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# On the Continuous Limit for a System of Classical Spins

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**Abstract.** The continuum limit of a cubic lattice of classical spins processing in the magnetic field created by their closest neighbours is considered. Results concerning existence, uniqueness and (for initially small spin deviation) long time behaviour, are presented.

## 1. Introduction

A classical model for an isotropic ferromagnet is provided by a collection of three-dimensional spin vectors with unit length and arbitrary directions, located at the nodes of a $d$-dimensional cubic lattice. We denote by $S_i$ or $S(x_i)$ the (classical) spin located at the point $x_i = n_{i_1}h_1 + \ldots + n_{i_d}h_d$, where the $n_{i_j}$'s are integers, and $h_j$ the mesh vector in the $j$-direction. We assume that all the mesh vectors have the same length $h$.

Concerning the dynamics, a simple hypothesis consists in assuming that each spin $S(x_i)$ processes in the local magnetic field $\sum_{j=1}^{d} S(x_i+h_j)+S(x_i-h_j)$ created by the closest neighbours. The equations of motion are written [1, 3, 10, 11]

$$\frac{dS}{dt^*}(x_i) = J\sum_{j=1}^{d} S(x_i)\wedge(S(x_i+h_j)+S(x_i-h_j)), \tag{1.1}$$

where $\wedge$ denotes vectorial product and $J$ the (positive) nearest neighbour exchange coupling constant. Equation (1.1) can be rewritten:

$$\frac{dS}{dt}(x_i) = \sum_{j=1}^{d} S(x_i)\wedge\frac{S(x_i+h_j)-2S(x_i)+S(x_i-h_j)}{h^2} \tag{1.2}$$

with

$$t = h^2 t^*/J\,.$$

When interested in phenomena at scales large compared to the lattice mesh size and with time scale $O(1/h^2)$, we are led to consider the limit $h \to 0$ in Eq. (1.2).

Taking formally the limit, we obtain the partial differential equation

$$\frac{\partial S}{\partial t} = S \wedge \Delta S, \tag{1.3}$$

where $S = S(x, t)$ is now a three-dimensional continuous vector field. The purpose of this paper is to justify this asymptotic procedure and to investigate the regularity properties of the limit solution. We also consider the special case where the initial conditions correspond to spins which are almost parallel. In that case, we show that the solution is smooth in the large and that for $t \to +\infty$, the dynamics is asymptotically linear in a stereographic representation.

## 2. Existence in the Large of Weak Solutions

We construct a solution of

$$\begin{cases} \dfrac{\partial S}{\partial t} = S \wedge \Delta S \\ S(x,0) = S_0(x) \end{cases} \tag{2.1}$$

as the limit, when $h$ tends to zero of sequences $S_h(x_i)$ solutions of Eq. (1.2). We define right and left approximations of the derivatives in the form

$$D_j^+ S_h(x_i) \equiv \frac{S_h(x_i + h_j) - S_h(x_i)}{h}, \tag{2.2a}$$

$$D_j^- S_h(x_i) \equiv \frac{S_h(x_i) - S_h(x_i - h_j)}{h}. \tag{2.2b}$$

$\nabla^\pm \equiv (D^{\pm}{}_1, \dots, D_d^\pm)$ is the right/left approximate gradient.

The Laplacian is approximated by

$$\tilde{\Delta} S_h(x_i) \equiv \sum_{j=1}^{d} D_j^- (D_j^+ S_h)(x_i) = \sum_{j=1}^{d} D_j^+ (D_j^- S_h)(x_i). \tag{2.2c}$$

It is easily checked that for two scalar sequences $\{u_h(x_i)\}$ and $\{v_h(x_i)\}$, we have

$$\begin{aligned} D_j^+ (u_h v_h)(x_i) &= u_h(x_i) D_j^+ v_h(x_i) + D_j^+ u_h(x_i) v_h(x_i + h_j) \\ &= u_h(x_i + h_j) D_j^+ v_h(x_i) + D_j^+ u_h(x_i) v_h(x_i) \\ &= \tfrac{1}{2}(u_h(x_i + h_j) + u_h(x_i)) D_j^+ v_h(x_i) \\ &\quad + \tfrac{1}{2} D_j^+ u_h(x_i)(v_h(x_i + h_j) + v_h(x_i)), \end{aligned} \tag{2.3}$$

and similar relations for $D_j^-(u_h v_h)(x_i)$.

Furthermore

$$\tilde{\Delta}(u_h v_h) = u_h \tilde{\Delta} v_h + \nabla^+ v_h \cdot \nabla^+ v_h + \nabla^- u_h \cdot \nabla^- v_h + \tilde{\Delta} u_h v_h, \tag{2.4}$$

and

$$\begin{cases} D_j^+ u_h(x_i - h_j) = D_j^- u_h(x_i) \\ D_j^- u_h(x_i + h_j) = D_j^+ u_h(x_i) \end{cases}. \tag{2.5}$$

For the sequences $\{u_h(x_i)\}$ and $\{v_h(x_i)\}$, decaying fast enough when $|x_i| \to +\infty$, the discrete formula of integration by parts reads

$$\sum_{x_i} v_h(x_i) \cdot D_j^+ u_h(x_i) = - \sum_{x_i} u_h(x_i) \cdot D_j^- v_h(x_i) . \tag{2.6}$$

We define the scalar product

$$(u_h, v_h)_h = h^d \sum_{x_i} u_h(x_i) \cdot v_h(x_i) , \tag{2.7}$$

together with the norms

$$|u_h|^2_{\tilde{L}^2_h} = (u_h, u_h)_h , \tag{2.8a}$$

$$|u_h|^2_{\tilde{H}^1_h} = |u_h|^2_{\tilde{L}^2_h} + \sum_{j=1}^{d} |D_j^+ u_h|^2_{\tilde{L}^2_h} , \tag{2.8b}$$

and

$$|u_h|_{\tilde{H}^{-1}_h} = \sup_{v_h} \frac{(u_h \cdot v_h)_h}{|v_h|_{\tilde{H}^1_h}} , \tag{2.8c}$$

which is the dual norm of $|\cdot|_{\tilde{H}^1_h}$ with respect to the $\tilde{L}^2_h$ scalar product $(\cdot, \cdot)_h$. For given $h > 0$, let $\{S_h(x_i, t)\}$ be the unique sequence which satisfies for all time the discrete equation (1.2) that we rewrite in the form:

$$\begin{cases} \dfrac{\partial S_h}{\partial t} = S_h \wedge \tilde{\Delta} S_h \equiv D_i^+ (S_h \wedge D_i^- S_h) \\ S_h(x_i, 0) = S_h^0(x_i) \end{cases} \tag{2.9}$$

with $|S_h^0(x_i)| = 1$. Multiplying Eq. (2.9) by $S_h$, we get:

$$\frac{d}{dt} |S_h(x_i, t)|^2 = 0 , \tag{2.10}$$

which ensures that $S_h(x_i, t)$ remains of unit length.

Taking the scalar product in $\tilde{L}^2_h$ with $\tilde{\Delta} S_h$, we obtain

$$\frac{d}{dt} \sum_{j=1}^{d} |D_j^+ S_h|^2_{\tilde{L}^2_h} = 0 , \tag{2.11}$$

and thus

$$\sum_{j=1}^{d} |D_j^+ S_h|^2_{\tilde{L}^2_h} = \sum_{j=1}^{d} |D_j^+ S_h^0|^2_{\tilde{L}^2_h} \tag{2.13}$$

with $S_h^0(x_i) = S_h(x_i, 0)$. Moreover, for any sequence $\{v_h(x_i)\}$ in $\tilde{H}^1_h$, we have

$$\left( \frac{\partial S_h}{\partial t}, v_h \right)_h = - \sum_{i=1}^{d} (S_h \wedge D_i^- S_h, D_i^- v_h)_h , \tag{2.14}$$

and consequently

$$\left| \frac{\partial S_h}{\partial t} \right|_{\tilde{H}^{-1}_h} \leqq |D_j^+ S_h^0|_{\tilde{L}^2_h} . \tag{2.15}$$

In order to pass to the limit $h \to 0$ in Eq. (2.9), we introduce the interpolation operators $q_h, p_h$, and $r_h^{(m)}$ [7]:

For any point $x$ which belongs to the cell $C_i = \{x_i, x_i + h_1\} \times \ldots \times \{x_i, x_i + h_d\}$ of the lattice, we define the piecewise continuous function

$$q_h S_h(x) = S(x_i). \tag{2.16}$$

We also define $p_h S_h$ as a piecewise linear function with respect to each variable. In the cell $C_i$,

$$\begin{aligned} p_h S_h(x) = S_h(x_i) + \sum_{j=1}^{d} D_j^+ S_h(x_i) \frac{hj}{h} \cdot (x - x_i) + \ldots \\ + \sum_{j=1}^{d} D_1^+ \ldots D_{j-1}^+ D_{j+1}^+ \ldots D_d^+ S_h(x_i) \prod_{\substack{l=1 \\ l \neq j}}^{d} \frac{h_l}{h} (x - x_i) \\ + D_1^+ \ldots D_d^+ S_h(x_i) \prod_{l=1}^{d} \frac{h_l}{h} \cdot (x - x_i). \end{aligned} \tag{2.17}$$

Finally,

$$\begin{aligned} r_h^{(m)} S_h(x) = S_h(x_i) + \sum_{\substack{j=1 \\ j \neq m}} D_i^+ S_h(x_i) \frac{h_j}{h} \cdot (x - x_i) \\ + D_1^+ \ldots D_{m-1}^+ D_{m+1}^+ \ldots D_d^+ S_h(x_i) \prod_{\substack{l=1 \\ l \neq m}} \frac{h_l}{h} \cdot (x - x_i). \end{aligned} \tag{2.18}$$

We have the relation [7]

$$\frac{\partial}{\partial x_m} p_h S_h = r_h^{(m)} (D_m^+ S_h). \tag{2.19}$$

Let the initial data $S_0(x)$ for Eq. (2.1) satisfy $|S_0(x)| = 1$ and $\frac{\partial S_0}{\partial x_i} \in L^2(\mathbb{R}^d)$. We construct a sequence $S_h^0$ such that $p_h S_h^0 \to S_0$ in $H^1_{\text{loc}}(\mathbb{R}^d)$. From estimates (2.10), (2.13), and (2.15), we deduce that [9]:

$$\frac{d}{dt} p_h S_h \text{ remains in a bounded set of } L^\infty(0, T, H^{-1}(\mathbb{R}^d)),$$

$$p_h S_h, q_h S_h, r_h^{(m)} S_h \text{ remain in a bounded set of } L^\infty(0, T, H^1_{\text{loc}}(\mathbb{R}^d)).$$

As a consequence, there exists a subsequence $S_h$ such that $p_h S_h$ converges strongly in $L^2_{\text{loc}}$ to $S$ and thus almost everywhere; $q_h S_h$ converges almost everywhere to the same limit. $\frac{\partial}{\partial t} p_h S_h$ converges to $\frac{\partial S}{\partial t}$ in $L^\infty(0, T, H^{-1})$ weak *.

