$. The quaternionic structure $\sigma: \tilde{E}_z \to \tilde{E}_{\tau z}$ maps L_z^+ to $L_{\tau z}^-$ and so on S the bundle $L(-k)$ has a quaternionic structure. Hence $L(k-1) = L(-k) \otimes O(2k-1)$ is real.

Since S is defined by the condition $L_z^+ = L_z^-$, this corresponds to the vanishing of the map $L^- \subset \tilde{E} \to \tilde{E}/L^+ \cong (L^+)^*$. But $L^- \cong L^*(-k)$ and $L^+ \cong L(-k)$, so S is the divisor of a section $\psi \in H^0(T\mathbb{P}_1, O(2k))$ and since it is compact, is defined by an equation

$$\psi = \eta^k + a_1(\zeta)\eta^{k-1} + \ldots + a_k(\zeta) = 0,$$

where $a_i(\zeta)$ is a polynomial of degree $2i$ in ζ (see [8]). The curve S may of course be singular or reducible.

To summarize, we have the following properties of S:

S is a compact algebraic curve in the linear system $|O(2k)|$ on $T\mathbb{P}_1$ such that S is real and L^2 is trivial on S.

In this section we shall first describe the space $H^1(C, O)$ for any curve C in the system $|O(2k)|$, use it to show that S has no multiple components, and then prove the important vanishing condition B4 of Sect. 1. For brevity we put $T = T\mathbb{P}_1$.

Recall that there is a canonical section $\eta \frac{d}{d\zeta} \in H^0(T, O(2))$ which we shall denote simply by η. We shall prove the following:

Proposition (3.1). *Let S be a curve in the linear system $|O(2k)|$. Then every element $c \in H^1(S, O)$ may be written uniquely in the form*

$$c = \sum_{i=1}^{k-1} \eta^i \pi^* c_i,$$

where $c_i \in H^1(\mathbb{P}_1, O(-2i))$ and $\pi: S \to \mathbb{P}_1$ is the projection.

Proof. We first compactify T to a compact non-singular algebraic surface $\hat{T}$ by replacing a line bundle by a bundle of projective lines. We take

$$\hat{T} = \mathbb{P}(O(2) \oplus O) = \mathbb{P}(O(1) \oplus O(-1)),$$

since $T = T\mathbb{P}_1 = O(2)$.

Let H_1 denote the line bundle $\pi^* O(1)$ on $\hat{T}$ and H_2 the tautological bundle over $\mathbb{P}(O(1) \oplus O(-1))$. Recall that any projective bundle $\mathbb{P}(V)$ has a tautological bundle H whose dual is defined by:

$$H^* = \{(x, y) \in \mathbb{P}(V) \times V \,|\, \pi(x) = \pi(y) \text{ and } y \in x\}.$$

The canonical bundle K of holomorphic 2-forms on $\hat{T}$ is then expressed, in additive notation, by

$$K = -2H_2 - 2H_1. \tag{3.2}$$

The section $\eta \in H^0(T, O(2))$ extends to a section $\hat{\eta} \in H^0(\hat{T}, H_1 + H_2)$, vanishing only on the zero section Z, and there is a section $\hat{\xi} \in H^0(\hat{T}, H_2 - H_1)$ vanishing only on the section at infinity. Thus $\hat{\xi}$ trivializes $H_2 - H_1$ on $T \subset \hat{T}$, and hence in a neighbourhood of S.

We shall consider first the restriction map

$$H^1(\hat{T}, (k-1)(H_2 - H_1)) \xrightarrow{\varrho_S} H^1(S, (k-1)(H_2 - H_1)),$$

and show that this is an isomorphism. Since $H_2 - H_1$ is trivial on S we shall then have a description of $H^1(S, O)$.

To do this, first note that the curve S is defined in $\hat{T}$ as the divisor of

$$\hat{\psi} = \hat{\eta}^k + a_1 \hat{\eta}^{k-1} \hat{\xi} + \ldots + a_k \hat{\xi}^k \in H^0(\hat{T}, k(H_1 + H_2)).$$

Thus we consider the exact sequence of sheaves

$$0 \to O_{\hat{T}}(-H_2 - (2k-1)H_1) \to O_{\hat{T}}((k-1)(H_2 - H_1)) \to O_S \to 0.$$

From the exact cohomology sequence, ϱ_S will be an isomorphism if

$$H^p(\hat{T}, -H_2 - (2k-1)H_1) = 0 \quad \text{for} \quad p = 1, 2. \tag{3.3}$$

Now on $\hat{T} = \mathbb{P}(O(2) \oplus O)$, the curvature of the natural connection on the tautological bundle is non-negative and positive in the fibre directions. In our notation this bundle is $H_1 + H_2$. Since H_1 is positive on the base,

$$(H_1 + H_2) + \varepsilon H_1 > 0 \quad \text{for} \quad \varepsilon > 0.$$

Hence,

$$(2k-1)H_1+H_2=(2k-2)H_1+(H_1+H_2)>0$$

for $k>1$, and so by Kodaira's vanishing theorem,

$$H^p(\hat{T}, -H_2-(2k-1)H_1)=0 \quad \text{for} \quad p=0,1.$$

On the other hand, we have the Riemann-Roch theorem for a line bundle with Chern class l:

$$h^0-h^1+h^2=\tfrac{1}{2}l^2+\tfrac{1}{2}c_1 l+\tfrac{1}{12}(c_1^2+c_2).$$

Since $\hat{T}$ is rational, $c_1^2+c_2=12$. Now taking $L=-H_2-(2k-1)H_1$ and using (3.2) and the intersection numbers: $H_1\cdot H_2=1$, $H_1^2=0$, $H_2^2=0$, we see that $h^2=(2k-1)-1-(2k-1)+1=0$. Hence (3.3) holds and ϱ_S is an isomorphism.

Consider next the map

$$\hat{\xi}: H^1(\hat{T},(l-1)(H_2-H_1))\to H^1(\hat{T}, l(H_2-H_1)).$$

Since $\hat{\xi}$ vanishes on the rational curve at infinity C, and $\hat{\eta}$ trivializes H_1+H_2 in a neighbourhood of C, we have the exact sequence of sheaves:

$$0\to O_{\hat{T}}((l-1)(H_2-H_1))\xrightarrow{\hat{\xi}} O_{\hat{T}}(l(H_2-H_1))\xrightarrow{\varrho_C} O_C(-2l)\to 0.$$

Since $H^0(\mathbb{P}_1, O(-2l))=0$ for $l>0$, we see that $\hat{\xi}$ is injective. Also, if we consider $c_l\in H^1(\mathbb{P}_1, O(-2l))$, then

$$\varrho_C(\hat{\eta}^l\pi^*c_l)=c_l. \tag{3.4}$$

This is because C is a section of $\pi:\hat{T}\to\mathbb{P}_1$.

Now, consider an element $a\in H^1(\hat{T},(k-1)(H_2-H_1))$ of the form

$$a=\sum_1^{k-1}\hat{\eta}^i\hat{\xi}^{k-1-i}\pi^*c_i.$$

If this vanishes, then from (3.4), $c_{k-1}=0$. However, since $\hat{\xi}$ is injective, this implies

$$\sum_1^{k-2}\hat{\eta}^i\hat{\xi}^{k-2-i}\pi^*c_i=0.$$

Repeating the argument we see that $c_i=0$ for all i. Hence the elements of the form $\hat{\eta}^i\hat{\xi}^{k-1-i}\pi^*c_i$ for $c_i\in H^1(\mathbb{P}_1, O(-2i))$ and $1\leqq i\leqq k-1$ are all linearly independent and span a subspace of $H^1(\hat{T},(k-1)(H_2-H_1))$ of dimension

$$\sum_1^{k-1}\dim H^1(\mathbb{P}_1, O(-2i))=\sum_1^{k-1}(2i-1)=(k-1)^2. \tag{3.5}$$

Thus, since ϱ is an isomorphism, this is a $(k-1)^2$-dimensional subspace of $H^1(S,O)$. However, from the exact sequence

$$0\to O_{\hat{T}}(-k(H_1+H_2))\to O_{\hat{T}}\to O_S\to 0$$

we see, using the Riemann-Roch theorem for $\hat{T}$, that

$$\dim H^0(S,O)-\dim H^1(S,O)=1-(k-1)^2.$$

Since S is connected, $\dim H^0(S,O)=1$, so $\dim H^1(S,O)$, the arithmetic genus, is $(k-1)^2$.

Hence from (3.5) every element $c\in H^1(S,O)$ is uniquely expressible as

$$c=\sum_{1}^{k-1} \eta^i \pi^* c_i,$$

and Proposition (3.1) is proved.

If we now specialize to the case of the spectral curve of a monopole, we may deduce the following property:

Proposition (3.6). *Let S be the spectral curve of a monopole. Then S has no multiple components.*

Proof. Let C be a reduced curve which occurs as a component of S with multiplicity greater than one. Then from the definition of S, $L^+=L^-$ on the first formal neighbourhood of $C\subset T$. Hence L^2 is trivial on the first order neighbourhood. The curve C is the divisor of a section ψ of $O(2l)$ for some $l<k$ (we may take C to be real). Now consider the exact sequence of sheaves for the first formal neighbourhood $O_C^{(1)}(2L)$,

$$0\to O_C(2L-2l)\to O_C^{(1)}(2L)\to O_C(2L)\to 0.$$

If a is the trivialization of $2L$ on S, and it extends to the first order, then the obstruction $\delta(a)\in H^1(C,O(-2l))$ in the exact cohomology sequence vanishes. In particular, if we take a family of sections $\psi(t)\in H^0(T,O(2l))$ with $\psi(1)=\psi$, then the obstruction to extending a in this direction vanishes, since it is the element

$$\left\langle \delta(a),\frac{\partial\psi}{\partial t}(1)\right\rangle \in H^1(C,O).$$

Take the family $\psi(t)=\psi(t\eta,\zeta)$, then $\eta\to t\eta$ defines a biholomorphic equivalence between the divisor C_t of $\psi(t)$ and C. The line bundle $2L$ on C_t is then given as a bundle on C by the transition function $e^{2t\eta/\zeta}$.

Hence the obstruction to extending the trivialization in this direction is the element in $H^1(C,O)$ represented by the cocycle $2\eta/\zeta$.

Now ζ^{-1} represents a non-trivial element of $H^1(\mathbb{P}_1,O(-2))$, thus by Proposition (3.1), the cocycle $2\eta/\zeta$ represents a non-trivial element in $H^1(C,O)$.

Hence $\delta(a)\neq 0$ and the Proposition is proved.

This is one condition on the spectral curve which is not included in [8]. We shall next prove a more fundamental condition which will turn out to be equivalent to the non-singularity of the monopole determined by S. It is a vanishing theorem analogous to the vanishing theorem for bundles on $\mathbb{P}_3$ which was so important in the ADHM construction of instantons [2, 7, 12].

We denote by L^z the line bundle over T corresponding to the solution $\Phi=zi$ of the $U(1)$ Bogomolny equations. Its transition function with respect to the covering $\tilde{U}_0, \tilde{U}_1$ is $e^{z\eta/\zeta}$.

Theorem (3.7). *Let S be the spectral curve of a monopole of charge k. Then,*

$$H^0(S,L^z(k-2))=0 \quad \text{if} \quad z\in(0,2).$$

Proof. First let us recall how the spectral curve determines the bundle $\tilde{E}$ over T, according to [8], Sect. 7.

The bundle L^2 is trivial on S, so the coboundary map

$$\delta: H^0(S, L^2) \to H^1(T, L^2(-2k))$$

defines an extension of line bundles $0 \to L(-k) \to E^+ \to L^*(k) \to 0$ by $\delta(a)$, where $a \in H^0(S, L^2)$ is a trivialization. Since S is connected, a is unique up to a scalar multiple, so the bundle E^+ is uniquely defined.

The real structure defines an antiholomorphic map $\sigma: L^2(-2k) \to L^{-2}(-2k)$ on T and so $\sigma\delta(a)$ defines another extension

$$0 \to L^*(-k) \to E^- \to L(k) \to 0.$$

The bundles E^+, E^- are both isomorphic, the isomorphism defining the quaternionic structure on $\tilde{E} = E^+ \cong E^-$. The two extensions then correspond to the two distinguished subbundles L^+ and L^-.

The coboundary map is at the source of the proof. The basic idea is to consider the composite map:

$$H^0(S, L^z(k-2)) \xrightarrow{\delta} H^1(T, L^z(-k-2)) \xrightarrow{i} H^1(T, L^{z-1}\tilde{E}(-2)),$$

where i is induced by the inclusion $L(-k) \to \tilde{E}$. If $s \in H^0(S, L^z(k-2))$, then $i\delta(s)$ represents, using the twistor interpretation of massless fields, a solution ϕ of a differential equation of Laplacian type on $\mathbb{R}^3$. We show that ϕ decays at infinity fast enough to ensure that it vanishes identically. It will then follow that s must itself vanish.

In order to obtain estimates, we need a good analytical control of the transforms involved, and so we pass from Čech cohomology to Dolbeault cohomology in order to represent sheaf cohomology classes.

Let $a \in H^0(S, L^2)$ be the trivialization and $\{V_i\}$ a sufficiently small covering of a neighbourhood of S by open balls. Since S is compact, this can be taken to lie in a compact subset of T.

A Čech representative for $\delta(a)$ consists of extending the covering to T and taking the cocycle $(a_i - a_j)/\psi$ on $V_i \cap V_j$, where a_i is some holomorphic extension of a to V_i. We take the cocycle zero on any other intersections. A Dolbeault representative may be obtained by taking a partition of unity $\{\phi_i\}$ subordinate to the covering and defining $\theta_j = \bar{\partial} \sum \phi_i(a_i - a_j)/\psi$. Then, on $V_j \cap V_k$

$$\theta_j - \theta_k = \bar{\partial}(\sum \phi_i(a_k - a_j)/\psi) = \bar{\partial}((a_k - a_j)/\psi) = 0.$$

Hence θ is a well-defined $\bar{\partial}$-closed $(0,1)$ form with values in $L^2(-2k)$. In fact,

$$\theta = \bar{\partial}(\sum \phi_i a_i)/\psi = \bar{\partial}\alpha/\psi, \tag{3.8}$$

where α is a C^∞ section of L^2 of compact support.

