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Research Article

On the Continuity of Hausdorff Dimension of Julia Sets Concerning Potts Models

Gang Liu[1] and Junyang Gao[2]

[1] *Department of Mathematics and Computational Science, Hengyang Normal University, Hengyang 421002, China*
[2] *School of Science, China University of Mining and Technology (Beijing), Beijing 100083, China*

Correspondence should be addressed to Gang Liu; liugangmath@sina.cn

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Considering the Julia sets of a family of rational maps concerning two-dimensional diamond hierarchical Potts models in statistical mechanics, we show the continuity of their Hausdorff dimension.

1. Introduction

The continuity of Hausdorff dimension of Julia sets is an important and interesting problem for rational maps with degree $d \geq 2$. In general, this problem adheres to the continuity of Julia sets which is response to the stability of system. It is well known that both the Julia set $J(R)$ and its Hausdorff dimension of a rational map R vary continuously in the parameter space Rat_d if R is hyperbolic [1, 2]. However, as we know, there are no direct relationship between them when R is not hyperbolic though there are many works devoted to the two problems [1, 3, 4].

In this paper, we discuss a family of rational maps $T_{n\lambda} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ for $n = 3$; here

$$T_{n\lambda}(z) = \left(\frac{z^2 + \lambda - 1}{2z + \lambda - 2}\right)^n \quad (1)$$

with two parameters $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. $T_{n\lambda}$ is a renormalization transformation of λ-state Potts models on the two-dimensional diamond-like hierarchical lattice with bifurcation number n in statistical mechanics [5]. In turn, the zeros of the partition function for the model with bifurcation number n condense to the Julia sets of $T_{n\lambda}$ [6]. It has been shown that there exists some relationship between the critical temperatures, the critical amplitudes, and the structures of the Julia sets [7]. Therefore, much interest has been devoted to these physical models, since they exhibit a connection between statistical mechanics and complex dynamics [6, 8–15].

We have known that, for any given $n \in \mathbb{N}$, the Julia set $J(T_{n\lambda})$ of $T_{n\lambda}$ is continuous in the Hausdorff distance for any $\lambda \in \mathbb{R}$ except two points [11]. Whether the Hausdorff dimension of $J(T_{n\lambda})$ is also continuous for any $\lambda \in R$ except two points? From the proof of the main result in [10, 11], for even integer n, it is easy to see that $T_{n\lambda}$ is hyperbolic in the real axis $\mathbb{R}$ except countable points. Except at most three points from those countable points, $T_{n\lambda}$ is subhyperbolic but not hyperbolic; though the dynamical property of $T_{n\lambda}$ is simple, it is difficult to compute all the iteration number of critical points which are eventually equal to the repelling fixed points in the iteration of $T_{n\lambda}$. Therefore, we cannot give a quantitative analysis for the corresponding critical points when the parameter is close to the above points. For any odd integer $n \geq 5$, there exist at least two real numbers $\lambda_1, \lambda \in (1, 2)$ such that $T_{n\lambda_1}$ and $T_{n\lambda_2}$ are Feigenbaum-like maps [15]. As we have seen, for the simplest Feigenbaum quadratic polynomials, the continuity of Hausdorff dimension of its Julia sets is unknown. Based on the above reason, we just consider the case for $n = 3$.

We define the following constants:

$$\alpha = 2 + \min_{0 \leq t \leq 1} \frac{t^6 - 2t^4 + 1}{t - 1},$$
$$\beta = 2 + \max_{-2 \leq t \leq 0} \frac{t^6 - 2t^4 + 1}{t - 1}. \quad (2)$$

We have the following result.