Now, for any $v \in L^\infty(0, T, H^1)$ and any sequence $v_h$ such that $p_h v_h$ converges to $v$ in $L^\infty(0, T, H^1)$, we have

$$\int p_h v_h \frac{\partial}{\partial t} p_h S_h = \int p_h D_i^+ (S_h \wedge D_i^- S_h) \cdot p_h v_h. \tag{2.20}$$

When $h \to 0$, the left-hand side of Eq. (2.20) tends to $\left(v, \frac{\partial S}{\partial t}\right)$.

To show that $p_h D_i^+(S_h \wedge D_i^- S_h)$ converges to $\frac{\partial}{\partial x_i}\left(S \wedge \frac{\partial S}{\partial x_i}\right)$ in $L^\infty(0, T, H_{\mathrm{loc}}^{-1})$ weak *, we first notice that since $S_h \wedge D_i^- S_h$ is in a bounded set of $L^\infty(0, T, \tilde{L}_h^2)$, $q_h(S_h \wedge D_i^- S_h)$, $p_h(S_h \wedge D_i^- S_h)$, and $r_h^{(m)}(S_h \wedge D_i^- S_h)$ converge to the same limit in $L^\infty(0, T, L^2)$ [7].

Now,

$$q_h(S_h \wedge D_i^- S_h) = q_h S_h \wedge q_h D_i^- S_h,$$

and

$$q_h S_h \to S \quad \text{almost everywhere},$$

$$q_h D_i^- S_h \to \frac{\partial S}{\partial x_i} \quad \text{in} \quad L^\infty(0, T, L^2) \quad \text{weak * [7]}.$$

Thus

$$p_h(S_h \wedge D_i^+ S_h) \to S \wedge \frac{\partial S}{\partial x_i} \quad \text{in } L^\infty(0, T, L^2) \text{ weak *};$$

$$\frac{\partial}{\partial x_i} p_h(S_h \wedge D_i^- S_h) \to \frac{\partial}{\partial x_i}\left(S \wedge \frac{\partial S}{\partial x_i}\right) \quad \text{in } L^\infty(0, T, H^{-1}) \text{ weak *}.$$

Now, from (2.19)

$$r_h^{(i)} D_i^+(S_h \wedge D_i^- S_h) = \frac{\partial}{\partial x_i} p_h(S_h \wedge D_i^- S_h),$$

and the right-hand side of Eq. (2.20) tends to $\sum_i \frac{\partial}{\partial x_i}\left(S \wedge \frac{\partial S}{\partial x_i}\right)$ in $L^\infty(0, T, H_{\mathrm{loc}}^{-1})$ weak *.

This leads to

**Theorem 2.1.** *For any $S_0$ such that $|S_0(x)| = 1$ almost everywhere and $\frac{\partial S}{\partial x_i}$ in $L^2(\mathbb{R}^d)$, there exists for all time a weak solution of Eq. (2.1) with $|S(x,t)| = 1$ almost everywhere and $\frac{\partial S}{\partial x_i} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^d))$. The solution is obtained as the (weak) limit when $h \to 0$ of sequences $\{S_h(x_i)\}$ satisfying the difference equation (2.9).*

*Remark.* We do not know if the weak solution obtained in Theorem 2.1 is unique.

## 3. Local Existence of Smooth Solutions

*Notations.* $W^{m,p}(\mathbb{R}^d)$ and $H^m(\mathbb{R}^d)$ denote the Sobolev space of (vectorial) functions equipped with the usual norms

$$|u|_{W^{m,p}} = \left(\sum_{0 \leq |k| \leq m} |D^k u|^p\right)^{1/p}, \tag{3.1a}$$

$$|u|_{H^m} = \left(\sum_{0 \leq |k| \leq m} |D^k u|^2\right)^{1/2}. \tag{3.1b}$$

For the sequences $\{S_h(x_i)\}$, we define the $\tilde{H}_h^m$-norm by

$$|S_h|_{\tilde{H}_h^m} = \Big(\sum_{0 \leq |k| \leq m} |\tilde{D}^k S_h|^2_{L_h^2}\Big)^{1/2}. \tag{3.2}$$

In Eq. (3.2)

$$\tilde{D}^k = \sum_{|\alpha| = k} \tilde{D}^\alpha,$$

where $\alpha$ is the multi index $(\alpha_1^+, \alpha_1^-, \dots, \alpha_d^+, \alpha_d^-)$. We use the notation

$$|\alpha| = \sum_{i=1}^{d} (\alpha_i^+ + \alpha_i^-)$$

and

$$\tilde{D}^d = (D_1^+)^{\alpha_1^+} (D_1^-)^{\alpha_1^-} \dots (D_d^+)^{\alpha_d^+} (D_d^-)^{\alpha_d^-}. \tag{3.3}$$

**Theorem 3.1.** *For initial condition $S_0(x)$ such that $|S_0(x)| = 1$ and $DS_0$ in $H^{m+1}(\mathbb{R}^d)$ $(m > d/2)$, there exists a constant $C$ such that on the time interval $[0, T_1[$ with $T_1 = C/|DS_0|^2_{H^{m+1}}$, Eq. (2.1) has a unique solution of unit length $S(x,t)$ with spatial derivatives $DS \in L^\infty(0, T_1, H^{m+1}(\mathbb{R}^d))$. The solution satisfies*

$$|DS(t)|^2_{H^{m+1}} \leq |DS_0|^2_{H^{m+1}} \exp \int_0^t |\mathrm{DS}(\tau)|^2_{W^{1,\infty}} d\tau. \tag{3.4}$$

*Proof of Theorem 3.1.* As in Sect. 2, we construct the solution of (2.1) as limits, when $h$ tends to zero of sequence $\{S_h(x_i)\}$ satisfying

$$\begin{cases} \dfrac{\partial S_h}{\partial t} = S_h \wedge \tilde{\Delta} S_h \\ S_h(x_i, 0) = S_h^0(x_i). \end{cases} \tag{3.5}$$

We choose $S_h^0(x_i)$ such that

$$p_h S_h^0 \to S_0 \tag{3.6}$$

and

$$p_h \tilde{D}^k S_h^0 \to D^k S_0 \quad \text{in } L^2 \text{ for any} \quad k = 1, \dots, m+1.$$

To obtain a priori estimates on $S_h$ and its "spatial derivatives," we first differentiate Eq. (3.5) with respect to $t$:

$$\frac{\partial^2 S_h}{\partial t^2} = (S_h \wedge \tilde{\Delta} S_h) \wedge \tilde{\Delta} S_h + S_h \wedge \tilde{\Delta}(S_h \wedge \tilde{\Delta} S_h). \tag{3.7}$$

Using identity (2.4) and the fact that

$$|S_h(x_i, t)| = |S_h^0(x_i)| = 1, \tag{3.8}$$

Eq. (3.7) is rewritten

$$\begin{aligned} \frac{\partial^2 S_h}{\partial t^2} + \tilde{\Delta}^2 S_h = {} & [-(\tilde{\Delta} S_h)^2 + (S_h \cdot \tilde{\Delta}^2 S_h)] S_h + (S_h \cdot \tilde{\Delta} S_h) \tilde{\Delta} S_h \\ & + (S_h \cdot D_i^+ \tilde{\Delta} S_h) D_i^+ S_h + (S_h \cdot D_i^- \tilde{\Delta} S_h) D_i^- S_h \\ & - (S_h \cdot D_i^+ \Delta S_h) D_i^+ \Delta S_h - (S_h \cdot D_i^- S_h) D_i^- \Delta S_h. \end{aligned} \tag{3.9}$$

One then deduces from (3.8) that

$$S_h \cdot \tilde{\Delta} S_h = -\tfrac{1}{2} \sum_{i=1}^{d} ((D_i^+ S_h)^2 + (D_i^- S_h)^2), \tag{3.10}$$

$$S_h \cdot \tilde{\Delta}^2 S_h = -\tfrac{1}{2} \tilde{\Delta}\{(D_i^+ S_h)^2 + (D_i^- S_h)^2\} - (\tilde{\Delta} S_h)^2 - D_i^+ S_h \cdot D_i^+ \tilde{\Delta} S_h - D_i^- S_h \cdot D_i^- \tilde{\Delta} S_h, \tag{3.11}$$

and (see Appendix A.a for details)

$$(S_h \cdot D_i^+ S_h) D_i^+ \tilde{\Delta} S_h + (S_h \cdot D_i^- S_h) D_i^- \tilde{\Delta} S_h = -\frac{h^2}{2} D_i^- \{(D_i^+ S)^2 (D_i^+ \Delta S)\}. \tag{3.12}$$

One also has (see Appendix A.b and A.c)

$$\begin{aligned} -(\tilde{\Delta} S_h)^2 + (S_h \cdot \tilde{\Delta}^2 S_h) = - \sum_{i,j=1}^{d} &\{\tfrac{1}{2} D_i^+ D_j^+ (D_i^- S_h \cdot D_j^- S_h) \\ &+ \tfrac{1}{2} D_i^- D_j^- (D_i^+ S_h \cdot D_j^+ S_h) + D_i^- D_j^+ (D_j^+ S_h \cdot D_i^- S_h)\}, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\{D_i^+ D_j^+ (D_i^- S_h \cdot D_j^- S_h)\} S_h(x_k) = D_i^+ D_j^+ ((D_i^- S_h \cdot D_j^- S_h) S_h) \\ &\quad + D_j^+ \{(D_i^+ S_h(x_k) \cdot D_j^- S_h(x_k + h_i)) D_i^+ S_h(x_k)\} \\ &\quad + \{D_i^+ D_j^+ S_h(x_k) \cdot D_j^+ S_h(x_k + h_i) + D_i^+ S_h(x_k) \cdot \tilde{\Delta} S_h(x_k + h_i)\} D_i^+ S_h(x_k), \end{aligned} \tag{3.14}$$

together with similar expressions for $D_j^{\pm} D_j^- (D_i^{\mp} S_j \cdot D_j^+ S_h)$.

