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ON THE CONTINUITY OF STATIONARY GAUSSIAN PROCESSES

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1. Introduction

Let us consider a stochastically continuous, separable and measurable stationary Gaussian process[1] $X = \{X(t), \ -\infty < t < \infty\}$ with mean zero and with the covariance function $\rho(t) = EX(t+s)X(s)$. The conditions for continuity of paths have been studied by many authors from various viewpoints. For example, Dudley [3] studied from the viewpoint of ε-entropy and Kahane [5] showed the necessary and sufficient condition in some special case, using the rather neat method of Fourier series.

In this note we shall discuss the continuity of paths of X, making use of the idea presented by Kahane. Our results are following: We express the covariance function ρ in the form

$$\rho(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dF(\lambda)$$

with a finite measure dF, symmetric with respect to origin.

Put $\qquad s_n = F(2^n, 2^{n+1}], \qquad n = 0, 1, 2, \cdots.$

THEOREM 1. *If* $E \sup_{t\in[0,1]} |X(t)| < \infty$, *then* $\sum_{n=0}^{\infty} \sqrt{s_n} < \infty$.

THEOREM 2. *Suppose that we can choose a decreasing sequence* $\{M_n\}$ *so that* $M_n \geq s_n$ *and* $\sum_{n=0}^{\infty} \sqrt{M_n} < \infty$. *Then* $E \sup_{t\in[0,1]} |X(t)| < \infty$.

THEOREM 3. *Suppose that* ρ *is convex on a small interval* $[0, \delta]$. *Then* $\sum_{n=0}^{\infty} \sqrt{s_n} < \infty$, *if* X *has continuous paths.*

By virtue of Theorem 2, we can easily see

COROLLARY. *Suppose that* ρ *is convex on a small interval* $[0, \delta]$ *and* s_n *is*

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[1] We mean a real valued process.

decreasing. Then X *has continuous paths if, and only if,* $E \sup_{t\in[0,1]} |X(t)| < \infty$, *which is equivalent to* $\sum_{n=0}^{\infty} \sqrt{s_n} < \infty$.

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2. Lemmas

Let $\{T_j, j = 1, 2, \cdots\}$ be a sequence of increasing positive numbers such that $\sum_{j=1}^{\infty} \frac{1}{T_j} < \infty$. According to [5, p. 69], we shall define following functions,

$$\chi(x) = \max(1 - |x|, 0), \qquad -\infty < x < \infty,$$

$$\theta_r(\lambda) = \prod_{j=r}^{\infty} \chi\Big(\frac{\lambda}{T_j}\Big), \qquad -\infty < \lambda < \infty,$$

$$K_r(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\lambda} \chi\Big(\frac{\lambda}{T_r}\Big) d\lambda$$

$$l_r(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\lambda} \theta_r(\lambda) d\lambda = \frac{1}{\sqrt{2\pi}} \int_{-T_r}^{T_r} e^{it\lambda} \theta_r(\lambda) d\lambda$$

$$l_r^*(t) = \frac{1}{\sqrt{2\pi}} \int_{-T_{r-1}}^{T_{r-1}} e^{it\lambda} \theta_r(\lambda) d\lambda.$$

As to these functions, we can easily see that θ_r is symmetric, non-negative and continuous, and l_r and l_r^* continuous. Since

$$K_r(t) = \frac{\sqrt{2}}{t^2 T_r \sqrt{\pi}} (1 - \cos T_r t) \geq 0, \tag{1}$$

l_r is non-negative as the convolution of K_n, $n \geq r$.
The following Lemma 1 is clear.

LEMMA 1.

$$l_r(t) = (l_{r+1}^* * K_r)(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} l_{r+1}^*(t-s) K_r(s) ds$$

$$l_r(t) = (l_{r+1} * K_r)(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} l_{r+1}(t-s) K_r(s) ds$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} l_r(t) dt = 1.$$

We express X in the form

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda} d\Phi(\lambda)$$

with a random measure $d\Phi$. Let $\boldsymbol{X}$ satisfy the condition of Theorem 1. We define stationary Gaussian processes $\boldsymbol{Y}_r$ and $\boldsymbol{Y}_r^*$ by

(2) $$Y_r(t,\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X(t-s,\omega)l_r(s)ds$$

and

(3) $$Y_r^*(t,\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X(t-s,\omega)l_r^*(s)ds$$

respectively. By virtue of the condition of Theorem 1, we can see that, for a. a. ω, the Lebesgue integral of the right side of (2), as well as (3), is a continuous function of t. Moreover, $\boldsymbol{Y}_r$ and $\boldsymbol{Y}_r^*$ are expressible in the form

(4) $$Y_r(t) = \int_{-T_r}^{T_r} e^{it\lambda}\theta_r(\lambda)d\Phi(\lambda)$$

and

(5) $$Y_r^*(t) = \int_{-T_{r-1}}^{T_{r-1}} e^{it\lambda}\theta_r(\lambda)d\Phi(\lambda).$$

As to the supremum value of these processes, we have Lemma 2,

LEMMA 2.

$$E\sup_{t\in[0,1]}|Y_r(t)|\leq a$$

$$E\sup_{t\in[0,1]}|Y_r^*(t)|\leq 2a$$

where $a = E\sup_{t\in[0,1]}|X(t)|$.