Theorem 1. *$T_{3\lambda}$ is defined in* (1) *and $\lambda \in \mathbb{R}$. Let $HD(J(T_{3\lambda}))$ be the Hausdorff dimension of $J(T_{3\lambda})$. Then $HD(J(T_{3\lambda}))$ is continuous at $\lambda \in \mathbb{R} \setminus \{\alpha, 0, \beta\}$.*

2. Some Notations and Preliminary Results

Let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with degree $\deg(R) \geq 2$. We denote by R^n the nth iteration of R. A point z is called *critical point* if $R'(z) = 0$. A point z is called *periodic point* if $R^k(z) = z$ for some $k \geq 1$; the minimal of such k is called the *period* of z. For a periodic point z_0, denote the *multiplier* of z_0 by $(R^k)'(z_0)$; the periodic point z_0 is either *attracting*, *indifferent*, or *repelling* according to $|(R^k)'(z_0)| < 1$, $|(R^k)'(z_0)| = 1$ or $|(R^k)'(z_0)| > 1$. In the indifferent case, we say z_0 is *parabolic* if $(R^k)'(z_0)$ is a root of unity.

The *Julia set*, denoted by $J(R)$, is the closure of repelling periodic points. Its complement is called *Fatou set*, denoted by $F(R)$; a connected component of $F(R)$ is called a *Fatou component*. A rational map R is called *hyperbolic*, if $P(R) \cap J(R) = \emptyset$, and *geometrically finite*, if the set $P(R) \cap J(R)$ is finite; here the *postcritical set* $P(R)$ of R is the closure of the forward orbits of critical points. A geometrically finite map is *subhyperbolic* (resp. *parabolic*) if it has no (resp. some) parabolic periodic points. It is called *critically nonrecurrent* if $c \notin \omega(c)$ for each critical point $c \in J(R)$, where $\omega(c)$ is the ω-limit set of c. A critically nonrecurrent map is *semihyperbolic* if it has no parabolic periodic points. For the classical results in complex dynamics, see [12, 16, 17].

Definition 2. A domain $D \subset \mathbb{C}$ is called a John domain if there exists $c > 0$ such that, for any $z_0 \in D$, there is an arc γ joining z_0 to some fixed reference point $w_0 \in D$ satisfying

$$\operatorname{dist}(z, \partial D) \geq c\left|z - z_0\right|, \quad z \in \gamma. \tag{3}$$

If $\infty \in \partial D$, we use the spherical metric to measure the distance.

Lemma 3 (see [18]). *Suppose R is semihyperbolic rational map, then every Fatou component of $F(R)$ is a John domain.*

Definition 4. A probability measure μ on the Julia set $J(R)$ is called t-conformal measure for a rational map $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ if $\mu(R(A)) = \int_A |R'|^t d\mu$ for every Borel set $A \subset J(R)$ such that $R|_A$ is injective; t is called the conformal exponent about μ.

Lemma 5 (see [19]). *Let h denote the Hausdorff dimension of $J(R)$ of a subhyperbolic rational map R, then there exists a unique invariant probability measure μ equivalent to the h-conformal measure; moreover, the normalized h-dimension Hausdorff measure is the only h-dimension conformal measure for R.*

Lemma 6 (see [1]). *Any normalized invariant conformal probability measure μ supported on the Julia set of a geometrically finite rational map R is either the conformal measure of Hausdorff dimension of $J(R)$, or an atomic measure supported on the inverse orbits of parabolic points and critical points.*

For simplicity, $T_\lambda = T_{3\lambda}$, and $A \sim B$ ($A, B \in \mathbb{R}$) means that $C^{-1}B < A < CB$ for some implicit constant C. By (1), for $\lambda \neq 0$, we have

$$T_\lambda'(z) = \frac{6(z-1)(z+\lambda-1)\left(z^2+\lambda-1\right)^2}{(2z+\lambda-2)^4}. \tag{4}$$

So, T_λ has ten critical points: 1, $1-\lambda$, $\pm\sqrt{\lambda-1}i$ (with the multiplicity 2), $(1-\lambda)/2$ (with the multiplicity 3), ∞. It is easy to see that $z = 1$ and ∞ are two superattracting fixed points.

