On the Construction of Paired Many-to-Many Disjoint Path Covers in Hypercube-Like Interconnection Networks with Faulty Elements*

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Abstract

A paired many-to-many k-disjoint path cover (k-DPC) of a graph G is a set of k disjoint paths joining k distinct source-sink pairs in which each vertex of G is covered by a path. This paper is concerned with paired many-to-many disjoint path coverability of hypercube-like interconnection networks, called restricted HL-graphs. The class includes twisted cubes, crossed cubes, multiply twisted cubes, Möbius cubes, Mcubes, and generalized twisted cubes. We show that every restricted HL-graph of degree m with f or less faulty elements has a paired many-to-many k-DPC for any f and $k \geq 2$ with $f + 2k \leq m$. The result improves the known bound of $f + 2k \leq m - 1$ by one.

1. Introduction

Various interconnection networks were proposed and their graph-theoretic properties have been investigated with their applications in parallel computing. Among the properties, finding parallel paths among nodes in interconnection networks is one of the important problems concerned with an efficient data transmission. Usually interconnection networks are represented as graphs and parallel paths are studied in terms of disjoint paths in graphs. In this paper, we will use standard terminology in graphs (see [1]).

Let $G = (V, E)$ be an undirected simple graph. A set of paths in G is called *disjoint* if they do not share any vertices. In disjoint path problems, one or more source vertices and one or more sink vertices are given to find disjoint paths between them. Depending on the number of sources or sinks, there are one-to-one[9, 2, 16], one-to-many[3, 10], and many-to-many disjoint path problems[11, 13]. Among them, many-to-many disjoint path problem is the most generalized one, and will be mainly discussed in this paper.

For a set $S = \{s_1, s_2, \ldots, s_k\}$ of k sources and a set $T = \{t_1, t_2, \ldots, t_k\}$ of k sinks in $V(G)$, the many-to-many k-disjoint path problem is to determine whether there exist k disjoint paths each joining a source and a sink. There are *paired* and *unpaired* types of many-to-many k-disjoint path problem. In paired type, each source should be joined to a specific sink, that is, s_j should be joined to t_j. In unpaired type, each source can be joined to an arbitrary sink. The sources and sinks are called *terminal* in general.

Disjoint path cover of a graph G is a set of disjoint paths covering all the vertices of G. The problem of finding disjoint path covers is closely related with well-known hamiltonian path problem and concerned with the application where the full utilization of vertices is important. Hamiltonian path problem can be viewed as a specific case of the disjoint path cover problem.

The disjoint path cover problem can be extended to a

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graph with some faulty elements (vertices and/or edges). Fault tolerance is one of the important measures in networks. Especially, fault-hamiltonicity of various interconnection networks was widely investigated in the literature[4, 5, 6, 12, 15, 17]. A graph G is called *f-fault hamiltonian* (resp. *f-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G\backslash F$ for any set F of faulty elements with $|F| \leq f$.

Considering all the above versions of disjoint path cover problems, we give definitions for a graph G with a set F of faulty elements.

Definition 1 *Given a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in $G\backslash F$ such that $S \cap T = \emptyset$, a* paired many-to-many k-disjoint path cover *joining S and T is a set of k fault-free disjoint paths P_j joining s_j and t_j, $1 \leq j \leq k$, that cover all the fault-free vertices of G.*

Definition 2 *Given a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in $G\backslash F$ such that $S \cap T = \emptyset$, an* unpaired many-to-many k-disjoint path cover *joining S and T is a set of k fault-free disjoint paths P_j joining s_j and t_{i_j}, $1 \leq j \leq k$, with an arbitrary permutation $(i_1, i_2, \ldots, i_k)$ of $\{1, 2, \ldots, k\}$ that cover all the fault-free vertices of G.*

In this paper, we consider a graph with faulty elements which has a k-DPC for arbitrary k sources and k sinks rather than fixed sources and sinks, which is called many-to-many k-disjoint path coverable graph. It is defined as follows.

Definition 3 *A graph G is called* f-fault paired (resp. unpaired) many-to-many k-disjoint path coverable *if $f + 2k \leq |V(G)|$ and for any set F of faulty elements with $|F| \leq f$, G has a paired (resp. unpaired) k-DPC for any set S of k sources and any set T of k sinks in $G\backslash F$ such that $S \cap T = \emptyset$.*

Many interconnection networks such as restricted HL-graphs and recursive circulant $G(2^m, 4)$ can be constructed by connecting two lower dimensional networks. We represent the construction as follows. Given two graphs G_0 and G_1 with n vertices each, we denote by V_j and E_j the vertex set and edge set of G_j, $j = 0, 1$, respectively. Let $V_0 = \{v_1, v_2, \ldots, v_n\}$ and $V_1 = \{w_1, w_2, \ldots, w_n\}$. With respect to a permutation $M = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we can "merge" the two graphs into a graph $G_0 \oplus_M G_1$ with $2n$ vertices in such a way that the vertex set $V = V_0 \cup V_1$ and the edge set $E = E_0 \cup E_1 \cup E_2$, where $E_2 = \{(v_j, w_{i_j}) | 1 \leq j \leq n\}$. We denote by $G_0 \oplus G_1$ a graph obtained by merging G_0 and G_1 w.r.t. an arbitrary permutation M. Here, H_0 and H_1 are called *components* of $H_0 \oplus H_1$.

Vaidya *et al.*[18] introduced a class of hypercube-like interconnection networks, called *HL-graphs*, which can be defined by applying the $\oplus$ operation repeatedly as follows: $HL_0 = \{K_1\}$; for $m \geq 1$, $HL_m = \{G_0 \oplus G_1 | G_0, G_1 \in HL_{m-1}\}$. Then, $HL_1 = \{K_2\}$; $HL_2 = \{C_4\}$; $HL_3 = \{Q_3,\ G(8, 4)\}$. Here, C_4 is a cycle graph with 4 vertices, Q_3 is a 3-dimensional hypercube, and $G(8, 4)$ is a recursive circulant which is defined as follows: the vertex set is $\{v_i | 0 \leq i \leq 7\}$ and the edge set is $\{(v_i, v_j) | i+1 \text{ or } i+4 \equiv j \pmod 8\}$. $G(8, 4)$ is isomorphic to twisted cube TQ_3 and Möbius ladder with four spokes.