With this explicit representative, we can define the holomorphic structure on the extension E^+. We take $E^+ = L(-k) \oplus L^*(k)$ with a new $\bar{\partial}$-operator given by

$$\bar{\partial}(x, y) = (\bar{\partial}x + \theta y, \bar{\partial}y). \tag{3.9}$$

Similarly the holomorphic structure on E^- is defined by

$$\bar{\partial}(x', y') = (\bar{\partial}x' + \theta^* y', \bar{\partial}y'), \tag{3.10}$$

where $\theta^* \in \Omega^{0,1}(L^{-2}(-2k))$ is the form conjugate to θ using the real structure. Thus, from (3.8),

$$\theta^* = \bar{\partial}\alpha^*/\psi^* = \bar{\partial}\alpha^*/\psi, \tag{3.11}$$

since $\psi \in H^0(T, O(2k))$ is real.

Now E^+ and E^- are holomorphically equivalent via an isomorphism which preserves the natural symplectic form on each:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $A \in \Gamma(T, L^{-2})$, $D \in \Gamma(T, L^2)$, $B \in \Gamma(T, O(-2k))$, and $C \in \Gamma(T, O(2k))$ and $AD - BC = 1$.

Now to be holomorphic the matrix above must intertwine the two $\bar{\partial}$-operators. Hence,

$$\begin{aligned} \bar{\partial}x' + \theta^* y' &= \bar{\partial}(Ax + By) + \theta^*(Cx + Dy) \\ &= A(\bar{\partial}x + \theta y) + B\bar{\partial}y, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \bar{\partial}y' &= \bar{\partial}(Cx + Dy) \\ &= C(\bar{\partial}x + \theta y) + D\bar{\partial}y. \end{aligned} \tag{3.13}$$

Now from (3.13) $\bar{\partial}C = 0$, so C is a holomorphic section of $O(2k)$. In fact, by the very definition of the spectral curve, $C = \psi$. Also from (3.13) we have $\bar{\partial}D = C\theta = \psi\bar{\partial}\alpha/\psi = \bar{\partial}\alpha$. Hence $D - \alpha$ is a holomorphic section of L^2. But if L^2 has a holomorphic section on T, then it must be trivial and define the trivial solution $\Phi = 0$ to the Bogomolny equations. Since for L^2, $\Phi = 2i$ we must therefore have $D = \alpha$.

Now from (3.12), $\bar{\partial}A + \theta^* C = 0$. Hence from (3.11) $\bar{\partial}A + \bar{\partial}\alpha^* = 0$, and again since L^{-2} has no holomorphic sections, $A = -\alpha^*$.

The final entry B is determined by the condition $AD - BC = 1$, hence the quaternionic structure on $\tilde{E}$, i.e. the isomorphism $E^+ \cong E^-$ is defined by

$$\begin{pmatrix} -\alpha^* & -(1 + \alpha\alpha^*)/\psi \\ \psi & \alpha \end{pmatrix}. \tag{3.14}$$

Now, returning to the theorem, take a section $s \in H^0(S, L^z(k-2))$ and form the element $i\delta(s) \in H^1(T, L^{z-1}\tilde{E}(-2))$. We can represent $i\delta(s)$, by a procedure analogous to the above, by a Dolbeault form

$$(\bar{\partial}\sigma/\psi, 0) \in \Omega^{0,1}(L^{z-1}E^+(-2)),$$

where σ is a section of $L^z(k-2)$, compactly supported around S.

But, from (3.14)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \bar{\partial}\sigma/\psi \\ 0 \end{pmatrix} = \begin{pmatrix} -\alpha^*\bar{\partial}\sigma/\psi \\ \bar{\partial}\sigma \end{pmatrix},$$

and [as a section of $L^{z-1}E^-(-2)$] we have

$$\bar{\partial}\begin{pmatrix} 0 \\ \sigma \end{pmatrix} = \begin{pmatrix} \theta^*\sigma \\ \bar{\partial}\sigma \end{pmatrix} = \begin{pmatrix} \sigma\bar{\partial}a^*/\psi \\ \bar{\partial}\sigma \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \bar{\partial}\sigma/\psi \\ 0 \end{pmatrix} = \bar{\partial}\begin{pmatrix} 0 \\ \sigma \end{pmatrix} - \begin{pmatrix} \bar{\partial}(\alpha^*\sigma)/\psi \\ 0 \end{pmatrix}. \tag{3.15}$$

In other words, the cohomology class $i\delta(s) \in H^1(T, L^{z-1}\tilde{E}(-2))$ is represented by two forms,

$$\theta^+ = \bar{\partial}\sigma/\psi \in \Omega^{0,1}(L^{z-1}L^+(-2)) \quad \text{and} \quad \theta^- = -\bar{\partial}(\alpha^*\sigma)/\psi \in \Omega^{0,1}(L^{z-1}L^-(-2)),$$

where, moreover, if $\gamma = \bar{\partial}(0, \sigma)$

$$\theta^+ - \theta^- = \bar{\partial}\gamma, \tag{3.16}$$

where θ^+, θ^- and γ are all supported in the same compact neighbourhood of S.

Recall now that if E is a vector bundle on $\mathbb{R}^4$ with a self-dual connection and $\tilde{E}$ is the corresponding holomorphic bundle on $\mathbb{P}_3 \backslash \mathbb{P}_1$, then there is an isomorphism [2, 7, 12]

$$H^1(\mathbb{P}_3 \backslash \mathbb{P}_1, \tilde{E}(-2)) \cong \{\phi \in \Gamma(\mathbb{R}^4, E) | \nabla^*\nabla\phi = 0\}.$$

It is straightforward to deduce that, for a self-dual connection which is invariant under x_0-translation, there is a corresponding isomorphism

$$H^1(T, \tilde{E}(-2)) \cong \{\phi \in \Gamma(\mathbb{R}^3, E) | (\nabla^*\nabla + \Phi^*\Phi)\phi = 0\}. \tag{3.17}$$

In our case we are considering the bundle $L^{z-1}\tilde{E}$ which corresponds to the $U(2)$ solution of the Bogomolny equations obtained by taking the original connection on E, but with a modified Higgs field $\Phi' = \Phi + (z-1)i$. Note that the eigenvalues of $i\Phi'$ at infinity are $-z$ and $(2-z)$ so that if $z \in (0, 2)$ they have opposite sign.

We must now examine the isomorphism (3.17) in more detail in order to estimate the growth of ϕ.

The value of the section ϕ at a point $x \in \mathbb{R}^3$ is obtained by restricting to the corresponding section $P_x \subset T$.

Now by Serre duality

$$\begin{aligned} H^1(P_x, L^{z-1}\tilde{E}(-2)) &\cong H^0(P_x, L^{1-z}\tilde{E})^* \\ &= E_x^* \cong E_x. \end{aligned}$$

We want to estimate the norm of $\phi(x)$ and this we do by considering all holomorphic sections of $L^{1-z}\tilde{E}$ on P_x and evaluating the class $i\delta(s)$ corresponding to ϕ on them.

Now from [8], a section f of $L^{1-z}\tilde{E}$ corresponds to a section $\hat{f}$ of E on the unit sphere bundle $S^2\times\mathbb{R}^3$ which satisfies the equation $(\nabla_U - i\Phi')\hat{f}=0$ along the geodesic flow.

Similarly a representative form $\theta\in\Omega^{0,1}(L^{z-1}\tilde{E}(-2))$ for $i\delta(s)$ corresponds to a form $\hat{\theta}$ on $S^2\times\mathbb{R}^3$ which satisfies an analogous equation along the flow. The evaluation $\langle\hat{\theta},\hat{f}\rangle$ is then invariant under the flow and defines the evaluation $\langle\theta,f\rangle\in\Omega^{0,1}(T,O(-2))$. Therefore, to evaluate ϕ at a point $x\in\mathbb{R}^3$, we take a basis $\{e_1,e_2\}$ for E_x, pull back to $S^2\times\mathbb{R}^3$ to obtain holomorphic sections $\{f_1,f_2\}$ of $L^{1-z}\tilde{E}$ on P_x and evaluate the integral to obtain

$$\langle\phi,e_i\rangle_x=\langle i\delta(s),f_i\rangle=\int_{S^2}\langle\hat{\theta}(x,u),\hat{f}_i(x,u)\rangle\,,$$

where $u\in S^2\cong P_x$ runs over the unit tangent vectors at $x\in\mathbb{R}^3$.

Now from (3.16) we have two representatives θ^+ and θ^- for $i\delta(s)$, where $\theta^\pm\in\Omega^{0,1}(L^{z-1}L^\pm(-2))$. But in [8], L^+ is defined as the space of solutions to $(\nabla_U-i\Phi)s=0$ along a line which decay like t^ke^{-t} as $t\to\infty$. We similarly obtain estimates

$$\hat{\theta}^+(y+tu,u)\sim t^ke^{(z-2)t}\quad\text{as}\quad t\to+\infty\,,$$
$$\hat{\theta}^-(y+tu,u)\sim|t|^ke^{-z|t|}\quad\text{as}\quad t\to-\infty\,,$$

using the Higgs field Φ'.

Let us parametrize T by

$$\{(u,x)\in S^2\times\mathbb{R}^3|\,\|u\|=1\text{ and }u\cdot x=0\}\,.$$

That is, define a straight line by its direction and shortest distance to the origin.

Then since θ^+ and θ^- have support contained in some disc bundle of radius R, we have for $z\in(0,2)$ estimates of the form

$$\|\hat{\theta}^+(y+tu,u)\|<Ke^{-\varepsilon t}\,,\qquad t\geqq0\,,$$
$$\|\hat{\theta}^-(y+tu,u)\|<Ke^{-\varepsilon|t|}\,,\qquad t<0\,,$$

if $y\cdot u=0$.

Equivalently, putting $x=y+tu$,

$$\begin{aligned}\|\hat{\theta}^+(x,u)\|&<Ke^{-\varepsilon(x\cdot u)}\quad\text{if}\quad x\cdot u\geqq0\\ \|\hat{\theta}^-(x,u)\|&<Ke^{-\varepsilon|x\cdot u|}\quad\text{if}\quad x\cdot u<0\,.\end{aligned}\tag{3.18}$$

Now if $\|x\|^2-(x\cdot u)^2>R^2$, the straight line through x in the direction u is always a distance greater than R from the origin. Hence from (3.18), if $\|x\|>R$ we have

$$\begin{aligned}\|\hat{\theta}^+(x,u)\|&<K\exp(-\varepsilon\sqrt{\|x\|^2-R^2})\,,\quad\text{if}\quad x\cdot u\geqq0\,,\\ \|\hat{\theta}^-(x,u)\|&<K\exp(-\varepsilon\sqrt{\|x\|^2-R^2})\,,\quad\text{if}\quad x\cdot u<0\,.\end{aligned}\tag{3.19}$$

We define open sets $V^\pm$ in S^2 by

$$V^\pm=\{u\in S^2|\,\pm x\cdot u>0\text{ and }(x\cdot u)^2>\|x\|^2-R^2\}\,.$$

If $\|x\|$ is sufficiently large, then $V^\pm$ are disjoint neighbourhoods of $\pm x/\|x\|\in S^2$ and $\hat{\theta}^+$ and $\hat{\theta}^-$ have support in $V^+\cup V^-$.

Hence, integrating gives us

$$\langle i\delta(s), f_i\rangle = \int_{V^+} \langle \hat\theta^+, \hat f_i\rangle + \int_{V^-} \langle \hat\theta^+, \hat f_i\rangle$$
$$= \int_{V^+} \langle \hat\theta^+, \hat f_i\rangle + \int_{V^-} \langle \hat\theta^-, \hat f_i\rangle + \int_{V^-} \langle \bar\partial\gamma, \hat f_i\rangle .$$

However, from (3.16), γ is supported in $V^+ \cup V^-$, and so the last term vanishes by Stokes' theorem. Therefore from (3.19) we obtain

$$|\langle i\delta(s), f_i\rangle|_x < 4\pi K \exp(-\varepsilon\sqrt{\|x\|^2 - R^2})\|f_i\| .$$

and so as $\|x\| \to \infty$, $\|\phi(x)\|$ decays exponentially. Since (see [8]) the derivatives of $\hat\theta^\pm$ also decay exponentially along each line we also have a similar estimate for $\|\nabla\phi\|$.

We now use the standard vanishing theorem argument. From (3.17) $(\nabla^*\nabla + \Phi'^*\Phi')\phi = 0$, so

$$0 = \int_{\|x\| \leq R} \langle (\nabla^*\nabla + \Phi'^*\Phi')\phi, \phi\rangle$$
$$= \int_{\|x\| \leq R} (\|\nabla\phi\|^2 + \|\Phi'(\phi)\|^2) + \int_{\|x\| = R} \mathrm{Re}(\nabla\phi, \phi) .$$

But the boundary term tends to zero as $R \to \infty$, by our estimates, hence $\nabla\phi = 0$ and as $\phi \to 0$ we must have $\phi = 0$, and so $i\delta(s) = 0$.

To complete the proof we must show that $i\delta$ is injective. From the exact sequence

$$0 \to O_T(L^z(-k-2)) \to O_T(L^z(k-2)) \to O_S(L^z(k-2)) \to 0 ,$$

the coboundary map δ will be injective if $H^0(T, L^z(k-2)) = 0$ and from the sequence

$$0 \to O_T(L^z(-k-2)) \to O_T(L^{z-1}\tilde E(-2)) \to O_T(L^{z-2}(k-2)) \to 0 ,$$

the map i will be injective if $H^0(T, L^{z-2}(k-2)) = 0$.

However, if $z \neq 0$ there can be no holomorphic sections of $L^z(k-2)$ on T. Indeed, such a section pulls back to a holomorphic section of $O(k-2)$ on $\mathbb{P}_3 \backslash \mathbb{P}_1$ under the quotient map $\mathbb{P}_3 \backslash \mathbb{P}_1 \to T$. This is because L^z is the bundle on T associated to the representation $w \to e^{zw}$ of $\mathbb{C}$, considering $\mathbb{P}_3 \backslash \mathbb{P}_1$ as a principal $\mathbb{C}$-bundle over T. Hence L^z is trivial on $\mathbb{P}_3 \backslash \mathbb{P}_1$.

