On the control through leadership of the Hegselmann–Krause opinion formation model

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Abstract

This paper deals with control strategies for the Hegselmann–Krause opinion formation model with leadership. In this system, the control mechanism is included in the leader dynamics and the feedback control functions are determined via a stabilization procedure and with a model predictive optimal control process. Correspondingly, the issues of global stabilization, controllability, and tracking are investigated. The model predictive control scheme requires to solve a sequence of open-loop optimality systems discretized by an appropriate Runge–Kutta scheme and solved by a nonlinear conjugate gradient method. Results of numerical experiments demonstrate the validity of the proposed control strategies and their ability to drive the system to attain consensus.

1 Introduction

Social networks have significant effect on people behaviour and on their opinion. For this reason, in order to investigate the role of these networks in social behaviour, several mathematical models of social networks have been developed in the past few years. In general, systems describing a great variety of network-structured phenomena stem from agent-based models as seen in Refs. [1, 8, 6, 5, 10, 11, 12, 13, 15, 19], see also Ref. [26] for a survey. In particular, one of the most representative class of agent-based models explaining phenomena in social sciences are the opinion formation models; see the seminal papers [4, 16, 22, 32] and, for further developments on these models, see Refs. [2, 7, 17, 24, 25].

We focus on the Hegselmann–Krause (HK) model [22] where the evolution of the opinion depends on interactions among the agents taking place in the bounded domain of confidence (see, for instance, Ref. [7]). Our purpose is to investigate control strategies for variants of this model that are "non-invasive", in the sense that the control function does not apply directly to the agent's opinion but indirectly through the interaction of a controlling leader. The motivation for our approach is twofold. First, it is closer to real social behaviours as, for example, in politics. Second, it can be easily implemented as soon as a leadership is available. We refer to Refs. [3, 6, 14, 17, 30, 33] for results on multi-agent systems with leadership.

Our contribution to the mathematical investigation of opinion formation systems focuses on a HK model with leadership where the control function is implemented on the leader dynamics. We develop a feedback control to globally achieve the consensus and then discuss the local controllability. Furthermore, we investigate optimal control problems governed by the HK model with leadership. In the latter case, we extend results of previous work on model predictive control through leadership of a flocking system [33]. In the present paper the analysis is completed by comparing the effectiveness of the control obtained with a feedback L^∞ approach and with the model predictive procedure.

Our work is organized as follows. In Sec. 2, we discuss a variant of the HK model with leadership where the control is implemented on the leader. In Sec. 3, we investigate the global stabilization of the system through leadership. Further, in Sec. 4, we discuss the local controllability problem. In Sec. 5, we formulate an optimal control problem governed by the HK model with an objective function related to the energy of the system. Correspondingly, we introduce the optimality system representing the first-order optimality conditions.

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Moreover, we illustrate a Runge–Kutta discretization scheme guarantees a high-order convergence rate of the numerical solution of the optimal control problem. Results of numerical experiments, presented in Sec. 6, demonstrate the validity of the proposed control strategies. A section of conclusions completes this work.

2 A Hegselmann–Krause model with leadership

We consider the control of the HK opinion formation model with leadership. The system of N interacting agents with one leader with opinions in $\mathbb{R}^d$ is given by

$$\begin{cases} \dot{x}_0(t) = & u(t), \\ \dot{x}_i(t) = & \displaystyle\sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)) + c_i(x_0(t) - x_i(t)), \qquad \text{for} \quad i = 1, ..., N, \end{cases} \tag{1}$$

with given initial positions $x_j(0) \in \mathbb{R}^d$, for $j = 0, 1, \ldots, N$. The state of the system, representing the agents' and the leader's opinions is $\mathbf{x} = (x_0, x_1, ..., x_N) \in \mathbb{R}^{(N+1)d}$. We denote the leader by the index 0 and with the index $i = 1, \ldots, N$ we denote the N agents. The control function, which is a measurable function $t \mapsto u(t) \in \mathbb{R}^d$ satisfying the constraint $\|u\| \leq M$, acts only on the leader.

The first term in the dynamics (1), $\sum_{j=1}^{N} a_{ij}(x_j - x_i)$, comes from the original HK model. The key idea of the HK model is that each agent updates his opinion by averaging the opinions of the neighbors with a rescaling factor

$$a_{ij} = a(\|x_i(t) - x_j(t)\|),$$

given by a function $a(r) : [0, \infty) \to [0, 1]$ of the distance r between the opinions and representing the interaction rate dependence on the limited confidence domain. The function $a = a(r)$ is the following smooth cut-off function

$$a(r) = a(r; \delta, \varepsilon) \quad := \quad \begin{cases} 1, & 0 \leq r \leq \delta, \\ \varphi(r), & \delta < r < (\delta + \varepsilon), \\ 0, & (\delta + \varepsilon) \leq r, \end{cases} \tag{2}$$

where δ is the bounded confidence, $\varphi(r)$ is a decreasing smooth function on $(\delta, \delta + \varepsilon]$, and $\varepsilon > 0$ is a parameter of the HK model defining the width of the region where the cut-off function decays to zero.

The second term in the dynamics (1) models the action of the leader on the ith agent. A leader can be defined as one agent with a high level of confidence and self-esteem, that has the ability to withstand criticism, so that its dynamics is not influenced by the other agents' opinions. The influence of the opinion of the leader on the group opinion in decision making is given by the term $c_i(x_0(t) - x_i(t))$. The parameter

$$c_i = \gamma \phi(\|x_i - x_0\|),$$

represents the rate of relationship between the leader and the other agents, where $\phi : [0, \infty) \to (0, 1]$ is a smooth non-increasing positive function such that $\phi(0) = 1$ and $\lim_{r \to \infty} \phi(r) = 0$, and where the strength of the opinion leader is represented by the parameter $\gamma > 0$. In other words the leader has the ability to influence every agent with a factor that is inversely proportional to its distance from the agent.

In Sec. 6 we present results of numerical simulations in the uncontrolled case, i.e. when the control $u(t)$ is taken to be constantly zero. In those cases we show the formation of clusters due to the limitation of confidence. In the next section, we present a strategy to create a stabilizing control in order to drive all agents' opinions to a common one exploiting the leader's dynamics.

3 Global Stabilization

The uncontrolled dynamics of system (1) is governed by local interactions and, in large time, leads to the formation of clusters. Although, from the mathematical point of view, clusters are a stable configuration for the system, we focus on the possibility to steer, using the leader's action on the group, all agents to a same unique opinion, that is, to have emergence of "consensus".

Definition 1 (Consensus)**.** *We call* consensus *a configuration in which the states of all agents are equal, i.e,*

$$\mathbf{x}^* = (x_0, x_1, ..., x_N) \in \mathbb{R}^{(N+1)d} \quad \textit{such that} \quad x_0 = x_1 = \cdots = x_N. \tag{3}$$

We say that a solution $\mathbf{x}(t)$ *of system* (1) tends to consensus *if there exists a consensus configuration* $\mathbf{x}^* \in \mathbb{R}^{(N+1)d}$ *such that* $\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{x}^*$*.*

Being consensus an equilibrium for system (1) the problem of steering asymptotically the system to consensus is, in fact, a stabilization problem.