In [12], a subclass of nonbipartite HL-graphs, called *restricted HL-graphs*, was introduced by the authors, which is defined recursively as follows: $RHL_m = HL_m$ for $0 \leq m \leq 2$; $RHL_3 = HL_3 \backslash Q_3 = \{G(8, 4)\}$; $RHL_m = \{G_0 \oplus G_1 | G_0, G_1 \in RHL_{m-1}\}$ for $m \geq 4$. A graph which belongs to RHL_m is called an *m-dimensional restricted HL-graph.* Many of the nonbipartite hypercube-like interconnection networks such as crossed cube, Möbius cube, twisted cube, multiply twisted cube, Mcube, generalized twisted cube, etc. proposed in the literature are restricted HL-graphs with the exception of recursive circulant $G(2^m, 4)$ and "near" bipartite interconnection networks such as twisted m-cube. In fact, every $G(2^m, 4)$ with odd m is an m-dimensional restricted HL-graph. Some works on HL-graphs and restricted HL-graphs were appeared in the literature; for example, hamiltonicity of HL-graphs[8], fault-hamiltonicity of restricted HL-graphs[12], and fault-panconnectivity and fault-pancyclicity of restricted HL-graphs[14].

Only a few works can be found for many-to-many k-disjoint path cover problem with $k \geq 2$. It was shown in [13] and [11], respectively, that every m-dimensional restricted HL-graph and recursive circulant $G(2^m, 4)$ are f-fault *paired* many-to-many k-disjoint path coverable for any f and $k \geq 1$ with $f + 2k \leq m - 1$, and every m-dimensional restricted HL-graph is f-fault *unpaired* many-to-many k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq m - 2$. Every m-dimensional crossed cube, $m \geq 5$, was shown to have a paired 2-DPC consisting of two paths of equal length by Lai *et al.* in [7].

In this paper, we show that every m-dimensional restricted HL-graph is f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq m$. The bound on $f + 2k$ is improved by one as compared with [13]. The necessary condition given in [13] says "$f + 2k \leq m + 1$." Thus, the gap between the bound achieved and the bound $m + 1$ of necessity is just one.

2. Construction of Paired Disjoint Path Covers

Let $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. Here, G_0 and G_1 are called *subcomponents* of $H_0 \oplus H_1$. The main problem studied in this section is how paired many-to-many disjoint path coverability and unpaired many-to-many disjoint path coverability of G_i's and H_j's are translated into paired many-to-many disjoint path coverability of $H_0 \oplus H_1$. To achieve simpler construction, we make an assumption that each G_i has 2^{m-2} vertices and is of degree $m-2$. Thus, H_j has 2^{m-1} vertices and is of degree $m-1$. The main theorem will be stated as follows. We denote by $\delta(G)$ the minimum degree of a graph G.

Theorem 1 *Let $m \geq 5$. Let G_i, $i = 0, 1, 2, 3$, be a graph of degree $m-2$ having 2^{m-2} vertices. Suppose each G_i is (a) f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq \delta(G_i)$ and (b) f-fault unpaired many-to-many k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq \delta(G_i) - 2$. Let $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. Furthermore, we suppose each H_j is (c) f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq \delta(H_i)$ and (d) f-fault unpaired many-to-many k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq \delta(H_i) - 2$. Then, $H_0 \oplus H_1$ is f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq \delta(H_0 \oplus H_1) = m$.*

For a vertex v in $H_0 \oplus H_1$, we denote by $\bar{v}$ the vertex adjacent to v which is in a component different from the component in which v is contained.

Definition 4 *A vertex v is called* free *if v is fault-free and not a terminal, that is, $v \notin F$ and $v \notin S \cup T$. An edge (v, w) is called* free *if v and w are free and $(v, w) \notin F$.*

We denote by $H[v, w|G, F]$ a hamiltonian path in $G \backslash F$ joining a pair of fault-free vertices v and w in a graph G with a fault set F, that is, 1-DPC$[\{(v, w)\}|G, F]$. A path in a graph is represented as a sequence of vertices. A v-w path refers to a path from vertex v to w, and a v-path refers to a path whose starting vertex is v.

2.1. Proof of Theorem 1

Given a fault set F, a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$, and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in a graph G, a paired many-to-many k-disjoint path cover joining S and T in $G \backslash F$ is denoted by k-DPC$[\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}|G, F]$. We are to construct a k-DPC$[\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}|H_0 \oplus H_1, F]$ for any given F with $|F| \leq f$, S and T with $|S| = |T| = k \geq 2$ such that $f + 2k \leq m$.

F_0 and F_1 denote the sets of faulty elements in H_0 and H_1, respectively, and F_2 denotes the set of faulty edges joining vertices in H_0 and vertices in H_1, so that $F = F_0 \cup F_1 \cup F_2$. Let $f_0 = |F_0|$, $f_1 = |F_1|$, and $f_2 = |F_2|$. We also denote by k_i the number of source-sink pairs in H_i, $i = 0, 1$, and by k_2 the number of source-sink pairs between H_0 and H_1. We assume w.l.o.g. that

$$k_0 \geq k_1, \text{ and if } k_0 = k_1, f_0 \geq f_1.$$

We let $I_0 = \{1, 2, \ldots, k_0\}$, $I_2 = \{k_0 + 1, k_0 + 2, \ldots, k_0 + k_2\}$, and $I_1 = \{k_0 + k_2 + 1, k_0 + k_2 + 2, \ldots, k_0 + k_2 + k_1\}$. We assume that $\{s_j, t_j | j \in I_0\} \cup \{s_j | j \in I_2\} \subseteq V(H_0)$ and $\{s_j, t_j | j \in I_1\} \cup \{t_j | j \in I_2\} \subseteq V(H_1)$.