Substituting in Eq. (3.9), and noticing that

$$\begin{aligned} (\tilde{\Delta} S_h(x_k + h_j) \cdot D_j^+ S(x_k)) + (S_h(x_k) \cdot D_j^+ \tilde{\Delta} S_h(x_k)) &= D_j^+ (S_h \cdot \tilde{\Delta} S_h) \\ &= -\tfrac{1}{2} D_j^+ ((D_i^+ S_h)^2 + (D_i^- S_h)^2), \end{aligned} \tag{3.15}$$

with

$$\tfrac{1}{2}(S_h \cdot \tilde{\Delta} S_h) \tilde{\Delta} S_h + \tfrac{1}{2} D_j^{\pm} (S_h \cdot \tilde{\Delta} S_h) D_j^{\pm} S_j = \tfrac{1}{2} D_j^{\pm} ((S_h \cdot \tilde{\Delta} S_h) D_j^{\mp} S_h), \tag{3.16}$$

Eq. (3.9) is rewritten:

$$\begin{aligned} \frac{\partial^2 S_h}{\partial t^2} + \tilde{\Delta}^2 S_h = \sum_{i,j} &- \{\tfrac{1}{2} D_i^+ D_j^+ [(D_i^- S_h \cdot D_j^- S_h) S_h] \\ &+ D_i^+ D_j^- [(D_i^- S_h \cdot D_j^+ S_h) S_h] + \tfrac{1}{2} D_i^- D_j^- [(D_i^+ S_h \cdot D_j^+ S_h) S_h]\} \\ &+ \tfrac{1}{2} D_j^+ \Big\{ (D_i^- S_h \cdot D_j^- S_h)(x_k + h_i) D_i^+ S_h + (D_i^+ S_h \cdot D_i^- S_h)(x_k - h_i) D_i^+ S_h D_i^- S_h \\ &- \frac{(D_i^+ S_h)^2 + (D_i^- S_h)^2}{2} D_j^- S_h \Big\} + \tfrac{1}{2} D_j^+ \Big\{ (D_i^- S_h \cdot D_i^+ S_h)(x_k + h_i) D_i^+ S_h \\ &+ (D_i^+ S_h \cdot D_j^+ S_h)(x_k - h_i) D_i^- S_h - \frac{(D_i^- S_h)^2 + (D_i^+ S_h)^2}{2} D_j^+ S_h \Big\} \\ &+ h^2 D_j^- \{\tfrac{1}{4}((D_i^+ D_j^+ S_h)^2 + (D_i^- D_j^+ S_h)^2) D_j^+ S_h - \tfrac{1}{2}(D_j^+ S_h)^2 D_j^+ \Delta S_h\}. \end{aligned} \tag{3.17}$$

To obtain a priori estimates on the "spatial derivatives" of $S_h$, one applies to both sides of Eq. (3.17) the operator $\tilde{D}^\alpha(|\alpha|=m)$ and take the $\tilde{L}_h^2$ scalar product with $\tilde{D}^\alpha \dfrac{\partial S_h}{\partial t}$. The left-hand side of the resulting equation reads:

$$\frac{1}{2}\frac{d}{dt}\left(\left|\tilde{D}^\alpha \frac{\partial S_h}{\partial t}\right|^2_{\tilde{L}_h^2} + |\tilde{D}^\alpha \tilde{\Delta} S_h|_{\tilde{L}_h^2}\right). \tag{3.18}$$

In the right-hand side we write

$$\begin{aligned}&\left(\tilde{D}^\alpha D_j^+\{(D_i^- S_h \cdot D_j^- S_h)(x_k+h_i) D_i^+ S_h)\}, \tilde{D}^\alpha \frac{\partial S_h}{\partial t}\right)_h \\ &\leqq |\tilde{D}^\alpha D_j^+\{(D_i^- S_h \cdot D_j^- S_h)(x_k+h_i) D_i^+ S_h\}|_{\tilde{L}_h^2} \left|\tilde{D}^\alpha \frac{\partial S_h}{\partial t}\right|_{\tilde{L}_h^2}.\end{aligned} \tag{3.19}$$

We then use Corollary (1.a) of Appendix B to obtain that the right-hand side of (3.19) is bounded from above by

$$C\{|\tilde{D}S_h|^2_{L_h^\infty} |\tilde{D}^{m+2} S_h|_{L_h^2} + |\tilde{D}S_h|_{\tilde{L}_h^\infty} |\tilde{D}^{m+1}(D_i^- S_h \cdot D_j^- S_h)(x_k+h_i)|_{\tilde{L}_h^2}\} \left|\tilde{D}^\alpha \frac{\partial S_h}{\partial t}\right|_{\tilde{L}_h^2}, \tag{3.20}$$

with

$$|\tilde{D}^{m+1}(D_i^- S_h \cdot D_j^- S_h)(x_k+h_i)|_{\tilde{L}_h^2} \leqq C|\tilde{D}S_h|_{\tilde{L}_h^\infty} |\tilde{D}^{m+2} S_h|_{\tilde{L}_h^2}. \tag{3.21}$$

It follows that the left-hand side of Eq. (3.19) is bounded from above by

$$C|\tilde{D}S_h|^2_{\tilde{L}_h^\infty} |\tilde{D}^{m+2} S_h|_{\tilde{L}_h^2} \left|\tilde{D}^m \frac{\partial S_h}{\partial t}\right|_{\tilde{L}_h^2}. \tag{3.22}$$

Analogous results are obtained for similar terms appearing in (3.17). Let's now turn to the terms of the form

$$\begin{aligned}&\left(\tilde{D}^\alpha D_i^+ D_j^+\{(D_i^- S_h \cdot D_j^- S_h) S_h\}, \tilde{D}^\alpha \frac{\partial S_h}{\partial t}\right)_h = \left(\tilde{D}^\alpha D_i^+ D_j^+\{(D_i^- S_h \cdot D_j^- S_h) S_h\}\right. \\ &\quad \left. - \tilde{D}^\alpha D_i^+ D_j^+ (D_i^- S_h \cdot D_j^- S_h) S_h, \tilde{D}^\alpha \frac{\partial S_h}{\partial t}\right)_h \\ &\quad + \left(\tilde{D}^\alpha D_i^+ D_j^+ (D_i^- S_h \cdot D_j^- S_h) S_h, \tilde{D}^\alpha \frac{\partial S_h}{\partial t}\right)_h.\end{aligned} \tag{3.23}$$

Using Corollary (1.b) of Appendix B, the first term of the right-hand side of Eq. (3.23) is bounded by

$$C(|\tilde{D}S_h|_{L_h^\infty} |\tilde{D}^{m+1}(D_i^- S_h \cdot D_j^- S_h)|_{\tilde{L}_h^2} + |\tilde{D}S_h|_{\tilde{L}_h^\infty} |\tilde{D}^{m+2} S_h|_{\tilde{L}_h^2}) \leqq C|\tilde{D}S_h|^2_{\tilde{L}_h^\infty} |\tilde{D}^{m+2} S_h|_{\tilde{L}_h^2}. \tag{3.24}$$

For the second term of right-hand side of Eq. (3.23), we write, using integration by parts:

$$h^d \sum_{x_k} \tilde{D}^\alpha D_i^+ D_j^+ (D_i^- S_h \cdot D_j^- S_h)\left(S_h \cdot \tilde{D}^\alpha \frac{\partial S_h}{\partial t}\right)$$
$$= -h^d \sum_{x_k} \tilde{D}^\alpha D_j^+ (D_i^- S_h \cdot D_j^- S_h) D_i^- \left(S_h \cdot \tilde{D}^\alpha \frac{\partial S_h}{\partial t}\right)$$
$$= -h^d \sum \tilde{D}^\alpha D_j^+ (D_i^- S_h \cdot D_j^- S_h)\left(D_i^- S_h(x_k) \cdot \tilde{D}^\alpha \frac{\partial S_h}{\partial t}(x_k - h_i)\right.$$
$$\left. + S_h(x_k) \cdot D_i^- \tilde{D}^\alpha \frac{\partial S_h}{\partial t}(x_k)\right). \tag{3.25}$$

The first term is similar to those already considered and is thus bounded from above by

$$C|\tilde{D}S_h|^2_{\tilde{L}_h^\infty} |\tilde{D}^{m+2} S_h|_{\tilde{L}_h^2} \left|\tilde{D}^m \frac{\partial S_h}{\partial t}\right|_{\tilde{L}_h^2}. \tag{3.26}$$

For the second term of (3.25) we use the fact that $S_h \cdot \frac{\partial S_h}{\partial t} = 0$ by writing

$$-S_h \cdot D_i^- \tilde{D}^\alpha \frac{\partial S_h}{\partial t} = D_i^- \tilde{D}^\alpha \left(S_h \cdot \frac{\partial S_h}{\partial t}\right) - S_h \cdot D_i^- \tilde{D}^\alpha \frac{\partial S_h}{\partial t}. \tag{3.27}$$

Using Eqs. (B.17) and (B.18) of Appendix B, its $\tilde{L}_h^2$-norm is bounded by

$$C\left(|\tilde{D}S_h|_{\tilde{L}_h^\infty} \left|\tilde{D}^m \frac{\partial S_h}{\partial t}\right|_{L_h^2} + \left|\frac{\partial S_h}{\partial t}\right|_{\tilde{L}_h^\infty} |\tilde{D}^{m+1} S_h|_{L_h^2}\right), \tag{3.28}$$

where

$$\left|\frac{\partial S_h}{\partial t}\right|_{\tilde{L}_h^\infty} \leqq |\tilde{\Delta} S_h|_{\tilde{L}_h^\infty}.$$

For the three last terms of Eq. (3.17), we use two factors $h$ to decrease the number of derivatives and thus obtain terms similar to those already bounded. For example,

$$h^2 D_j^- \{(D_i^+ D_j^+ S_h)^2 D_j^+ S_h\} = D_j^- \{(D_i^+ S_h(x_k + h_j) - D_i^+ S_h(x_k))^2 D_j^+ S_h\}. \tag{3.29}$$