Proof. By Lemma 1, we have

$$E\sup_{t\in[0,1]}|Y_r(t)| < \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} E\sup_{t\in[0,1]}|X(t-s)|\,l_r(s)ds = a.$$

Put $Z_r(t) = Y_r(t) - Y_r^*(t)$. Then $\boldsymbol{Z}_r$ has continuous paths and is expressible in the form

$$Z_r(t) = \int_{T_{r-1}<|\lambda|\leq T_r} e^{it\lambda}\theta_r(\lambda)d\Phi(\lambda).$$

Therefore $\boldsymbol{Z}_r$ and $\boldsymbol{Y}_r^*$ are mutually independent. So, for any topological Borel set A of $C[0,1]$,

$$P(\tilde{\boldsymbol{Y}}_r \in A) = \int_{C[0,1]} P(\tilde{\boldsymbol{Y}}_r^* \in A - \xi)\, P(\tilde{\boldsymbol{Z}}_r \in d\xi)$$

where $\tilde{f}$ stands for the restriction on $[0,1]$ of f. Hence, for $\varepsilon > 0$,

$$P(\sup_{t\in[0,1]} |Y_r(t)| < c) \leq \sup_{\xi\in C[0,1]} P(\sup_{t\in[0,1]} |Y_r^*(t) + \xi(t)| < c)$$

$$\leq P(\sup_{t\in[0,1]} |Y_r^*(t) + \xi_c(t)| < c) + \varepsilon$$

with $\xi_c \in C[0,1]$. On the other hand, by virtue of the symmetricty of $\boldsymbol{Y}_r^*$,

$$P(\sup_{t\in[0,1]} |Y_r^*(t) + \eta(t)| < c) = P(\sup_{t\in[0,1]} |Y_r^*(t) - \eta(t)| < c), \quad \eta \in C[0,1].$$

Therefore, we have

$$1 - (\sup_{t\in[0,1]} |Y_r^*(t)| \geq c) = P(2 \sup_{t\in[0,1]} |Y_r^*(t)| < 2c)$$

$$\geq P(\sup_{t\in[0,1]} |Y_r^*(t) + \xi_c(t)| + \sup_{t\in[0,1]} |Y_r^*(t) - \xi_c(t)| < 2c)$$

$$\geq P(\sup_{t\in[0,1]} |Y_r^*(t) + \xi_c(t)| < c, \ \sup_{t\in[0,1]} |Y_r^*(t) - \xi_c(t)| < c)$$

$$\geq 1 - 2P(\sup_{t\in[0,1]} |Y_r^*(t) + \xi_c(t)| \geq c)$$

$$\geq 2P(\sup_{t\in[0,1]} |Y_r(t)| < c) - 2\varepsilon - 1.$$

Tending ε to 0, we get

$$P(\sup_{t\in[0,1]} |Y_r^*(t)| \geq c) \leq 2P(\sup_{t\in[0,1]} |Y_r(t)| \geq c).$$

Hence

$$\sum_{n=0}^{N} \frac{n}{2^k} P\Big(\frac{n}{2^k} \leq \sup_{t\in[0,1]} |Y_r^*(t)| < \frac{n+1}{2^k}\Big) \tag{6}$$

$$= \frac{1}{2^k} \sum_{n=1}^{N} P(\sup_{t\in[0,1]} |Y_r^*(t)| \geq \frac{n}{2^k}\Big) - \frac{N}{2^k} P\Big(\sup_{t\in[0,1]} |Y_r^*(t)| \geq \frac{N+1}{2^k}\Big)$$

$$\leq 2\sum_{n=0}^{N+1} \frac{n+1}{2^k} P\Big(\frac{n}{2^k} \leq \sup_{t\in[0,1]} |Y_r(t)| < \frac{n+1}{2^k}\Big) + \frac{N+1}{2^{k-1}} P\Big(\sup_{t\in[0,1]} |Y_r(t)| \geq \frac{N+1}{2^k}\Big).$$

Appealing to the former half of Lemma 2, we have $NP\Big(\sup_{t\in[0,1]} |Y_r(t)| \geq \frac{N+1}{2^k}\Big)$ tends to 0, as $N \uparrow \infty$. So, (6) implies the latter half of Lemma 2.