We have $|F| \leq f$, $k = k_0 + k_1 + k_2 \geq 2$, and $f + 2k \leq m$. Observe that a paired many-to-many k-disjoint path cover in $H_0 \oplus H_1$ with a *virtual* fault set $F \cup F'$, where F' is a set of arbitrary $m - 2k - |F|$ fault-free edges, is also a paired many-to-many k-disjoint path cover in $H_0 \oplus H_1$ with the fault set F. Thus, we can assume

$$f + 2k = m \text{ and } |F| = f.$$

By the condition (d), each H_i is $m-4$-fault hamiltonian-connected, or equivalently, $f + 2k - 4$-fault hamiltonian-connected. Since $m \geq 5$ and $k \geq 2$, we have that

H_i is 1-fault hamiltonian-connected and
f-fault hamiltonian-connected.

Hereafter in this section, an f-fault k-DPC refers to an f-fault paired many-to-many k-disjoint path cover joining the set of sources and the set of sinks. There are four cases, Cases I through IV.

Case I: $k_1 \geq 1$ or $f_0 \leq f - 1$.

In this case, H_0 is f_0-fault paired many-to-many $k_0 + k_2$-disjoint path coverable. By the assumption of $k_0 \geq k_1$, if $k_1 + k_2 \geq 1$, H_1 is f_1-fault paired many-to-many $k_1 + k_2$-disjoint path coverable.

Procedure PairedDPC-A$(H_0 \oplus H_1, S, T, F)$
/* under the condition of $k_1 \geq 1$ or $f_0 \leq f - 1$ */

1. Pick up k_2 free edges joining vertices in H_0 and vertices in H_1. Let the free edges be (x_j, y_j), $j \in I_2$, with $x_j \in V(H_0)$.

2. Find $k_0 + k_2$-DPC$[\{(s_j, t_j) | j \in I_0\} \cup \{(s_j, x_j) | j \in I_2\} | H_0, F_0]$.

3. Case $k_1 + k_2 \geq 1$:

   (a) Find $k_1 + k_2$-DPC$[\{(s_j, t_j) | j \in I_1\} \cup \{(y_j, t_j) | j \in I_2\} | H_1, F_1]$.

   (b) Merge the two DPC's with the k_2 free edges.

4. Case $k_1 + k_2 = 0$:

   (a) Let (x, y) be an edge on some path in the $k_0 + k_2$-DPC such that all the $\bar{x}$, $(x, \bar{x})$, $\bar{y}$, and $(y, \bar{y})$ are fault-free.

(b) Find $H[\bar{x},\bar{y}|H_1,F_1]$.

(c) Merge the k_0+k_2-DPC and the hamiltonian path with edges $(x,\bar{x})$ and $(y,\bar{y})$. Discard the edge (x,y).

Lemma 1 *When $k_1 \geq 1$ or $f_0 \leq f-1$, Procedure PairedDPC-A constructs an f-fault k-DPC.*

Proof: We claim the k_2 free edges in Step 1 exist. There are 2^{m-1} candidate free edges and $f+2k$ blocking elements (f faults and $2k$ terminals). The number of nonblocked candidates is at least $2^{m-1}-(f+2k)=2^{m-1}-m>m>k_2$ for any $m\geq 5$. Thus, the claim is proved. The k_0+k_2-DPC in H_0 exists when $k_0+k_2\geq 2$, if $k_1\geq 1$, we have $f_0+2(k_0+k_2)\leq f+2(k-1)\leq m-1$, and if $f_0\leq f-1$, we have $f_0+2(k_0+k_2)\leq (f-1)+2k\leq m-1$. When $k_0+k_2=1$, the k_0+k_2-DPC is a hamiltonian path between two vertices in H_0. The hamiltonian path exists since H_0 is f-fault hamiltonian-connected and $f_0\leq f$. Similarly, we can show the existence of k_1+k_2-DPC in Step 3(a) and the hamiltonian path in Step 4(b). We claim the edge (x,y) in Step 4(a) exists. There are at least $|V(H_0)|-f_0-k$ candidate edges, and at most f_1+f_2 elements can block the candidates. Since each element blocks at most two candidates, the number of nonblocked candidates is at least $|V(H_0)|-f_0-k-2(f_1+f_2)\geq 2^{m-1}-k-2f > 2^{m-1}-2m\geq 6$ for any $m\geq 5$. Note that $f+2k=m$. ■

Case II: $k_1=0$, $f_0=f$, $k_0\geq 1$, $k_2\geq 1$, and for some $a\in I_2$, $\bar{s_a}$ is not a terminal.

All the sources and all the faulty elements, if any, are contained in H_0. Notice that H_0 may not be f_0-fault many-to-many k_0+k_2-disjoint path coverable since $f_0+2(k_0+k_2)=f+2k\not\leq m-1$. Nevertheless, if $k\geq 3$, there always exists an f_0+1-fault k_0+k_2-1-DPC in H_0 with s_a being a *virtual* fault. The k_0+k_2-1-DPC (instead of k_0+k_2-DPC) can be utilized to construct an f-fault k-DPC in $H_0\oplus H_1$. In fact, $(s_a,\bar{s_a})$ plays a role of the free edge for s_a-t_a path.

When $k=2$, this approach will not be applied since the existence of f_0+1-fault k_0+k_2-1-DPC, or equivalently f_0+1-fault hamiltonian path in H_0 is not guaranteed. We consider the subcase $k=2$ first, as shown in the following Procedure PairedDPC-B. The procedure is applicable for the case $k_1=0$, $f_0=f$, and $k_0=k_2=1$, regardless of whether the $\bar{s_2}$, $2\in I_2$, is a terminal or not. It utilizes fault-hamiltonicity of components H_0 and H_1. Its correctness is straightforward since each H_i is f-fault hamiltonian-connected and 1-fault hamiltonian-connected.