Now by Hartog's theorem any such section on $\mathbb{P}_3 \backslash \mathbb{P}_1$ extends to $\mathbb{P}_3$. However, the action of $\mathbb{C}$ on such sections is algebraic and there are no sections which transform with the transcendental multiplier e^{zw} [see also (5.4)].

Thus if $z \neq 0$ or 2 the map $i\delta$ is injective and so $s = 0$, concluding the theorem.

We have thus established the conditions B1–B4 for the spectral curve S.

The condition $L^2 = 1$ together with the antiholomorphic isomorphism $L \cong L^*$ defined by the real structure means that the element $2[\eta/\zeta] \in H^1(S, O)$ is an imaginary lattice point with respect to $H^1(S, \mathbb{Z}) \subset H^1(S, O)$ and hence the straight line through 0 and $[2\eta/\zeta] \in H^1(S, O)$ defines a homomorphism

$$h : S^1 \to H^1(S, O)/H^1(S, \mathbb{Z}) = \mathrm{Pic}(S) .$$

Now we may identify the Picard group with the Jacobian Jac(S) of divisor classes of degree $g-1$ by $L \to L(k-2)$, (recall that $\deg(k-2)=k(k-2)=(k-1)^2-1$), and then the vanishing condition of Theorem (3.7) is that $h^{-1}(\Theta)=1$, where Θ is the theta divisor of line bundles of degree $(g-1)$ with at least one section. In particular h is injective, so $[2\eta/\zeta]$ can not be a multiple of any element in $H^1(S,\mathbb{Z})$.

4. Solving Nahm's Equations

Let us suppose now that S is a curve in $T\mathbb{P}_1$ satisfying conditions B1–B4, i.e.

(i) S is defined by an equation

$$\eta^k + a_1\eta^{k-1} + \ldots + a_k = 0,$$

where $a_i \in H^0(\mathbb{P}_1, O(2i))$ are real.

(ii) S has no multiple components and $L(k-1)$ is real.

(iii) The line bundle L is of order 2 on S.

(iv) $H^0(S, L^z(k-2))=0$ for $z \in (0,2)$.

We shall show how S defines canonically a solution of Nahm's equations, satisfying the conditions C1–C4. First we derive some consequences of (iv).

Consider the exact sequence of sheaves

$$0 \to O_{\hat{T}}(lH_1 - k(H_1+H_2)) \xrightarrow{\psi O_{\hat{T}}} (lH_1) \to O_S(lH_1) \to 0$$

on the compactification $\hat{T}$ of T.

From the exact cohomology sequence and the Riemann-Roch theorem for $\hat{T}$ we deduce that

$$\begin{aligned}
&\dim H^0(S,O(l)) - \dim H^1(S,O(l)) \\
&= \tfrac{1}{2}l^2H_1^2 - \tfrac{1}{2}lH_1\cdot K - \tfrac{1}{2}(lH_1 - k(H_1+H_2))^2 + \tfrac{1}{2}(lH_1 - k(H_1+H_2))\cdot K \\
&= l + lk - k^2 - l + 2k \\
&= k(l-k) + 2k\,.
\end{aligned}$$

Hence for the flat line bundle L^z, by invariance under deformation,

$$\dim H^0(S, L^z(l)) - \dim H^1(S, L^z(l)) = k(l-k) + 2k\,. \tag{4.1}$$

Thus in particular $\dim H^0(S, L^z(k-2)) = \dim H^1(S, L^z(k-2))$, and from (iv),

$$H^1(S, L^z(k-2)) = 0 \quad \text{for} \quad z \in (0,2)\,. \tag{4.2}$$

Now let F be a fibre of T which intersects S in k distinct points [its existence is assured by (ii)] and consider the exact sequence

$$0 \to O_S(L^z(l)) \to O_S(L^z(l+1)) \to O_{S\cap F}(L^z(l+1)) \to 0\,.$$

We see immediately that

$$H^0(S, L^z(l)) \to H^0(S, L^z(l+1))$$

is injective and

$$H^1(S, L^z(l)) \to H^1(S, L^z(l+1))$$

is surjective. Hence from (iv) and (4.2)

$$\begin{aligned} H^0(S, L^z(l)) &= 0 \quad \text{for} \quad l \leqq k-2, \\ H^1(S, L^z(l)) &= 0 \quad \text{for} \quad l \geqq k-2. \end{aligned} \tag{4.3}$$

Using (4.1) this implies in particular that

$$\dim H^0(S, L^z(k-1)) = k. \tag{4.4}$$

When $z=0$ (or $z=2$ since L^z is trivial), the line bundle $O(l)$ has more sections:

Proposition (4.5). *If $l<2k$, then every section $s \in H^0(S, O(l))$ may be written uniquely in the form*

$$s = \sum_{i=0}^{[l/2]} \eta^i \pi^* c_i,$$

where $c_i \in H^0(\mathbb{P}_1, O(l-2i))$ and $\eta \in H^0(S, O(2))$ is the tautological section on T.

Proof. Take first the case when l is even, $l=2n$, say, and consider the exact sequence

$$0 \to O_{\hat{T}}((n-k)(H_1+H_2)) \to O_{\hat{T}}(n(H_1+H_2)) \to O_S(2n) \to 0.$$

Now $H^0(\hat{T}, (n-k)(H_1+H_2)) = 0$ since $(n-k)<0$, so we shall have

$$H^0(\hat{T}, n(H_1+H_2)) \cong H^0(S, O(2n)), \tag{4.6}$$

if $H^1(\hat{T}, (n-k)(H_1+H_2)) = 0$. But if Z is the zero section, there is an exact sequence

$$0 \to O_{\hat{T}}(-m(H_1+H_2)) \to O_{\hat{T}}((1-m)(H_1+H_2) \to O_Z(2-2m) \to 0.$$

Now if $m>1$, $H^0(\mathbb{P}_1, O(2-2m)) = 0$, so

$$H^1(\hat{T}, -m(H_1+H_2)) \to H^1(\hat{T}, (1-m)(H_1+H_2))$$

is injective and, repeating, injects into $H^1(\hat{T}, -(H_1+H_2))$. However, $\hat{T}$ is rational so $H^1(\hat{T}, O) = 0$, and as $H^0(\hat{T}, O) \to H^0(Z, O)$ is an isomorphism, it follows from the exact cohomology sequence with $m=1$, that $H^1(\hat{T}, -(H_1+H_2)) = 0$. Consequently $H^1(\hat{T}, -m(H_1+H_2))$ vanishes too.

Thus we have established (4.6). Restricting from T to $\hat{T}$ we have

$$H^0(\hat{T}, n(H_1+H_2)) \cong H^0(T, O(2n)) \cong H^0(S, O(2n)),$$

and by the result of [8] Sect. 7 every section of $H^0(T, O(2n))$ is of the form $s = a_0\eta^n + a_1\eta^{n-1} + \ldots + a_n$, so the proposition is verified.

If $l = 2n+1$ we use a similar argument with the exact sequence

$$0 \to O_{\hat{T}}(H_1 + (n-k)(H_1+H_2)) \to O_{\hat{T}}(H_1 + n(H_1+H_2)) \to O_S(2n+1) \to 0.$$

In this case, we need to prove $H^1(\hat{T}, -H_2) = 0$. But by Serre duality, using (3.2),

$$H^1(\hat{T}, -H_2) \cong H^1(\hat{T}, H_2 - 2H_1 - 2H_2)^*.$$

This, however, is zero by Kodaira's vanishing theorem.

An important special case of the proposition is where $l=2$. It follows then, that if $k>1$, the space $H^0(S, O(2))$ is spanned by

$$\eta\frac{d}{d\zeta}, \frac{d}{d\zeta}, \zeta\frac{d}{d\zeta}, \zeta^2\frac{d}{d\zeta},$$

where $\eta\frac{d}{d\zeta}$ is the tautological section already introduced and the other three form a basis for the holomorphic vector fields on $\mathbb{P}_1$.

As a consequence of (4.5) the vector space $H^0(S, L^z(k-1))$ jumps in dimension from k to $\frac{1}{4}(k+1)^2$ or $\frac{1}{4}k(k+2)$ (where k is odd or even respectively) as z tends to 0 and 2. We can nevertheless define a holomorphic family of vector spaces over $\mathbb{C}$ – a vector bundle V – by taking the direct image sheaf π_*M of the line bundle M over $\mathbb{C}\times S$ whose fibre at $(z, w)\in\mathbb{C}\times S$ is $L^z(k-1)_w$. The direct image sheaf is torsion free on the 1-dimensional space $\mathbb{C}$ and hence is locally free. Since $H^0(S, L^z(k-2))=0$ generically [it vanishes for $z\in(0,2)$], the bundle V is of rank k and its fibre at $z\in(0,2)$ is simply $V_z=H^0(S, L^z(k-1))$. We shall investigate the fibres at the endpoints later on. The principal object we shall consider in this section is the product map:

$$H^0(S, O(2))\otimes H^0(S, L^z(k-1))\xrightarrow{m} H^0(S, L^z(k+1)), \tag{4.7}$$

and in particular its kernel.

Proposition (4.8). *Let K_z denote the kernel of m in* (4.7). *Then the map $h: K_z\to V_z$ defined by*

$$h(\eta\otimes s_0+1\otimes s_1+\zeta\otimes s_2+\zeta^2\otimes s_3)=s_0$$

is an isomorphism if $z\in(0,2)$.

Proof. The sections of $O(2)$ embed T into a quadric cone in $\mathbb{P}_3$:

$$\{(z_0, z_1, z_2, z_3)|z_0=\eta, z_1=1, z_2=\zeta, z_3=\zeta^2\}.$$

Let us denote by H the hyperplane line bundle on $\mathbb{P}_3$, then $H|_T\cong O(2)$.

There is an exact sequence of vector bundles on $\mathbb{P}_3$ (the Euler sequence): $0\to\Omega^1_P(H)\to\mathbb{C}^4\to H\to 0$, where Ω^1_P is the cotangent bundle, and

$$\mathbb{C}^4\cong H^0(\mathbb{P}_3, H)\cong H^0(T, O(2))\cong H^0(S, O(2))$$

by (4.5). Thus restricting to S and tensoring with $L^z(k-1)$, we have an exact sequence

$$0\to\Omega^1_P L^z(k+1)\to H^0(S, O(2))\otimes L^z(k-1)\to L^z(k+1)\to 0. \tag{4.9}$$

The exact cohomology sequence of (4.9) gives

$$0\to H^0(S, \Omega^1_P L^z(k+1))\to H^0(S, O(2))\otimes H^0(S, L^z(k-1))$$
$$\xrightarrow{m} H^0(S, L^z(k+1))\to H^1(S, \Omega^1_P L^z(k+1))\to \dots.$$

We shall show that m is surjective and identify its kernel.

Consider $(z_0, z_1, z_2, z_3) \in \mathbb{P}_3$. The tangent plane to the cone touching the line $\zeta = \infty$ is $z_1 = 0$. On the complement of this we have trivialization of Ω_P^1 given by:

$$\omega_1 = d(z_0/z_1); \quad \omega_2 = d(z_2/z_1); \quad \omega_3 = d(z_3/z_1).$$

On the complement of the tangent plane along $\zeta = 0$ we have a trivialization

$$\tilde{\omega}_1 = d(z_0/z_3); \quad \tilde{\omega}_2 = d(z_2/z_3); \quad \tilde{\omega}_3 = d(z_1/z_3),$$

and so

$$\begin{aligned} \tilde{\omega}_1 &= d(z_0/z_1 \cdot z_1/z_3) = \omega_1 \zeta^{-2} + \eta \tilde{\omega}_3, \\ \tilde{\omega}_2 &= d(z_2/z_1 \cdot z_1/z_3) = \omega_2 \zeta^{-2} + \zeta \tilde{\omega}_3, \\ \tilde{\omega}_3 &= -\zeta^{-4} \omega_3. \end{aligned}$$

Hence on T, with respect to the standard covering $\tilde{U}_0, \tilde{U}_1$, Ω_P^1 is defined by the transition matrix

$$\begin{pmatrix} \zeta^{-2} & 0 & 0 \\ 0 & \zeta^{-2} & 0 \\ -\eta\zeta^{-4} & -\zeta^{-3} & -\zeta^{-4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}. \tag{4.10}$$

In particular, we may consider the map $f: \Omega_P^1 \to \Omega_T^1 \to \Omega_F^1 \cong O(-2)$, where Ω_F^1 is the cotangent bundle along the fibres of T. Since $\omega_1 = d\eta$ and $\tilde{\omega}_1$ are local trivializations of Ω_F^1, the kernel of $f: \Omega_P^1 \to \Omega_F^1$ is the extension of $O(-2)$ by $O(-4)$ given by the transition matrix

$$\begin{pmatrix} \zeta^{-2} & 0 \\ -\zeta^{-3} & -\zeta^{-4} \end{pmatrix},$$

which (since $[\zeta^{-1}] \in H^1(\mathbb{P}_1, O(-2))$ is non-trivial) is the non-trivial extension on $\mathbb{P}_1$. Hence

$$\ker f \cong O(-3) \oplus O(-3).$$

Consider then the exact sequence

$$0 \to L^z(k-2) \oplus L^z(k-2) \to \Omega_P^1 L^z(k+1) \xrightarrow{f} L^z(k-1) \to 0. \tag{4.11}$$

From (4.3)

$$H^0(S, L^z(k-2)) = H^1(S, L^z(k-2)) = 0,$$

and

$$H^1(S, L^z(k-1)) = 0.$$

Hence from the exact cohomology sequence of (4.11),

$$f: H^0(S, \Omega_P^1 L^z(k+1)) \to H^0(S, L^z(k-1))$$

is an isomorphism and $H^1(S, \Omega_P^1 L^z(k+1)) = 0$ if $z \in (0, 2)$. Consequently m is onto, with k-dimensional kernel isomorphic to $H^0(S, L^z(k-1))$ under the map f.