Theorem 1. *For every initial condition* $\mathbf{x}(0) \in \mathbb{R}^{(N+1)d}$ *and every* M *there exists a control* $t \mapsto u(t) \in \mathbb{R}^d$ *satisfying* $\|u\| \le M$ *such that the associated solution* $\mathbf{x}(t)$ *with initial data* $\mathbf{x}(0)$ *tends to consensus.*

Proof. For every t let $\bar{\imath} = \bar{\imath}(t)$ be the smallest index in $\{1, \dots, N\}$ such that

$$\|x_{\bar{\imath}}(t) - x_0(t)\| \ge \|x_j(t) - x_0(t)\|, \qquad \text{for every } j = 1, \dots, N.$$

Let

$$\alpha(t) = \frac{1}{2} \min\left\{ \frac{\phi(\|x_{\bar{\imath}}(t) - x_0(t)\|)}{N - \phi(\|x_{\bar{\imath}}(t) - x_0(t)\|)}, \frac{2M}{\gamma \sum_i \|x_i(t) - x_0(t)\|} \right\}.$$

Note that $\alpha(t) > 0$ for every $t \ge 0$. Consider the control law

$$u(t) = \alpha(t)\gamma \sum_{j=1}^{N} \phi(\|x_j(t) - x_0(t)\|)(x_j(t) - x_0(t)).$$

The control u is admissible since

$$\|u\| \le \alpha\gamma \sum_{j=1}^{N} \phi(\|x_j - x_0\|)\|x_j - x_0\| \le \alpha\gamma \sum_{j=1}^{N} \|x_j(t) - x_0(t)\| \le M.$$

Consider $t \ge 0$ and assume, for simplicity of notation, that $\bar{\imath} = 1$, $c_i = \gamma\phi(\|x_i(t) - x_0(t)\|)$, and $a_{ij} = a(\|x_i(t) - x_j(t)\|)$. Then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|x_1 - x_0\|^2 &= \langle \dot{x}_1 - \dot{x}_0, x_1 - x_0 \rangle \\
&= \langle \dot{x}_1, x_1 - x_0 \rangle - \langle \dot{x}_0, x_1 - x_0 \rangle \\
&= \sum_{j=2}^{N} a_{1j} \langle x_j - x_1, x_1 - x_0 \rangle - c_1 \|x_1 - x_0\|^2 \\
&\quad - \alpha \sum_{j=2}^{N} c_j \langle x_j - x_0, x_1 - x_0 \rangle - \alpha c_1 \|x_1 - x_0\|^2 \\
&= \sum_{j=2}^{N} a_{1j} \langle x_j - x_0, x_1 - x_0 \rangle - \sum_{j=2}^{N} a_{1j} \|x_1 - x_0\|^2 - c_1 \|x_1 - x_0\|^2 \\
&\quad - \alpha \sum_{j=2}^{N} c_j \langle x_j - x_0, x_1 - x_0 \rangle - \alpha c_1 \|x_1 - x_0\|^2 \\
&\le \sum_{j=2}^{N} |a_{1j} - \alpha c_j| \|x_j - x_0\| \|x_1 - x_0\| \\
&\quad - \left(\sum_{j=2}^{N} a_{1j} + (1+\alpha) c_1 \right) \|x_1 - x_0\|^2 \\
&\le \left(\sum_{j=2}^{N} (|a_{1j} - \alpha c_j| - a_{1j}) - (1+\alpha) c_1 \right) \|x_1 - x_0\|^2.
\end{aligned}$$

Now if j is such that $a_{1j} - \alpha c_j \geq 0$ then $|a_{1j} - \alpha c_j| - a_{1j} = -\alpha c_j < 0$. Otherwise, if $a_{1j} - \alpha c_j < 0$ then $|a_{1j} - \alpha c_j| - a_{1j} = \alpha c_j - 2a_{1j} \leq \alpha c_j$. Hence

$$
\begin{aligned}
\sum_{j=2}^{N}(|a_{1j} - \alpha c_j| - a_{1j}) - (1+\alpha)c_1 &\leq (N-2)\alpha - (1+\alpha)c_1 \\
&\leq (N - c_1)\alpha - c_1 \\
&\leq -\frac{c_1}{2},
\end{aligned}
$$

for the choice of α. In particular $\max_i \|x_i(t) - x_0(t)\|$ is decreasing for every t. Therefore

$$\phi(\max_i \|x_i(t) - x_0(t)\|) \geq \phi(\max_i \|x_i(0) - x_0(0)\|),$$

and denoting by $I_0 = \max_i \|x_i(0) - x_0(0)\|$ we have that

$$\max_i \|x_i(t) - x_0(t)\|^2 \leq \exp\left(-t\frac{\gamma\phi(I_0)}{2}\right) I_0,$$

which gives that

$$\|x_i(t) - x_0(t)\| \to 0, \quad \text{as } t \to \infty, \quad \text{for every } i = 1, \ldots, N,$$

in other words the system tends to consensus. □

We discuss global stabilization of the HK model by using design tools of feedback control law based on an L^∞ approach (see Section 2 in Ref. [26]). In Sec. 6, we present test cases and the corresponding results of numerical simulations.

4 Local Controllability

In this section, we discuss a local controllability strategy for the controlled HK model. No information can be deducted on the local controllability around consensus from the linearized systems since, as the following example shows, it is not controllable.

Example 1. *Consider the linearization of system* (1) *around the consensus* $\mathbf{x}^*$. *The linearized system for the variable* $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$ *is given by*

$$\dot{\tilde{\mathbf{x}}} = A\tilde{\mathbf{x}} + Bu, \tag{4}$$

where A is a block matrix and B is a block vector as follows

$$A = \begin{pmatrix} 0_d & 0_d & \cdots & 0_d \\ c_1 I_d & -(\sum_{j\neq 1} a_{1j} + c_1)I_d & \cdots & a_{1N}I_d \\ \vdots & \vdots & \ddots & \vdots \\ c_N I_d & a_{N1}I_d & \cdots & -(\sum_{j\neq N} a_{Nj} + c_N)I_d \end{pmatrix}, \qquad B = \begin{pmatrix} I_d \\ 0_d \\ \vdots \\ 0_d \end{pmatrix},$$

where 0_d is the null matrix and I_d is the identity.
Consider the simple case $N + 1 = 3$ and $d = 1$. In this case the eigenvalues of A are given by

$$
\begin{aligned}
\lambda_1 &= 0, \\
\lambda_2 &= -1 - \frac{d_1}{2} - \frac{d_2}{2} - \frac{\sqrt{(d_1 - d_2)^2 + 4}}{2}, \\
\lambda_3 &= -1 - \frac{d_1}{2} - \frac{d_2}{2} + \frac{\sqrt{(d_1 - d_2)^2 + 4}}{2}.
\end{aligned}
$$

where $d_1 = \gamma\phi(\|x_1 - x_0\|)$ and $d_2 = \gamma\phi(\|x_2 - x_0\|)$. The corresponding eigenvectors are

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ \frac{d_2}{2} - \frac{d_1}{2} - \frac{\sqrt{(d_1 - d_2)^2 + 4}}{2} \\ 1 \end{pmatrix},$$

and

$$w_3 = \begin{pmatrix} 0 \\ \dfrac{d_2}{2} - \dfrac{d_1}{2} + \dfrac{\sqrt{(d_1-d_2)^2+4}}{2} \\ 1 \end{pmatrix}.$$

In particular, by classical controllability results, such as the Hautus Lemma (see, for instance, Lemma 3.3.7 in Ref. [31]) system (4) *is not controllable since* $w_2^T B = 0$ *(and* $w_3^T B = 0$*).*

Following this example, it is possible to prove that if ϕ is constant on an interval $[0, \delta]$, as in Sections 5 and 6 below, then system (1) is not locally controllable. Indeed in this case, system (1) is linear whenever $\|\mathbf{x}-\mathbf{x}^*\| < \delta$ for some consensus $\mathbf{x}^*$ and it does not verify the rank condition, necessary for controllability. For more general situation the local controllability depends on the communication rate ϕ between the leader and the agents and, a priori, we cannot infer the local controllability of the system.
However the system verifies a "weaker" local controllability property, that is the local controllability to consensus. As the following lemma states, if the agents are sufficiently close to each other they are attracted by the leader. The proof relies on the Lyapunov stability of the leader's state x_0.