Procedure PairedDPC-B($H_0\oplus H_1,S,T,F$)
/* under the condition of $k_1=0$, $f_0=f$, and $k_0=k_2=1$ */

1. Regarding s_1 as a *virtual* free vertex, find a hamiltonian path $P_h=H[s_2,t_1|H_0,F_0]$. Let $P_h=(s_2,P_x,x,s_1,P'_1,t_1)$.
2. Case $\bar{x}\neq t_2$:
   (a) Find a hamiltonian path $P'_h=H[\bar{x},t_2|H_1,\emptyset]$.
   (b) Let $P_1=(s_1,P'_1,t_1)$ and $P_2=(s_2,P_x,x,P'_h)$.
3. Case $\bar{x}=t_2$:
   (a) Pick up an arbitrary edge (u,v) on P_h with $u,v\neq x$.
   (b) Find a hamiltonian path $P'_h=H[\bar{u},\bar{v}|H_1,\{t_2\}]$.
   (c) Let $P_1=(s_1,P'_1,t_1)$ and $P_2=(s_2,P_x,x,t_2)$, and then replace the edge (u,v) with (u,P'_h,v).

Procedure PairedDPC-C($H_0\oplus H_1,S,T,F$)
/* under the condition of $k_1=0$, $f_0=f$, $k_0\geq 1$, $k_2\geq 1$, $k\geq 3$, and there exists a source s_a, $a\in I_2$, with $\bar{s_a}$ being not a terminal */

1. Pick up k_2-1 free edges joining vertices in H_0 and vertices in H_1. Let the free edges be (x_j,y_j), $j\in I_2\backslash a$, with $x_j\in V(H_0)$.
2. Regarding s_a as a *virtual* fault, find k_0+k_2-1-DPC$[\{(s_j,t_j)|j\in I_0\}\cup\{(s_j,x_j)|j\in I_2\backslash a\}|H_0,F_0\cup\{s_a\}]$.
3. Find k_2-DPC$[\{(\bar{s_a},t_a)\}\cup\{(y_j,t_j)|j\in I_2\backslash a\}|H_1,\emptyset]$.
4. Merge the two DPC's with $(s_a,\bar{s_a})$ and the k_2-1 free edges.

Lemma 2 *When $k_1=0$, $f_0=f$, $k_0\geq 1$, $k_2\geq 1$, $k\geq 3$, and there exists a source s_a, $a\in I_2$, with $\bar{s_a}$ being not a terminal, Procedure PairedDPC-C constructs an f-fault k-DPC.*

Proof: The existence of k_2-1 free edges can be proved in the same way as in the proof of Lemma 1. The k_0+k_2-1-DPC exists since $f_0+1+2(k_0+k_2-1)=f+1+2(k-1)=m-1$. The existence of k_2-DPC is obvious. ■

Case III: $k_1=0$, $f_0=f$, $k_0\geq 1$, either $k_2=0$ or $k_2\geq 1$ and for every $j\in I_2$, $\bar{s_j}$ is a terminal.

This is one of the hardest cases. An f_0-fault k_0+k_2-disjoint path coverability of H_0 is not guaranteed. The construction of an f-fault k-DPC relies on the construction of $k-1$-DPC in H_1 or when $f\geq 1$, k-DPC in H_1. Notice that if v is a free vertex or a terminal in $\{s_j,t_j|j\in I_0\}$, then $\bar{v}$ is always a free vertex. We consider the subcase $k_0\geq 2$ first. In this case, fault-hamiltonicity of H_0 and $k-1$-disjoint path coverability of H_1 are employed.

Procedure PairedDPC-D($H_0\oplus H_1,S,T,F$)
/* under the condition of $k_1=0$, $f_0=f$, $k_0\geq 2$, and either $k_2=0$ or $k_2\geq 1$ and $\bar{s_j}$ is a sink for every $j\in I_2$ */

1. Pick up k_2 free edges (x_j, y_j), $j \in I_2$, with $x_j \in V(H_0)$ and $y_j \in V(H_1)$ such that (s_j, x_j) is an edge and fault-free.

2. Regarding s_1 and t_1 as *virtual* free vertices, find a hamiltonian path $H[s_2, t_2|H_0, F_0 \cup F' \cup F'']$, where $F' = \{s_j, x_j|j \in I_2\}$ and $F'' = \{s_j, t_j|j \in I_0\backslash\{1,2\}\}$. Here, F' and F'' are *virtual* fault sets. Let the hamiltonian path be $(s_2, P_u, u, s_1, P_1', t_1, v, P_v, t_2)$.

3. Find k_0+k_2-1-DPC$[\{(y_j, t_j)|j \in I_2\} \cup \{(\bar{s_j}, \bar{t_j})|j \in I_0\backslash\{1,2\}\} \cup \{(\bar{u}, \bar{v})\}|H_1, \emptyset]$.

4. Merge the hamiltonian path and the DPC with $\{(s_j, x_j, y_j)|j \in I_2\}$, $\{(s_j, \bar{s_j}), (t_j, \bar{t_j})|j \in I_0\backslash\{1,2\}\}$, and $\{(u, \bar{u}), (v, \bar{v})\}$. Discard edges (s_1, u) and (t_1, v).

Lemma 3 *When $k_1 = 0$, $f_0 = f$, $k_0 \geq 2$, and either $k_2 = 0$ or $k_2 \geq 1$ and $\bar{s_j}$ is a terminal for every $j \in I_2$, Procedure PairedDPC-D constructs an f-fault k-DPC.*

Proof: For each $j \in I_2$, we can pick up a free edge (x_j, y_j) one by one since there are $\delta(H_0) = m-1$ candidates and at most $f+2(k-1) = m-2$ blocking elements (f faulty elements, $2k_0$ terminals, k_2-1 sources, and k_2-1 free edges picked up). The hamiltonian path in H_0 exists since $f_0 + 2(k_0-2) + 2k_2 = f + 2k - 4 = m - 4$. Obviously, the $k_0 + k_2 - 1$-DPC exists in H_1. ■

We come to the case that $k_1 = 0$, $f_0 = f$, $k_0 = 1$, and either $k_2 = 0$ or $k_2 \geq 1$ and $\bar{s_j}$ is a terminal for every $j \in I_2$. By the assumption of $k \geq 2$, we have $k_2 \geq 1$. Furthermore, the case $k_2 = 1$ was already considered in Procedure PairedDPC-B, and thus we assume $k_2 \geq 2$. Therefore, we have $k \geq 3$ and $m \geq 6$. Remember $t_1 \in V(H_0)$ and $t_j \in V(H_1)$ for all $j \geq 2$. There are two procedures depending on whether $f \geq 1$ or not. For the case $f \geq 1$, we utilize fault-hamiltonicity of H_0 and 0-fault k-disjoint path coverability of H_1.