Putting together all the estimates, we finally obtain

$$\frac{d}{dt}\left\{\left|\frac{\partial S_h}{\partial t}\right|^2_{\tilde{H}_h^m} + |\tilde{D}S_h|^2_{\tilde{H}_h^{m+1}}\right\} \leqq C(|\tilde{D}S|^2_{\tilde{L}_h^\infty} + |\tilde{D}^2 S|^2_{\tilde{L}_h^\infty}) \times \left(\left|\frac{\partial S_h}{\partial t}\right|^2_{\tilde{H}_h^m} + |\tilde{D}S|^2_{\tilde{H}_h^{m+1}}\right). \tag{3.30}$$

By using an extension to sequences of Sobolev-type inequalities (see Appendix B, Proposition 2), it follows that, for $m > d/2$,

$$\frac{d}{dt}\left\{\left|\frac{\partial S_h}{\partial t}\right|^2_{\tilde{H}_h^m} + |\tilde{D}S_h|^2_{\tilde{H}_h^{m+1}}\right\} \leqq C\left(\left|\frac{\partial S_h}{\partial t}\right|^2_{\tilde{H}_h^m} + |\tilde{D}S|^2_{\tilde{H}_h^{m+1}}\right)^2. \tag{3.31}$$

Thus $p_h\tilde{D}^k\frac{\partial S_h}{\partial t}$ and $p_h\tilde{D}^{k+1}S_h$ $(0\leqq k\leqq m)$ remain in a bounded set of $L^\infty(0, T_h, L^2)$ with $T_h=(C_1/|DS_h^0|^2_{\tilde{H}^{m+1}})$. For $h<1$, this time can be bounded from below by $C/|DS_0|^2_{H^{m+1}}$. We then pass to the limit in Eq. (2.1).

To prove (3.3), we differentiate Eq. (2.1) with respect to $t$, and get $(\partial_i=\partial/\partial x_i)$

$$\frac{\partial^2 S}{\partial t}+\Delta^2 S=-(\Delta S)^2 S+(S\cdot\Delta S)\Delta S+(S\cdot\Delta^2 S)S+2(S\cdot\partial_i\Delta S)\partial_i S\,. \quad (3.32)$$

Using that $S$ is of unit length, we have

$$\begin{aligned} S\cdot\Delta^2 S-(\Delta S)^2 &= -\Delta((\nabla S)^2)-2(\partial_i S\cdot\partial_i\Delta S)-2(\Delta S)^2\\ &= -2\{(\partial_i S\cdot\partial_i\Delta S)+(\partial^2_{ij}S)^2+(\Delta S)^2\}\\ &= -2\{\partial^2_{ij}(\partial_i S\cdot\partial j S)\}S\,. \end{aligned} \quad (3.33)$$

Thus, Eq. (3.32) reads

$$\begin{aligned} \frac{\partial^2 S}{\partial t^2}+\Delta^2 S &= -2\partial_{ij}(\partial_i S\cdot\partial_j S)S-(\nabla S)^2\Delta S+2(S\cdot\partial_i\Delta S)\partial_i S\\ &= -2\partial_{ij}[(\partial_i S\cdot\partial_j S)S]+2\partial_j[(\partial_i S\cdot\partial_j S)\partial_i S]+2(\Delta S\cdot\partial_j S)\partial_j S\\ &\quad+\partial_j(\nabla S)^2\partial_j S-(\nabla S)^2\Delta S+2(S\cdot\partial_j\Delta S)\partial_j S\\ &= -2\partial_{ij}[(\partial_i S\cdot\partial_j S)S]+2\partial_j[(\partial_i S\cdot\partial_j S)\partial_i S]-2\partial_j((\nabla S)^2\partial_j S)\,. \end{aligned}$$

On this equation, one establishes estimates (3.3) using the same method we used for the sequences. Uniqueness readily results from regularity.

## 4. The Special Case of Initially Quasiparallel Spins

*a) The Equations of Motion in the Stereographic Representation.* As long as it exists, the solution $S(x,t)$ remains on the sphere $S^2$ of radius 1. Therefore, we shall rewrite the equation of motion using the stereographic representation of the unit sphere $S^2$ on the plane $x_3=1$. Each point $S=(S_1,S_2,S_3)$ of the unit sphere [except the south pole $P=(0,0,-1)$] has an image $Q(\alpha,\beta,1)$ obtained as the intersection of the straight line $PS$ with the plane $x_3=1$. Defining

$$S^\pm=S_1\pm iS_2 \quad (4.1)$$

and

$$z=\alpha+i\beta\,, \quad (4.2)$$

we have the relations

$$S^+=\frac{4z}{4+|z|^2}\,, \quad (4.3)$$

$$S_3=\frac{4-|z|^2}{4+|z|^2}\,.$$

Equation (2.1) is first rewritten

$$\begin{cases} \dfrac{\partial S^+}{\partial t} = i(S_3 \Delta S^+ - S^+ \Delta S_3), \\ \dfrac{\partial S_3}{\partial t} = \dfrac{i}{2}(S^+ \Delta S^- - S^- \Delta S^+). \end{cases} \tag{4.4}$$

Substituting (4.3) in Eqs. (4.4), we have

$$\begin{cases} \dfrac{\partial S^+}{\partial t} = 4\left(\dfrac{z_t}{|z|^2+4} - z\dfrac{z\bar{z}_t + \bar{z}z_t}{(|z|^2+4)^2}\right) \\ \dfrac{\partial S_3}{\partial t} = -8\dfrac{z\bar{z}_t + \bar{z}z_t}{(|z|^2+4)^2}. \end{cases} \tag{4.5}$$

By elimination and substitution, we get

$$\begin{aligned} \frac{\partial z}{\partial t} &= \tfrac{1}{8}(|z|^2+4)\left(2\frac{\partial S^+}{\partial t} - z\frac{\partial S_3}{\partial t}\right) \\ &= \left\{-\Delta\left(\frac{1}{|z|^2+4}\right) z(|z|^2+4) + \frac{4}{|z|^2+4}\Delta z - \frac{z^2}{|z|^2+4}\Delta\bar{z} \right. \\ &\quad \left. + 8\nabla z \cdot \nabla\left(\frac{1}{|z|^2+4}\right) - 2z^2\nabla\bar{z} \cdot \nabla\left(\frac{1}{|z|^2+4}\right)\right\}. \end{aligned} \tag{4.6}$$

Computing $\nabla\left(\frac{1}{|z|^2+4}\right)$ and $\Delta\left(\frac{1}{|z|^2+4}\right)$, we finally get for the representative point $z$, the equation

$$i\frac{\partial z}{\partial t} + \Delta z = F(z, \nabla z), \tag{4.7}$$

with

$$F(z \cdot \nabla z) = \frac{2\bar{z}(\nabla z)^2}{4+|z|^2}.$$

*b) Existence of Global Solutions.* Equation (4.7) is a nonlinear Schrödinger equation which for small $z$ is essentially cubic. It however does not directly enter in the framework considered by Klainerman and Ponce [6] and Shatah [14] because $F'_{z_{x_i}}$ is not real. We shall consider initial conditions such that $z$ is "small." In the primitive variables, this means that initially the components $S_1^0(x)$ and $S_2^0(x)$ are "small" and $S_3^0(x)$ is "close" to 1. In other words, the spins are initially almost parallel.

**Proposition 4.1.** *(See for example* [6].*) The solution of the linear Schrödinger equation*

$$\begin{cases} i\dfrac{\partial z}{\partial t} + \Delta z = 0 \\ z(x,0) = z_0(x) \end{cases} \tag{4.8}$$

*satisfies*

$$|z(t)|_{L^q} \leqq C(1+t)^{-\frac{d}{2}+\frac{d}{q}} |z_0|_{W^{N_p,p}} \tag{4.9}$$

*for* $\frac{1}{p}+\frac{1}{q}=1$, $q \geqq 2$, *and* $N_p > d\frac{2-p}{p}$.

*Proof.* We recall elements of the proof for the sake of completeness. The solution of (4.8) satisfies.

$$|z(t)|_{L^\infty} \leqq C t^{-d/2} |z_0|_{L^1}, \tag{4.10}$$

and

$$|z(t)|_{L^2} = |z_0|_{L^2}. \tag{4.11}$$

By interpolation between $L^2$ and $L^\infty$, we have

$$|z(t)|_{L^q} \leqq C t^{\frac{-d}{2}+\frac{d}{q}} |z_0|_{L^p} \quad \text{for} \quad \frac{1}{p}+\frac{1}{q}=1,\ q \geqq 2. \tag{4.12}$$

To avoid divergence at $t=0$, we write, using Sobolev embedding theorems

$$|z(t)|_{L^q} \leqq C_1 |z(t)|_{W^{m,2}} = C_1 |z_0|_{W^{m,2}} \leqq C_2 |z_0|_{W^{N_p,p}} \tag{4.13}$$

with

$$\frac{1}{q} > \frac{1}{2} - \frac{m}{d} \quad \text{and} \quad \frac{1}{2} > \frac{1}{p} - \frac{N_p - m}{d} \quad \text{or} \quad N_p > d\frac{2-p}{p}.$$

Estimate (4.9) follows from (4.12) and (4.13).

**Lemma 4.1.** *For* $z \in W^{k,p}(\mathbb{R}^d)$ *and* $|z|_{L^\infty} < 1$, *then*

$$\textit{for} \quad i=1 \ \textit{or} \ 2 \ |D^k S_i|_{L^p} \leqq C |D^k z|_{L^p} \qquad (k \geqq 0) \tag{4.14}$$

$$|D^k S_3|_{L^p} \leqq C |D^k z|_{L^p} \qquad (k \geqq 1). \tag{4.15}$$

*Proof.* This is a particular case of a result of Moser [12], (see also [4]).