Lemma 7 (see [6]). *$\alpha \in (-2, 0)$, $\beta \in (2, 3)$, and*

(1) *T_λ has only two real fixed points q, 1 ($q < -1$) for $\lambda \in (-\infty, \alpha)$;*

(2) *T_λ has only two real fixed points 1, q ($q > 1$) for $\lambda \in (\beta, +\infty)$;*

(3) *T_λ has only three real fixed points q_1, q_2, 1 ($q_1 < -1$, $0 < q_2 < 1$) for $\lambda = \alpha$ or $\lambda = 0$;*

(4) *T_λ has only three real fixed points q_1, 1, q_2 ($q_1 < -1$, $q_2 > 1$) for $\lambda = \beta$;*

(5) *T_λ has only four real fixed points q_1, 0, 1, q_2 ($q_1 < -1$, $q_2 > 1$) for $\lambda = 1$;*

(6) *T_λ has only four real fixed points q_1, q_2, 1, q_3 ($q_1 < q_2 < 0$, $q_3 > 1$) for $\lambda \in (1, \beta)$;*

(7) *T_λ has only four real fixed points q_1, q_2, 1, q_3 ($q_1 < -1$, $q_2 \in (0, 1)$, $q_3 > 1$) for $\lambda \in (0, 1)$;*

(8) *T_λ has only four real fixed points q_1, 0, 1, q_3 ($q_1 < -1$, q_2, $q_3 \in (0, 1)$) for $\lambda \in (\alpha, 0)$.*

Lemma 8 (see [10]). *T_λ is hyperbolic for $\lambda \in \mathbb{R} \setminus \{\alpha, \beta, 3 \pm \sqrt{2}\}$, $T_{3\pm\sqrt{2}}$ is subhyperbolic, and T_α and T_β are parabolic.*

3. The Proof of Theorem 1

In the following, we denote $T_\lambda^2(\pm\sqrt{\lambda-1}i) = T_\lambda(0) = v_\lambda$, $q_\lambda = q_1$ is the repelling fixed point for λ close but not equal to $3-\sqrt{2}$, and $q_\lambda = q$ is also the repelling fixed point for λ close but not equal to $3+\sqrt{2}$. It is easy to see that $v_\lambda - q_\lambda \to v_{\lambda_0} - q_{\lambda_0} = 0$ when $\lambda \to \lambda_0$, $\lambda_0 \in \{3-\sqrt{2}, 3+\sqrt{2}\}$.

Proposition 9. *Consider*

$$q_\lambda = \left(\frac{\lambda_0 - 1}{\lambda_0 - 2}\right)^3 + k\left(\lambda - \lambda_0\right) + O\left(\left(\lambda - \lambda_0\right)^2\right) \tag{5}$$

as $\lambda \to \lambda_0$; here $k = (78 + 36\sqrt{2})/97$ for $\lambda_0 = 3 - \sqrt{2}$ and $k = (78 - 36\sqrt{2})/97$ for $\lambda_0 = 3 + \sqrt{2}$.

Proof. Considering the real fixed points of T_λ and taking $t = \sqrt[3]{x}$, from the equation $T_\lambda(x) = x$, it follows that

$$\lambda = 2 + \frac{t^6 - 2t^4 + 1}{t - 1}. \tag{6}$$

When λ is close but not equal to λ_0, denote that

$$q_\lambda = \left(\frac{\lambda_0 - 1}{\lambda_0 - 2}\right)^3 + k(\lambda - \lambda_0) + O\left((\lambda - \lambda_0)^2\right). \quad (7)$$

(1) If $\lambda_0 = 3 - \sqrt{2}$, $q_{\lambda_0} = -2\sqrt{2}$. By the continuity, $q_\lambda < 0$. By (6) and $\lambda \in \mathbb{R}$, it satisfies

$$(\lambda - \lambda_0 + \lambda_0 - 2)\left(\sqrt[3]{q_\lambda} - 1\right) = q_\lambda^2 - 2q_\lambda \sqrt[3]{q_\lambda} + 1. \quad (8)$$