Procedure PairedDPC-E($H_0 \oplus H_1, S, T, F$)
/* under the condition of $k_1 = 0$, $f_0 = f \geq 1$, $k_0 = 1$, $k_2 \geq 2$, and $\bar{s_j}$ is a sink for every $j \in I_2$ */

1. Pick up $k_2 - 1$ free edges (x_j, y_j), $j \in I_2\backslash 2$, with $x_j \in V(H_0)$ and $y_j \in V(H_1)$ such that (s_j, x_j) is an edge and fault-free.

2. Regarding s_2 as a *virtual* free vertex, find a hamiltonian path $P_h = H[s_1, t_1|H_0, F_0 \cup F']$, where $F' = \{s_j, x_j|j \in I_2\backslash 2\}$.

3. There exists a free vertex x_2 such that (s_2, x_2) is an edge of P_h. Removing s_2 and x_2 from P_h results two subpaths (s_1, P_u, u) and (v, P_v, t_1). Let $y_2 = \bar{x_2}$.

4. Find $k_0 + k_2$-DPC$[\{(y_j, t_j)|j \in I_2\} \cup \{(\bar{u}, \bar{v})\}|H_1, \emptyset]$.

5. Merge the hamiltonian path and the DPC with $\{(s_j, x_j, y_j)|j \in I_2\}$ and $\{(u, \bar{u}), (v, \bar{v})\}$.

Lemma 4 *When $k_1 = 0$, $f_0 = f \geq 1$, $k_0 = 1$, $k_2 \geq 2$, and $\bar{s_j}$ is a sink for every $j \in I_2$, Procedure PairedDPC-E constructs an f-fault k-DPC.*

Proof: The existence of $k_2 - 1$ free edges can be proved in a very similar way as in the proof of Lemma 3. The hamiltonian path P_h exists since $f_0 + 2(k_2-1) = f + 2k - 4 = m - 4$. The $k_0 + k_2$-DPC exists in H_1 since $2(k_0 + k_2) = m - f \leq m - 1$. ■

Finally, we have $f = 0$. We will show that for 'some' k_2 free edges joining vertices in H_0 and vertices in H_1, there exist two DPC's: a $k_0 + k_2$-DPC from sources to the union of sink t_1 and endvertices of the free edges in H_0 and k_2-DPC between sinks and endvertices of the free edges in H_1. The construction of a $k_0 + k_2$-DPC in H_0 is a little complicated. It consists of two subcases, as shown in Steps 1 and 2 of the following procedure.

For a vertex v in G_0 (resp. G_1), $\hat{v}$ denotes the vertex in G_1 (resp. G_0) which is adjacent to v. Let $I_2' = \{j \in I_2|s_j \in V(G_0)\}$ and $I_2'' = I_2\backslash I_2'$, and let $k_2' = |I_2'|$ and $k_2'' = |I_2''|$, so that $k_2' + k_2'' = k_2$. It is assumed w.l.o.g. that $k_2' \geq k_2''$.

Procedure PairedDPC-F($H_0 \oplus H_1, S, T, F$)
/* under the condition of $k_1 = 0$, $f = 0$, $k_0 = 1$, $k_2 \geq 2$, and $\bar{s_j}$ is a sink for every $j \in I_2$ */

1. Case $k_2'' \geq 1$ or $k_2'' = 0$ and $\hat{s_a}$ is a free vertex for some $a \in I_2'$:
   (a) Let x_a be a vertex in H_0 such that $(s_a, x_a) \in E$ and $(s_b, x_a) \notin E$ for some $a, b \in I_2$.
   (b) Pick up $k_2 - 2$ free edges (x_j, y_j), $j \in I_2\backslash\{a, b\}$, with $x_j \in V(H_0)$ and $y_j \in V(H_1)$ such that $x_j \neq x_a$.
   (c) Find $k_0 + k_2 - 1$-DPC$[\{(s_1, t_1), (s_b, x_a)\} \cup \{(s_j, x_j)|j \in I_2\backslash\{a, b\}\}|H_0, F']$, where $F' = \{s_a\}$. Let the s_b-path in the DPC be (s_b, P', x_b, x_a).
   (d) Let s_a-x_a path be (s_a, x_a) and let s_b-x_b path be (s_b, P', x_b). Let $y_a = \bar{x_a}$ and $y_b = \bar{x_b}$.

2. case $k_2'' = 0$ and $\hat{s_i}$ is a terminal for every $i \in I_2'$:
   /* $k_2 = 2$, $s_2, s_3 \in V(G_0)$, and $s_1, t_1 \in V(G_1)$ */
   (a) Pick up two free edges (x_2, y_2) and (x_3, y_3) with $x_2, x_3 \in V(G_0)$ and $y_2, y_3 \in V(H_1)$.
   (b) Find 2-DPC$[\{(s_2, x_2), (s_3, x_3)\}|G_0, \emptyset]$.
   (c) Find $H[s_1, t_1|G_1, \emptyset]$.