Lemma 1. *Let* $u(t) = 0$ *for every* $t \geq 0$*. If* $\|x_i(0) - x_0\| \leq \delta/N$ *for every* $i = 1, \ldots, N$ *then*

$$\lim_{t\to\infty} x_i(t) = x_0, \qquad \text{for every } i = 1, \ldots, N.$$

Proof. Let $T \geq 0$ be the maximal time such that $\sum_i \|x_i(t) - x_0\| \leq \delta$ on $t \in [0, T]$, with the convention that $T = +\infty$ if $\sum_i \|x_i(t) - x_0\| \leq \delta$ for every $t \geq 0$. Then for every $t \in [0, T]$ the interaction coefficients between agents are

$$a_{ij} = a(\|x_i - x_j\|) = 1, \qquad \text{for every } i, j = 1, \ldots, N.$$

For simplicity we set $c_i = \gamma\phi(\|x_i(t) - x_0(t)\|)$ and we drop the dependence on t. Hence

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2N}\sum_{i=1}^N \|x_i - x_0\|^2 &= \frac{1}{N}\sum_{i=1}^N \langle \dot{x}_i, x_i - x_0\rangle \\
&= \frac{1}{N}\sum_i\sum_j \langle x_j - x_i, x_i - x_0\rangle - \frac{1}{N}\sum_i c_i\|x_i - x_0\|^2 \\
&= \frac{1}{N}\sum_i\sum_j \langle x_j - x_0, x_i - x_0\rangle - \sum_i \|x_i - x_0\|^2 \\
&\quad - \frac{1}{N}\sum_i c_i\|x_i - x_0\|^2 \\
&\leq \frac{1}{N}\sum_i \|x_i - x_0\| \sum_j \|x_j - x_0\| - \sum_i (1 + c_i/N)\|x_i - x_0\|^2 \\
&\leq \sum_i \|x_i - x_0\|^2 - \sum_i (1 + c_i/N)\|x_i - x_0\|^2 \\
&= -\sum_i c_i/N\|x_i - x_0\|^2 \\
&\leq -\gamma\phi(\delta/2)\frac{1}{N}\sum_i \|x_i - x_0\|^2. \qquad (5)
\end{aligned}$$

In particular the function $V(t) = \frac{1}{2N}\sum_{i=1}^N \|x_i - x_0\|^2$ is a Lyapunov function for the system. Moreover for every $t \geq 0$

$$\sum_i \|x_i(t) - x_0\| \leq \sum_i \|x_i(0) - x_0\| \leq \delta.$$

In particular the estimate (5) holds true for every $t \geq 0$ and

$$V(t) \leq \exp(-2\gamma\phi(\delta/2)t)\, V(0),$$

which gives that $x_i(t) \to x_0$ as $t \to +\infty$ for every $i = 1, \ldots, N$. □

As a consequence of this lemma and of Theorem 1, we have the following result of global stabilization and partial controllability.

Corollary 1. *For every initial condition* $\mathbf{x}(0)$ *and for every consensus* $\mathbf{x}^*$ *there exists a control* u *such that the associated solution* $\mathbf{x}(t)$ *of* (1) *with initial condition* $\mathbf{x}(0)$ *tends to* $\mathbf{x}^*$.

Proof. By Theorem 1 there exist a consensus configuration $\bar{\mathbf{x}} = (\bar{x}, \dots, \bar{x})$ and a control u steering the solution $\mathbf{x}(t)$ of (1) with initial condition $\mathbf{x}(0)$ to $\bar{\mathbf{x}}$. Let $t_1 > 0$ be a sufficiently large time such that

$$\|x_i(t_1) - \bar{x}\| \leq \frac{\delta}{2N}, \qquad \text{for every } i = 0, \dots, N.$$

Now consider the desired consensus configuration $\mathbf{x}^* = (x^*, \dots, x^*)$ and a finite sequence of points $z_0 = \bar{x}, z_1, \dots, z_\ell, z_{\ell+1} = x^*$ such that

$$\|z_k - z_{k+1}\| \leq \frac{\delta}{2N} \text{ for every } k = 0, \dots, \ell.$$

We apply iteratively Lemma 1 in order to construct a control steering the solution to every consensus configuration associated with the sequence $z_0, z_1, \dots, z_\ell, z_{\ell+1}$. For $k \in \{0, \dots, \ell\}$ let $\tau > 0$ be such that

$$\|x_i(\tau) - z_k(\tau)\| \leq \frac{\delta}{2N}, \qquad \text{for every } i = 0, \dots, N,$$

then consider the control

$$u(t) = \begin{cases} M \frac{z_{k+1} - x_0(t)}{\|z_{k+1} - x_0(t)\|}, & \text{if } z_{k+1} \neq x_0, \\ 0, & \text{if } z_{k+1} = x_0. \end{cases}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|x_0(t) - z_{k+1}\|^2 = \langle u, x_0(t) - z_{k+1}\rangle = -M\|x_0(t) - z_{k+1}\|,$$

in particular x_0 reaches z_{k+1} in finite time, say τ_k. Now $\|x_i(\tau + \tau_k) - z_{k+1}\| \leq \delta/N$ for every $i = 1, \dots, N$. Indeed for every $t \in [\tau, \tau + \tau_k]$ let $i = i(t)$ be such that $\|x_i(t) - z_{k+1}\|$ is maximal, then we have

$$\begin{aligned} \frac{d}{dt}\|x_i(t) - z_{k+1}\|^2 &= \langle \dot{x}_i(t), x_i(t) - z_{k+1}\rangle \\ &= \sum_j \langle x_j(t) - x_i(t), x_i(t) - z_{k+1}\rangle \\ &= \sum_j \langle x_j(t) - z_{k+1}, x_i(t) - z_{k+1}\rangle - N\|x_i(t) - z_{k+1}\|^2 \\ &\leq \sum_j \|x_j(t) - z_{k+1}\|\|x_i(t) - z_{k+1}\| - N\|x_i(t) - z_{k+1}\|^2, \end{aligned}$$

which is smaller than or equal to 0 for the maximality of the index i. Therefore

$$\max_j \|x_j(t) - z_{k+1}\| \leq \max_j \|x_j(0) - z_{k+1}\| \leq \frac{\delta}{N},$$

for every $t \in [\tau, \tau + \tau_k]$. Then set the control $u(t) = 0$ for $t > \tau + \tau_k$ and by Lemma 1 we have that

$$\lim_{t \to \infty} x_i(t) = z_{k+1}, \qquad \text{for every } i = 1, \dots, N.$$

The statement follows by induction on $k = 0, \dots, \ell$. □

Remark 1. *The idea of the proof of Corollary 1 is the following. Once the system is sufficiently close to consensus the leader starts moving very slowly to the new desired consensus, reaching it in finite time (possibly very large). When the leader move sufficiently slow, the distance between leader and followers is always smaller than* δ/N. *Lemma 1 guarantees that the followers tend asymptotically to the leader. This strategy can be adapted to prove an analogous of Theorem 1 in the case in which the interaction function* ϕ *between the leader and the* *agents vanishes, in other words in the bounded confidence case. Indeed it is known[22, 7] that the system in free evolution tends to clusters. The leader can move towards the nearest agent amongst the ones distant more than δ/N reaching its position in finite time. If the motion of the leader is sufficiently slow then Lemma 1 (which is in fact a local result) guarantees that the agents in the leader's cluster tends to the leader and, in particular, do not leave the cluster. Hence the population in the leader's clusters increases by one. Finally the system tends to consensus.*