Substituting (8) with (7), by a calculation, we can deduce that

$$\begin{aligned}(\lambda - \lambda_0 + 1 - \sqrt{2})&\left(-\sqrt{2} - 1 + \frac{k(\lambda - \lambda_0)}{6}\right) \\ &+ O\left((\lambda - \lambda_0)^2\right) \\ &= \left(-2\sqrt{2} + k(\lambda - \lambda_0)\right)^2 - 2\left(-2\sqrt{2} + k(\lambda - \lambda_0)\right) \\ &\times \left(-\sqrt{2} + \frac{k(\lambda - \lambda_0)}{6}\right) + 1 + O\left((\lambda - \lambda_0)^2\right),\end{aligned} \quad (9)$$

then $k = (78 + 36\sqrt{2})/97$.

(2) If $\lambda_0 = 3 + \sqrt{2}$, $q_{\lambda_0} = 2\sqrt{2}$. By the similar method used in Case (1), we can deduce that $k = (78 - 36\sqrt{2})/97$. □

Proposition 10. *$HD(J(T_\lambda))$ is continuous for $\lambda \in \{3 + \sqrt{2}, 3 - \sqrt{2}\}$.*

Proof. By Lemma 8, T_λ is hyperbolic for λ close but not equal to λ_0. Then there exists a unique conformal probability measure μ_λ for T_λ supported in $J(T_\lambda)$; μ_λ has exponent $d_\lambda = \mathrm{HD}(J(T_\lambda))$. This means that, for every measurable set $V \subset J(T_\lambda)$ where T_λ is injective, $\mu_\lambda(T_\lambda(V)) = \int_V |(T_\lambda)'|^{d_\lambda} d\mu_\lambda$. Furthermore the measure of a point is zero for μ_λ; that is, μ_λ is not atomic.

Since T_{λ_0} is subhyperbolic, by Lemma 5, there exists a unique conformal probability measure for T_{λ_0} supported in $J(T_{\lambda_0})$. By cases (6) and (10) in the proof of Theorem 1 of the paper [10], we know that $1 - \lambda_0 \in F(T_{\lambda_0})$ for $\lambda_0 = 3 \pm \sqrt{2}$. By Lemma 6, the unique conformal probability measure has exponent $d_{\lambda_0} = \mathrm{HD}(J(T_{\lambda_0}))$ or is atomic, supported in $\{T_{\lambda_0}^{-k}(\pm\sqrt{\lambda_0 - 1}i)\}_{k\geq 0}$. By a similar discussion used in [4], in order to prove that

$$\lim_{\lambda \to \lambda_0} \mathrm{HD}(J(T_\lambda)) = \mathrm{HD}\left(J\left(T_{\lambda_0}\right)\right), \quad (10)$$

it is enough to prove that

$$\lim_{r \to 0} \lim_{\lambda \to \lambda_0} \mu_\lambda\left(B_r\left(\pm\sqrt{\lambda_0 - 1}i\right)\right) = 0; \quad (11)$$

here $B_r(x) = \{z \mid |z - x| < r\}$. Noting that $J(T_\lambda)$ and $F(T_\lambda)$ ($\lambda \in \mathbb{R}$) are symmetry with the real axis, it suffices to prove that

$$\lim_{r \to 0} \lim_{\lambda \to \lambda_0} \mu_\lambda\left(B_r\left(\sqrt{\lambda_0 - 1}i\right)\right) = 0. \quad (12)$$

In fact, if μ_{λ_0} is any weak limit of $\{\mu_\lambda\}$, then μ_{λ_0} is a conformal probability measure for T_{λ_0} supported in $J(T_{\lambda_0})$. The previous limit implies that the measure μ_{λ_0} is not atomic at $\sqrt{\lambda_0 - 1}i$, so, it has exponent $d_{\lambda_0} = \mathrm{HD}(J(T_{\lambda_0}))$. Noting that $\mu_{\lambda_0}(T_{\lambda_0}(V)) = \int_V |(T_{\lambda_0})'|^{d_{\lambda_0}} d\mu_{\lambda_0}$ and $\mu_\lambda(T_\lambda(V)) \to \mu_{\lambda_0}(T_{\lambda_0}(V))$ as $\lambda \to \lambda_0$ for any measurable set V, it follows that $d_\lambda \to d_{\lambda_0}$. Next we set that λ is close but not equal to λ_0.