It is straightforward to see that f is the map h in the Proposition, which is thus proved.

From Proposition (4.8) the sections s_1, s_2, s_3 are all determined linearly by s_0. Thus there exist well-defined endomorphisms $\tilde{A}_0(z), \tilde{A}_1(z), \tilde{A}_2(z) \in \operatorname{End} V_z$ for $z \in (0,2)$, such that

$$(\eta + \tilde{A}_0 + \zeta \tilde{A}_1 + \zeta^2 \tilde{A}_2)s = 0. \tag{4.12}$$

The endomorphisms $\tilde{A}_i$ almost define the solutions to Nahm's equations but they are not yet matrices – we have to trivialize the vector bundle V over $\mathbb{C}$ in order to obtain such a form. We shall define a connection in order to obtain such a trivialization.

Recall that one approach to connections is the notion of covariant derivative of a section of a vector bundle. Suppose then that $s(z)$ is a local holomorphic section of V. We can represent s by a pair of holomorphic functions $f_0 : U \to \mathbb{C}^k$, $f_1 : U' \to \mathbb{C}^k$ (where $U = \tilde{U}_0 \cap S$, $U' = \tilde{U}_1 \cap S$), such that $f_0 = e^{z\eta/\zeta} \zeta^{(k-1)} f_1$ on $U \cap U'$. If we now naively differentiate with respect to z, we obtain

$$\frac{\partial f_0}{\partial z} = \frac{\eta}{\zeta} f_0 + e^{z\eta/\zeta} \zeta^{k-1} \frac{\partial f_1}{\partial z}, \tag{4.13}$$

which clearly does not transform as a section of V. However, by virtue of (4.12) we may write

$$\frac{\eta}{\zeta} s = -(\tilde{A}_0 \zeta^{-1} + \tfrac{1}{2}\tilde{A}_1)s - (\tfrac{1}{2}\tilde{A}_1 + \tilde{A}_2 \zeta)s,$$

and then

$$\begin{aligned} &\frac{\partial f_0}{\partial z} + (\tfrac{1}{2}\tilde{A}_1 s + \zeta \tilde{A}_2 s)|_{U \cap U'} \\ &\quad = \frac{\partial f_0}{\partial z} - \frac{\eta}{\zeta} f_0 - (\zeta^{-1}\tilde{A}_0 s + \tfrac{1}{2}\tilde{A}_1 s)|_{U \cap U'} \\ &\quad = e^{z\eta/\zeta} \zeta^{k-1} \left[\frac{\partial f_1}{\partial z} - (\zeta^{-1}\tilde{A}_0 s + \tfrac{1}{2}\tilde{A}_1 s)\right]\Bigg|_{U \cap U'}. \end{aligned} \tag{4.14}$$

Since $\frac{1}{2}\tilde{A}_1 s + \zeta \tilde{A}_2 s$ is regular in U and $\zeta^{-1}\tilde{A}_0 s + \frac{1}{2}\tilde{A}_1 s$ is regular in U', we have a well-defined connection on V over $(0,2)$, whose covariant derivative is defined by

$$\nabla_z s = \frac{\partial f_0}{\partial z} + (\tfrac{1}{2}\tilde{A}_1 s + \zeta \tilde{A}_2 s)|_U. \tag{4.15}$$

We now take a basis $s_1, \ldots, s_k$ of covariant constant sections along $(0,2)$, and we may then represent the endomorphism $\tilde{A}_i$ by the matrix A_i.

Nahm's equations come from the following proposition:

Proposition (4.16). *Let* $A_0 = T_1 + iT_2$, $A_1 = -2iT_3$, $A_2 = T_1 - iT_2$, *then* T_1, T_2, T_3 *satisfy Nahm's equations*

$$\frac{dT_i}{dz} = \tfrac{1}{2} \sum \varepsilon_{ijk} [T_j, T_k] \quad \textit{for} \quad z \in (0,2).$$

Proof. Let us write $A=A_0+\zeta A_1+\zeta^2 A_2$ and $A_+=\frac{1}{2}A_1+\zeta A_2$. Then from (4.12)

$$(\eta+A)s=0, \tag{4.17}$$

and since V was trivialized using covariant constant sections, from (4.15)

$$\frac{\partial s}{\partial z}+A_+s=0 \quad \text{on } U. \tag{4.18}$$

Hence from (4.17), $(\eta+A)\frac{\partial s}{\partial z}+\frac{dA}{dz}s=0$, and from (4.18), $-(\eta+A)A_+s+\frac{dA}{dz}s=0$, hence $-A_+\eta s-AA_+s+\frac{dA}{dz}s=0$, so by (4.17) again

$$\left([A_+,A]+\frac{dA}{dz}\right)s=0. \tag{4.19}$$

Now take a fibre F of T for which $S\cap F$ consists of k distinct points. If F is the divisor of a section u of $O(1)$ on T, consider the exact sequence

$$0\to O_S L^z(k-2)\xrightarrow{u} O_S L^z(k-1)\to O_{S\cap F}\to 0.$$

Since $H^0(S,L^z(k-2))=H^1(S,L^z(k-2))=0$, then from the exact cohomology sequence, the restriction map $\varrho: H^0(S,L^z(k-1))\to H^0(S\cap F,O)$ is an isomorphism, hence there is a basis of sections $s_1,\ldots,s_k$ of $H^0(S,L^z(k-1))$, such that $s_i(x_j)=0$, if $i\neq j$ but $s_i(x_i)\neq 0$, where $\{x_1,\ldots,x_k\}=S\cap F$.

Now Eq. (4.19) is independent of η, hence we have a matrix $B=[A_+A]+\frac{dA}{dz}$ such that $\sum_j B_{ij}s_j(x_l)=0$, $\forall i,l$, and so $B_{ij}=0$, $\forall i,j$. Thus

$$\frac{dA}{dz}=[A,A_+]. \tag{4.20}$$

Since the property of the fibre we chose is generic, Eq. (4.20) is true for all ζ, and equating coefficients we obtain

$$\frac{dA_0}{dz}=\tfrac{1}{2}[A_0,A_1]; \quad \frac{dA_1}{dz}=[A_0,A_2]; \quad \frac{dA_2}{dz}=\tfrac{1}{2}[A_1,A_2],$$

and substituting for T_i, we immediately obtain Nahm's equations.

Remark. Equation (4.20) is in Lax form. It follows immediately that $\frac{d}{dz}\mathrm{Tr}A^n=0$, $\forall n\geqq 0$, and hence that the spectrum of the matrix A is independent of z. We may write this as $\det(\eta+A(\zeta))=0$, which is a curve in T, a conserved quantity of Nahm's equations.

From Eq. (4.12) it follows that if $(\eta,\zeta)\in S$, then $\det(\eta+A(\zeta))=0$.

Since $\det(\eta+A)=\eta^k+b_1\eta^{k-1}+\ldots+b_k$ defines a divisor of $|O(2k)|$ just like S, and S has no multiple components, it follows that $S=\{(\eta,\zeta)\in T|\det(\eta+A(\zeta))=0\}$. Thus S is the invariant curve of our solution, and it is clear that the general solution to the equations is obtained by essentially the same procedure as above.

The next problem is to determine the behaviour of our solution to Nahm's equations at $z=0$.

5. Boundary Conditions

We first identify the fibre V_0 at $z=0$ of the vector bundle V.

Proposition (5.1). *Let $V_0 \subset H^0(S, O(k-1))$ be the fibre of V at $z=0$. Then*

$$V_0 = \pi^* H^0(\mathbb{P}_1, O(k-1)).$$

Proof. We know that V_0, like $H^0(\mathbb{P}_1, O(k-1))$, is k-dimensional. The proof consists of showing that any section of $O(k-1)$ on S which is pulled back from $\mathbb{P}_1$ may be extended locally to a section of $L^z(k-1)$ over $\mathbb{C} \times S$. If there is such an extension, then there exist power series expansions

$$s(\eta, \zeta, z) = s_0 + z s_1 + z^2 s_2 + \dots,$$
$$s'(\eta, \zeta, z) = s'_0 + z s'_1 + z^2 s'_2 + \dots,$$

on $W \times U$ and $W \times U'$ respectively, where W is a neighbourhood of $0 \in \mathbb{C}$ and $U = \tilde{U}_0 \cap S$, $U' = \tilde{U}_1 \cap S$, from the standard covering of T by two open sets, such that $s = \zeta^{k-1} e^{z\eta/\zeta} s'$ on $U \cap U'$. Thus, equating first the coefficients of z, we have $s_1 = \zeta^{k-1} \frac{\eta}{\zeta} s'_0 + \zeta^{k-1} s'_1$, so that the class $[\eta s_0/\zeta] \in H^1(S, O(k-1))$ comes from a coboundary and thus vanishes. There are analogous obstructions for higher order extensions, all lying in the group $H^1(S, O(k-1))$, whose structure we investigate next.

Lemma (5.2). *Every element $c \in H^1(S, O(k-1))$ may be written uniquely in the form*

$$c = \sum_{i=[k+2/2]}^{k-1} \eta^i \pi^* c_i,$$

where $c_i \in H^1(\mathbb{P}_1, O(k-1-2i))$.

Proof. The proof is similar to (3.1). By Kodaira's vanishing theorem and the Riemann-Roch theorem, $H^p(\hat{T}, -H_2 - kH_1) = 0$ for all p, hence the restriction map

$$\varrho : H^1(\hat{T}, (k-1)H_2) \to H^1(S, O(k-1)) \tag{5.3}$$

is an isomorphism. Next, the map

$$\hat{\xi} : H^1(\hat{T}, (k-1)H_1 + (l-1)(H_2 - H_1)) \to H^1(\hat{T}, (k-1)H_1 + l(H_2 - H_1))$$

is injective if $-2l + (k-1) \leqq 0$, by considering the exact sequence for the curve at infinity C.

We now consider an element

$$a = \sum_{i=[k+2/2]}^{k-1} \hat{\eta}^i \hat{\xi}^{k-1-i} \pi^* c_i \in H^1(\hat{T}, (k-1)H_2).$$

If this vanishes, then restricting to C we have $c_{k-1}=0$ [cf. (3.4)]. However, since $\hat{\xi}$ is injective, then

$$\sum \hat{\eta}^i \hat{\xi}^{k-2-i} \pi^* c_i = 0$$

and repeating we find $c_i=0$ for all i. This provides, taking into account (5.3), a space of sections of $O(k-1)$ on S of dimension

$$\begin{aligned} \tilde{h}^1 &= \sum_{m=1}^{l-1} 2m, \qquad \text{if} \quad k=2l, \\ &= \sum_{m=1}^{l-1} 2m-1, \quad \text{if} \quad k=2l-1. \end{aligned}$$

However, by the Riemann-Roch formula (4.1), $h^0-h^1=k$, and from (4.5), $h^0=\sum_{i=0}^{[k-1/2]}(k-2i)$.

It follows that $\tilde{h}^1=h^1$ and the lemma is proved.

We consider next the higher order obstructions. An extension of a section of $O(k-1)$ on S to a section of $L^z(k-1)$ to the m^{th} order can be defined, relative to the open sets u and u' by holomorphic functions

$$\begin{aligned} s &= s_0 + z s_1 + z^2 s_2 + \dots + z^m s_m, \qquad s_i \in H^0(U,O), \\ s' &= s'_0 + z s'_1 + \dots + z^m s'_m, \qquad s'_i \in H^0(U',O), \end{aligned}$$

such that

$$s = \zeta^{k-1} e^{z\eta/\zeta} s' \bmod z^{m+1} \quad \text{on} \quad U \cap U'.$$

One particular type of extension is to consider functions s and s' which are functions of $z\eta$ on U (and $z\eta/\zeta^2$ on U'), since the transition function $e^{z\eta/\zeta}$ is itself of this form. We then seek

$$\begin{aligned} p &= p_0 + z\eta p_1 + \dots + z^m \eta^m p_m, \qquad p_i \in H^0(U_0, O(k-1-2i)), \\ p' &= p'_0 + z\eta p'_1 + \dots + z^m \eta^m p'_m, \qquad p'_i \in H^0(U_1, O(k-1-2i)), \end{aligned}$$

where $\mathbb{P}_1 = U_0 \cup U_1$ is the standard covering, such that $p = e^{z\eta/\zeta} p' \bmod z^{m+1}$ on $U_0 \cap U_1$. This, replacing $z\eta$ by η, is the condition for extending a section of $O(k-1)$ on the zero section Z of $T\mathbb{P}_1$ to a section of $L(k-1)$ on the m^{th} formal neighbourhood. We have the following lemma:

Lemma (5.4). *Every section of $L(m)$ on $Z \subset T\mathbb{P}_1$ can be extended uniquely to the m^{th} formal neighbourhood, but no section can be extended to the $(m+1)^{\text{th}}$ neighbourhood.*

Proof. A section of $L(m)$ on the m^{th} neighbourhood of Z consists of sections $p_i \in H^0(U_0, O(m-2i))$ and $p'_i \in H^0(U_1, O(m-2i))$, such that

$$p_0 + \eta p_1 + \dots + \eta^m p_m = e^{\eta/\zeta}(p'_0 + \eta p'_1 + \dots + \eta^m p'_m) \bmod \eta^{m+1}.$$

In other words, we seek functions p_i on U_0 and p_i' on U_1 such that on $U_0 \cap U_1$,

$$\begin{pmatrix} \zeta^m & 0 & 0 & \dots & 0 \\ \zeta^{m-1} & \zeta^{m-2} & 0 & \dots & 0 \\ \frac{1}{2}\zeta^{m-2} & \zeta^{m-3} & \zeta^{m-4} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m!} & \frac{\zeta^{-1}}{(m-1)!} & \dots & \dots & \zeta^{-m} \end{pmatrix} \begin{pmatrix} p_0' \\ \cdot \\ \cdot \\ \cdot \\ p_m' \end{pmatrix} = \begin{pmatrix} p_0 \\ \cdot \\ \cdot \\ \cdot \\ p_m \end{pmatrix}. \tag{5.5}$$

Now

$$\begin{pmatrix} p_0' \\ \cdot \\ \cdot \\ \cdot \\ p_m' \end{pmatrix} = \begin{pmatrix} c_0\zeta^{-m+l} \\ c_1\zeta^{-m+l+1} \\ \vdots \\ c_{m-l} \\ \vdots \\ 0 \end{pmatrix}$$

is a solution if

$$\sum_{i=0}^{m-l} c_i/(n-i)! = 0 \quad \text{for} \quad l+1 \leqq n \leqq m. \tag{5.6}$$

However, a determinant of the form

$$\Delta = \begin{vmatrix} \frac{1}{m!} & \dots & \frac{1}{n!} \\ \vdots & & \vdots \\ \frac{1}{n!} & \dots & \frac{1}{(2n-m)!} \end{vmatrix}, \quad \text{where} \quad n \leqq m,$$

is always non-zero. Indeed, clearing the denominators and dividing the i^{th} column by $(i-1)!$, Δ vanishes if and only if $\tilde{\Delta}$ vanishes, where

$$\tilde{\Delta} = \begin{vmatrix} 1 & \binom{m}{1} & \binom{m}{2} & \dots & \binom{m}{m-n} \\ \vdots & & & & \vdots \\ 1 & \binom{n}{1} & \binom{n}{2} & \dots & \end{vmatrix}.$$

But a linear relation amongst the rows of $\tilde{\Delta}$ implies, by the binomial theorem, that there exist $\lambda_i \in \mathbb{R}$, such that $q(x) = \sum_{i=n}^{m} \lambda_i (1+x)^i$ is divisible by x^{m-n+1}. But $q(x) = (1+x)^n r(x)$, where $\deg r \leqq m-n$, hence x divides $(1+x)^n$, which is absurd.