5 Optimal control

In Sec. 3, we consider the problem of regrouping the system of agents to a desired consensus by designing a feedback control for the leader. There are various tools to design the stabilizing feedback. In this section, we study how to enforce consensus in the "best" possible way. First, the formulation of optimal control problems for the HK model with the presence of a leader are considered. Further, we illustrate a Runge–Kutta (RK) discretization scheme to accurately solve the corresponding optimal control system. In addition, we consider a model predictive control (MPC) scheme to construct a feedback control strategy. This section extends previous results on optimal control through leadership of a flocking system[33] to the case of our HK model.
From now on, we set the problem in a more general framework in which we allow the communication rate of the leader with the followers to be zero. To this purpose we introduce two functions: c_i^1 and c_i^2. In the first case, $c_i^1 : \mathbb{R} \to [0, 1]$ represents a cut-off smooth function of the bounded confidence δ_0 as follows

$$c_i^1(r) = c_i^1(r; \delta_0, \varepsilon_0) \quad = \quad \begin{cases} 1, & 0 \le r \le \delta_0, \\ \tilde{\varphi}(r_{i0}), & \delta_0 < r < (\delta_0 + \varepsilon_0), \\ 0, & (\delta_0 + \varepsilon_0) \le r, \end{cases} \tag{6}$$

for a smooth decreasing function $\tilde{\varphi}(r)$ between $(\delta_0, \delta_0 + \varepsilon_0]$. We then denote by $c_i^2(r) = \phi(r)$ the function of the distance between the leader and the followers as in the previous sections, i.e. a non-increasing positive function such that $\phi(0) = 1$ and $\lim_{r\to\infty} \phi(r) = 0$.
We consider the cost functional

$$\begin{aligned} J(\mathbf{x}, u) \quad := \quad & \frac{\mu}{2}|x_0(T) - x_{des}(T)|^2 + \frac{1}{2N^2}\int_0^T \sum_{i,j=1}^N |x_i(t) - x_j(t)|^2 dt \\ & + \frac{1}{2}\int_0^T \sum_{i=1}^N |x_0(t) - x_i(t)|^2 dt + \frac{\nu}{2}\int_0^T |u(t)|^2 dt, \end{aligned}$$

where the positive parameters μ and ν in the cost function are weight constants. The second and third term in the cost functional account for the energy of the system. In addition, the tracking functional at final time requires the leader to approach a desired target configuration x_{des}. This term has also the property to stabilize the MPC scheme[18]. The last term in the functional represents the cost of the control.
Our optimal control problem is stated as follows

$$\min J(\mathbf{x}, u) \tag{7}$$

subject to

$$\begin{cases} \dot{x}_0(t) = & u(t) \\ \dot{x}_i(t) = & \displaystyle\sum_{j\neq 0,i} a_{ij}(x_j - x_i) + c_i^{\tilde{n}}(x_0 - x_i), \quad \text{for} \quad i = 1, ..., N, \end{cases} \tag{8}$$

with given initial conditions and for $\tilde{n} = 1, 2$.
We associate with (7)-(8), the equation

$$\begin{cases} \dot{\hat{x}}(t) = & \displaystyle\frac{1}{2N^2}\sum_{i,j=1}^N |x_i(t) - x_j(t)|^2 + \frac{1}{2}\sum_{i=1}^N |x_0(t) - x_i(t)|^2 + \frac{\nu}{2}|u(t)|^2, \\ \hat{x}(t_0) = & 0. \end{cases} \tag{9}$$

In such a way the optimal control problem can be written in the form

$$\min\left(\frac{1}{2}|x_0(T)-x_{des}(T)|^2+\hat{x}(T)\right), \tag{10}$$

$$\begin{aligned}\text{subject to}\quad \dot{x}_0 &= u(t)\\ \dot{x}_i &= \sum_{j\neq 0,i} a_{ij}(x_j-x_i)+c_i^{\tilde{n}}(x_0-x_i), \quad \text{for} \quad i=1,...,N.\\ \dot{\hat{x}} &= \frac{1}{2N^2}\sum_{i,j=1}^{N}|x_i(t)-x_j(t)|^2+\frac{1}{2}\sum_{i=1}^{N}|x_0(t)-x_i(t)|^2+\frac{\nu}{2}|u(t)|^2,\end{aligned}$$

with given initial conditions.
For the simplicity, we write (10) in the more compact form

$$\min\left(\frac{1}{2}|x_0(T)-x_{des}(T)|^2+\hat{x}(T)\right), \tag{11}$$
$$\text{subject to}\quad \dot{\tilde{\mathbf{x}}}=\tilde{\mathbf{f}}(\tilde{\mathbf{x}},u),$$

where $\tilde{\mathbf{x}}=(x_0,x_1,...,x_N,\hat{x})^T\in\mathbb{R}^{(N+2)d}$ and $\tilde{\mathbf{f}}$ is the right hand side of (9).
We consider the discretization of the optimal control problem (11) by a RK scheme on a uniform time mesh, with the following time-step size

$$h=\frac{t_f-t_0}{n},$$

where n is the total number of discrete time intervals in (t_0,t_f) and the value of $\tilde{\mathbf{x}}(t)$ at the discrete time t_k is denoted with

$$\tilde{\mathbf{x}}_k=\tilde{\mathbf{x}}(t_k), \quad t_k=kh, \quad \text{for} \quad k=0,\dots,n.$$

Corresponding to the RK discretization setting, the optimal control problem (11) with s-stage RK scheme becomes the following

$$\min\left(\frac{1}{2}|x_0(T)-x_{des}(T)|_2^2+\hat{x}(T)\right),$$
$$\begin{aligned}\text{subject to}\quad \tilde{\mathbf{x}}_{k+1}&=\tilde{\mathbf{x}}_k+h\sum_{i=1}^{s}b_i\tilde{\mathbf{f}}(\mathbf{y}_{ki},u_{ki}), \quad \tilde{\mathbf{x}}(t_0)=\tilde{\mathbf{x}}_0,\\ \mathbf{y}_{ki}&=\tilde{\mathbf{x}}_k+h\sum_{j=1}^{s}a_{ij}\tilde{\mathbf{f}}(\mathbf{y}_{kj},u_{kj}),\end{aligned} \tag{12}$$

for $1\le i,j\le s$, and $0\le k\le n-1$. The vector $u_k\in\mathbb{R}^{sd}$ represents the s-stages of the RK discrete control vector at time step k. We have

$$u_k=(u_{k1},u_{k2},...,u_{ks})\in\mathbb{R}^{sd}.$$

The discrete optimality system corresponding to (12) is given by

$$\begin{aligned}\tilde{\mathbf{x}}_{k+1} &= \tilde{\mathbf{x}}_k+h\textstyle\sum_{i=1}^{s}b_i\tilde{\mathbf{f}}(\mathbf{y}_{ki},u_{ki}), \qquad \tilde{\mathbf{x}}(t_0)=\tilde{\mathbf{x}}_0\\ \mathbf{y}_{ki} &= \tilde{\mathbf{x}}_k+h\textstyle\sum_{j=1}^{n}a_{ij}\tilde{\mathbf{f}}(\mathbf{y}_{kj},u_{kj})\\ \Psi_k &= \Psi_{k+1}+\textstyle\sum_{i=1}^{s}b_i\chi_{ki}, \qquad \Psi_n=-\nabla_x\Phi(\tilde{\mathbf{x}}(T))\\ \chi_{ki} &= (\nabla_x\tilde{\mathbf{f}}(\mathbf{y}_{ki},u_{ki}))^\top\left(\Psi_{k+1}+\textstyle\sum_{j=1}^{s}\frac{b_ja_{ji}}{b_i}\chi_{kj}\right).\end{aligned} \tag{13}$$

From this system, the following gradient results

$$\nabla_{u_{ki}}J(u)=-(\nabla_u\tilde{\mathbf{f}}(\mathbf{y}_{ki},u_{ki}))^\top\left(\Psi_{k+1}+\sum_{j=1}^{s}\frac{b_ja_{ji}}{b_i}\chi_{kj}\right), \tag{14}$$

for $1 \leq i, j \leq s$, and $0 \leq k \leq n-1$.