Define stationary Gaussian processes $\boldsymbol{V}_r$ and $\boldsymbol{V}_r^*$ by

$$V_r(t)=\frac{1}{\sqrt{2\pi}}\int_{|s|>\frac{1}{\sqrt{T_r}}} Y_{r+1}(t-s)K_r(s)ds$$

and

$$V_r^*(t)=\frac{1}{\sqrt{2\pi}}\int_{|s|>\frac{1}{\sqrt{T_r}}} Y_{r+1}^*(t-s)K_r(s)ds.$$

Then we can easily see, by Lemma 2,

LEMMA 3.

$$E\sup_{t\in[0,1]}|V_r(t)|\leq\frac{4\sqrt{2}\,a}{\sqrt{\pi}\,T_r}$$

$$E\sup_{t\in[0,1]}|V^*(t)|\leq\frac{8\sqrt{2}\,a}{\sqrt{\pi}\,T_r}\ .$$

3. **Proof of Theorem 1.**

To prove Theorem 1, we shall firstly show the following proposition,

PROPOSITION. *Let $\{T_r\}$ be a sequence of increasing positive numbers such that* $\sum_{r=1}^{\infty}\frac{1}{\sqrt{T_r}}<\infty$. *Then*

$$\sum_{j=1}^{\infty}\Big(\int_{T_j<|\lambda|\leq T_{j+1}}\prod_{k=j+1}^{\infty}\Big(1-\frac{|\lambda|}{T_k}\Big)^2 dF(\lambda)\Big)^{\frac{1}{2}}<\infty.$$

Proof.

We define successively random variables S_j, S_j' and H_j, $j=1,2,\cdots$, as follows,

$$S_1(\omega)\equiv 0$$

$$H_1(\omega)\equiv Y_1(S_1(\omega),\omega)$$

$$S_1'(\omega)\equiv\begin{cases}\min\{t;\ |t|\leq\tau_1,\ Y_2^*(t,\omega)=\min\limits_{|s|\leqslant\tau_1}Y_2^*(s,\omega),\ \text{if } H_1(\omega)<\min\limits_{|s|\leqslant\tau_1}Y_2^*(s,\omega)\\ \min\{t;\ |t|\leq\tau_1,\ Y_2^*(t,\omega)=\max\limits_{|s|\leqslant\tau_1}Y_2^*(s,\omega),\ \text{if } H_1(\omega)>\max\limits_{|s|\leqslant\tau_1}Y_2^*(s,\omega)\\ \min\{t;\ |t|\leq\tau_1,\ Y_2^*(t,\omega)=H_1(\omega),\quad \text{otherwise}\end{cases}$$

where $\tau_1=1+\frac{1}{\sqrt{T_1}}$. We can easily see that S_1' is measurable with respect to the Borel field, $\mathscr{B}_1$, spanned by $\{d\Phi(\lambda),\ |\lambda|\leq T_1\}$.

$$S_{j+1}(\omega) \equiv \begin{cases} S'_j(\omega), \text{ if } Y_{j+1}(S'_j(\omega), \omega) \geq H_j(\omega) \\ \min\{t;\ |t| \leq \tau_j,\ Y_{j+1}(t,\omega) = \max\limits_{|s|\leq\tau_j} Y_{j+1}(s,\omega), \text{ if } H_j(\omega) > \max\limits_{|s|\leq\tau_j} Y_{j+1}(s,\omega) \\ \min\{t;\ |t| \leq \tau_j,\ Y_{j+1}(t,\omega) = H_j(\omega), \text{ otherwise.} \end{cases}$$

$H_{j+1}(\omega) \equiv Y_{j+1}(S_{j+1}(\omega), \omega)$.

$$S'_{j+1}(\omega) \equiv \begin{cases} \min\{t;\ |t| \leq \tau_{j+1},\ Y^*_{j+2}(t,\omega) = \min\limits_{|s|\leq\tau_{j+1}} Y^*_{j+2}(s,\omega)\}, \\ \qquad\qquad \text{if } H_{j+1}(\omega) < \min\limits_{|s|\leq\tau_{j+1}} Y^*_{j+2}(s,\omega) \\ \min\{t;\ |t| \leq \tau_{j+1},\ Y^*_{j+2}(t,\omega) = \max\limits_{|s|\leq\tau_{j+1}} Y^*_{j+2}(t,\omega)\}, \\ \qquad\qquad \text{if } H_{j+1}(\omega) > \max\limits_{|s|\leq\tau_{j+1}} Y^*_{j+2}(s,\omega) \\ \min\{t;\ |t| \leq \tau_{j+1},\ Y^*_{j+2}(t,\omega) = H_{j+1}(\omega)\}, \text{ otherwise,} \end{cases}$$

where $\tau_j = 1 + \frac{1}{\sqrt{T_1}} + \cdots\cdots + \frac{1}{\sqrt{T_j}}$. Successively, we can prove that S_j and S'_j are measurable w. r. to the Borel field, $\mathscr{B}_j$, spanned by $\{d\Phi(\lambda), |\lambda| \leq T_j\}$.

We shall show the boundedness of H_j.