3. Find k_2-DPC$[\{(y_j, t_j)|j \in I_2\}|H_1, \emptyset]$.

4. Merge the two DPC's with edges (x_j, y_j), $j \in I_2$.

Lemma 5 *When $k_1 = 0$, $f = 0$, $k_0 = 1$, $k_2 \geq 2$, and $\bar{s_j}$ is a sink for every $j \in I_2$, Procedure PairedDPC-F constructs an f-fault k-DPC.*

Proof: We first claim the existence of x_a in Step 1(a). When $k_2'' \geq 1$, let $a \in I_2'$ and $b \in I_2''$. Then, s_a and s_b are sources contained in G_0 and G_1, respectively. There are $m-2$ candidates for x_a in G_0 and at most $2k_0 + (k_2 - 1)$ blocking terminals. Since $2k_0 + (k_2 - 1) = k = m - k \leq m - 3$, there exists such a vertex x_a. When $k_2'' = 0$ and $\hat{s_a}$ is a free vertex for some $a \in I_2'$, let s_b be an arbitrary source in G_0 with $b \in I_2 \backslash a$. By the structure of $G_0 \oplus G_1$, $(s_b, x_a) \notin E$. Thus, the claim is proved. The existence of the $k_2 - 2$ free edges in Step 1(b) is straightforward. The $k_0 + k_2 - 1$-DPC in Step 1(c) exists since $1 + 2(k_0 + k_2 - 1) = 2k - 1 = m - 1$. By the choice of x_a, x_b is a free vertex different from x_a. Thus, a $k_0 + k_2$-DPC in H_0 is constructed successfully in Step 1. If $k_2'' = 0$ and $\hat{s_i}$ is a terminal for every $i \in I_2'$, we can see that $k_2 = 2$ and $\{\hat{s_2}, \hat{s_3}\} = \{s_1, t_1\}$. Since G_0 is paired many-to-many $k-1$-disjoint path coverable and G_1 is hamiltonian-connected, a $k_0 + k_2$-DPC can be constructed in Step 2. Existence of the k_2-DPC in Step 3 is due to $k_2 < k$, precisely speaking, due to $2k_2 = 2(k-1) \leq m-1$. This completes the proof. ■

Case IV: $k_2 = k$ and $f_0 = f$.

To construct an f-fault k-DPC in this case, we mainly utilize *unpaired* many-to-many disjoint path coverability of H_0 and *paired* many-to-many disjoint path coverability and hamiltonicity of *subcomponents* G_2 and G_3. By virtue of unpaired many-to-many disjoint path coverability, we are able to keep out of some troublesome subcases although this is one of the hardest cases.

However, there is an exceptional case in which we cannot apply unpaired many-to-many disjoint path coverability of H_0, the case of $k = 2$. We consider the exceptional case first in the following Procedure PairedDPC-G. Its correctness is straightforward since each H_i is f-fault hamiltonian-connected and 0-fault paired many-to-many 2-disjoint path coverable.

Procedure PairedDPC-G$(H_0 \oplus H_1, S, T, F)$
/* under the condition of $k_2 = k = 2$, $f_0 = f$ */

1. Find $H[s_1, s_2|H_0, F_0]$. Let the hamiltonian path be $(s_1, P_u, u, v, P_v, s_2)$ for some edge (u, v) with $\{\bar{u}, \bar{v}\} \cap \{t_1, t_2\} = \emptyset$.

2. Find 2-DPC$[\{(\bar{u}, t_1), (\bar{v}, t_2)\}|H_1, \emptyset]$.

3. Merge the hamiltonian path and 2-DPC with edges $(u, \bar{u})$ and $(v, \bar{v})$.

We assume $k \geq 3$ and thus $m \geq 6$. For a vertex v in G_2 (resp. G_3), $\hat{v}$ denotes the vertex in G_3 (resp. G_2) which is adjacent to v. We let $I_2' = \{j \in I_2 | t_j \in V(G_2)\}$ and $I_2'' = I_2 \backslash I_2'$, and let $k_2' = |I_2'|$ and $k_2'' = |I_2''|$. We assume w.l.o.g. either $2 \leq k_2' \leq k_2''$ or $k_2' \geq k_2 - 1$.

Procedure PairedDPC-H$(H_0 \oplus H_1, S, T, F)$
/* under the condition of $k_2 = k \geq 3$, $f_0 = f$, and $f \geq 1$ or $2 \leq k_2' \leq k_2''$ */

1. Pick up k_2 free edges (x_j, y_j), $j \in I_2$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$ such that $\hat{y_j}$ is not a terminal.

2. Find an f_0-fault *unpaired* many-to-many k_2-disjoint path cover joining $\{s_j | j \in I_2\}$ and $\{x_j | j \in I_2\}$ in H_0. Let s_j-path in the unpaired k_2-DPC join s_j and x_{i_j}, $j \in I_2$.

3. Case $f \geq 1$: Find k_2-DPC$[\{(y_{i_j}, t_j)|j \in I_2\}|H_1, \emptyset]$.

4. Case $f = 0$ and $2 \leq k_2' \leq k_2''$: Find k_2-DPC in H_1 as follows.
   (a) Find k_2'-DPC$[\{(y_{i_j}, t_j)|j \in I_2'\}|G_2, F']$, where $F' = \{y_{i_j} | j \in I_2''\}$.
   (b) Find k_2''-DPC$[\{(\hat{y_{i_j}}, t_j)|j \in I_2''\}|G_3, \emptyset]$.
   (c) Merge the k_2'-DPC and k_2''-DPC with edges $(y_{i_j}, \hat{y_{i_j}})$, $j \in I_2''$.

5. Merge the *unpaired* k_2-DPC in H_0 and k_2-DPC in H_1 with edges (x_{i_j}, y_{i_j}), $j \in I_2$.

Lemma 6 *When $k_2 = k \geq 3$, $f_0 = f$, and $f \geq 1$ or $2 \leq k_2' \leq k_2''$, Procedure PairedDPC-H constructs an f-fault k-DPC.*

Proof: The k_2 free edges in Step 1 exist since there are 2^{m-2} candidates and at most $f + 2k$ elements (f faults and $2k$ terminals) block the candidates. Of course, $2^{m-2} - (f + 2k) = 2^{m-2} - m \geq m \geq k_2$ for any $m \geq 6$. The existence of unpaired k_2-DPC is due to that $f_0 + k_2 = f + k = m - k \leq m - 3$. The k_2-DPC in Step 3 exists since $2k_2 \leq (f-1) + 2k_2 = f + 2k - 1 = m - 1$. The existence of k_2'-DPC in Step 4(a) is due to $|F'| + 2k_2' = k_2'' + 2k_2' = 2k - k_2'' \leq 2k - 2 \leq m - 2$. The k_2''-DPC in Step 4(b) also exists since $2k_2'' = 2k - 2k_2' \leq m - 2$. ■

Now, we have $k_2 = k \geq 3$, $f = 0$, and $k_2' \geq k_2 - 1$. The subcase $k_2' = k_2 - 1$ is considered first in the following. The vertex α in G_2, which is adjacent to the sink in G_3, plays an extraordinary role in the construction. Unpaired many-to-many disjoint path coverability of H_0, hamiltonicity of G_2, and paired many-to-many disjoint path coverability of G_3 are utilized.