Thus, up to a constant multiple, there exists a unique solution to (5.6) and furthermore c_0 and c_{m-k} are non-zero.

Adopting a slightly different point of view, the column vectors produced above provide a trivialization of the vector bundle E_m over $\mathbb{P}_1$ defined by the transition function which is the matrix in (5.5). We then have an exact sequence $0 \to E_{m-1}(-1) \to E_m \xrightarrow{p_0} O(m) \to 0$ of vector bundles on $\mathbb{P}_1$. Since E_{m-1} is trivial and $H^p(\mathbb{P}_1, O(-1)) = 0$, we obtain from the exact cohomology sequence the isomorphism

$$H^0(\mathbb{P}_1, E_m) \xrightarrow[\cong]{p_0} H^0(\mathbb{P}_1, O(m)). \tag{5.7}$$

Thus any section of $O(m)$ has a unique extension to a section of E_m and hence a section of $L(m)$ on the m^{th} formal neighbourhood.

An extension to the $(m+1)^{\text{th}}$ order is given by pulling back a section of $O(m)$ in the following exact sequence

$$0 \to E_m(-2) \to E_{m+1}(-1) \to O(m) \to 0.$$

However, since E_{m+1} is trivial $H^0(\mathbb{P}_1, E_{m+1}(-1)) = 0$, so there are no such extensions.

We shall call this extension $\bar{s}$ of $s \in H^0(\mathbb{P}_1, O(m))$ to the m^{th} neighbourhood the *canonical* extension. In the language of formal neighbourhoods it defines an isomorphism $H^0(\mathbb{P}_1, L^{(m)}(m)) \cong H^0(\mathbb{P}_1, O(m))$, which is, by uniqueness, invariant under $\mathrm{SL}(2, \mathbb{C})$, the group of biholomorphic transformations of $T\mathbb{P}_1$ which preserves the zero section Z.

Now if we consider the exact sequences, $0 \to O(m-2k) \to L^{(k)}(m) \to L^{(k-1)}(m) \to 0$, it is clear that the $\mathrm{SL}(2, \mathbb{C})$-module $S^m \cong H^0(\mathbb{P}_1, O(m))$ occurs with multiplicity 1 in $H^0(\mathbb{P}_1, L^{(k)}(m))$ for $k \leq m$. Hence we may recognise the restriction of the canonical extension to the k^{th} neighbourhood as the unique $\mathrm{SL}(2, \mathbb{C})$-invariant extension. The group invariance also shows that from the exact sequence

$$0 \to O(-m-2) \to L^{(m+1)}(m) \to L^{(m)}(m) \to 0,$$

the coboundary map $\delta: H^0(\mathbb{P}_1, L^{(m)}(m)) \to H^1(\mathbb{P}_1, O(-m-2))$ defines an $\mathrm{SL}(2, \mathbb{C})$-invariant homomorphism h from $H^0(\mathbb{P}_1, O(m))$ to $H^1(\mathbb{P}_1, O(-m-2))$ by

$$hs = \delta \bar{s}. \tag{5.8}$$

By Lemma (5.4) this is non-zero, and since both spaces are irreducible representations, h must be an isomorphism.

We may now complete the proof of Proposition (5.1).

Let s be a section of $O(k-1)$ on $\mathbb{P}_1$, and take the canonical extension as defined by (5.4) of $\pi^* s \in H^0(S, O(k-1))$ to the $(k-1)^{\text{th}}$ order as a section of $L^z(k-1)$ on $S \times \mathbb{C}$. The obstruction to extending to the k^{th} order is the element

$$c = \eta^k \pi^* hs \in H^1(S, O(k-1)). \tag{5.9}$$

However, on S we have the relation $\eta^k + a_1 \eta^{k-1} + \ldots + a_k = 0$, so (5.9) may be written as

$$c = -\sum_{i=1}^{l} a_i \eta^{k-i} \pi^* hs, \qquad l = k - [k+2/2],$$

or

$$c = \sum \eta^{k-i} \pi^* h_i, \tag{5.10}$$

where $h_i \in H^1(\mathbb{P}_1, O(2i-k-1))$.

By (5.8) each term $\eta^{k-i}\pi^* h_i$ is the obstruction to extending the canonical extension of a section $\pi^* s_i \in H^0(S, O(k-1-2i))$ to the $(k-2i)^{\text{th}}$ order as a section of $L^z(k-1-2i)$. Equivalently it is the obstruction to extending the canonical extension of $\eta^i \pi^* s_i \in H^0(S, O(k-1))$ to the $(k-2i)^{\text{th}}$ order, or of extending $z^{2i}\eta^i \pi^* s_i$ from the $(k-1)^{\text{th}}$ order to the k^{th} order.

Hence, if $\bar{s}$ denotes a canonical extension,

$$\bar{s} - z^2 \eta \bar{s}_1 - z^4 \eta^2 \bar{s}_2 - \ldots - z^{2l} \eta^l \bar{s}_l \tag{5.11}$$

extends from the $(k-1)^{\text{th}}$ to the k^{th} order in z.

A further extension will be obstructed by some element $c' \in H^1(S, O(k-1))$. However, from (5.2) and (5.8), c' canonically determines sections $s'_1, \ldots, s'_l$ such that

$$c' = \sum \eta^{k-i} h(s'_i). \tag{5.12}$$

Hence, modifying the section to the k^{th} order by multiples of the $\bar{s}'_i$ terms as in (5.11) we may remove the obstruction, and continue. Note that, since each term in (5.12) is the obstruction to extending a section to at most the $(k-2)^{\text{th}}$ order, the modifications to (5.11) necessary to extend to the $(k+1)^{\text{th}}$ order will all be multiples of z^3. Thus, proceeding as above, each coefficient of z^n requires a finite number of modifications, and we obtain a power series in z which defines a formal extension.

We may now appeal to the theorem of Wavrik [14] which shows that if a formal extension exists, then an actual (i.e. convergent) one does also.

Consequently, in the notation of the proposition, $\pi^* H^0(\mathbb{P}_1, O(k-1)) \subset V_0$. However, since both are k-dimensional spaces we obtain equality and (5.11) is proved.

We shall next investigate in a similar fashion the behaviour of the kernel $K_z \subset H^0(S, O(2)) \otimes H^0(S, L^z(k-1))$ of the product map m of (4.7), as z tends to 0. From (5.9) this is equivalent to considering which sections of the vector bundle $\Omega_P^1(k+1)$ over S extend to sections of $\Omega_P^1 L^z(k+1)$. Since the direct image sheaf $\pi_* \Omega_P^1 L^z(k+1)$ over $\mathbb{C}$ is locally free, and from (4.8) $\dim K_z = k$ if $z \in (0, 2)$, then there is a well-defined k-dimensional subspace $K_0 \subset H^0(S, O(2)) \otimes V_0$, which extends. From (5.1) and (4.5), $K_0 \subset H^0(T\mathbb{P}_1, O(2)) \otimes \pi^* H^0(\mathbb{P}_1, O(k-1))$. Now if $\eta s_0 + s_1 + \zeta s_2 + \zeta^2 s_3 = 0$ for $s_i \in \pi^* H^0(\mathbb{P}_1, O(k-1))$, it follows from (4.5) that $s_0 = 0$. Hence, $K_0 \subset \pi^*(H^0(\mathbb{P}_1, O(2)) \otimes H^0(\mathbb{P}_1, O(k-1)))$. Let X_0, X_1, X_2 be a basis for the Lie algebra of SL(2, $\mathbb{C}$), dual with respect to the Killing form to the basis $\frac{d}{d\zeta}, \zeta \frac{d}{d\zeta}, \zeta^2 \frac{d}{d\zeta}$. Each X_i acts as an endomorphism of $H^0(\mathbb{P}_1, O(k-1))$.

Proposition (5.13). *Every element $s \in K_0$ can be expressed uniquely in the form*

$$s = \pi^*(1 \otimes X_0 \tilde{s} + \zeta \otimes X_1 \tilde{s} + \zeta^2 \otimes X_2 \tilde{s})$$

for some $\tilde{s} \in H^0(\mathbb{P}_1, O(k-1))$.

Proof. If we consider the action of SL$(2,\mathbb{C})$, then $H^0(\mathbb{P}_1, O(2))\otimes H^0(\mathbb{P}_1, O(k-1))$ is the representation space $S^2\otimes S^{k-1}\cong S^{k+1}\oplus S^{k-1}\oplus S^{k-3}$. The proposition says that K_0 is the unique submodule isomorphic to S^{k-1}.

We seek local sections $s_0, s_1, s_2, s_3\in H^0(S, L^z(k-1))$, such that

$$\eta s_0+s_1+\zeta s_2+\zeta^2 s_3=0.$$

From Proposition (5.1) $s_0=zt_0$, hence $\eta z t_0+s_1+\zeta s_2+\zeta^2 s_3=0$.

As in the proof of (5.1) we try first extensions which are functions of ηz, which we may relate to the formal neighbourhoods of the zero section $Z\subset T\mathbb{P}_1$.

Lemma (5.14). *Let s be an element of*

$$S^{k-1}\subset H^0(\mathbb{P}_1, O(2))\otimes H^0(\mathbb{P}_1, O(k-1)).$$

Then there is a unique extension of s of order $(k-1)$ to the kernel of m in

$$H^0(Z, O^{(k-1)}(2))\otimes H^0(Z, L^{(k-1)}(k-1)),$$

where $L^{(k-1)}(k-1)$ is the restriction of $L(k-1)$ to the $(k-1)^{\text{th}}$ formal neighbourhood of $Z\subset T\mathbb{P}_1$.

Proof. From [8], $H^0(Z, O^{(k-1)}(2))\cong H^0(T\mathbb{P}_1, O(2))$ if $k>1$, and from Lemma (5.4), $H^0(Z, L^{(k-1)}(k-1))\cong S^{k-1}$ as an SL$(2,\mathbb{C})$ module. Hence as SL$(2,\mathbb{C})$ spaces,

$$\begin{aligned} H^0(Z, O^{(k-1)}(2))\otimes H^0(Z, L^{(k-1)}(k-1)) &\cong (S^0\oplus S^2)\otimes S^{k-1}\\ &\cong S^{k-1}\oplus S^{k+1}\oplus S^{k-1}\oplus S^{k-3}. \end{aligned} \tag{5.15}$$

Now restricting Ω^1_P to $T\mathbb{P}_1\subset\mathbb{P}_3$, we obtain an exact sequence of vector bundles $0\to N^*\to\Omega^1_P\to\Omega^1_T\to 0$, where $N^*\cong O(-4)$ is the conormal bundle and Ω^1_T the cotangent bundle of $T\mathbb{P}_1$. Hence on the $(k-1)^{\text{th}}$ formal neighbourhood, we obtain a sequence $0\to H^0(Z, L^{(k-1)}(k-3))\to H^0(Z, \Omega^1_P L^{(k-1)}(k+1))\to H^0(Z, \Omega^1_T L^{(k-1)}(k+1))\to\dots$. But from Lemma (5.4), $H^0(Z, L^{(k-1)}(k-3))=0$, hence $H^0(Z, \Omega^1_P L^{(k-1)}(k+1))$ is a subspace of $H^0(Z, \Omega^1_T L^{(k-1)}(k+1))$. Next restricting to the fibres of $T\mathbb{P}_1$, we obtain an exact sequence $0\to O(-2)\to\Omega^1_T\to O(-2)\to 0$, and hence a cohomology sequence $0\to H^0(Z, L^{(k-1)}(k-1))\to H^0(Z, \Omega^1_T L^{(k-1)}(k+1))\to H^0(Z, L^{(k-1)}(k-1))\to\dots$.

Now from (5.4), $H^0(Z, L^{(k-1)}(k-1))\cong S^{k-1}$, so $H^0(Z, \Omega^1_P L^{(k-1)}(k+1))$ is a submodule of $S^{k-1}\oplus S^{k-1}$, and in particular has no irreducible components of type S^{k+1} or S^{k-3}. Hence in (5.15) the kernel of m is a subspace of $S^{k-1}\oplus S^{k-1}$.

If $\bar{s}$ is the canonical extension of $s\in H^0(\mathbb{P}_1, O(k-1))$ to $H^0(Z, L^{(k-1)}(k-1))$, then $\eta\bar{s}\in H^0(Z, L^{(k-1)}(k+1))$ is non-zero, hence one of the S^{k-1} components in (5.15) maps non-trivially under m. Hence, as an SL$(2,\mathbb{C})$-module, $\ker m\cong S^{k-1}$, or is zero. It cannot, however, be zero as from Lemma (5.4) we have $H^0(Z, L^{(k-1)}(k+1))\cong S^{k+1}\oplus S^{k-1}\oplus S^{k-3}$, which contains S^{k-1} only once.