**Lemma 4.2.** *For* $z \in H^{m+N_{5/6}+2}(\mathbb{R}^d) \cap W^{m+1,6}(\mathbb{R}^d)$ $(m \geqq N_{5/6})$ *the functional* $F$ *defined in* (4.7) *satisfies:*

$$|F(z, \nabla z)|_{W^{m+N_{5/6}+1,5/6}} \leqq C |z|^2_{W^{m+1,6}} |\nabla z|_{H^{m+N_{5/6}+1}}. \tag{4.16}$$

*Proof.*

$$F(z, \nabla z) = 2\frac{\bar{z}(\nabla z)^2}{|z|^2+4}, \tag{4.17}$$

$$\partial_i F(z, \nabla z) = 2\left\{\frac{2\bar{z}\nabla z \cdot \partial_i \nabla z}{|z|^2+4} + \frac{\partial_i \bar{z}(\nabla z)^2}{|z|^2+4} - \frac{\bar{z}(\nabla z)^2 (z\partial_i \bar{z} + \bar{z}\partial_i z)}{(|z|^2+4)^2}\right\}. \tag{4.18}$$

Let us consider the first term of the right-hand side of (4.18) and write

$$D^{m+N_{5/6}}\left(\frac{\bar{z}\nabla z \cdot \partial_i \nabla z}{|z|^2+4}\right) = \sum_{j=0}^{m+N_{5/6}} C^j_{m+N_{5/6}} D^j\left(\frac{\bar{z}\nabla z}{|z|^2+4}\right) D^{m+N_{5/6}-j} \partial_i \nabla z. \tag{4.19}$$

Using Hölder inequalities, we have, for $0 \leqq j \leqq m$,

$$\left| D^j\left(\frac{\bar{z}\nabla z}{|z|^2+4}\right) D^{m+N_{5/6}-j}\partial_i \nabla z\right|_{L^{5/6}} \leqq C\left|\frac{z}{|z|^2+4}\right|_{W^{m,6}} |\nabla z|_{W^{m,6}} |\nabla \partial_i z|_{H^{m+N_{5/6}}}$$

(by Lemma 4.1)

$$\leqq C|z|^2_{W^{m+1,6}} |\nabla \partial_i z|_{H^{m+N_{5/6}}}, \tag{4.20a}$$

and for $j$ between $m+1$ and $m+N_{5/6}$:

$$\left| D^j\left(\frac{\bar{z}\nabla z}{|z|^2+4}\right) D^{m+N_{5/6}-j}\partial_i \nabla z\right|_{L^{5/6}} \leqq C\left| D^j\left(\frac{\bar{z}\nabla z}{|z|^2+4}\right)\right|_{L^{3/2}} |\nabla \partial_i z|_{W^{N_{5/6}-1,6}} \tag{4.20b}$$

with

$$D^j\left(\frac{\bar{z}\nabla z}{|z|^2+4}\right) = \sum_{k=0}^{j} C_k^j D^k\left(\frac{\bar{z}}{|z|^2+4}\right) D^{j-k}\nabla z\,. \tag{4.21}$$

The $(m+1)$ first terms in the summation (4.21) are bounded from above in $L^{3/2}$ by

$$C\left|\frac{z}{|z|^2+4}\right|_{W^{m,6}} |\nabla z|_{H^{m+N_{5/6}}}\,. \tag{4.22}$$

The $(j-m)$ last terms in the summation (4.21) are bounded from above in $L^{3/2}$ by

$$C\left| D^{m+1}\left(\frac{z}{|z|^2+4}\right)\right|_{H^{N_{5/6}-1}} |\nabla z|_{W^{N_{5/6}-1,6}}\,. \tag{4.23}$$

Putting together inequalities (4.19)–(4.23), one gets,

$$\left| D^{m+N_{5/6}}\left(\bar{z}\frac{(\nabla z)^2}{|z|^2+4}\right)\right|_{L^{5/6}} \leqq C|z|^2_{W^{m+1,6}} |\nabla z|_{W^{m+1+N_{5/6}}} \tag{4.24}$$

provided $m \geqq N_{5/6} > \dfrac{7d}{5}$.

Estimates for the other terms appearing in $F$ or $\nabla F$ are similar.

**Proposition 4.2.** *For initial data $S_0=(S_1^0, S_2^0, S_3^0)$ such that $|S_0(x)|=1$, $|S_1^0|_{H^{m+2}}$ and $|S_2^0|_{H^{m+2}}<\delta$ with $|S_3^0-1|_{H^{m+2}}<\delta$, where $m>d/2$ and $\delta$ sufficiently small, there exists a finite interval $[0, T_1]$ such that*

$$|S(x,t)|=1\,, \qquad \nabla S \in L^\infty(0, T_1, H^{m+1}(\mathbb{R}^d))$$

*and*

$$|z|_{L^\infty} = 2\left|\frac{S_1+iS_2}{S_3+1}\right|_{L^\infty} < 1\,.$$

During this time interval, $S$ satisfies estimate (3.3).

**Proposition 4.3.** *In dimension $d \geqq 2$, and under the hypothesis*

$$\begin{cases} |z_0|_{W^{m+1,6}} < \delta & (m > d/2) \\ |\nabla S_0|_{H^{m_1}} < \delta & m_1 \geqq m + N_{5/6} + 1 \end{cases}$$

*the quantity*

$$M(T_1) = \sup_{0 \leqq t \leqq T_1} (1+t)^{d/3} |z(t)|_{W^{m+1,6}} \tag{4.25}$$

*remains bounded by a constant $M_0$ independent of $T_1$.*

*Proof.* From the above propositions, we deduce that the solution of (4.7) satisfies, for $t \in [0, T_1]$

$$|z(t)|_{W^{m+1,6}} \leqq C(1+t)^{-d/3} |z_0|_{W^{m_1,5/6}} + \int_0^t (1+t-\tau)^{-d/3} |z(\tau)|^2_{W^{m+1,6}} |\nabla z(\tau)|_{H^{m_1}} d\tau . \tag{4.26}$$

with

$$|\nabla z(\tau)|_{H^{m_1}} \leqq C |\nabla S(\tau)|^2_{H^{m_1}} \leqq C |\nabla S_0|_{H^{m_1}} \exp \int_0^\tau |DS(\tau')|^2_{W^{1,\infty}} d\tau' . \tag{4.27}$$

But

$$\begin{aligned} |DS(\tau')|^2_{W^{1,\infty}} &\leqq C|z(\tau)|^2_{W^{2,\infty}} \leqq C|z(\tau')|^2_{W^{m+1,6}} \quad \text{for} \quad m+1 > \frac{d}{6} \\ &\leqq CM(T_1)^2 (\tau'+1)^{-2d/3} , \end{aligned} \tag{4.28}$$

and thus

$$\begin{aligned} \exp \int_0^\tau |DS(\tau')|^2_{W^{1,\infty}} d\tau' &\leqq \exp(CM(T_1)^2) \left( \int_0^\tau (\tau'+1)^{-2d/3} d\tau' \right) \\ &\leqq \exp(KM(T_1)^2) \quad \text{if} \quad d \geqq 2 . \end{aligned} \tag{4.29}$$

Substituting in (4.26), we obtain

$$M(T_1) \leqq \delta \left\{ 1 + M(T_1)^2 \exp(KM(T_1)^2) \int_0^t \frac{(1+t)^{d/3} d\tau}{(1+t-\tau)^{d/3} (1+\tau)^{2d/3}} \right\} . \tag{4.30}$$

For $d \geqq 2$ this integral is uniformly bounded in $t$, and $M(T_1)$ satisfies the estimate:

$$M(T_1) \leqq \delta(1 + M(T_1)^2 \exp(KM(T_1)^2)) . \tag{4.31}$$

Proceeding like in [6], one shows that $M(T_1)$ is uniformly bounded by a constant $M_0$ provided $\delta$ is sufficiently small.

**Theorem 4.1.** *In dimension $d \geqq 2$, for initial data $S_0$, such that $|S_0(x)| = 1$, $|S_1^0|_{W^{m+1,6}}$ and $|S_2^0|_{W^{m+1,6}} < \delta$ $(m > d/2)$ and $|\nabla S_0|_{H^{m_1+1}} < \delta$ $\left( m_1 \geqq m + \frac{7d}{5} + 1 \right)$, there exists a unique solution $S$ of Eq. (2.1) of unit length, with $S_1$ and $S_2$ in $L^\infty(\mathbb{R}^+, W^{m+1,6}(\mathbb{R}^d))$.*

*Moreover,*

$$|\nabla S(t)|_{W^{m,6}} = O(t^{-d/3}).$$

*Proof.* During the time $[O, T_1]$ where Theorem 3.1 holds, we have been using the above propositions:

$$\begin{aligned} |\nabla S(t)|_{H^{m_1}} &\leqq C|\nabla S^0|_{H^{m_1}} \exp\left\{C\int_0^t |\nabla S(\tau)|^2_{W^{1,\infty}} d\tau\right\} \\ &\leqq C|\nabla S^0|_{H^{m_1}} \exp\left\{C\int_0^t |\nabla z(\tau)|^2_{W^{1,\infty}} d\tau\right\} \\ &\leqq C|\nabla S^0|_{H^{m_1}} \exp\left\{C\int_0^t |\nabla z(\tau)|^2_{W^{m,6}} d\tau\right\} \\ &\leqq C|\nabla S^0|_{H^{m_1}} \exp\{CM_0^2\} \leqq C\delta \exp\{CM_0^2\}, \end{aligned} \tag{4.32}$$

and

$$|z(t)|_{L^\infty} \leqq \frac{CM_0}{(1+t)^{d/3}}.$$

Now, choose $\delta$ sufficiently small, such that $\dfrac{CM_0}{\left(1+\dfrac{T_1(\delta)}{2}\right)^{d/3}} < 1$. We then reapply the local existence theorem to obtain the existence of a smooth solution satisfying (3.3) and $|z|_{L^\infty} < 1$ for time $t \in [T_1, T_2]$. During this period of time, we have $M(t) \leqq M_0$ with the same $M_0$ and

$$|z|_{L^\infty} \leqq \frac{CM_0}{(1+t)^{d/3}}.$$

We reapply the local existence theorem to obtain the global solution.