To investigate the well-posedness of the optimal control problem (12), we remark that the smoothness and coercivity conditions stated in Ref. [20] can be verified for the optimal control problem governed by the equations (12). It follows that the accuracy results stated in Ref. [20] (Theorem 2.1) for the RK scheme apply to (12).
We then discuss a MPC scheme (see, for instance Ref. [18]) implementing a closed-loop control strategy for the HK model in order to track a given sequence of desired positions in time. Let $(0, T)$ be the time interval where the evolution is considered. We assume time windows of size $\Delta t = T/M$ for positive integer M. Let $t_m = m\Delta t$, $m = 0, 1, \dots, M$, at time t_0, we have the initial conditions denoted by $\tilde{\mathbf{x}}_0$ and the desired positions at the end of each time window denoted by $x_{des}(t_m)$, $m = 1, \dots, M$. The MPC strategy starts at time t_0 and solves the open-loop optimal control problem (12) defined in the interval (t_0, t_1). Then, the results $\tilde{\mathbf{x}}$ of system measured in time $t = t_1$ will be a initial value for the subsequent optimization problem defined in the interval (t_1, t_2). This procedure is repeated by receding the time horizon until the last time window is reached. We notice that the closed-loop system with the MPC scheme is nominally asymptotically stable, see Refs. [18, 28].
The MPC procedure is summarized in the following algorithm.

Algorithm 1 (MPC Control)**.** *Set* $m = 0$, $\tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0$*;*

1. *measure the state* $\tilde{\mathbf{x}}(t_m) = \tilde{\mathbf{x}}_m$ *and the target* $x_{des}(t_{m+1})$*;*
2. *in* (t_m, t_{m+1})*, set initial condition* $\tilde{\mathbf{x}}_m^0 = \tilde{\mathbf{x}}_m$*;*
3. *solve (12), thus obtain the optimal pair* $(\tilde{\mathbf{x}}, u)$*;*
4. *if* $t_{m+1} < T$*, set* $m := m + 1$, $\tilde{\mathbf{x}}_m = \tilde{\mathbf{x}}(t_m)$*, go to 1;*
5. *end.*

Concerning the step (3) of Algorithm 1, which consists in solving the optimal control problem (12), the solution of the state equation in (12) gives the mapping $u \to \tilde{\mathbf{x}}(u)$, that allows to transform the constrained optimization problem in an unconstrained one as follows

$$\min_{u \in U} J(u) := J(\mathbf{x}(u), u). \tag{15}$$

We solve these problem implementing a nonlinear conjugate gradient (NCG) strategy. The evaluation of the corresponding gradient is given in (14): For a given u, we solve first the forward HK equation and then the adjoint problem. This procedure is implemented with the RK scheme and is summarized in the following

Algorithm 2 (Evaluation of the gradient at u)**.**

1. *Solve the discrete optimal control system (13) with the given initial conditions;*
2. *solve the discrete adjoint equation in (13) with the computed terminal condition;*
3. *compute the gradient* $\nabla_u J(u)$ *using (14);*
4. *end.*

We solve the optimization problem (12) by computing the gradient using Algorithm 2 and implementing it in a NCG scheme; see, e.g., Ref. [4]. For details on NCG implementation see, for instance, Refs. [20, 21].

6 Numerical experiments

In this section, we present numerical simulations with system (1) in the one dimensional case, i.e. for $d = 1$. First, we consider the uncontrolled HK model where the control $u = 0$. This model is solved using the explicit

Table 1: Parameters for the HK model with leadership.

connectivity function (a_{ij})	δ	2.5
	ε	0.05
connectivity function (c_i^1)	δ_0	5
	ε_0	0.05
connectivity function (c_i^2)	l_a	20
strength of leader power	γ	10

fourth-order Runge–Kutta method illustrated in previous section. In our experiments, the connectivity functions $a_{ij} = a(\|x_i - x_j\|)$ are given by

$$a(r) = a(r;\delta,\varepsilon) \quad = \quad \begin{cases} 1, & 0 \le r \le \delta, \\ \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{r-\delta} + \frac{1}{r-(\delta+\varepsilon)}\right), & \delta < r < (\delta+\varepsilon), \\ 0, & (\delta+\varepsilon) \le r. \end{cases}$$

The connectivity coefficient accounting for the relationship between the leader and the other agents is chosen as follows

$$c_i^1(r) = c_i^1(r;\delta_0,\varepsilon_0) \quad = \quad \begin{cases} 1, & 0 \le r \le \delta_0, \\ \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{r-\delta_0} + \frac{1}{r-(\delta_0+\varepsilon_0)}\right), & \delta_0 < r < (\delta_0+\varepsilon_0), \\ 0, & (\delta_0+\varepsilon_0) \le r, \end{cases}$$

where r is the distance from the leader and δ_0 is the bounded confidence. Alternatively, we take

$$c_i^2(r_{i0}) = e^{-\frac{|x_i - x_0|^2}{l_a}},$$

where $l_a = 20$ is a parameter that scales the region of attraction. The comparison of the connectivity function is shown in Fig. 1.

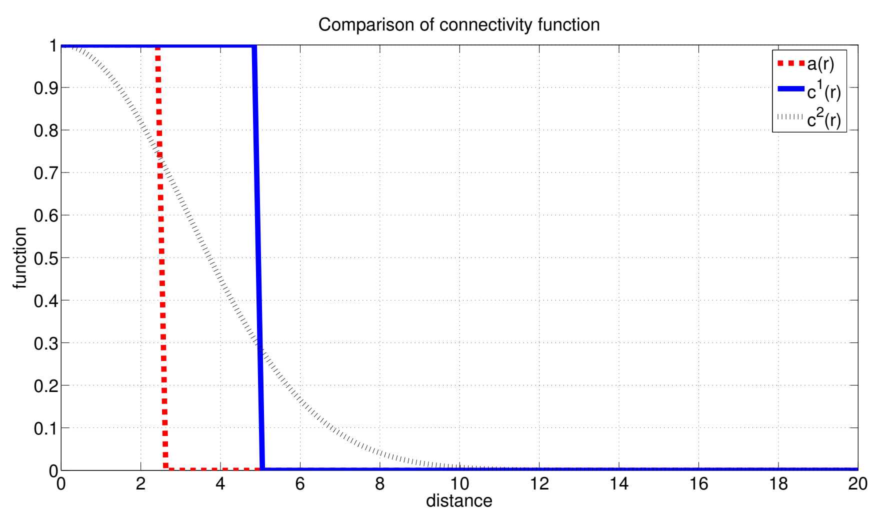

Figure 1: Comparison of the connectivity function a_{ij} , c_i^1 ,c_i^2

Fig. 2 shows numerical results with the HK model and an uncontrolled external leader. The initial positions of the agent's opinion are distributed randomly in $[-20, 20]$ while the leader's initial opinion is $x_0(0) = 30$. For these initial conditions $\mathbf{x}(0) = (x_0(0), x_1(0), ..., x_N(0))$, $N = 19$, and bounded confidence $\delta = 2.5, \delta_0 = 5$, it