Since q_{λ_0} and q_λ are the real repelling fixed points of T_{λ_0} and T_λ, respectively, by the continuity, $q_\lambda \to q_{\lambda_0}$ as $\lambda \to \lambda_0$. By the Koenig's Theorem [16], there exist a neighborhood U_0 of q_{λ_0} with diameter not more than a $\delta > 0$ and a conformal map $\phi_{\lambda_0} : U_0 \to B_{\delta_1}(0)$ for some $\delta_1 > 0$ such that ϕ_{λ_0} conjugates T_{λ_0} on U_0 to the scaling function $z \to T'_{\lambda_0}(q_{\lambda_0})z$ on $B_{\delta_1}(0)$. Similarly, there exists a conformal map $\phi_\lambda : U_0^\lambda \to B_{\delta'_1}(0)$ which conjugates T_λ to the scaling function $z \to T'_\lambda(q_\lambda)z$. It is easy to construct a quasiconformal map $\phi : A_{\delta_1} = \{z \mid \delta_1 < |z| < \delta_2\} \to A_{\delta_2} = \{z \mid \delta'_1 < |z| < \delta'_2\}$; here $\delta_2 = |T'_{\lambda_0}(q_{\lambda_0})|\delta_1$ and $\delta'_2 = |T'_\lambda(q_\lambda)|\delta'_1$, such that $\phi(T'_{\lambda_0}(q_{\lambda_0})z) = T'_\lambda(q_\lambda)\phi(z)$ for $|z| = \delta_1$. Pull back by the scaling function; we can extend ϕ to a quasiconformal map $\phi : B_{\delta_2}(0) \to B_{\delta'_2}(0)$ which conjugates $z \to T'_{\lambda_0}(q_{\lambda_0})z$ to $z \to T'_\lambda(q_\lambda)z$. For every $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, define

$$j_\lambda = \phi_\lambda^{-1} \circ \phi \circ \phi_{\lambda_0} : U_0 \to U_0^\lambda. \quad (13)$$

Hence, j_λ is a conjugation between T_{λ_0} on U_0 and T_λ on U_0^λ. Let $z(\lambda) = j_\lambda(q_{\lambda_0})$, by definition, $z(\lambda_0) = q_{\lambda_0}$ and $z(\lambda) = q_\lambda$.

Reducing $\epsilon > 0$ if necessary, there are constants $C_0 > 0$ and $\theta_0 \in (0, 1)$ such that, for all $m \geq 1$, all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, and all q_λ,

$$\left|\left(T_\lambda^m\right)'(q_\lambda)\right|^{-1} \leq C_0\theta_0^m. \quad (14)$$

On the other hand, for every $k \geq 1$, let U_k be the preimage of $B_\delta(q_{\lambda_0})$ under $T_{\lambda_0}^k$ containing q_{λ_0}, and let V_k be the pullback of U_k by $T_{\lambda_0}^2$ containing $\sqrt{\lambda_0 - 1}i$. Moreover, we denote $j_\lambda(U_k)$ by U_k^λ containing q_{λ_0} and let V_k^λ be the pullback of U_k^λ by T_λ^2 containing $\sqrt{\lambda_0 - 1}i$. By Koebe Distortion Theorem, reducing $\delta > 0$ if necessary, there is an implicit constant $K > 1$ such that, for all $\omega \in U_k^\lambda \subset B_\delta(q_{\lambda_0})$ and all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$,

$$\frac{1}{K} \leq \left|\frac{\left(T_\lambda^k\right)'(q_\lambda)}{\left(T_\lambda^k\right)'(\omega)}\right| \leq K. \quad (15)$$

So, $|(T_\lambda^m)'(\omega)|^{-1} \leq KC_0\theta_0^m$; that is, the distortion of T_λ^k in U_k is bounded by K; denote this property as the uniform Bounded Distortion Property.