*c) Long-Time Behaviour of the Solutions*

**Theorem 4.2.** *Under the hypothesis of Theorem 4.1, the solution of* (4.7) *exists for* $-\infty < t < +\infty$. *Moreover, there exist two solutions* $z^\pm$ *of the linear Schrödinger equation such that*

$$|z - z^\pm|_{H^{m_1}} \to 0, \quad \text{where} \quad t \to \pm\infty.$$

*Proof.* The existence of solutions $-\infty < t < +\infty$ follows from changing $t$ to $-t$ and applying Theorem 4.1. Proceeding as in [14], we define $z^+$ and $z^-$ by

$$z^\pm(t) = z(t) + \int_t^{\pm\infty} U_0(t-\tau) F(z, \nabla z)(\tau)\, d\tau, \tag{4.33}$$

where $U_0$ denotes the Green function of the linear Schrödinger equation,

$$\int_{-\infty}^{+\infty} |U_0(t-\tau) F(z, \nabla z)|_{H^{m_1}}(\tau)\, d\tau \leqq \int_{-\infty}^{+\infty} |F(z, \nabla z)_{H^{m_1}}(\tau)\, d\tau,$$

where

$$|F(z, \nabla z)|_{H^{m_1}} \leqq C|z|^2_{W^{1,\infty}} |\nabla z|_{H^{m_1-1}} \leqq C|z|^2_{W^{m,6}} |\nabla z|_{H^{m_1-1}} \quad \text{for} \quad m > d/6$$
$$\leqq \frac{C}{(1+|\tau|)^{2d/3}} \sup_\tau |\nabla S|^{(\tau)}_{H^{m_1-1}} \text{ (from 4.32)}$$
$$\leqq \frac{C}{(1+|\tau|)^{2d/3}}, \tag{4.34}$$

where $C$ denotes various constant. Therefore

$$|z(t) - z^{\pm}(t)|_{H^{m_1}} \leqq \frac{C}{(1+|t|)^{2d/3-1}}. \tag{3.35}$$

Finally, it is easily checked that $z^{\pm}$ satisfy the linear Schrödinger equation.

## 5. The One-Dimensional Problem

The one-dimensional problem is specific in the sense that it is completely integrable. Lakshmanan et al. [8] interpreted the vector $S$ as a unit vector tangent to a curve having curvature $\kappa = \left|\frac{\partial S}{\partial x}\right|$ and torsion $\tau = \frac{1}{\kappa^2} S \cdot \left(\frac{\partial S}{\partial x} \wedge \frac{\partial^2 S}{\partial x^2}\right)$. They showed that the function

$$\psi(x, t) = \kappa(x, t) \exp i \int^x \tau(x, t)\, dx \tag{5.1}$$

satisfies the cubic Schrödinger equation which can be solved by inverse scattering (Zakharov and Shabat [16]). A different approach was used by Takhajan [15] who introduced the matrix

$$S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \tag{5.2}$$

and put Eq. (2.1) in the form of the Lax representation

$$\frac{\partial L}{\partial t} = i[L, M] \tag{5.3}$$

with

$$L = S\frac{\partial}{\partial x} \quad \text{and} \quad M = 2S\frac{\partial^2}{\partial x^2} + \frac{\partial S}{\partial x}\frac{\partial}{\partial x}. \tag{5.4}$$

From the general local existence theorem proved in Sect. 3, we know that for initial data $S_0(x)$ such that $|S_0(x)| = 1$ and $|\nabla S_0| \in H^{m+1}(\mathbb{R})$ $(m > \frac{1}{2})$, there exists a unique solution $S$ of (1.3) with unit length and $\nabla S \in L^\infty([O, T_1], H^{m+1})$, where $T_1 \sim C/|\nabla S_0|^2_{H^{m+1}}$. Our aim in this section is to show that in one dimension the local solution $S$ can be continued for all time and that it is the limit as $h \to 0$ of the solution $S_h$ of (1.2) with initial conditions $S_h^0$ such that

$$p_h S_h^{(0)} \to S_0$$

and

$$p_h \tilde{D}^k S_h^{(0)} \to D^k S_0 \text{ in } L^2 \quad \text{for} \quad k=1,\dots,m.$$

**Proposition 5.1.** *In one dimension, the solution of* (2.3) *satisfies for all time* $t \in [O, T_1]$,

$$\left|\frac{\partial S}{\partial t}\right|_{L^2}^2 + \left|\frac{\partial^2 S}{\partial x^2}\right|_{L^2}^2 - \frac{3}{4}\left|\frac{\partial S}{\partial x}\right|_{L^4}^4 = E_2, \tag{5.5}$$

*where $E_2$ is a constant.*

*Proof.* We take the $L^2$ scalar product of Eq. (3.32) with $S_t$ and observe that

$$2\int\left(S \cdot \frac{\partial^3 S}{\partial x^3}\right)\left(\frac{\partial S}{\partial x} \cdot \frac{\partial S}{\partial t}\right) dx = \frac{3}{4}\frac{d}{dt}\int\left(\frac{\partial S}{\partial x}\right)^4 dx. \tag{5.6}$$

**Theorem 5.1.** *For initial data $S_0(x)$ with $|S_0(x)|=1$ and $\frac{dS_0}{dx} \in H^m(\mathbb{R})$, there exists for all time $T$ a unique solution of* (2.3) *with unit length such that $\frac{\partial S}{\partial x} \in L^\infty(O, T, H^m)$.*

*Proof.* It follows immediately from (5.5) that

$$\left|\frac{\partial S}{\partial t}\right|_{L^2}^2 + \left|\frac{\partial^2 S}{\partial x^2}\right|_{L^2}^2 \leqq |E_2| + C\left|\frac{\partial S_0}{\partial x}\right|_{L^2}^3 \left|\frac{\partial^2 S}{\partial x^2}\right|_{L^2},$$

and thus

$$\left|\frac{\partial S}{\partial t}\right|_{L^2} \quad \text{and} \quad \left|\frac{\partial^2 S}{\partial x^2}\right|_{L^2} \quad \text{bounded uniformly in } [O, T_1].$$

Differentiating $m$ times Eq. (2.3), we can see easily that all the derivatives of $S$ are bounded independently of $T_1$. This enables us to apply repeatedly the local existence theorem and prove global existence. Uniqueness of solutions results from regularity properties.

**Theorem 5.2.** *In dimension $d=1$, for any $T>0$, the solution $S_h$ of the approximate equation* (1.2) *converges in $L^\infty(O, T, H^1)$ to the solution $S$ of Eq.* (1.3), *with initial conditions satisfying*

$$\begin{cases} p_h S_h^{(0)} \to S_0 \\ p_h \tilde{D}^k S_h^{(0)} \to D^k S_0 \end{cases} \text{in } L^2 \quad \text{for} \quad h=1,\dots,m.$$

*Proof.* The difference $u_h(x_i) = S(x_i) - S_h(x_i)$ defined at the points of the lattice satisfies:

$$\frac{\partial}{\partial t} u_h(x_i) = u_h(x_i) \wedge \Delta S(x_i) + S_h(x_i) \wedge \tilde{\Delta} u_h(x_i) + S_h(x_i) \wedge (\Delta - \tilde{\Delta}) S(x_i). \tag{5.7}$$

Taking the $\tilde{L}_h^2$ scalar product with $u_h$, one obtains

$$\frac{1}{2}\frac{d}{dt}|u_h|_{\tilde{L}_h^2}^2 = (S_h \wedge \tilde{\Delta} u_h, u_h)_h + (S_h \wedge (\Delta - \tilde{\Delta}) S, u_h)_h, \tag{5.8}$$

where

$$(S_h \wedge \tilde{\Delta} u_h, u_h)_h = \left( D^+ S_h \wedge D^+ u_h, \frac{u_h(x_i) + u_h(x_{i+1})}{2} \right)_h$$

$$\leqq |D^+ S_h^0|_{\tilde{L}_h^2} |u_h|_{\tilde{L}_h^\infty} |D^+ u_h|_{\tilde{L}_h^2} \leqq |D^+ S_h^0|_{\tilde{L}_h^2} |u_h|^2_{\tilde{H}_h^1}, \tag{5.9a}$$

and

$$\leqq (S_h \wedge (\Delta - \tilde{\Delta}) S, u_h)_h \leqq |u_h|_{\tilde{L}_h^2} |(\Delta - \tilde{\Delta}) S|_{\tilde{L}_h^2}. \tag{5.9b}$$

Taking the $\tilde{L}_h^2$ scalar of Eq. (5.7) with $\tilde{\Delta} u_h$, one obtains

$$\frac{1}{2} \frac{d}{dt} |D^+ u_h|^2_{\tilde{L}_h^2} = (u_h \wedge \Delta S, \tilde{\Delta} u_h)_h + (S_h \wedge (\Delta - \tilde{\Delta}) S, \tilde{\Delta} u_h)_h, \tag{5.10a}$$

$$\begin{aligned}(u_h \wedge \Delta S, \Delta u_h)_h &= -\tfrac{1}{2}((D^+(u_h \wedge D^- u_h) + D^-(u_h \wedge D^+ u_h)), \Delta S)_h \\ &= \tfrac{1}{2}(u_h \wedge D^- u_h, D^- \Delta S)_h + \tfrac{1}{2}(u_h, D^+ u_h, D^+ \Delta S)_h \end{aligned} \tag{5.10b}$$

$$(u_h \wedge D^- u_h, D^- \Delta S)_h \leqq |u_h|^2_{\tilde{H}_h^1} |D^- \Delta S|_{\tilde{L}_h^\infty}.$$

From Proposition 5.1, we have for all $T$

$$\sup_{0 \leqq t \leqq T} |D^- \Delta S|_{\tilde{L}_h^\infty} \leqq \kappa(T).$$

Putting together Eq. (5.8), Eq. (5.10), we have for any $t \in [O, T]$,

$$\frac{d}{dt} |u_h|_{\tilde{H}_h^1} \leqq C_1 (\kappa(T) |u_h|_{\tilde{H}_h^1} + |\nabla^+ (\Delta - \tilde{\Delta}) S|_{\tilde{L}_h^2}),$$

$$|\nabla^+ (\Delta - \tilde{\Delta}) S|_{\tilde{L}_h^2} \leqq C h^2 |D^5 S|_{L^2} \leqq C h^2 \kappa_1(T).$$

Thus

$$|u_h|_{\tilde{H}_h^1} \leqq |u_h^0|_{\tilde{H}_h^1} \exp(C_1 \kappa(T) t) + \frac{C h^2 \kappa_1(T)}{C_1 \kappa(T)} (\exp(C_1 \kappa(T) t) - 1).$$

When $h \to 0$, $|u_h|_{\tilde{H}_h^1} \to 0$.

## Appendix A

In this appendix, we give some details on the obtention of Eqs. (3.12)–(3.14).