Procedure PairedDPC-I$(H_0 \oplus H_1, S, T, F)$
/* under the condition of $k_2 = k \geq 3$, $f = 0$, and $k_2' = k_2 - 1$ */

1. Let t_{k_2} be the sink in G_3, and let $\alpha = \hat{t_{k_2}}$.

2. (a) Case α is a sink:
   Pick up k_2 free edges (x_j, y_j), $j \in I_2$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$.

   (b) Case α is a free vertex and $\bar{\alpha}$ is a source, say s_p:
   Pick up k_2-1 free edges (x_j, y_j), $j \in I_2 \backslash p$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$.

   (c) Case both α and $\bar{\alpha}$ are free vertices:
   Inclusive of $(\bar{\alpha}, \alpha)$, pick up k_2 free edges (x_j, y_j), $j \in I_2$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$.

3. (a) Case α is a sink or both α and $\bar{\alpha}$ are free vertices:
   Find an *unpaired* k_2-DPC joining $\{s_j | j \in I_2\}$ and $\{x_j | j \in I_2\}$ in H_0. Let s_j-path in the unpaired DPC join s_j and x_{i_j}, $j \in I_2$. We let $t_p = \alpha$ if α is a sink, and let $y_{i_p} = \alpha$ if both α and $\bar{\alpha}$ are free vertices.

   (b) Case α is a free vertex and $\bar{\alpha}$ is a source s_p:
   Regarding s_p as a *virtual* fault, find an *unpaired* k_2-1-DPC joining $\{s_j | j \in I_2 \backslash p\}$ and $\{x_j | j \in I_2 \backslash p\}$ in H_0. Let s_j-path in the unpaired DPC join s_j and x_{i_j}, $j \in I_2 \backslash p$. Let s_p-path be (s_p), and let $x_{i_p} = s_p$ and $y_{i_p} = \alpha$.

4. (a) Case $p \neq k_2$:
   Let $q \in I_2$ with $q \neq p, k_2$. Find $H[y_{i_q}, t_q | G_2, F']$, where $F' = \{y_{i_j}, t_j | j \in I_2 \backslash \{p, q, k_2\}\} \cup \{y_{i_{k_2}}\}$. Let the hamiltonian path be $(y_{i_q}, P_u, u, y_{i_p}, P', t_p, v, P_v, t_q)$. Find k_2-1-DPC$[\{(\hat{u}, \hat{v}), (\hat{y_{i_{k_2}}}, t_{k_2})\} \cup \{(\hat{y_{i_j}}, \hat{t_j}) | j \in I_2 \backslash \{p, q, k_2\}\} | G_3, \emptyset]$. Merge the hamiltonian path and k_2-1-DPC with edges $(u, \hat{u})$, $(v, \hat{v})$, $(y_{i_{k_2}}, \hat{y_{i_{k_2}}})$, and $(y_{i_j}, \hat{y_{i_j}})$, $(t_j, \hat{t_j})$, $j \in I_2 \backslash \{p, q, k_2\}$.

   (b) Case $p = k_2$:
   Let $q, r \in I_2$ with $q, r \neq k_2$. Find $H[y_{i_q}, t_q | G_2, F']$, where $F' = \{y_{i_j}, t_j | j \in I_2 \backslash \{p, q, r\}\} \cup \{y_{i_p}\}$. Let the hamiltonian path be $(y_{i_q}, P_u, u, y_{i_r}, P', t_r, v, P_v, t_q)$. Find k_2-2-DPC$[\{(\hat{u}, \hat{v})\} \cup \{(\hat{y_{i_j}}, \hat{t_j}) | j \in I_2 \backslash \{p, q, r\}\} | G_3, F'']$, where $F'' = \{t_{k_2}\}$. Merge the hamiltonian path and k_2-2-DPC with edges $(u, \hat{u})$, $(v, \hat{v})$, (y_{i_p}, t_{k_2}), and $(y_{i_j}, \hat{y_{i_j}})$, $(t_j, \hat{t_j})$, $j \in I_2 \backslash \{p, q, r\}$.

5. Merge the k_2 disjoint paths joining s_j and x_{i_j} in H_0 and k_2 disjoint paths joining y_{i_j} and t_j in H_1 with edges (x_{i_j}, y_{i_j}), $j \in I_2$.

Lemma 7 *When $k_2 = k \geq 3$, $f = 0$, and $k_2' = k_2 - 1$, Procedure PairedDPC-I constructs a k-DPC.*

Proof: The existence of free edges in Step 2 can be shown in a similar way to the proof of Lemma 6. Both the unpaired k_2-DPC in Step 3(a) and 1-fault unpaired k_2-1-DPC in Step 3(b) exist since $k_2 = k = m - k \leq m - 3$. When $p \neq k_2$ (Step 4(a)) the hamiltonian path between y_{i_q} and t_q in G_2 exists since $|F'| \leq 2(k_2 - 3) + 1 = 2k - 5 = m - 5$. By the construction, $t_{k_2} \notin \{\hat{u}, \hat{v}, \hat{y_{i_{k_2}}}\} \cup \{\hat{y_{i_j}}, \hat{t_j} | j \in I_2 \backslash \{p, q, k_2\}\}$. The k_2-1-DPC in G_3 exists since $2(k_2 - 1) = 2k - 2 = m - 2$. Similarly, when $p = k_2$ (Step 4(b)), we can see $t_{k_2} \notin \{\hat{u}, \hat{v}\} \cup \{\hat{y_{i_j}}, \hat{t_j} | j \in I_2 \backslash \{p, q, r\}\}$ and existence of the hamiltonian path in G_2 and 1-fault k_2-2-DPC in G_3. ■

When $k_2 = k \geq 3$, $f = 0$, and $k_2' = k_2$, the following Procedure PairedDPC-J constructs a k_2-DPC. The procedure is very similar to Procedure PairedDPC-I. Its correctness can be shown similar to Lemma 7, and omitted in this paper.