We also denote the largest $k = p$ such that $B_r(\sqrt{\lambda_0 - 1}i) \subset V_k^\lambda$ for $r > 0$ small enough and all λ sufficiently close to λ_0. It follows that, for $r \to 0$, $p = p(r) \to \infty$. The following suffices to prove that

$$\lim_{p \to \infty} \lim_{\lambda \to \lambda_0} \mu_\lambda\left(V_p^\lambda\right) = 0. \quad (16)$$

Step 1. Let D be a disc containing $\sqrt{\lambda_0 - 1}i$, small enough such that $\deg T_\lambda|_D = 3$, since $\sqrt{\lambda_0 - 1}i$ is a critical point with the multiplicity 2. Reducing $\epsilon > 0$ if necessary, such that $U_1^\lambda \subset T_\lambda^2(D)$. T_λ is hyperbolic when λ is close to λ_0, then the probability measure μ_λ is not atomic; we have

$$\mu_\lambda\left(V_p^\lambda\right) = \sum_{m\geq p} \mu_\lambda\left(V_m^\lambda - V_{m+1}^\lambda\right) \tag{17}$$

for all $p \geq 1$. By the construction of the t-conformal measure μ of a rational map R ([2]), we know that $\mu(A_{-1}) = \int_{A_{-1}} |(R^{-1})'|^t d\mu$ for every Borel set $A \subset J(R)$ such that $R : A_{-1} \to A$ is conformal. For $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, we have

$$\begin{aligned}\mu_\lambda\left(V_m^\lambda - V_{m+1}^\lambda\right) &\leq 3\mu_\lambda\left(U_m^\lambda - U_{m+1}^\lambda\right) \\ &\quad\times \inf_{z\in(V_m^\lambda - V_{m+1}^\lambda)\cap J(T_\lambda)} \left|\left(T_\lambda^2\right)'(z)\right|^{-d_\lambda}.\end{aligned} \tag{18}$$

By the uniform Bounded Distortion Property, note that $z(\lambda) = q_\lambda$ and μ_λ is a probability measure, then

$$\mu_\lambda\left(U_m^\lambda - U_{m+1}^\lambda\right) \leq K^{d_\lambda}\left|\left(T_\lambda^m\right)'(z(\lambda))\right|^{-d_\lambda}. \tag{19}$$

Furthermore, we claim that there exists $C_1 > 0$ such that, for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ and $z \in V_1^\lambda$,

$$\left|\left(T_\lambda^2\right)'(z)\right| \geq C_1\left|T_\lambda^2(z) - v_\lambda\right|^{2/3}. \tag{20}$$

In fact, (20) is obvious for $z = \sqrt{\lambda - 1}i$, since $T_\lambda'(\sqrt{\lambda - 1}i) = 0$ and $v_\lambda = T_\lambda^2(\sqrt{\lambda - 1}i)$. Suppose $z \neq \sqrt{\lambda - 1}i$; by the uniform Bounded Distortion Property and Koebe Distortion Theorem, it follows that

$$\begin{gathered}\operatorname{dist}\left(v_\lambda, \partial U_1^\lambda\right) \sim \operatorname{diam}\left(U_1^\lambda\right), \\ \operatorname{dist}\left(\sqrt{\lambda - 1}i, \partial V_1^\lambda\right) \sim \operatorname{diam}\left(V_1^\lambda\right) \sim \left(\operatorname{diam}\left(T_\lambda\left(V_1^\lambda\right)\right)\right)^{1/3} \\ \sim \left(\operatorname{diam}\left(U_1^\lambda\right)\right)^{1/3},\end{gathered} \tag{21}$$

since $\deg T_\lambda|_{V_1^\lambda} = 3$ and $\deg T_\lambda|_{T_\lambda(V_1^\lambda)} = 1$. Then

$$\left|\left(T_\lambda^2\right)'(z)\right| \sim \left(\operatorname{diam}\left(U_1^\lambda\right)\right)^{2/3} \sim \left(\operatorname{dist}\left(v_\lambda, \partial U_1^\lambda\right)\right)^{2/3}, \tag{22}$$

so, we get (20).