(a) We first show that

$$E \equiv (S_h \cdot D_j^+ S_h) D_j^+ \tilde{\Delta} S_h + (S_h \cdot D_j^- S_h) D_j^- \tilde{\Delta} S_h = -\frac{h^2}{2} D_j^- \{(D_j^+ S_h)^2 D_j^+ \tilde{\Delta} S_h\}. \tag{A.1}$$

In the following, we shall drop the indices $h$ for simplicity. From $|S(x_i)| = 1$, we deduce

$$S \cdot D_j^+ S = -\frac{h}{2} (D_j^+ S)^2, \tag{A.2}$$

and

$$S \cdot D_j^- S = \frac{h}{2}(D_j^- S)^2 ,$$

$$\begin{aligned}(S \cdot D_j^+ S) D_j^+ \tilde{\Delta} S &= -\frac{h}{2}(D_j^+ S)^2 D_j^+ \tilde{\Delta} S \\ &= -\frac{h}{2}\{D_j^- [(D_j^+ S)^2 (x_i) \tilde{\Delta} S(x_i + h_j)] - D_j^- (D_j^+ S)^2 (x_i) \tilde{\Delta} S(x_i)\} ,\end{aligned}$$

$$\begin{aligned}(S \cdot D_j^- S) D_j^- \tilde{\Delta} S &= \frac{h}{2}\{D_j^+ [(D_j^- S)^2 (x_i) \tilde{\Delta} S(x_i - h_j)] - D_j^+ (D_j^- S)^2 (x_i) \tilde{\Delta} S(x_i)\} \\ &= \frac{h}{2}\{D_j^- [(D_j^+ S)^2 (x_i) \tilde{\Delta} S(x_i)] - D_j^- (D_j^+ S)^2 (x_i) \tilde{\Delta} S(x_i)\} . \qquad \text{(A.3)}\end{aligned}$$

This leads to (A.1).

(b) To prove Eq. (3.13), one first writes:

$$\begin{aligned}-(\tilde{\Delta} S)^2 + S_h \cdot \tilde{\Delta}^2 S_h = {} & \tfrac{1}{2} \tilde{\Delta}((D_i^+ S)^2 + (D_i^- S)^2) \\ & -2(\tilde{\Delta} S)^2 - (D_i^+ S \cdot D_i^+ \tilde{\Delta} S) - (D_i^- S \cdot D_i^- \tilde{\Delta} S) , \qquad \text{(A.4)}\end{aligned}$$

with

$$\begin{aligned}\tfrac{1}{2} \tilde{\Delta}((D_i^- S)^2 + (D_i^- S)^2) = {} & (D_i^+ S \cdot D_i^+ \tilde{\Delta} S) + (D_i^- S \cdot D_i^- \tilde{\Delta} S) \\ & + \sum_{i,j} \{\tfrac{1}{2}(D_i^+ D_j^+ S)^2 + \tfrac{1}{2}(D_i^- D_j^- S)^2 + (D_i^+ D_j^- S)^2\} , \qquad \text{(A.5)}\end{aligned}$$

and checks that

$$\begin{aligned}&\sum_{i,j} D_i^+ D_j^+ (D_i^- S \cdot D_j^- S) = (\tilde{\Delta} S)^2 + \sum_i (D_i^+ S \cdot D_i^+ \tilde{\Delta} S) + \sum_{i,j} (D_i^+ D_j^+ S)^2 , \\ &\sum_{i,j} D_i^- D_j^- (D_i^+ S \cdot D_j^+ S) = (\tilde{\Delta} S)^2 + \sum_i (D_i^- S \cdot D_i^- \tilde{\Delta} S) + \sum_{i,j} (D_i^- D_j^- S)^2 , \qquad \text{(A.6)} \\ &\sum_{i,j} D_i^- D_j^+ (D_i^- S \cdot D_j^+ S) = (\tilde{\Delta} S)^2 + \sum_i (D_i^- S \cdot D_i^- \tilde{\Delta} S) \\ &\qquad + \sum_i D_i^+ S \cdot D_i^+ \tilde{\Delta} S + \sum_{i,j} (D_i^+ D_j^- S)^2 .\end{aligned}$$

Combining (A.4)–(A.6) one obtains Eq. (3.13).

(c) To prove Eq. (3.14) one computes $D_i^+ D_j^+ [(D_i^- S \cdot D_j^- S) S]$ and obtains

$$\begin{aligned}\sum_{i,j} D_i^+ D_j^+ [(D_i^- S \cdot D_j^- S) S] &= D_i^+ \{D_j^+ (D_i^- S \cdot D_j^- S) S + (D_i^- S(x_k + h_j) \cdot D_j^+ S(x_k)) D_j^+ S\} \\ &= D_i^+ D_j^+ (D_i^- S \cdot D_j^- S) S + D_j^+ (D_i^+ S(x_k) \cdot D_j^- S(x_k + h_i)) D_i^+ S \\ &\quad + D_i^+ (D_i^- S(x_k + h_j) \cdot D_j^+ S(x_k)) D_j^+ S + (D_i^+ S(x_k + h_j) \cdot D_j^+ (x_k + h_i)) D_i^+ D_j^+ S \\ &= D_i^+ D_j^+ (D_i^- S \cdot D_j^- S) S + \{D_j^+ (D_i^+ S(x_k) \cdot D_j^- S(x_k + h_i)) D_i^+ S\} \\ &\quad + \{\Delta S(x_k + h_j) \cdot D_j^+ S(x_k) + D_i^+ S(x_k + h_j) \cdot D_i^+ D_j^+ S(x_k)\} D_j^+ S . \qquad \text{(A.7)}\end{aligned}$$

In the last expression

$$D_i^+ S(x_k+h_j) \cdot D_i^+ D_j^+ S(x_k) = D_j^+ \frac{(D_i^+ S)^2}{2} + \frac{h}{2}(D_i^+ D_j^+ S)^2 . \qquad (A.8)$$

Similarly,

$$\begin{aligned}\sum_{i,j} D_i^- D_j^- [(D_i^+ S \cdot D_j^+ S)S] &= D_i^- \{D_j^- (D_i^+ S \cdot D_j^+ S)S + (D_i^+ S(x_k-h_j) \cdot D_j^- S(x_k)) D_j^- S\} \\ &= D_i^- D_j^- (D_i^+ S \cdot D_j^+ S)S + D_j^- (D_i^- S(x_k) \cdot D_j^+ S(x_k-h_i)) D_i^- S \\ &\quad + D_i^- (D_i^+ S(x_h-h_j) \cdot D_j^- S(x_k)) D_j^- S + (D_i^- S(x_k-h_j) \cdot D_j^+ S(x_k+h_i)) D_i^- D_j^- S \\ &= D_i^- D_j^- (D_i^+ S \cdot D_j^+ S)S + \{D_j^- (D_i^- S(x_k) \cdot D_j^- S(x_k-k_i)) D_i^- S\} \\ &\quad + \{\tilde{\Delta} S(x_k-h_j) \cdot D_j^- S(x_k) + D_i^- S(x_k-h_j) \cdot D_i^- D_j^- S(x_k)\} D_j^- S , \qquad (A.9)\end{aligned}$$

with

$$D_i^- S(x_k-h_j) \cdot D_i^- D_j^- S(x_k) = \tfrac{1}{2} D_j^- (D_i^- S)^2 - \frac{h}{2}(D_i^- D_j^- S)^2 , \qquad (A.10)$$

and

$$\begin{aligned}D_i^+ D_j^- [(D_i^- S \cdot D_j^+ S)S] &= D_i^+ \{D_j^- (D_i^- S \cdot D_j^+ S)S + (D_i^- S(x_k-h_j) \cdot D_j^- S(x_k)) D_j^- S\} \\ &= D_i^+ D_j^- (D_i^+ S \cdot D_j^+ S)S + D_j^- (D_i^+ S(x_k) \cdot D_j^+ S(x_k+h_i)) D_i^+ S \\ &\quad + D_i^+ (D_i^- S(x_k-h_j) \cdot D_j^- S(x_k)) D_j^- S + (D_i^+ S(x_k-h_j) \cdot D_j^- (x_k+h_i)) D_i^+ D_j^- S \\ &= D_i^+ D_j^- (D_i^- S \cdot D_j^+ S)S + D_j^+ \{(D_i^+ S(x_k) \cdot D_j^+ S(x_k+h_i)) D_i^+ S\} \\ &\quad + \{\tilde{\Delta} S(x_k-h_j) \cdot D_j^- S(x_k) + D_i^+ S(x_k-h_j) \cdot D_i^+ D_j^- S(x_k)\} D_j^- S \qquad (A.11)\end{aligned}$$

with

$$D_i^+ S(x_k-h_j) \cdot D_i^+ D_j^- S(x_k) = D_j^- \frac{(D_i^+ S)^2}{2} - \frac{h}{2}(D_i^+ D_j^- S)^2 . \qquad (A.12)$$

Also

$$\begin{aligned}D_i^+ D_j^- [(D_i^- S \cdot D_j^+)S] &= D_j^- D_i^+ [(D_i^- S \cdot D_j^+)S] \\ &= D_j^- \{D_i^+ (D_i^- S \cdot D_j^- S)S + (D_i^+ S(x_k) \cdot D_j^+ S(x_k+h_i)) D_i^+ S\} \\ &= D_j^- D_i^+ (D_i^- S \cdot D_j^+ S)S + D_i^+ (D_i^- S(x_k-h_j) \cdot D_j^- S(x_k)) D_j^- S \\ &\quad + D_j^- (D_i^+ S(x_k) \cdot D_j^+ S(x_k+h_i)) D_i^+ S + (D_i^+ S(x_k-h_j) \cdot D_j^- S(x_k+h_i)) D_j^- D_i^+ S \\ &= D_j^- D_i^+ (D_i^- S \cdot D_j^+ S)S + D_i^+ \{(D_i^- S(x_k-h_j) \cdot D_j^- (x_k)) D_j^- S\} \\ &\quad + \{(D_i^+ S(x_k) \cdot \tilde{\Delta} S(x_k+h_i)) + (D_j^- D_i^+ S(x_k) \cdot D_j^- S(x_k+h_i))\} D_i^+ S \qquad (A.13)\end{aligned}$$

with

$$D_j^- D_i^+ S(x_k) \cdot D_j^- S(x_k+h_i) = \tfrac{1}{2} D_i^+ (D_j^- S)^2 + \frac{h}{2}(D_i^+ D_j^- S)^2 . \qquad (A.14)$$