Thus $\ker m\cong S^{k-1}$. If we restrict now to the 0^{th} order neighbourhood, then the S^{k-1} component in (5.15) of the form $\eta\otimes\bar{s}$ vanishes, hence the remaining one, $\ker m$, maps non-trivially and hence isomorphically on the S^{k-1} component in $H^0(\mathbb{P}_1, O(2))\otimes H^0(\mathbb{P}_1, O(k-1))$. This then is the required extension.

We can now find canonical extensions of s_1, s_2, s_3 and t_0, such that

$$\eta z\overline{t_0} + \bar{s}_1 + \zeta\bar{s}_2 + \zeta^2\bar{s}_3 = 0 \bmod z^k, \tag{5.16}$$

so long as $1\otimes s_1 + \zeta\otimes s_2 + \zeta^2\otimes s_3 \in S^{k-1}$.

To proceed further we make modifications to s_i as in (5.11) to extend each to the k^{th} order. Since each modification is of the form $z\eta u$ we simply subtract $z\eta u$ from $\bar{s}_i$ and add u to $\overline{t_0}$ to obtain an extension to k^{th} order satisfying (5.16). Proceeding this way, we obtain a formal extension. This defines a formal section of $\Omega^1_P L^z(k+1)$ and so, applying Wavrik's theorem again, there exists an actual extension. This proves (5.13).

Let us now take an element $\tilde{s}\in H^0(\mathbb{P}_1, O(k-1))$ and extend $s = \pi^*(1\otimes X_0\tilde{s} + \zeta\otimes X_1\tilde{s} + \zeta^2\otimes X_2\tilde{s})$, according to Proposition (5.13). To make the formal extension it was necessary to perform modifications to each $X_i\tilde{s}$ but these were all of degree $\geqq 2$ in z. Thus the first order term of the formal extension is the same as the canonical extension. To obtain an actual extension from the formal extension, modifications may be necessary, but we can always find a convergent extension which agrees with the formal extension up to order n, for any given n.

We shall compute the first order term next. From Lemma (5.4) it is equivalent to determining the ratio c_0/c_1 for

$$\begin{pmatrix} p_0' \\ \cdot \\ \cdot \\ \cdot \\ p_{k-1}' \end{pmatrix} = \begin{pmatrix} c_0\zeta^{-1} \\ c_1 \\ \vdots \\ 0 \end{pmatrix},$$

which satisfies (5.5). We shall, however, use the alternative description of the canonical extension as the SL(2, ℂ)-invariant extension.

We require, then, from (5.5) the SL(2, ℂ) invariant splitting of the space of sections of the bundle over $\mathbb{P}_1$ defined by the transition function

$$S_{01} = \begin{pmatrix} \zeta^{k-1} & 0 \\ \zeta^{k-2} & \zeta^{k-3} \end{pmatrix},$$

and defining the invariant extension given by the class $[\zeta^{-1}]\in H^1(\mathbb{P}_1, O(-2))$.

A model for this extension is provided by the 1-jet extension $J_1(k-1)$ of $O(k-1)$, $0\to O(k-3)\to J_1(k-1)\to O(k-1)\to 0$. In this case, the invariant splitting is simply the derivative or 1-jet map:

$$j_1 : H^0(\mathbb{P}_1, O(k-1)) \to H^0(\mathbb{P}_1, J_1(k-1)).$$

The transition function for $O(k-1)$ is ζ^{k-1} so the derivative of a section $f_0 = \zeta^{k-1} f_1$ transforms as $f_0' = (k-1)\zeta^{k-2} f_1 + \zeta^{k-1} f_1'$, and defines $J_1(k-1)$ by the transition function

$$T_{01} = \begin{pmatrix} \zeta^{k-1} & 0 \\ (k-1)\zeta^{k-2} & \zeta^{k-3} \end{pmatrix}.$$

Thus, for a monomial section $\zeta^l \in H^0(\mathbb{P}_1, O(k-1))$, we have

$$j_1(\zeta^l) = \begin{pmatrix} \zeta^l \\ l\zeta^{-1} \end{pmatrix}$$

relative to T_{01}. Changing the basis for the transition function S_{01} gives the canonical extension, in the notation of (5.4), as

$$\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} \zeta^l \\ \dfrac{l}{k-1}\zeta^{l-1} \end{pmatrix}. \tag{5.17}$$

Now the basis $\{X_0, X_1, X_2\}$ of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ dual to $\left\{\frac{d}{d\zeta}, \zeta\frac{d}{d\zeta}, \zeta^2\frac{d}{d\zeta}\right\}$ is easily seen to be $X_0 = -\frac{1}{2}\zeta^2\frac{d}{d\zeta}$, $X_1 = \frac{1}{2}\zeta\frac{d}{d\zeta}$, $X_2 = -\frac{1}{2}\frac{d}{d\zeta}$, and the action of X_0, X_1, X_2 on $\zeta^l \in H^0(\mathbb{P}_1, O(k-1))$ is

$$\begin{aligned} X_0 \cdot \zeta^l &= -\tfrac{1}{2}(l-k+1)\zeta^{l+1}, \\ X_1 \cdot \zeta^l &= \tfrac{1}{2}(2l-k+1)\zeta^l, \\ X_2 \cdot \zeta^l &= -\tfrac{1}{2} l \zeta^{l-1}. \end{aligned} \tag{5.18}$$

Hence if we take $p_0 = \zeta^l$, then from (5.17) and (5.18) we find

$$\begin{aligned} X_0 \cdot p_0 + \zeta X_1 \cdot p_0 + \zeta^2 X_2 \cdot p_0 &= 0, \\ X_0 \cdot p_1 + \zeta X_1 \cdot p_1 + \zeta^2 X_2 \cdot p_1 &= \tfrac{1}{2} \cdot \zeta^l. \end{aligned} \tag{5.19}$$

Thus, in (5.16), we have

$$t_0 = -\tfrac{1}{2}\tilde{s} \bmod z, \tag{5.20}$$

where $s_i = X_{i-1}\tilde{s}$.

We now evaluate the endomorphisms $\tilde{A}_i(z) \in \operatorname{End} V_z$ as $z \to 0$. From (4.12), putting $s_0 = z t_0$, $z\tilde{A}_i t_0 = s_{i+1}$. Thus $z\tilde{A}_i$ is regular and from (5.20)

$$\lim_{z\to 0} z\tilde{A}_i(\tilde{s}) = -2s_{i+1}(0) = -2X_i \cdot \tilde{s}. \tag{5.21}$$

Thus $\tilde{A}_i$ has a simple pole at $z=0$ with residue given by (5.21).

To obtain the matrices A_i we must consider the behaviour of the connection on V as $z \to 0$. From (4.15) the covariant derivative is defined by

$$V_z = \frac{\partial f_0}{\partial z} + (\tfrac{1}{2}\tilde{A}_1 s + \zeta \tilde{A}_2 s)|_u.$$

Now from (5.21) and (5.18),

$$\tfrac{1}{2}\tilde{A}_1 + \zeta\tilde{A}_2 = \frac{(k-1)}{2z} + B, \tag{5.22}$$

where B is holomorphic in a neighbourhood of $z=0$.

The matrices $A_i(z)$ are defined by taking a trivialization of V along $(0,2)$ by covariant constant sections s_i, $1 \leqq i \leqq k$, and setting

$$\sum A_i^{jk}(z) s_k = \tilde{A}_i(z) s_j . \tag{5.23}$$

From (5.22) the sections $t_i = z^{(k-1)/2} s_i$ are regular and so from (5.23), the matrix $A_i(z)$ has the same pole and residue as the endomorphism $\tilde{A}$ – because the residue of the connection is a scalar. Hence we finally obtain the required boundary conditions:

Proposition (5.24). *Let S be a curve in $T\mathbb{P}_1$ satisfying conditions* B1–B4. *Then the matrices T_i produced from Proposition (4.16) have simple poles at $z=0$ and $z=2$, whose residues define an irreducible representation of* SU(2).

Proof. From (5.21) the residues of A_0, A_1, A_2 define the standard representation of $SL(2,\mathbb{C})$ on $H^0(\mathbb{P}_1, O(k-1))$, which is irreducible. Setting $A_0 = T_1 + iT_2$, $A_1 = -2iT_3$, $A_2 = T_1 - iT_2$ as in (4.16) gives for the residues of T_i, the representation restricted to SU(2), which is still irreducible.

Condition B3, that L^2 is trivial on S, implies that at $z=2$, the behaviour of $A_i(z)$ is identical to $z=0$.

6. Reality Conditions

It remains to check the reality conditions on the matrices T_i, that is,

C1. $T_i^*(z) = -T_i(z)$,

C2. $T_i(z) = -\bar{T}_i(2-z)$.

For C1, we must define a hermitian structure on the vector bundle V. Recall that for $z \in (0,2)$, $V_z = H^0(S, L^z(k-1))$.

The real structure on S defines an antilinear isomorphism $\sigma: H^0(S, L^z(k-1)) \to H^0(S, L^{-z}(k-1))$. As in the proof of (3.7) we shall denote this conjugation operation by $\sigma(s) = s^*$. Now consider $s, t \in H^0(S, L^z(k-1))$, and $st^* \in H^0(S, O(2k-2))$. From Proposition (4.5) we can write this uniquely as

$$st^* = c_0 \eta^{k-1} + c_1 \eta^{k-2} + \dots + c_{k-1} , \tag{6.1}$$

where $c_i \in \pi^* H^0(\mathbb{P}_1, O(2i))$.

We define a hermitian inner product on V by

$$\langle s, t \rangle = c_0 . \tag{6.2}$$

This clearly has the correct anti-linearity. It is not obvious yet that it is positive definite.

Now from the definition of $\tilde{A}_i$ we have

$$\eta s + \tilde{A}_0 s + \zeta \tilde{A}_1 s + \zeta^2 \tilde{A}_2 s = 0 \in H^0(S, L^z(k+1)) .$$

Apply the real structure σ, and we find

$$-\eta \sigma(s) + \zeta^2 \sigma(\tilde{A}_0 s) - \zeta \sigma(\tilde{A}_1 s) + \sigma(\tilde{A}_2 s) = 0 , \tag{6.3}$$

using the real structure on $H^0(\mathbb{P}_1, O(2))$. On the other hand we have

$$\eta \sigma(s) + \tilde{A}_0(\sigma s) + \zeta \tilde{A}_1(\sigma s) + \zeta^2 \tilde{A}_2(\sigma s) = 0 . \tag{6.4}$$

Hence adding (6.3) to (6.4) and using Proposition (4.8) we obtain:

$$\begin{aligned} \sigma\tilde{A}_0 &= -\tilde{A}_2\sigma, \\ \sigma\tilde{A}_2 &= -\tilde{A}_0\sigma, \\ \sigma\tilde{A}_1 &= \tilde{A}_1\sigma. \end{aligned} \tag{6.5}$$

Now consider the inner products $\langle\tilde{A}_i s, t\rangle$. We have $\eta s + \tilde{A}_0 s + \zeta\tilde{A}_1 s + \zeta^2\tilde{A}_2 s = 0$, and hence

$$\eta s t^* + (\tilde{A}_0 s)t^* + \zeta(\tilde{A}_1 s)t^* + \zeta^2(\tilde{A}_2 s)t^* = 0. \tag{6.6}$$

Similarly,

$$\eta t s^* + (\tilde{A}_0 t)s^* + \zeta(\tilde{A}_1 t)s^* + \zeta^2(\tilde{A}_2 t)s^* = 0,$$

and applying the real structure to this,

$$-\eta t^* s + \zeta^2(\tilde{A}_0 t)^* s - \zeta(\tilde{A}_1 t)^* s + (\tilde{A}_2 t)^* s = 0. \tag{6.7}$$

Now adding (6.7) to (6.6) we find

$$\{(\tilde{A}_0 s)t^* + s(\tilde{A}_2 t)^*\} + \zeta\{(\tilde{A}_1 s)t^* - s(\tilde{A}_1 t)^*\} + \zeta^2\{(\tilde{A}_2 s)t^* + s(\tilde{A}_0 t)^*\} = 0. \tag{6.8}$$

Now from the exact sequence [see (4.5)], $0 \to O_{\hat{T}} \xrightarrow{\psi} O_{\hat{T}}(k(H_1 + H_2)) \to O_S(2k) \to 0$ we see, since $H^1(\hat{T}, O) = 0$, that every section of $O(2k)$ on S can be written in the form $s = c_0\eta^k + c_1\eta^{k-1} + \ldots + c_k$, and the only linear relation among the sections is the vanishing of a multiple of ψ.

Hence, since (6.8) involves no power of η^k, and hence no multiple of ψ, we may deduce by considering the coefficient of η^{k-1}, that

$$\begin{aligned} \langle\tilde{A}_0 s, t\rangle &= -\langle s, \tilde{A}_2 t\rangle, \\ \langle\tilde{A}_1 s, t\rangle &= \langle s, \tilde{A}_1 t\rangle. \end{aligned} \tag{6.9}$$

Thus, if we write $\tilde{A}_0 = \tilde{T}_1 + i\tilde{T}_2$, $\tilde{A}_1 = 2i\tilde{T}_3$, $\tilde{A}_2 = \tilde{T}_1 - i\tilde{T}_2$, then each $\tilde{T}_i$ is skew-adjoint with respect to the hermitian form.