LEMMA 4.

$$\sup_{j=1,2,\cdots} |H_j(\omega)| < \infty, \qquad \text{a. a. } \omega.$$

Proof. By virtue of Lemma 3, we have

$$\sum_{r=1}^{\infty} E \sup_{|t|\leq\tau} |V_r(t)| < \infty, \tag{7}$$

where $\tau = \lim\limits_{j\to\infty} \tau_j$. On the other hand,

$$\begin{aligned} &\sup_{|t|\leq\tau} \left| \frac{1}{\sqrt{2\pi}} \int_{|s|\leq\frac{1}{\sqrt{T_r}}} X(t-s) l_r(s) ds \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{|s|\leq\frac{1}{\sqrt{T_r}}} \sup_{|u|\leq 2\tau} |X(u)| l_r(s) ds \\ &\leq \sup_{|u|\leq 2\tau} |X(u)| < \infty, \qquad \text{a. a. } \omega. \end{aligned} \tag{8}$$

Therefore, we see

$$\sup_{r=1,2,\cdots} \sup_{|t|\leqslant r} |Y_r(t)| < \infty. \qquad \text{a. a. } \omega.$$

Recalling the definition of H_j, we have Lemma 4.

$$\text{(9)} \qquad H_{j+1}(\omega) - H_j(\omega)$$
$$= \{(H_{j+1}(\omega) - H_j(\omega)) \vee 0\} - \{(Y_j(S_j(\omega), \omega) - \sup_{|s|\leqslant \tau_j} Y_{j+1}(s, \omega)) \vee 0\}^{2)}.$$

On the other hand, for $t \in [-\tau_{j-1}, \tau_{j-1}]$,

$$Y_j(t) = \frac{1}{\sqrt{2\pi}} \int_{|s|\leq \frac{1}{\sqrt{T_j}}} Y_{j+1}(t-s)K_j(s)ds + \frac{1}{\sqrt{2\pi}} \int_{|s|> \frac{1}{\sqrt{T_j}}} Y_{j+1}(t-s)K_j(s)ds$$
$$\leq \sup_{|t|\leqslant \tau_j} Y_{j+1}(t) + \sup_{|t|\leqslant \tau_{j-1}} V_j(t).$$

So,

$$Y_j(t) - \sup_{|s|\leqslant \tau_j} Y_{j+1}(s) \leq \sup_{|s|\leqslant \tau_{j-1}} V_j(t), \qquad |t| \leq \tau_{j-1}.$$

Therefore,

$$(Y_j(S_j) - \sup_{|s|\leqslant \tau_j} Y_{j+1}(s)) \vee 0 \leq \sup_{|t|\leqslant \tau_{j-1}} |V_j(t)|.$$

Appealing to Lemma 3, we have

$$\text{(10)} \qquad \sum_{j=1}^{\infty} E\{(Y_j(S_j) - \sup_{|s|\leqslant \tau_j} Y_{j+1}(s)) \vee 0\} < \infty.$$

As to the first term of the right side of (9),

$$\sum_{j=1}^{n} (H_{j+1} - H_j) \vee 0$$
$$= H_{n+1} - H_1 + \sum_{j=1}^{n} (Y_j(S_j) - \sup_{|s|\leq \tau_j} Y_{j+1}(s)) \vee 0.$$

Therefore, using Lemma 4 and (10), we get

$$\text{(11)} \qquad \sum_{j=1}^{\infty} (H_{j+1} - H_j) \vee 0 < \infty, \qquad \text{a. a. } \omega.$$

On the other hand, recalling the definition of H_j, we see

$$\text{(12)} \qquad (H_{j+1} - H_j) \vee 0 = (Y_{j+1}(S_j') - H_j) \vee 0$$

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2) $a \vee b = \max(a, b)$.

$$\geq \{(Y_{j+1}(S'_j) - Y^*_{j+1}(S'_j)) \vee 0\} - \{(H_j - \sup_{|t|\leq\tau_j} Y^*_{j+1}(t)) \vee 0\}.$$

So, using the similar method as (10), we get

$$\sum_{j=1}^{\infty} E\{(H_j - \sup_{|t|\leq\tau_j} Y^*_{j+1}(t)) \vee 0\} < \infty. \tag{13}$$

Therefore, combining (11) and (13) to (12), we have

$$\sum_{j=1}^{\infty} (Y_{j+1}(S'_j) - Y^*_{j+1}(S'_j)) \vee 0 < \infty, \qquad \text{a. a. } \omega. \tag{14}$$

Put $\gamma_j = Y_{j+1}(S'_j) - Y^*_{j+1}(S'_j)$ and

$$v_j = \int_{T_j<|\lambda|\leq T_{j+1}} \prod_{k=j+1}^{\infty}\Big(1-\frac{|\lambda|}{T_k}\Big)^2 dF(\lambda).$$