Procedure PairedDPC-J$(H_0 \oplus H_1, S, T, F)$
/* under the condition of $k_2 = k \geq 3$, $f = 0$, and $k_2' = k_2$ */

1. Let $\alpha = \hat{t_{k_2}}$. Here, α is a free vertex in G_3.

2. (a) Case $\bar{\alpha}$ is a free vertex:
   Let $(x_1, y_1) = (\bar{\alpha}, \alpha)$. Pick up k_2-1 free edges (x_j, y_j), $j \in I_2 \backslash 1$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$.

   (b) Case $\bar{\alpha}$ is a source, say s_p:
   Pick up k_2-1 free edges (x_j, y_j), $j \in I_2 \backslash p$, with $x_j \in V(H_0)$ and $y_j \in V(G_2)$.

3. (a) Case $\bar{\alpha}$ is a free vertex:
   Find an *unpaired* k_2-DPC joining $\{s_j | j \in I_2\}$ and $\{x_j | j \in I_2\}$ in H_0. Let s_j-path in the unpaired k_2-DPC join s_j and x_{i_j}, $j \in I_2$. We let $y_{i_p} = \alpha$.

   (b) Case $\bar{\alpha}$ is a source s_p:
   Regarding s_p as a *virtual* fault, find an *unpaired* k_2-1-DPC joining $\{s_j | j \in I_2 \backslash p\}$ and $\{x_j | j \in I_2 \backslash p\}$ in H_0. Let s_j-path in the unpaired DPC join s_j and x_{i_j}, $j \in I_2 \backslash p$. Let s_p-path be (s_p), and let $x_{i_p} = s_p$ and $y_{i_p} = \alpha$.

4. (a) Case $p \neq k_2$:
   Let $q \in I_2$ with $q \neq p, k_2$. Find $H[y_{i_q}, t_q | G_2, F']$, where $F' = \{y_{i_j}, t_j | j \in I_2 \backslash \{p, q, k_2\}\} \cup \{t_p\}$. Let the hamiltonian path be $(y_{i_q}, P_u, u, y_{i_{k_2}}, P', t_{k_2}, v, P_v, t_q)$. Find k_2-1-DPC$[\{(\hat{u}, \hat{v}), (y_{i_p}, \hat{t_p})\} \cup \{(\hat{y_{i_j}}, \hat{t_j}) | j \in I_2 \backslash \{p, q, k_2\}\} | G_3, \emptyset]$. Merge the hamiltonian path and k_2-1-DPC with edges $(u, \hat{u})$, $(v, \hat{v})$, $(t_p, \hat{t_p})$, and $(y_{i_j}, \hat{y_{i_j}})$, $(t_j, \hat{t_j})$, $j \in I_2 \backslash \{p, q, k_2\}$.

(b) Case $p = k_2$:
Let $q, r \in I_2$ with $q, r \neq k_2$. Find $H[y_{i_q}, t_q|G_2, F']$, where $F' = \{y_{i_j}, t_j | j \in I_2 \backslash \{p, q, r\}\} \cup \{t_{k_2}\}$. Let the hamiltonian path be $(y_{i_q}, P_u, u, y_{i_r}, P', t_r, v, P_v, t_q)$. Find $k_2 - 2$-DPC$[\{(\hat{u}, \hat{v})\} \cup \{(\hat{y_{i_j}}, \hat{t_j}) | j \in I_2 \backslash \{p, q, r\}\} | G_3, F'']$, where $F'' = \{y_{i_p}\}$. Merge the hamiltonian path and $k_2 - 2$-DPC with edges $(u, \hat{u})$, $(v, \hat{v})$, (y_{i_p}, t_{k_2}), and $(y_{i_j}, \hat{y_{i_j}})$, $(t_j, \hat{t_j})$, $j \in I_2 \backslash \{p, q, r\}$.

5. Merge the k_2 disjoint paths joining s_j and x_{i_j} in H_0 and k_2 disjoint paths joining y_{i_j} and t_j in H_1 with edges (x_{i_j}, y_{i_j}), $j \in I_2$.

2.2. Restricted HL-graphs

In this subsection, we are to construct an f-fault paired many-to-many k-DPC in an m-dimensional restricted HL-graph for any f and $k \geq 2$ with $f + 2k \leq m$ by employing Theorem 1. For our purpose, we need unpaired many-to-many disjoint path coverability of restricted HL-graphs with faulty elements. It was considered in [11] as follows.

Lemma 8 *[11] Every m-dimensional restricted HL-graph, $m \geq 3$, is f-fault unpaired many-to-many k-disjoint path coverable for any f and $k \geq 1$ with $f + k \leq m - 2$.*

The existence of a paired many-to-many 2-DPC in 4-dimensional restricted HL-graphs is checked by a computer program for each $G(8,4) \oplus G(8,4)$ in RHL_4, sources s_1 and s_2, and sinks t_1 and t_2. Thus, we have the lemma.

Lemma 9 *Every 4-dimensional restricted HL-graph is 0-fault paired many-to-many 2-disjoint path coverable.*

Now, we are ready to state paired many-to-many disjoint path coverability of restricted HL-graphs.

Theorem 2 *Every m-dimensional restricted HL-graph, $m \geq 3$, is f-fault paired many-to-many k-disjoint path coverable for any f and $k \geq 2$ with $f + 2k \leq m$.*

Proof: The proof is by induction on m. For $m = 3$, the theorem is vacantly true since $f + 2k \geq 4 > m$. For $m = 4$, the theorem holds true by Lemma 9. Let $m \geq 5$. Theorem 1 and Lemma 8 lead to the theorem. ■

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