can be see that the opinions $x_i(t), \quad$ for $\;i = 0, ..., N$, converge to a limiting opinion $\mathbf{x}^*$. Moreover, due to the limitation of confidence, clusters of agents' opinions are formed, that is, agents with different opinions form groups or *clusters*, namely $x_i^* \neq x_j^*$ when i and j are not in the same cluster, while in the other case, $x_i^* = x_j^*$ whenever i and j are in the same cluster.
We investigate the HK model with a leader subject to the global stabilizing feedback control given by Theorem 1. We allow the connectivity function with the leader ϕ to be zero in the bounded confidence case (that is with connectivity functions c_1^i). We consider two test cases, corresponding to two different initial conditions. In the bounded confidence case, for some initial configuration, this feedback fails to be a global stabilizer (see Fig. 4(a)). For all tests, the parameters related to the model are set up as stated in Table 1.
Case I The initial conditions are taken as in the previous experiment, i.e. a group of 20 agents including the leader are examined where the opinion leader is at $x_0(0) = 30$, while the other opinions are chosen randomly in [-20,20]. In the Figures 3(a), we report the results with the HK system with c_i^1, and in Fig. 3(b), we report results obtained with c_i^2, when the control with different functions c_i^1 and c_i^2 is applied. We see that by applying the input control to the HK system, the leader forces the agent's opinions to achieve consensus.
Case II The initial position of the leader opinion is placed at the center of the group of agents, while the opinions of the other agents are randomly distributed in a neighborhood of the leader's position where the distance between the leader opinion and the next nearest agent opinion is greater than the length of bounded confidence δ_0. With this set of initial conditions, we get the results as shown in Fig.4, where Fig. 4(a) and 4(c) show the solution of the HK system with the c_i^1 function and Fig. 4(b) and 4(d) show results using c_i^2. As we see in Fig. 4(a), because the value of c_i^1 is initially zero, the feedback control u is equal to zero. Therefore the leader has no influence on the other agents and fails in steering the agent to consensus. On the other hand, with the c_i^2 function consensus is obtained.

To complete this section, we present results of numerical experiments obtained with the optimal control problem (12) in the time interval $[0, 10]$. We consider two series of experiments; in the first one, we consider the consensus problem and in the second one, we focus on the tracking problem. In both tests, we solve the optimal control problem (7) with $N + 1 = 10$. The parameters related to the model are given as shown in Table 1. In addition, the parameters in the objective function are given by $\mu = 1, \nu = 0.001$. The initial opinions of the agents are randomly chosen in [-5,10] and the opinion of the leader is at $x_0(t_0) = 20$. Furthermore, the target is $x_{des} = \cos([0, 10])$. To apply the MPC strategy, the time horizon is divided into subintervals of size $\Delta t = 0.25$. From Fig. 5 and Fig. 6 we see that the resulting optimal control is able to steer the system to achieve the objective.

7 Conclusions

In this work, different control strategies for the Hegselmann–Krause opinion formation model with leadership were investigated. On the one hand, the control function was determined as a feedback control obtained via a stabilization procedure. On the other hand, it was obtained using a model predictive optimal control strategy. In both cases, the control function was included in the leader dynamics.
For the proposed control schemes, the issues of global stabilization, controllability, and tracking are investigated. Furthermore, in order to implement the model predictive control scheme, an appropriate Runge–Kutta scheme and a nonlinear conjugate gradient method were discussed. Results of numerical experiments demonstrated the validity and limitation of the proposed control strategies to drive the Hegselmann–Krause model to attain consensus.

Acknowledgment

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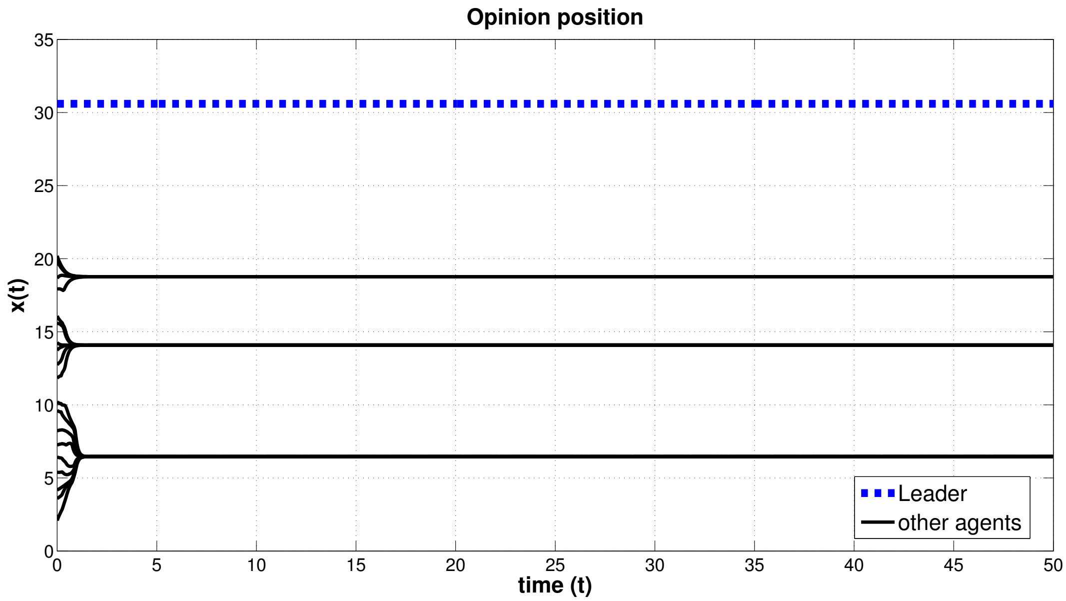
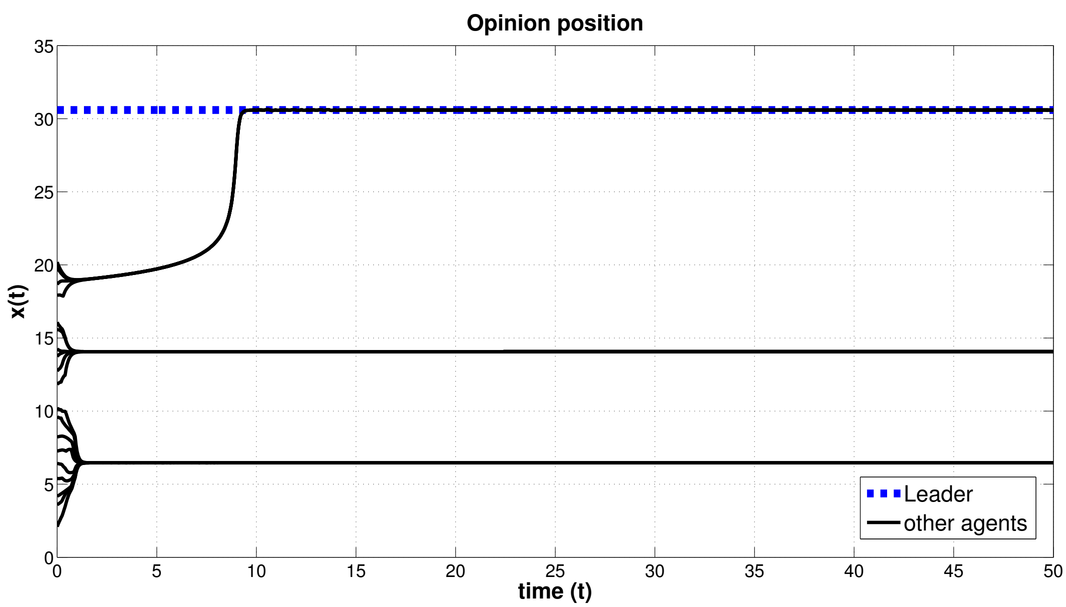
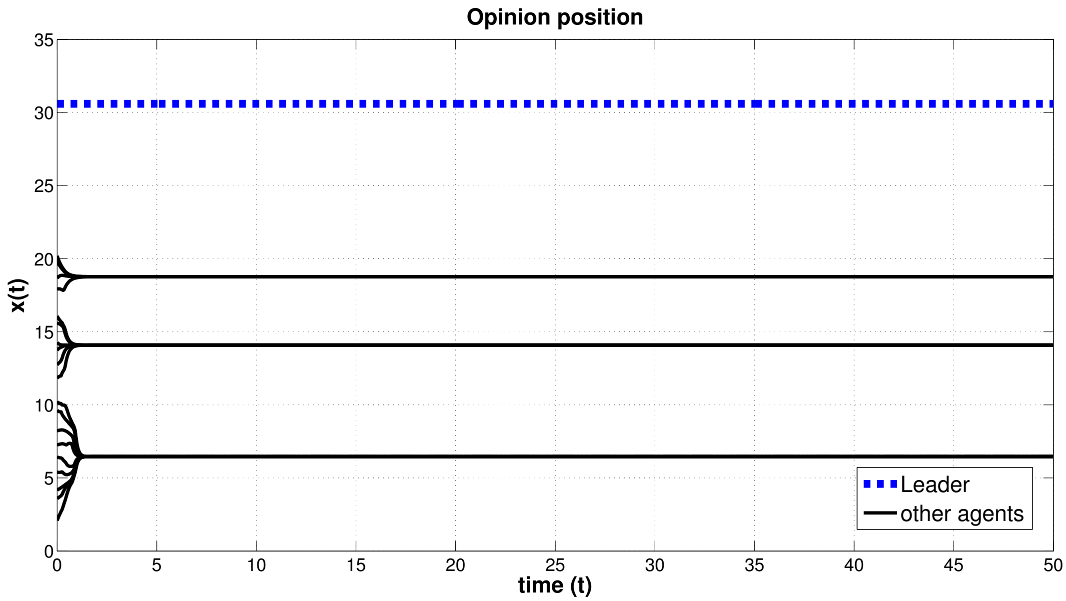