Step 2. Let $k = k(\lambda)$ be the largest integer such that $v_\lambda \in U_k^\lambda$ and let $m \geq 1$. Then there are three cases.

Case 1. ($k-1 \leq m \leq k+1$). By the uniform Bounded Distortion Property, it follows that $|(T_\lambda^m)'(z(\lambda))|^{-1} \sim |z(\lambda) - v_\lambda|$, since $k \to \infty$ as $\lambda \to \lambda_0$. By Proposition 9, it follows that

$$\begin{aligned}\left|z(\lambda) - v_\lambda\right| &\sim \left|\left(\frac{\lambda - 1}{\lambda - 2}\right)^3 - \left(\frac{\lambda_0 - 1}{\lambda_0 - 2}\right)^3\right| \\ &\sim \left|\frac{\lambda - 1}{\lambda - 2} - \frac{\lambda_0 - 1}{\lambda_0 - 2}\right| \sim \left|\lambda - \lambda_0\right|,\end{aligned} \tag{23}$$

since $\lambda_0 \neq 1 + (k/2)$. So, we get $|(T_\lambda^m)'(z(\lambda))|^{-1} \sim |\lambda - \lambda_0|$ with constant independent of λ; hence, $|(T_\lambda^m)'(z(\lambda))|^{-1} \leq C_2|\lambda - \lambda_0|$ for some constant $C_2 > 0$ independent of λ, but on the other hand,

$$\operatorname{dist}\left(v_\lambda, \left(U_m^\lambda - U_{m+1}^\lambda\right) \cap J\left(T_\lambda\right)\right) \geq \operatorname{dist}\left(v_\lambda, J\left(T_\lambda\right)\right). \tag{24}$$

Then for all $z \in (V_m^\lambda - V_{m+1}^\lambda) \cap J(T_\lambda)$, by (20), it follows that

$$\left|\left(T_\lambda^2\right)'(z)\right| > C_1 \operatorname{dist}\left(v_\lambda, J\left(T_\lambda\right)\right)^{2/3}, \tag{25}$$

so,

$$\mu_\lambda\left(V_m^\lambda - V_{m+1}^\lambda\right) \leq C_3\left|\lambda - \lambda_0\right|^{d_\lambda} \operatorname{dist}\left(v_\lambda, J\left(T_\lambda\right)\right)^{-(2/3)d_\lambda}, \tag{26}$$

where $C_3 = 3(KC_2(C_1)^{-1})^{d_\lambda}$.

Case 2. ($m < k - 1$). Noting that

$$\operatorname{dist}\left(v_\lambda, \left(U_m^\lambda - U_{m+1}^\lambda\right)\right) \geq \operatorname{dist}\left(\partial U_{m+1}^\lambda, U_{m+2}^\lambda\right), \tag{27}$$

then by the uniform Bounded Distortion Property, we have

$$\operatorname{dist}\left(v_\lambda, \left(U_m^\lambda - U_{m+1}^\lambda\right)\right) > C_4\left|\left(T_\lambda^m\right)'(z(\lambda))\right|^{-1}. \tag{28}$$

As in Case 1, we have

$$\begin{aligned}\left|\left(T_\lambda^2\right)'(z)\right| &> C_1\left(\operatorname{dist}\left(v_\lambda, U_m^\lambda - U_{m+1}^\lambda\right)\right)^{2/3} \\ &\geq C_1C_4^{2/3}\left|\left(T_\lambda^m\right)'(z(\lambda))\right|^{-2/3}.\end{aligned} \tag{29}$$

It follows that

$$\begin{aligned}\mu_\lambda\left(V_m^\lambda - V_{m+1}^\lambda\right) &\leq 3K^{d_\lambda}\left|\left(T_\lambda^m\right)'(z(\lambda))\right|^{-d_\lambda} \\ &\quad\times\left(C_1C_4^{2/3}\right)^{-d_\lambda}\left|\left(T_\lambda^m\right)'(z(\lambda))\right|^{(2/3)d_\lambda} \\ &= C_5\left|\left(T_\lambda^m\right)'(z(\lambda))\right|^{-d_\lambda/3}.\end{aligned} \tag{30}$$

By (14), $\mu_\lambda(V_m^\lambda - V_{m+1}^\lambda) \leq C_5\theta_0^{md_\lambda/3}$, where $C_5 = 3K^{d_\lambda}(C_1C_4^{2/3})^{-d_\lambda}C_0^{d_\lambda/3}$.