Then, we see, appealing to the independence of $d\Phi$,

$$P(\gamma_j \leq x/\mathscr{B}_j) = \frac{1}{\sqrt{2\pi v_j}}\int_{-\infty}^{x} e^{-\frac{y^2}{2v_j}}\,dy,$$

since S'_j is $\mathscr{B}_j$-measurable.
Hence

$$E(\gamma_j \vee 0) = \frac{\sqrt{v_j}}{\sqrt{2\pi}} \tag{15}$$

and

$$E(\gamma_j \vee 0)^2 = \frac{v_j}{2}\,. \tag{16}$$

Appealing to the following Lemma

LEMMA. [5, p. 64]. *If X is a non-negative random variable with mean finite, then*

$$P(X > \lambda E(X)) \geq (1-\lambda)^2 \frac{(EX)^2}{EX^2}\,, \qquad \forall\lambda \in (0,1),$$

we can derive

$$P\Big(\sum_{j=1}^{\infty}(\gamma_j \vee 0) > \frac{\sum_{j=1}^{n}\sqrt{v_j}}{2\sqrt{2\pi}}\Big) \geq P\Big(\sum_{j=1}^{n}(\gamma_j \vee 0) > \frac{\sum_{j=1}^{n}\sqrt{v_j}}{2\sqrt{2\pi}}\Big) \geq \frac{1}{4\pi}\,.$$

So,

$$P\Big(\sum_{j=1}^{\infty}(\gamma_j \vee 0) \geqq \frac{\sum_{j=1}^{\infty}\sqrt{v_j}}{2\sqrt{2\pi}}\Big) \geqq \frac{1}{4\pi}\ .$$

By virtue of (14), we conclude

$$\sum_{j=1}^{\infty}\sqrt{v_j} < \infty.$$

This completes the proof of Proposition.

Making use of Proposition, we can easily prove Theorem 1. Put $T_k=2^k$ and $\alpha = \prod_{k=0}^{\infty}(1-3\cdot 2^{-k-2})^2$. Then we have

$$2\alpha F(2^j,\ 3\cdot 2^{j-1}] = \alpha\int_{2^j<|\lambda|\leqslant\frac{3}{2}2^j} dF(\lambda) \leq \int_{2^j<|\lambda|\leqslant\frac{3}{2}2^j}\prod_{k=j+1}^{\infty}\Big(1-\frac{|\lambda|}{2^k}\Big)^2 dF(\lambda)$$

$$\leq \int_{2^j<|\lambda|\leqslant 2^{j+1}}\prod_{k=j+1}^{\infty}\Big(1-\frac{|\lambda|}{2^k}\Big)^2 dF(\lambda).$$

So, by Proposition,

$$\sum_{j=0}^{\infty} F(2^j,\ 3\cdot 2^{j-1}]^{\frac{1}{2}} < \infty. \tag{17}$$

Put $T_k = 3\cdot 2^{k-1}$ and $\alpha = \prod_{k=0}^{\infty}\Big(1-\frac{1}{3}2^{-k+1}\Big)^2$. Then we have

$$2\alpha F(3\cdot 2^{j-1},\ 2^{j+1}] \leq \int_{3\cdot 2^{j-1}<|\lambda|\leqslant 2^{j+1}}\prod_{k=j+1}^{\infty}\Big(1-\frac{|\lambda|}{3\cdot 2^{k-1}}\Big)^2 dF(\lambda)$$

$$\leq \int_{3\cdot 2^{j-1}<|\lambda|\leqslant 3\cdot 2^j}\prod_{k=j+1}^{\infty}\Big(1-\frac{|\lambda|}{3\cdot 2^{k-1}}\Big)^2 dF(\lambda).$$

So,

$$\sum_{j=0}^{\infty} F(3\cdot 2^{j-1},\ 2^{j+1}]^{\frac{1}{2}} < \infty. \tag{18}$$

By virtue of (17) and (18), we have Theorem 1.

4. Proof of Theorem 2

We shall first assume that s_n is decreasing and $\sum_{n=0}^{\infty}\sqrt{s_n} < \infty$. We put $c(j) = 2^{2^j}$ and define ξ_j and η_j by

$$\xi_j(t) = \int_{c(j-1)<|\lambda|\leq c(j)} e^{it\lambda}\, d\Phi(\lambda)$$

and

$$\eta_j = \max_{k=0,\cdots,c(j+1)} \left| \xi_j\left(\frac{k}{c(j+1)}\right)\right|, \quad j = 1, 2, \cdots$$

respectively. Then the process ξ_j has continuous paths. Appealing to the following Lemma,

LEMMA. [5. Proposition 2].

$$E\eta_j \leq h + \sum_{k=0}^{c(j+1)} \int_h^\infty |x|\, d\mu_{\xi_j\left(\frac{k}{c(j+1)}\right)}(x), \quad h>0$$

where μ_ξ is the probability law of ξ, we have

$$E\eta_j \leq h + (c(j+1)+1)\sqrt{\frac{2\sigma_j}{\pi}}\, e^{-\frac{h^2}{2\sigma_j}}$$

where $\sigma_j = 2F(c(j-1),\ c(j)]$. Let $h = h(j) = \sqrt{2\sigma_j \log c(j+1)}$.