(a) t=50 with c_i^1

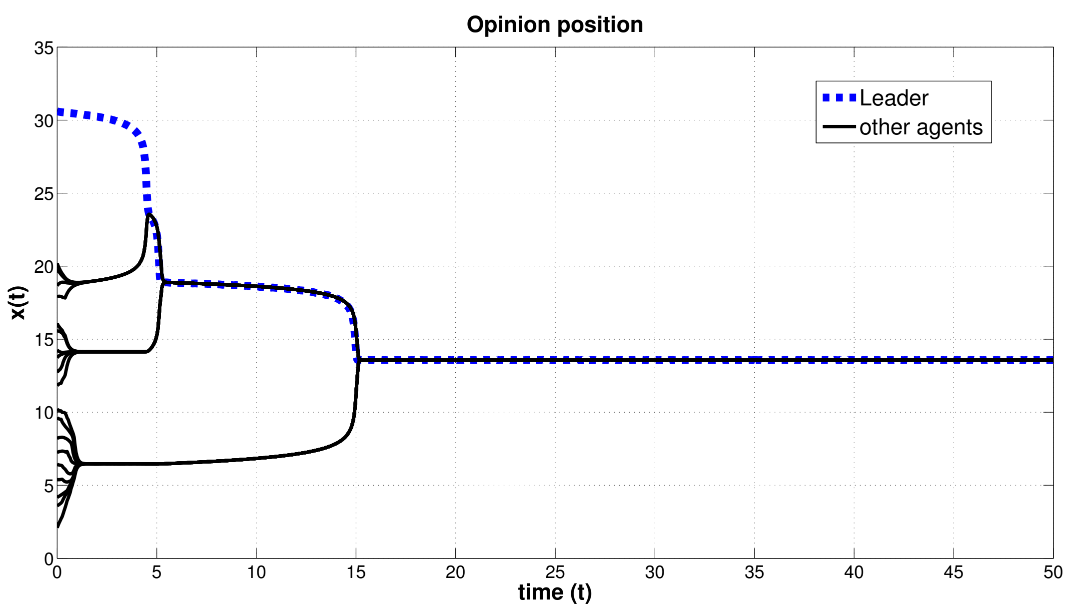


(b) t=50 with c_i^2

Figure 2: Free evolution of the system (uncontrolled case). Simulation with $N+1=20$ agents. The initial position values are chosen randomly in $[-20, 20]$ and leader opinion is at $x_0(0) = 30$. Fig.(a) shows with c_i^1. Fig.(b) result is with c_i^2. The evaluation of the opinion is denoted by a line and the dashed line represent the opinion of the leader.

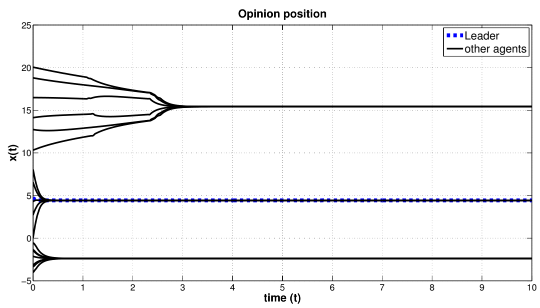


(a) t=50 with c_i^1

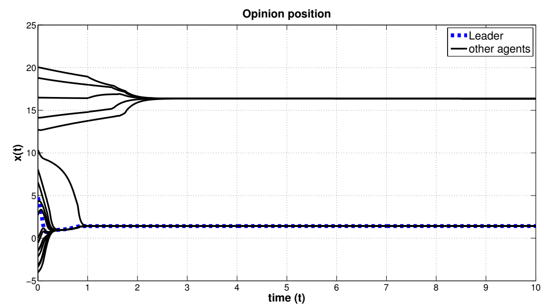


(b) t=50 with c_i^2

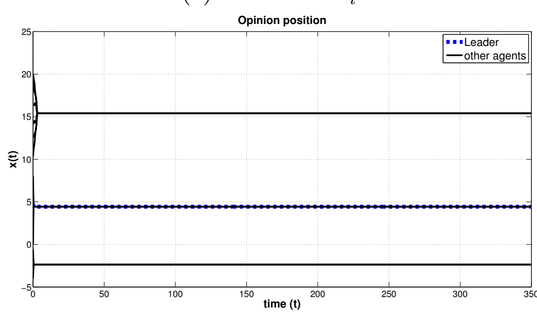
Figure 3: Stabilizing control. Simulation with $N+1=20$ agents. The initial position values are chosen randomly in $[-20, 20]$ and leader opinion is at $x_0(0) = 30$. Fig.(a) show results with c_i^1, results (b) with c_i^2. The opinion evaluation of each agent is denoted by a continuous line and the dashed line presents the opinion of the leader.