Case 3. ($m > k + 1$). We have

$$\operatorname{dist}\left(v_\lambda, \left(U_m^\lambda - U_{m+1}^\lambda\right)\right) \geq \operatorname{dist}\left(\partial U_{m-1}^\lambda, U_m^\lambda\right). \tag{31}$$

By a similar discussion as used in Case 2,

$$\operatorname{dist}\left(v_\lambda, \left(U_m^\lambda - U_{m+1}^\lambda\right)\right) \geq C_4\left|\left(T_\lambda^m\right)'(z(\lambda))\right|^{-1}, \tag{32}$$

then $\mu_\lambda(V_m^\lambda - V_{m+1}^\lambda) \leq C_5\theta_0^{md_\lambda/3}$.

Step 3. Since T_λ is hyperbolic when λ is close but not equal to λ_0, by Lemma 3, every Fatou component of $F(T_\lambda)$ is a John domain. Noting that $F(T_\lambda)$ is symmetry with the real axis $\mathbb{R}$ and $q_\lambda \in J(T_\lambda)$, then the angle at q_λ of two curves γ_1 and γ_2 of $\partial A_\lambda(\infty)$ (or $\partial A_\lambda(1)$) is positive. Since $v_\lambda \to q_\lambda$ as $\lambda \to \lambda_0$, it follows that $\operatorname{dist}(v_\lambda, J(T_\lambda)) \sim \operatorname{dist}(v_\lambda, q_\lambda)$ as $\lambda \to \lambda_0$. On the other hand, by Proposition 9, it follows that $\operatorname{dist}(v_\lambda, q_\lambda) \sim |z(\lambda) - v_\lambda| \sim |\lambda - \lambda_0|$. Thus, $\operatorname{dist}(v_\lambda, J(T_\lambda)) \sim |\lambda - \lambda_0|$ as $\lambda \to \lambda_0$.

By Steps 1 and 2, for $p \geq 1$, we have

$$\mu_\lambda\left(V_p^\lambda\right) \leq 3C_3|\lambda - \lambda_0|^{d_\lambda} \operatorname{dist}\left(v_\lambda, J\left(T_\lambda\right)\right)^{-(2/3)d_\lambda} + C_5 \sum_{m\geq p, m\neq k-1,k,k+1} \theta_0^{md_\lambda/3}. \tag{33}$$

Since

$$\sum_{m\geq p,} \theta_0^{md_\lambda/3} = \frac{\left(\theta_0^{d_\lambda/3}\right)^p}{1-\theta_0^{d_\lambda/3}}, \tag{34}$$

we conclude that

$$\lim_{p\to\infty}\lim_{\lambda\to\lambda_0} \mu_\lambda\left(V_p^\lambda\right) = 0. \tag{35}$$

So, $\operatorname{HD}(J(T_\lambda))$ is continuous at $\lambda \in \{3-\sqrt{2}, 3+\sqrt{2}\}$. □

The Proof of Theorem 1. Since the Hausdorff dimension $\operatorname{HD}(J(T_\lambda))$ varies continuously in Rat_d if T_λ is hyperbolic [1, Theorem 11.1] and $\deg(T_0) \neq \deg(T_\lambda) = 6$ for $\lambda \neq 0$, by Lemma 8 and Proposition 10, $\operatorname{HD}(J(T_\lambda))$ is continuous for $\lambda \in \mathbb{R} \setminus \{\alpha, 0, \beta\}$.

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