Putting together (A.7)–(A.14), we obtain:

$$\begin{aligned}
&\sum_{i,j}\tfrac{1}{2}D_i^+D_j^+[(D_i^-S\cdot D_j^-S)S]+\tfrac{1}{2}D_i^-D_j^-[(D_i^+S\cdot D_j^+S)S]+D_i^+D_j^-[(D_i^-S\cdot D_j^+S)S]\\
&\quad=\{\tfrac{1}{2}D_i^+D_j^+(D_i^-S\cdot D_j^-S)+\tfrac{1}{2}D_i^-D_j^-(D_i^+S\cdot D_j^+S)+D_i^+D_j^-(D_i^-S\cdot D_j^+S)\}S\\
&\qquad+\tfrac{1}{2}D_j^+\{(D_i^-S\cdot D_j^-S)(x_k+h_i)D_i^+S+(D_j^-S\cdot D_i^+S)(x_k-h_i)D_i^-S\}\\
&\qquad+\tfrac{1}{2}D_j^-\{(D_i^+S\cdot D_j^+S)(x_k-h_i)D_i^-S+(D_i^-S\cdot D_j^+S)(x_k+h_i)D_i^+S\}\\
&\qquad+(\tilde{\Delta}S(x_k+h_j)\cdot D_j^+S(x_k))D_j^+S+((\tilde{\Delta}S(x_k-h_j)\cdot D_j^-S(x_k))D_j^-S\\
&\qquad+\tfrac{1}{4}D_j^+((D_i^+S)^2+(D_i^-S)^2)D_j^+S+\tfrac{1}{4}D_j^-((D_i^+S)^2+(D_i^-S)^2)D_j^-S\\
&\qquad+\frac{h}{4}(D_i^+D_j^+S)^2D_j^+S-\frac{h}{4}(D_i^+D_j^-S)^2D_j^-S+\frac{h}{2}(D_i^-D_j^+S)^2D_j^+S\\
&\qquad-\frac{h}{2}(D_i^-D_j^-S)^2D_j^-S\,. \qquad\qquad (A.15)
\end{aligned}$$

In the last expression,

$$\begin{aligned}
&(D_i^+D_j^+S)^2D_j^+S-(D_i^+D_j^-S)^2D_j^-S\\
&\quad=(D_i^+D_j^+S)^2(x_k)D_j^+S(x_k)-(D_i^+D_j^+S)^2(x_k-h_j)D_j^+S(x_k-h_j)\\
&\quad=hD_j^-\{(D_i^+D_j^+S)^2D_j^+S\} \qquad\qquad (A.16)
\end{aligned}$$

and

$$(D_i^-D_j^+S)^2D_j^+S-(D_i^-D_j^-S)^2D_j^-S=hD_j^-\{(D_i^-D_j^+S)^2D_j^+S\}\,. \qquad (A.17)$$

## Appendix B

Discrete versions of Sobolev type inequalities were established by Ladysenskaya [7] using interpolation procedures. She proved in particular

$$|f_h|_{\tilde{L}_h^p}\leqq C|f_h|_{\tilde{W}_h^{1,q}}\frac{1}{p}=\frac{1}{q}-\frac{1}{d}\,. \qquad (B.1)$$

One also has

$$|f_h|_{\tilde{L}_h^\infty}\leqq C|f_h|_{\tilde{W}_h^{1,q}}q>d\,. \qquad (B.2)$$

The idea of the proof is to show the "equivalence" of the norms

$$\left|\frac{\partial}{\partial x_i}p_hf_h\right|_{L^q}\quad\text{and}\quad |D_i^\pm f_h|_{\tilde{L}_h^q}$$

and apply the Sobolev Gagliardo-Nirenberg inequalities to the function $p_hf_h$.

As a consequence, we have (by induction)

$$|f_h|_{\tilde{L}_h^\infty}\leqq C|f_h|_{\tilde{H}_h^m}\quad\text{for}\quad m>d/2\,. \qquad (B.3)$$

Another embedding property we need we is

$$|\tilde{D}^jf_h|_{\tilde{L}_h^p}\leqq C|f_h|_{\tilde{L}_h^q}^{1-j/m}|\tilde{D}^mf_h|_{\tilde{L}_h^r}^{j/m} \qquad (B.4)$$

with

$$\frac{1}{p}=\frac{j}{m}\frac{1}{r}+\left(1-\frac{j}{m}\right)\frac{1}{q}. \tag{B.5}$$

The proof of (B.4) is similar to that given by Nirenberg [13] (see also Friedman [2]) in the case of the functions. We shall thus only reformulate a lemma which requires some adjustments.

**Lemma.** *Any one-dimensional sequence* $f=\{f_i\}_{i=-\infty}^{i=+\infty}$ *satisfies*

$$\sum_{i=K}^{K+N}\left|\frac{f_{i+1}-f_i}{h}\right|^p \leqq C^p(Nh)^{1+p-\frac{p}{r}}\left(\sum_{i=K}^{K+N}\left|\frac{f_{i+1}-2f_i+f_{i-1}}{h^2}\right|\right)^{p/r} + C^p(Nh)^{-(1+p-p/r)}\left(\sum_{i=K}^{K+N}|f_i|^q\right)^{p/q}, \tag{B.6}$$

*where C is a numerical constant independent of N.*

*Proof.* We write $(l<k)$

$$\begin{aligned} f_{k+1}-f_k &= [(f_{k+1}-f_k)-(f_k-f_{k-1})]+[(f_k-f_{k-1})-(f_{k-1}-f_{k-2})]+\dots \\ &\quad +[(f_{l+1}-f_l)-(f_l-f_{l-1})]+f_l-f_{l-1} \end{aligned} \tag{B.7}$$

(and similar equality when $l>k$).

With the notations defined in Sect. 2, Eq. (B.7) reads:

$$(D^+f)_k=h\sum_{i=l}^{k}(\tilde{\Delta}f)_i+(D^-f)_l. \tag{B.8}$$

Writing Eq. (B.8) for any $l$ between $l_1+1$ and $l_2$ with $K<l_1<K+\frac{N}{4}$ and $K+\frac{3N}{4}<l_2<N$ and summing the resulting equations, one obtains:

$$(l_2-l_1)(D^+f)_k=(l_2-l_1)h\sum_{i=K}^{K+N}(\tilde{\Delta}f)_i+\frac{f_{l_1}-f_{l_2}}{h}. \tag{B.9}$$

Since $|l_2-l_1|>N/2$, we have

$$|(D^+f)_k|\leqq h\sum_{i=K}^{K+N}|(\tilde{\Delta}f)_i|+\frac{2(|f_{l_1}|+|f_{l_2}|)}{hN}. \tag{B.10}$$

Summing for all $l_1$ with $K\leqq l_1\leqq K+\frac{N}{4}$ and all $l_2$ with $K+\frac{3N}{4}\leqq l_2\leqq K+N$, one gets

$$\left(\frac{Nh}{4}\right)^2|(D^+f)_k|\leqq\left(\frac{Nh}{4}\right)^2 h\sum_{i=K}^{K+N}|(\tilde{\Delta}f)_i|+\frac{h}{2}\sum_{i=K}^{K+N}|f_i|, \tag{B.11}$$

and thus

$$\left(\frac{Nh}{4}\right)^{2p}|(D^+f)_k|^p\leqq C_p\left\{\left(\frac{Nh}{4}\right)^{2p}h^p\left(\sum_{i=K}^{K+N}|(\tilde{\Delta}f)_i|\right)^p+\frac{h^p}{2^p}\left(\sum_{i=K}^{K+N}|f_i|\right)^p\right\}. \tag{B.12}$$

By Hölder inequality

$$\sum_{i=K}^{K+N} |f_i| \leqq N^{1-\frac{1}{q}} \left( \sum_{i=K}^{K+N} |f_i|^q \right)^{1/q}, \tag{B.13}$$

$$h^p \left( \sum_{i=K}^{K+N} |f_i| \right)^p \leqq (hN)^{p\left(1-\frac{1}{q}\right)} \left( h \sum_{i=K}^{K+N} |f_i|^q \right)^{p/q}, \tag{B.14}$$

$$h^p \left( \sum_{i=K}^{K+N} |(\tilde{\Delta} f)_i| \right)^p \leqq (hN)^{p\left(1-\frac{1}{r}\right)} \left( h \sum_{i=K}^{K+N} 1 |(\tilde{\Delta} f)_i|^r \right)^{p/r}. \tag{B.15}$$

Substituting in (B.12) and assuming on $k$, $(K \leqq k \leqq N+K)$, we have $\left( \frac{2}{p} = \frac{1}{q} + \frac{1}{r} \right)$

$$h \sum_{i=K}^{K+N} |(D^+ f)_k|^p \leqq C_p \left\{ (Nh)^{1+p-p/r} \left( h \sum_{i=K}^{K+N} |(\tilde{\Delta} f)_i|^r \right)^{p/r} + (Nh)^{-1-p+p/r} \left( h \sum_{i=K}^{K+N} |f_i|^q \right)^{p/q} \right\}. \tag{B.16}$$

This lemma is used to prove (B.4) in the case $d=1, j=1$, and $m=2$. Extension to dimension $d>1$ is done by applying the previous result to each $D_k^{\pm} f$ treating all the indices $i \neq k$ as parameters. We then sum all these inequalities and use Hölder inequalities. Extension to other values of $j$ and $m$ satisfying (B.5) is done by induction.

As consequences of (B.4) we have the following inequalities used in Sect. 3:

$$|\tilde{D}^\alpha (f_h g_h)|_{\tilde{L}_h^2} \leqq C(|f_h|_{\tilde{L}_h^\infty} |\tilde{D}^m g|_{\tilde{L}_h^2} + |g_h|_{\tilde{L}_h^\infty} |\tilde{D}^m f|_{\tilde{L}_h^2}), \tag{B.17}$$

and

$$|\tilde{D}^\alpha (f_h g_h) - f_h \tilde{D}^\alpha g|_{\tilde{L}_h^2} \leqq C(|\tilde{D} f_h|_{\tilde{L}_h^\infty} |\tilde{D}^{m-1} g_h|_{\tilde{L}_h^2} + |g_h|_{\tilde{L}_h^\infty} |\tilde{D}^m f_h|_{\tilde{L}_h^2}), \tag{B.18}$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is the multi-index with $|\alpha| = m$.

The proof of (B.17) and (B.18) is analogous to that given in the case of functions by Moser [12] and Klainerman and Majda [5].

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