We investigate next the effect of the connection on the hermitian form. Consider $st^* = c_0(z)\eta^{k-1} + \ldots$, and restrict to the open set $\tilde{U}_0 \cap S$. Then

$$\frac{ds_0}{dz}t_0^* + s_0\frac{dt_0^*}{dz} = \frac{dc_0}{dz}\eta^{k-1} + \ldots. \tag{6.10}$$

Now

$$\frac{dt_0^*}{dz} = \left(\frac{dt_1}{dz}\right)^* = (\nabla_z t + \zeta^{-1}\tilde{A}_0 t + \tfrac{1}{2}\tilde{A}_1 t)^*$$

from (4.15), and similarly

$$\frac{ds_0}{dz} = \nabla_z s - \tfrac{1}{2}\tilde{A}_1 s - \zeta\tilde{A}_2 s.$$

Hence, substituting in (6.10),

$$(\nabla_z s)t^* + s(\nabla_z t)^* - \tfrac{1}{2}(\tilde{A}_1 s)t^* - \zeta(\tilde{A}_2 s)t^* - \zeta s\sigma\tilde{A}_0\sigma^{-1}t^* + \tfrac{1}{2}s\sigma\tilde{A}_1\sigma^{-1}t^*$$
$$= \frac{dc_0}{dz}\eta^{k-1} + \dots,$$

and from (6.5)

$$(\nabla_z s)t^* + s(\nabla_z t)^* = \frac{dc_0}{dz}\eta^{k-1} + \dots .$$

Hence, equating coefficients of η^{k-1}, we have from the definition (6.2)

$$\langle \nabla_z s, t\rangle + \langle s, \nabla_z t\rangle = \frac{d}{dz}\langle s, t\rangle, \tag{6.11}$$

and so the connection preserves the hermitian structure.

Thus, trivializing V with the connection, we obtain a hermitian inner product for which the matrices T_i are skew-adjoint. In particular, the residues at $z=0,2$ of the T_i's are skew-adjoint with respect to this inner product. However, since they define an *irreducible* representation of SU (2), there is, up to a scalar, a unique non-trivial hermitian inner product for which they are skew-adjoint, and in particular it is positive definite. Thus $\langle s,t\rangle$ is either zero or definite.

Let $s\in H^0(S, L^z(k-1))$ be a section which vanishes at only $(k-1)$ points of the fibre $\pi^{-1}(\zeta_0)$ as in the proof of (4.16). Suppose $\langle s,s\rangle=0$, then $ss^*=c_1\eta^{k-2} + \dots + c_{k-1}$, and for $\zeta=\zeta_0$ this vanishes for $(k-1)$ values of η. This means that $c_1(\zeta_0)=c_2(\zeta_0)= \dots =c_{k-1}(\zeta_0)=0$, and so ss^* vanishes for all k points in the fibre. But then s vanishes on the fibre, which is a contradiction. Hence the inner product is non-zero and so is definite.

Thus the matrices $T_i(z)$ satisfies C1.

The reality condition C2 involves the triviality of L^2. Let $a\in H^0(S, L^2)$ be a trivialization, and consider the real constant function $c=a\sigma(a)$, and the antilinear map

$$\sigma' : H^0(S, L^z(k-1)) \to H^0(S, L^{2-z}(k-1)),$$

defined by $\sigma'=a\sigma$. Then $\sigma'^2(s)=a\sigma(a)s=cs$.

Hence after normalization by $|c|$, σ' defines a real structure if $c>0$, and a quaternionic structure if $c<0$ on the bundle V. However, by hypothesis $\sigma' : H^0(S, L(k-1)) \to H^0(S, L(k-1))$ is *real*, so V has a real structure. It is easy to see that this is compatible with the connection, so if we trivialize V with covariant constant sections which are both real and unitary, then condition C2 follows directly.

Hence the solution of Nahm's equations generated by the curve S satisfies all the required conditions for Nahm's construction.

7. The Spectral Curve for Nahm's Construction

We have now shown that a solution to Nahm's equations determines a monopole, that a monopole determines an algebraic curve, and that an algebraic curve determines a solution to Nahm's equations. To complete the circle of ideas, we shall show that if we solve Nahm's equations as in Sect. 4 from an algebraic curve S, and then produce a monopole by the procedure of Sect. 2, then its spectral curve is S itself.

Recall, then, from Sect. 4 the characterization of S in terms of the matrices T_i. The curve S is the divisor of $\psi \in H^0(\mathbb{P}_1, O(2k))$, where

$$\psi = \det(\eta + (T_1 + iT_2) + 2iT_3\zeta + (T_1 - iT_2)\zeta^2).$$

Without loss of generality let us suppose that $(\eta, \zeta) = (0, 0)$ is in S, and then

$$\det(T_1 + iT_2) = 0 \quad \text{for all} \quad z \in (0, 2). \tag{7.1}$$

Now the oriented straight line in $\mathbb{R}^3$ corresponding to $(\eta, \zeta) = (0, 0)$ is

$$\underline{x}(t) = (0, 0, t). \tag{7.2}$$

We must show that $(0, 0)$ lies on the spectral curve, that is that there is a solution to $(\nabla_3 + i\Phi)s = 0$ on this line which decays at both ends. The method we use was suggested by Nahm.

Note first that Nahm's equations imply in particular that

$$\left[\frac{d}{dz} - iT_3, T_1 + iT_2\right] = 0. \tag{7.3}$$

Hence $T_1 + iT_2$ acts on the full k-dimensional space of solutions to $\frac{df}{dz} = iT_3 f$, $f:(0, 2) \to \mathbb{C}^k$, with no boundary conditions. Since this is a first order equation, at each point $z \in (0, 2)$, the values of a basis of solutions $f_\alpha(z)$ are linearly independent. However $\det(T_1 + iT_2) = 0$ for all $z \in (0, 2)$, so there is a solution f_+ such that

$$(T_1 + iT_2)f_+ = 0, \qquad \forall z \in (0, 2). \tag{7.4}$$

Similarly since $(T_1 + iT_2)^* = -T_1 + iT_2$, there is a solution f_- of $\frac{df}{dz} = -iT_3 f$, such that

$$(T_1 - iT_2)f_- = 0. \tag{7.5}$$

Now consider Nahm's operator (2.2): $\Delta^* = \bar{x} + i\frac{d}{dz} - i\sum T_j e_j$. Along the straight line we are considering, $\Delta^* = -te_3 + i\frac{d}{dz} - i\sum T_j e_j$. If we decompose $\mathbb{C}^k \otimes \mathbb{C}^2$ into eigenspaces of e_3, then the null space of Δ^* is described by the equations

$$\begin{aligned} \left(it + i\frac{d}{dz} - T_3\right) f_1 &= i(T_1 + iT_2) f_2, \\ \left(-it + i\frac{d}{dz} + T_3\right) f_2 &= i(-T_1 + iT_2) f_1. \end{aligned} \tag{7.6}$$

Thus

$$(f_1, f_2) = (e^{-tz} f_-, 0),$$
$$(f_1, f_2) = (0, e^{tz} f_+), \tag{7.7}$$

are two linearly independent solutions to $\Delta^* f = 0$. They are also $\mathscr{L}^2$ solutions, as we shall see next.

Since $T_3(z) = a_3/z + S(z)$, where S is regular near $z = 0$,

$$\frac{df_+}{dz} = \frac{ia_3}{z} f_+ + iS(z) f_+ . \tag{7.8}$$

The residue ia_3 comes from an irreducible representation of SU(2), and so is diagonalizable with distinct eigenvalues, (essentially the weights of the representation) $\lambda_1, \ldots, \lambda_k$. From the theory of ordinary differential equations there exists a basis of solutions $\{f_\alpha\}$ to (7.8), such that $z^{-\lambda_\alpha} f_\alpha \to e_\alpha$, as $z \to 0$, where e_α is a unit eigenvector corresponding to the eigenvalue λ_α.

Now in the representation S^{k-1}, the matrix $a_1 + ia_2$ acts on the eigenvectors of a_3 by

$$(a_1 + ia_2) e_\alpha = n_\alpha e_{\alpha+1} , \tag{7.9}$$

where n_α is non-zero unless $\alpha = k$ and the eigenvalues λ_i are ordered so that $\lambda_1 < \lambda_2 < \ldots < \lambda_k$. Now

$$T_1 + iT_2 = (a_1 + ia_2)/z + U(z),$$

where $U(z)$ is regular in a neighbourhood of $z = 0$. Hence from (7.4) $(a_1 + ia_2) f_+ = -zU(z) f_+$ and if $f_+ = \sum_{\alpha=1}^{k} c_\alpha f_\alpha$, then

$$\sum c_\alpha (a_1 + ia_2) f_\alpha = -zU(z) \sum c_\alpha f_\alpha . \tag{7.10}$$

Multiplying by $z^{-\lambda_1}$ and letting $z \to 0$, we see that $c_1 n_1 e_2 = 0$, hence $c_1 = 0$. Now multiplying by $z^{-\lambda_2}$ and repeating we find $c_\alpha = 0$ for $1 \leq \alpha < k$. Hence $z^{-\lambda_k} f_+ \to c e_k$, and as λ_k is positive, f_+ is certainly in $\mathscr{L}^2$ near $z = 0$.

By a similar analysis near $z = 2$ we find f_+ (and analogously f_-) are square integrable.

Thus the solutions (7.7) provide a basis, in Nahm's version of the ADHM construction, for the fibre E_x of the vector bundle on which we define the solution of the Bogomolny equations.

Now take an arbitrary section s of E along the line $x(t) = (0, 0, t)$. We may write this, by the discussion above, as

$$s = (a_1(t) e^{-tz} f_-(z), a_2(t) e^{tz} f_+(z)) .$$

The connection on E is given by orthogonal projection of the ordinary derivative and the Higgs field by projection of $i(1 - z)$ (see Sect. 2). Thus

$$(\nabla_3 - i\Phi) s = P\left(\left(\frac{\partial}{\partial t} + (1 - z)\right) s\right). \tag{7.11}$$

Now

$$\frac{\partial s}{\partial t}+(1-z)s=((a_1'+(1-2z)a_1)e^{-tz}f_-,(a_2'+a_2)e^{tz}f_+).$$

So taking $a_2(t)=0$, s gives zero in (7.11) if

$$(a_1'+a_1)\int_0^2 e^{-2tz}\|f_-(z)\|^2dz=2a_1\int_0^2 ze^{-2tz}\|f_-(z)\|^2dz. \tag{7.12}$$

We may rewrite this as

$$\frac{\partial}{\partial t}\left(a_1\int_0^2 e^{-2tz}\|f_-\|^2dz\right)+a_1\int_0^2 e^{-2tz}\|f_-\|^2dz=0,$$

and so

$$a_1=ce^{-t}\Big/\int_0^2 e^{-2tz}\|f_-\|^2dz.$$

Thus

$$s=\left(e^{-t(1+z)}f_-\Big/\int_0^2 e^{-2tz}\|f_-\|^2dz,0\right)$$

is a solution to $(\nabla_3-i\Phi)s=0$. Moreover,

$$\|s\|^2=e^{-2t}\Big/\int_0^2 e^{-2tz}\|f_-\|^2dz=\left(\int_0^2 e^{2t(1-z)}\|f_-\|^2dz\right)^{-1},$$

and this [see (2.12)] tends to zero as $t\to\pm\infty$. Hence the line we are considering is indeed a line belonging to the spectral curve.

Thus S is contained in the spectral curve, but they have the same degree and the spectral curve has no multiple components. Hence S is the spectral curve.

The conclusion we draw from this is that any monopole may be constructed by Nahm's method using the spectral curve to generate a solution of Nahm's equations.

8. Remarks

1. It has been pointed out several times to the author that Nahm's equations (2.4) are essentially Euler's equations for a spinning top in the case $k=2$. Equations of this type have been studied intensively quite recently, and in fact the linearization by a flow on the Jacobian of a curve, which we described in Sect. 4 is contained in Theorem (1) of Adler and van Moerbeke [1]. The author is indebted to P. Griffiths for pointing this out. It is perhaps interesting to note that the linearization of Sect. 4 arises from the monad construction of vector bundles on $\mathbb{P}_3$, which forms the algebraic geometric foundation of the ADHM construction.

In this paper we were considering the module structure of

$$M=\bigoplus_{m=k-1}^{\infty} H^0(S,L^z(m))$$

over the ring

$$R=\bigoplus_{m=0}^{\infty} H^0(S, O(2m)).$$

However, we have the coboundary map

$$\delta: H^0(S, L^z(m)) \to H^1(T\mathbb{P}_1, L^z(m-2k)),$$

and the inclusion

$$i: H^1(T\mathbb{P}_1, L^z(m-2k)) \to H^1(T\mathbb{P}_1, L^{z-1}\tilde{E}(m-k)),$$

and the pull-back map

$$p^*: H^1(T\mathbb{P}_1, L^{z-1}\tilde{E}(m-k)) \to H^1(\mathbb{P}_3 \backslash \mathbb{P}_1, p^*\tilde{E}(m-k))$$

(recall that L^z is trivial on $\mathbb{P}_3\backslash\mathbb{P}_1$).
Composing these we see that M is a subspace of

$$\bigoplus_{l=-1}^{\infty} H^1(\mathbb{P}_3\backslash\mathbb{P}_1, p^*\tilde{E}(l)),$$

and the ring R is the subring of

$$\bigoplus_{m=0}^{\infty} H^0(\mathbb{P}_3\backslash\mathbb{P}_1, O(m)),$$

which is invariant under the action of $\mathbb{C}$ on $\mathbb{P}_3\backslash\mathbb{P}_1$, whose quotient is $T\mathbb{P}_1$. This is the standard monad set-up of Horrocks and Barth [2] restricted to $\mathbb{P}_3\backslash\mathbb{P}_1$.

2. There is a hierarchy of equations like Nahm's equations corresponding to linear flows in other directions of the Jacobian (again treated in [1]). From our point of view, any direction in the Jacobian corresponds to an element of $H^1(S, O)$, and from Proposition (3.1) this is of the form

$$c=\sum_{i=1}^{k-1} \eta^i \pi^* c_i,$$

where $c_i \in H^1(\mathbb{P}_1, O(-2i))$.

Using the standard open covering of $\mathbb{P}_1$, c_i is represented by a cocycle $c_i=[p_i(\zeta)/\zeta^i]$, where $p_i(\zeta)$ is a polynomial of degree $(2i-2)$.