Then we see

$$E\eta_j \leq 2h(j). \tag{19}$$

Since $\sigma_j = \sum_{k=2^{j-1}}^{2^j-1} s_k$, we get

$$2^j\sigma_j = 2^j \sum_{k=2^{j-1}}^{2^j-1} s_k \leq 2^{2j} s_{2^{j-1}}.$$

Hence,

$$\sqrt{2^j\sigma_j} \leq 2^j\sqrt{s_{2^{j-1}}} \leq 4\sum_{k=2^{j-2}}^{2^{j-1}} \sqrt{s_k}.$$

Consequently, by (19), we have

$$\sum_{j=1}^\infty E\eta_j < \infty. \tag{20}$$

Define ζ and θ by

$$\zeta(j,k,p,q,r) = \xi_j\left(\frac{k}{c(j+1)} + \frac{q}{c(j+1)c(p)} + \frac{r}{c(j+1)c(p+1)}\right)$$
$$- \xi_j\left(\frac{k}{c(j+1)} + \frac{q}{c(j+1)c(p)}\right),$$

$r=1,\dots,c(p)$, $q=1,\dots,c(p)$, $k=0,\dots,c(j+1)$, $p=1,2,\dots,$ $j=1,2,\dots,$ and

$$\theta(j,p)=\max_{k,q,r}|\zeta(j,k,p,q,r)|.$$

Then we see

$$E\zeta^2(j,k,p,q,r)=2\int_{c(j-1)<\lambda\leq c(j)}\left(1-\cos\frac{r}{c(j+1)c(p+1)}\lambda\right)dF(\lambda)$$
$$\leq\frac{1}{2}\frac{\sigma_j}{c^2(j)c^2(p)}.$$

Again, using the same Lemma, we have

$$E\theta(j,p)\leq 2\sqrt{\log c(p+1)\,c(j+1)}\,\frac{\sqrt{\sigma_j}}{c(j)c(p)}.$$

Therefore

$$\sum_{j=1}^{\infty}\sum_{p=1}^{\infty}E\theta(j,p)<\infty. \tag{21}$$

By virtue of the separability of X and ξ_j, we have

$$\sup_{t\in[0,1]}|X(t)|\leq\sum_{j=1}^{\infty}\sup_{t\in[0,1]}|\xi_j(t)|+|d\Phi(0)|,\qquad \text{a. a. } \omega,$$

and

$$\sup_{t\in[0,1]}|\xi_j(t)|\leq\eta_j+\sum_{p=1}^{\infty}\theta(j,p),\qquad \text{a. a. } \omega.$$

So, taking (20) and (21) into account, we complete the proof of Theorem 2 in the first case.

Define a symmetric finite measure G by

$$G(A)=F(A)+\sum_{n=0}^{\infty}(M_n-s_n)\delta_{2^{n+1}}(A),\qquad A\subset[0,\infty),$$

where δ_a is the delta measure concentrated at a. Let X_1 and X_2 be the mutually independent stationary Gaussian processes whose covariance function has the spectral measure F and $\sum_{n=0}^{\infty}(M_n-s_n)\delta_{2^n}(A)$, respectively. Then G is the spectral measure of the covariance function of X_1+X_2 and $G(2^n,2^{n+1}]=M_n$. So, using the result, we just proved,

$$E\sup_{t\in[0,1]}|X_1(t)+X_2(t)|<\infty.$$

Repeating the same method as Lemma 2, we have

$$E \sup_{t \in [0,1]} |X_1(t)| < \infty.$$

This completes the proof of Theorem 2.

5. **Proof of Theorem 3**

To prove Theorem 3, we shall first show the following Lemma,

LEMMA 5. *Assume that a symmetric, positive continuous function R is convex and decreasing on* $[0, \pi]$. *Then any Fourier coefficient* a_n, *i. e.* $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}R(t)dt$, *is non-negative. Moreover,* $\sum_{h=-\infty}^{\infty} a_n = R(0)$.

Proof. By symmetricity of R, for $n \geq 1$,

$$a_{-n} = a_n = \frac{1}{\pi}\int_0^{\pi} R(t) \cos nt\, dt = \frac{1}{n\pi}\int_0^{n\pi} R\left(\frac{s}{n}\right)\cos s\, ds. \tag{22}$$

$$\int_{2k\pi}^{2(k+1)\pi} R\left(\frac{s}{n}\right)\cos s\, ds$$

$$=\int_0^{\frac{\pi}{2}}\left(R\left(\frac{2k\pi+s}{n}\right)-R\left(\frac{2k\pi+\pi-s}{n}\right)-R\left(\frac{2k\pi+\pi+s}{n}\right)+R\left(\frac{2k\pi+2\pi-s}{n}\right)\right) \cos s\, ds.$$

By virtue of the convexity of R, the integrand is non-negative, and we have

$$\int_{2k\pi}^{2(k+1)\pi} R\left(\frac{s}{n}\right) \cos s\, ds \geq 0.$$

On the other hand, by the monotonicity of R,

$$\int_{2k\pi}^{2k\pi+\pi} R\left(\frac{s}{n}\right) \cos s\, ds = \int_0^{\frac{\pi}{2}}\left(R\left(\frac{2k\pi+s}{n}\right) - R\left(\frac{2k\pi+\pi-s}{n}\right)\right) \cos s\, ds \geq 0.$$

Therefore, appealing to (22), $a_n \geq 0$.