Following the procedure of Sect. 4, we define $A(\zeta)=A_0+\zeta A_1+\zeta^2 A_2$, and

$$B(\zeta)=\sum_{i=1}^{k-1} p_i(\zeta)(-A(\zeta)/\zeta)^i,$$

and then express $B(\zeta)=A_+(\zeta)+A_-(\zeta^{-1})$, where $A_+(\zeta)$ is the polynomial part of $B(\zeta)$. The arguments of Sect. 4 then show directly that a linear flow on the Jacobian in the direction of c $\left(\text{i.e. replacing } L^z \text{ by the flat line bundle with transition function } \exp\left(z\sum_{i=1}^{k-1} p_i(\zeta)(\eta/\zeta)^i\right)\right)$ leads to the matrix equation:

$$\frac{dA}{dz}+[A_+, A]=0. \tag{8.1}$$

Comparing coefficients of ζ, we obtain a higher order system of equations for T_i.

It is important to note that the flows on the space of matrices

$$(T_1, T_2, T_3) \in \mathbb{R}^3 \otimes \mathrm{su}(n),$$

defined by (8.1) do not necessarily commute, although clearly the linear flows on the Jacobian torus do. This is because, as in (4.15), we use A_+ and A_- to define a *connection* on the vector bundle V (over the complement of the theta-divisor in the Jacobian), defined by $V_L = H^0(S, L(k-1))$ for a flat line bundle L.

The matrix flows (8.1) are obtained by lifting the linear flows on Jac(S) horizontally relative to the connection. However, the connection is easily seen to have curvature which is an obstruction to the commutativity of horizontal vector fields. This connection does, however, appear to depend on the choice of covering of $\mathbb{P}_1$ and it may be that a more intelligent choice would lead to more invariant equations giving the higher order flows.

3. S. Katz has pointed out that Nahm's equations are closely connected with the equations studied by W. Schmid concerning the variation in Hodge structure of a degenerating family of algebraic varieties [13]. The only difference is one of real structure – Schmid's equations arise from a curve S in $T\mathbb{P}_1$ which is real relative to the real structure $(\eta, \zeta) \to (\bar{\eta}, \bar{\zeta})$. The linearization of these equations appears not to have been studied in this context.

4. Perhaps the most useful result of the circle of ideas presented here is the condition on the spectral curve which assures regularity of the monopole. This is condition B4: $H^0(S, L^z(k-2)) = 0$ for $z \in (0, 2)$. In general, this may still be a difficult condition to determine, depending as it does on the geometry of the theta-divisor. However, as an illustration of the power of the result, we prove finally that the axially symmetric solutions of Prasad and Rossi [11] are non-singular.

Theorem (8.2). *Let S be the curve in $T\mathbb{P}_1$ defined by $\psi(\eta, \zeta) = 0$, where*

$$\psi = \eta \prod_{l=1}^{n} (\eta^2 + l^2 \pi^2 \zeta^2) \quad \text{for} \quad k = 2n+1,$$

or

$$\psi = \prod_{l=0}^{n} (\eta^2 + (l + \tfrac{1}{2})^2 \pi^2 \zeta^2) \quad \text{for} \quad k = 2n+2.$$

Then S is the spectral curve of a non-singular solution of the Bogomolny equations.

Proof. We must verify that S satisfies conditions B1–B4. The first two are clearly satisfied, so we must consider just B3 and B4.

Now the curve S is reducible and has singularities at $(\eta, \zeta) = (0, 0)$ and $(0, \infty)$, so in order to consider holomorphic sections of line bundles on S, we must understand what is a holomorphic function in a neighbourhood of a singularity of S. This will be a local section of the sheaf $O/\mathscr{I}$ on $T\mathbb{P}_1$, where $\mathscr{I}$ is the ideal generated by ψ, thus of the form

$$\mathscr{I} = [(\eta - a_1\zeta)(\eta - a_2\zeta) \dots (\eta - a_k\zeta)].$$

A local section of $O/\mathscr{I}$ near $(0, 0)$ may then be represented uniquely by a function

$$f = g_1(\zeta) + \eta g_2(\zeta) + \dots + \eta^{k-1} g_k(\zeta), \tag{8.3}$$

and on each component $\eta = a_i\zeta$, we obtain a function

$$f_i = g_1(\zeta) + a_i\zeta g_2(\zeta) + \dots (a_i\zeta)^{k-1} g_k(\zeta).$$

Now, given k holomorphic functions $f_i(\zeta)$, we wish to know what conditions to impose in order that they define a local section of the sheaf $O/\mathscr{I}$. Clearly from (8.3) this is

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix} = \begin{pmatrix} 1 & a_1\zeta \dots a_1^{k-1}\zeta^{k-1} \\ 1 & a_2\zeta \dots a_2^{k-1}\zeta^{k-1} \\ \vdots & \vdots \\ 1 & a_k\zeta \dots a_k^{k-1}\zeta^{k-1} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_k \end{pmatrix}. \tag{8.4}$$

Now if the a_i's are distinct, the Vandermonde matrix V is invertible, where

$$V = \begin{pmatrix} 1 & a_1 \dots a_1^{k-1} \\ 1 & a_2 \dots a_2^{k-1} \\ \vdots & \\ 1 & a_k \dots a_k^{k-1} \end{pmatrix}.$$

Hence by taking $g_i = \zeta^{k-i} h_i$ for suitable holomorphic functions h_i in (8.4), it is clear that the condition depends only on the $(k-2)$-jet of f_i. Thus, if

$$f_i(\zeta) = f_{i1} + f_{i2}\zeta + \dots f_{ik-2}\zeta^{k-2} \bmod \zeta^{k-1},$$

then the condition (8.4) becomes:

$$f_{i1} = g_{11},$$
$$f_{i2} = g_{12} + a_i g_{21},$$
$$f_{i3} = g_{13} + a_i g_{22} + a_i^2 g_{31}, \text{ etc.}$$

i.e.

$$f_{ij} = \sum_l V_{il} G_{lj}, \tag{8.5}$$

where G_{ij} is a $k \times (k-1)$ matrix with $G_{ij} = 0$ if $i > j$.

Consider now the condition that L^2 be trivial on S. A section of L^2 is described by a holomorphic function f on U and f' on U', such that (see (5.1)) $f = e^{2\eta/\zeta} f'$ on $U \cap U'$. On each component $\eta = a_i\zeta$, we then require (constant) functions f_i, f_i', such that

$$f_i = e^{2a_i} f_i', \tag{8.6}$$

but also satisfying the compatibility condition (8.5) and the analogous one at $(\eta, \zeta) = (0, \infty)$. This implies

$$f_1 = f_2 = \dots = f_k,$$
$$f_1' = f_2' = \dots = f_k',$$

and so L^2 has a section on S if and only if $e^{2a_i} = e^{2a_j}$, $1 \leqq i, j \leqq k$, i.e. $a_i - a_j = mi\pi$, $m \in \mathbb{Z}$. Clearly this is satisfied by S above, so L^2 is trivialized by a section a.

We require also that $L(k-1)$ be real. If k is odd, this is equivalent to L being real, and if k is even to L being quaternionic, and this (see Sect. 6) depends on whether $c=a\sigma(a)$ is positive or negative. Now if we take a to be defined by the functions $(f_i=e^{2a_i}, f_i'=1)$ on $\eta-a_i\zeta=0$, then (recalling that the real structure interchanges U and U') we have

$$\sigma(a)=(f_i=1, f_i'=e^{2\bar{a}_i}) \quad \text{on} \quad \eta-a_i\zeta=0.$$

Thus

$$\begin{aligned} a\sigma(a) &= e^{2a_i} \\ &= e^{2il\pi}=1 \qquad \text{if } k \text{ is odd} \\ &= e^{2i(l+1/2)\pi}=-1 \text{ if } k \text{ is even}. \end{aligned}$$

Hence the reality condition is satisfied and condition B3 holds.

To verify the nonsingularity condition B4, we shall consider a section of $L^{-z}(k-2)$ on S. On the components of S this defines polynomials f_i of degree $(k-2)$ such that (8.4) is satisfied and

$$f_i'=e^{z\eta/\zeta}\zeta^{-(k-2)}f_i=e^{za_i}\zeta^{-(k-2)}f_i$$

is a polynomial in ζ^{-1} which satisfies a condition analogous to (8.4), namely

$$\begin{pmatrix} f_1' \\ f_2' \\ \vdots \\ f_k' \end{pmatrix} = \begin{pmatrix} 1 & a_1\zeta^{-1} \dots a_1^{k-1}\zeta^{-(k-1)} \\ 1 & a_2\zeta^{-1} \\ \vdots & \vdots \\ 1 & a_k\zeta^{-1} \dots a_k^{k-1}\zeta^{-(k-1)} \end{pmatrix} \begin{pmatrix} g_1' \\ g_2' \\ \vdots \\ g_k' \end{pmatrix}$$

for holomorphic functions g_i' of ζ^{-1}.

We obtain then a matrix equation

$$\sum_l D_{il}f_{lj}=\sum_l V_{il}G_{lj}', \tag{8.7}$$

where $D_{ij}=0$ if $i\neq j$ and $D_{ii}=e^{za_i}$ and G_{ij}' is a $k\times(k-1)$ matrix with $G_{ij}'=0$ if $k-j<i$.

Thus, from (8.5) and (8.7), $L^{-z}(k-2)$ has a non-zero section if and only if we can find non-zero matrices G and G' as above with

$$DVG=VG', \quad \text{i.e.} \quad V^{-1}DVG=G'. \tag{8.8}$$

Now if G (and hence G') is not identically zero, the j^{th} column of G must be non-zero for some j. Since $G_{ij}=0$ for $i>j$, only the first j entries of that column are non-zero. On the other hand $G_{ij}'=0$ for $k-j<i\leqq k$, so if (8.8) is satisfied, the $j\times j$ minor of $V^{-1}DV$ in the bottom left hand corner must be singular.

Hence the condition that $L^{-z}(k-2)$ should have *no* sections on S is that all the minors of $V^{-1}DV$ leading in from the bottom left hand corner should be non-singular.

The columns of V^{-1} are the coefficients of the polynomials $p_1, \dots, p_k$ of degree $(k-1)$ which satisfy $p_i(a_j)=0$, $i\neq j$, $p_i(a_i)=1$. Hence the columns of $V^{-1}DV$ are the coefficients of the polynomials $q_i(\zeta)$ of degree $(k-1)$ defined by

$$q_i(\zeta)=\sum_{l=1}^{k} p_l(\zeta)a_l^i e^{za_l}.$$

Thus $q_0(\zeta)$ is the Lagrange interpolation polynomial which satisfies $q_0(\zeta)=e^{z\zeta}$ at $\zeta=a_i$ and $q_i(\zeta)=\dfrac{\partial^i q_0}{\partial z^i}$. Let

$$q_0(\zeta)=c_0\zeta^{k-1}+c_1\zeta^{k-2}+\ldots+c_{k-1},$$

then $V^{-1}DV$ is the matrix

$$\begin{pmatrix} c_{k-1} & c'_{k-1} & c^{(k-1)}_{k-1} \\ \vdots & \vdots & \vdots \\ c_1 & c'_1 & \vdots \\ c_0 & c'_0 & \ldots c^{(k-1)}_0 \end{pmatrix}, \tag{8.9}$$

where $c'=\partial c/\partial z$.

Now at $\zeta=a_i$, $\dfrac{\partial q_0}{\partial z}=a_i e^{za_i}=\zeta q_0$ hence,

$$\frac{\partial q_0}{\partial z}-\zeta q_0=-c_0\prod_{i=1}^{k}(\zeta-a_i). \tag{8.10}$$

Thus $\dfrac{\partial c_i}{\partial z}-c_{i+1}=(-1)^i c_0\sigma_{i+1}(a_1,\ldots,a_k)$, and so each c_i is a linear combination with constant coefficients:

$$c_i=\sum_{l=0}^{i}\lambda_l\frac{\partial^l c_0}{\partial z^l}. \tag{8.11}$$

Thus the determinant of the $j\times j$ minor we are seeking is, from (8.9) and (8.11)

$$\begin{vmatrix} c_0^{(j-1)} & \ldots & c_0^{(2j-2)} \\ \vdots & & \vdots \\ c'_0 & c''_0\ldots & \vdots \\ c_0 & c'_0\ldots & c_0^{(j-1)} \end{vmatrix},$$

and this vanishes at $z=z_0$ if and only if there exist constants μ_l such that

$$f(z)=\sum_{l=0}^{j-1}\mu_l\frac{\partial^l c_0}{\partial z^l} \tag{8.12}$$

vanishes with multiplicity j at z_0.

Now the Lagrange polynomial $q_0(\zeta)$ is given by

$$q_0(\zeta)=\sum_{i=1}^{k}\prod_{j\neq i}\frac{(\zeta-a_j)}{(a_i-a_j)}e^{za_i},$$

so

$$c_0=\sum_{i=1}^{k}\frac{e^{za_i}}{\prod\limits_{j\neq i}(a_i-a_j)}. \tag{8.13}$$

If $k=2n+1$, this is

$$\begin{aligned} c_0&=\sum_{-n}^{+n}\frac{e^{zi\pi l}(-1)^l}{\pi^{2n}(2n)!}\binom{2n}{l}\\ &=\frac{2^{2n}(-1)^n}{\pi^{2n}(2n)!}(\sin z\pi/2)^{2n}, \end{aligned} \tag{8.14}$$

and if $k=2n+2$

$$c_0=\sum_{-n-1}^{+n}\frac{e^{zi\pi(l+1/2)}(-1)^l}{(2n+1)!i\pi^{2n+1}}\binom{2n+1}{l}$$
$$=\frac{2^{2n+1}(-1)^n}{\pi^{2n+1}(2n+1)!}(\sin z\pi/2)^{2n+1}. \tag{8.15}$$

Now from (8.12), (8.14), and (8.15)

$$f(z)=\sum_{l=0}^{j-1}\mu_l\frac{\partial^l c_0}{\partial z^l}=(\sin z\pi/2)^{2n-2j+2}P(\cot z\pi/2),$$

where P is a polynomial of degree $(j-1)$. But if $z\in(0,2)$, then $\sin z\pi/2\neq 0$, hence if $f(z)$ has a zero of order j, then f vanishes identically. It is impossible however, for c_0 to satisfy a linear differential equation of order $<2n$ since it vanishes with multiplicity $2n$ at $z=0$. Thus all the relevant minors of $V^{-1}DV$ are non-singular and condition B4 is satisfied.

Hence S gives a non-singular monopole via Nahm's construction.

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