Since R is continuous and bounded variation, its Fourier series converges to R uniformly on any closed subset of $(-\pi, \pi)$. Hence $\sum_{n=-\infty}^{\infty} a_n = R(0)$.

LEMMA 6. *Let R be a continuous, symmetric and positive definite function on* $(-\infty, \infty)$. *Assume that each Fourier coefficient* a_n, *i.e.* $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}R(t)dt$, *is non-negative. Then, the spectral measure dG of R satisfies*

$$\sum_{n=0}^{\infty}\sqrt{G(2^n,\ 2^{n+1}]}<\infty$$

if $\sum_{n=0}^{\infty}\sqrt{\sum_{k=2^n+1}^{2^{n+1}} a_k}<\infty.$

Proof. By the symmetry of dG,

$$a_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-int}\Big(\int_{-\infty}^{\infty}e^{it\lambda}\,dG(\lambda)\Big)\,dt \tag{23}$$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{\sin(\lambda-n)\pi}{\lambda-n}\,dG(\lambda)+\frac{1}{\pi}\int_0^{\infty}\frac{\sin(\lambda+n)\pi}{\lambda+n}\,dG(\lambda),$$

where $\frac{\sin 0\pi}{0}$ is read as $\lim_{x\to 0}\frac{\sin \pi x}{x}=\pi.$

Put $f(\lambda)=\sum_{n=2^k-4}^{2^{k+1}+3}\frac{\sin(\lambda-n)\pi}{\lambda-n},\qquad \lambda\geq 0,\quad (k=4,5,\ldots).$

Then we have, for $m=2^{k-1}+1,\ldots,2^k,\qquad \mu\in(0,1],$

$$f(2m-1+\mu)=\sum_{j=0}^{2m-2^k+3}\frac{\sin(j+\mu)\pi}{j+\mu}+\sum_{l=1}^{2^{k+1}+4-2m}\frac{\sin(l-\mu)\pi}{l-\mu}$$

$$\geq\Big(\frac{1}{\mu}-\frac{1}{1+\mu}+\frac{1}{2+\mu}+\frac{1}{1-\mu}-\frac{1}{2-\mu}+\frac{1}{3-\mu}\Big)\sin\mu\pi\geq\frac{7}{12}\,\frac{\sin\mu\pi}{\mu(1-\mu)},$$

and, by the same method,

$$f(2m+\mu)\geq\frac{7}{12}\,\frac{\sin\mu\pi}{\mu(1-\mu)},\quad \mu\in(0,1],\ \ m=2^{k-1},\ldots,2^k-1.$$

Therefore,

$$\int_{2^k+}^{2^{k+1}}f(\lambda)\,dG(\lambda)\geq\frac{7}{12}\sum_{l=2^k}^{2^{k+1}-1}\int_{0+}^{1}\frac{\sin\pi\mu}{\mu(1-\mu)}\,dG(l+\mu). \tag{24}$$

On the other hand, we have the following inequalities,

$$f(\lambda)\geq 0,\qquad \lambda\in[2^{k+1},\ 2^{k+1}+4]\cup[2^k-5,\ 2^k]. \tag{25}$$

$$f(2^{k+1}+4+\mu)=\frac{-\sin\mu\pi}{1+\mu}+\frac{\sin\mu\pi}{2+\mu}-\frac{\sin\mu\pi}{3+\mu}+\cdots+\frac{\sin\mu\pi}{2^k+8+\mu} \tag{26}$$

$$\geq\Big(-\frac{1}{1+\mu}+\frac{1}{2+\mu}-\frac{1}{3+\mu}\Big)\sin\mu\pi\geq-\frac{5}{6}\,\frac{\sin\mu\pi}{1+\mu}\geq-\frac{5}{6}(3-\sqrt{8})\,\frac{\sin\mu\pi}{\mu(1-\mu)},$$

$$\mu\in[0,1].$$

$$f(2^{k+1}+j+\mu)\geq-\frac{5}{6}(3-\sqrt{8})\frac{\sin\mu\pi}{\mu(1-\mu)},\ \ \mu\in[0,1],\ j=5,\ldots,2^{k+1}-1, \tag{27}$$

$$(28)\qquad f(2^k-j+\mu)\geq -\frac{5}{6}(3-\sqrt{8})\frac{\sin\mu\pi}{\mu(1-\mu)},\quad \mu\in[0,1],\ \ j=6,\ldots,2^{k-1}+1.$$

Hence, by (25) and (28),

$$(29)\qquad \int_{2^{k-1}+}^{2^k} f(\lambda)dG(\lambda)\geq -\frac{5}{6}(3-\sqrt{8})\sum_{l=2^{k-1}}^{2^k-1}\int_{0+}^{1}\frac{\sin\mu\pi}{\mu(1-\mu)}dG(l+\mu)$$

and, by (25) and (27),

$$(30)\qquad \int_{2^{k+1}+}^{2^{k+2}} f(\lambda)dG(\lambda)\geq -\frac{5}{6}(3-\sqrt{8})\sum_{l=2^{k+1}}^{2^{k+2}-1}\int_{0+}^{1}\frac{\sin\mu\pi}{\mu(1-\mu)}dG(l+\mu).$$

As to the value of integral of f on the remainder set of λ, we see

$$(31)\qquad \left|\int_0^{2^{k}-1} f(\lambda)dG(\lambda)\right|\leq \sum_{m=2^k-2}^{\infty}\int_0^{2^{k-1}}\frac{1}{(2m-\lambda)(2m+1-\lambda)}dG(\lambda)$$
$$\leq \sum_{m=2^k-2}^{\infty}\frac{G[0,2^{k-1}]}{(2m-2^{k-1})^2}\leq\frac{R(0)}{2^{k-1}-5},$$

and, similarly

$$(32)\qquad \left|\int_{2^{k+2}}^{\infty} f(\lambda)dG(\lambda)\right|\leq\frac{R(0)}{2^{k+1}-5}$$

On the other hand,

$$(33)\qquad \left|\sum_{n=2^k-4}^{2^{k+1}+3}\int_0^{\infty}\frac{\sin(\lambda+n)\pi}{\lambda+n}dG(\lambda)\right|$$
$$\leq\sum_{m=2^{k-1}-2}^{2^k+2}\frac{R(0)}{(2m)^2}\leq\frac{R(0)}{2^k-5}$$

Consequently, taking (23) into account, we have

$$(34)\qquad \delta_k+\frac{3R(0)}{2^{k-1}-5}\geq\frac{7}{12}\Delta_k-\frac{5}{6}(3-\sqrt{8})(\Delta_{k+1}+\Delta_{k-1}),$$

where $\delta_k=\sum_{n=2^k-4}^{2^{k+1}+3}\frac{a_n}{\pi}$ and $\Delta_k=\sum_{l=2^k}^{2^{k+1}-1}\int_{0+}^{1}\frac{\sin\mu\pi}{\mu(1-\mu)}dG(l+\mu)$.

Since $\Delta_k\leq\pi\cdot G(2^k,2^{k+1}]$, Δ_k tends to 0 as $n\uparrow\infty$. Therefore (34) implies

$$\sum_{k\geq5}\sqrt{\delta_k}+\sum_{k\geq5}\frac{\sqrt{3R(0)}}{\sqrt{2^{k-1}-5}}+\sqrt{\Delta_4}\geq\left(\sqrt{\frac{7}{12}}-2\sqrt{\frac{5}{6}(3-\sqrt{8})}\right)\sum_{k\geq5}\sqrt{\Delta_k}\geq\left(\sqrt{\frac{7}{12}}-\sqrt{\frac{6.88}{12}}\right)\sum_{k\geq5}\sqrt{\Delta_k}.$$

By the assumption of Lemma 6, i.e., $\sum_k\sqrt{\delta_k}<\infty$, we have

$$\sum_{k=1}^{\infty} \sqrt{\Delta_k} < \infty. \tag{35}$$

Appealing to the following inequality

$$\frac{\sin \mu\pi}{\mu(1-\mu)} \geq 1, \quad \text{on } [0,1],$$

we have $\Delta_k \geq G(2^k, 2^{k+1}]$ and, by (35), we complete the proof of Lemma 6.

Using Lemmas 5 and 6, we can easily prove Theorem 3. By the assumption of Theorem 3, we can choose a positive Δ, so that ρ is positive convex and decreasing on $[0, \Delta]$. Define a Gaussian process $\tilde{X}$ by $\tilde{X}(t) = X\left(\frac{\pi t}{\Delta}\right)$. Then the covariance function $\tilde{\rho}$ of $\tilde{X}$ is $\tilde{\rho}(t) = \rho\left(\frac{\pi t}{\Delta}\right)$, and its spectral measure $\tilde{F}$ is $\tilde{F}(A) = F\left(\frac{\Delta}{\pi} A\right)$ for any Borel set A. Since $\tilde{\rho}$ satisfies the condition of Lemma 5, we can construct a periodic covariance function R by

$$R(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}, \quad -\infty < t < \infty,$$

where $a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\rho}(t) e^{-int} dt$. Let Y be a stationary Gaussian process with mean zero and with the covariance function R. Since $R = \tilde{\rho}$ on $[-\pi, \pi]$, Y has the locally same probability law as $\tilde{X}$. So, Y has continuous paths. Hence Kahane's Theorem [5, p. 73], [3, p. 300] tells us that

$$\sum_{k=0}^{\infty} \sqrt{\sum_{n=2^k+1}^{2^{k+1}} a_n} < \infty.$$

Therefore, by Lemma 6, we have

$$\sum_{n=0}^{\infty} \sqrt{F\left(\frac{\Delta}{\pi} 2^n, \frac{\Delta}{\pi} 2^{n+1}\right]} < \infty.$$

This implies Theorem 3.

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