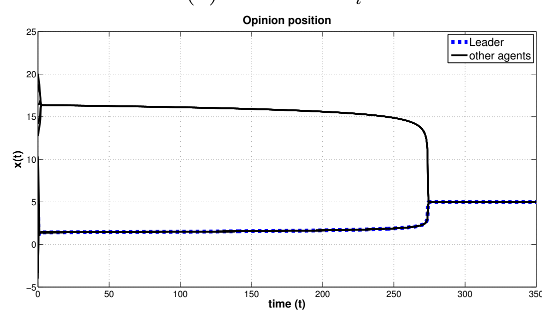
(a) t=10 with c_i^1



(b) t=10 with c_i^2

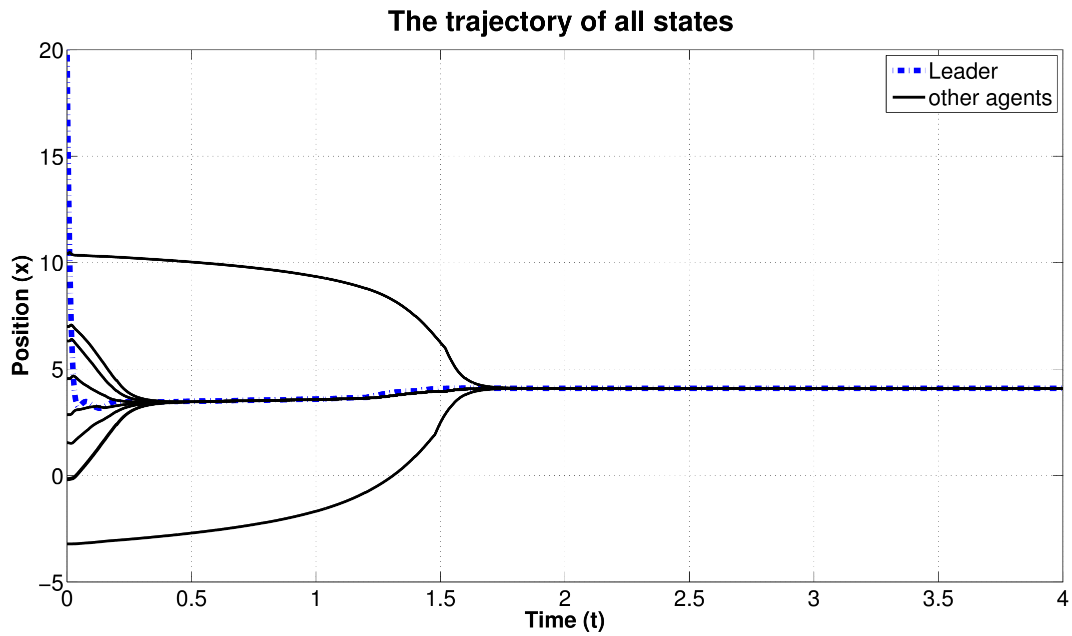


(c) t=350 with c_i^1

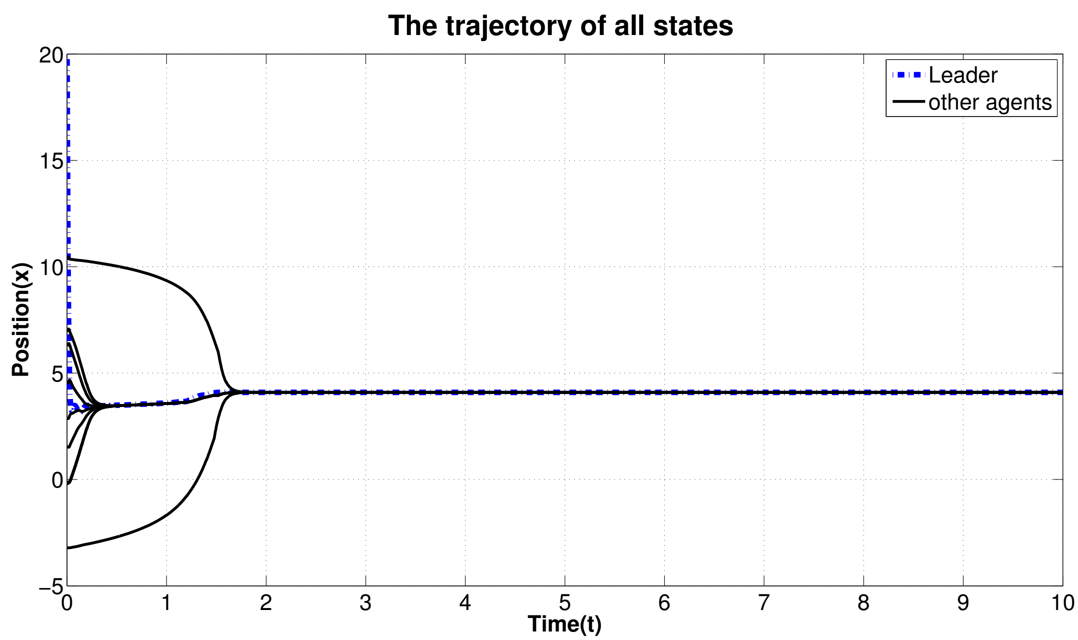


(d) t=350 with c_i^2

Figure 4: Stabilizing control. Simulation with $N + 1 = 20$ agents. The leader's initial opinion is at center of group, other initial positions are randomly distributed on both sides of leader's position. Fig.(a) and (c) show the results with c_i^1 and Fig.(b) and (d) results with c_i^2. The opinion evaluation of each agent is denoted by a continuous line and the dashed line presents the opinion leader.

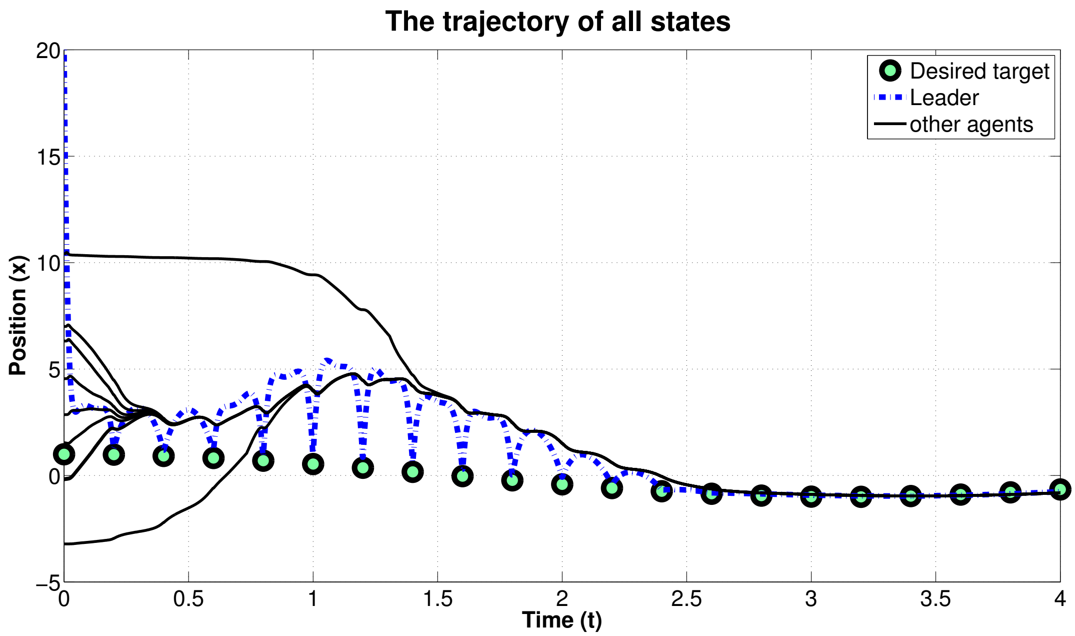


(a) t=[0,4]

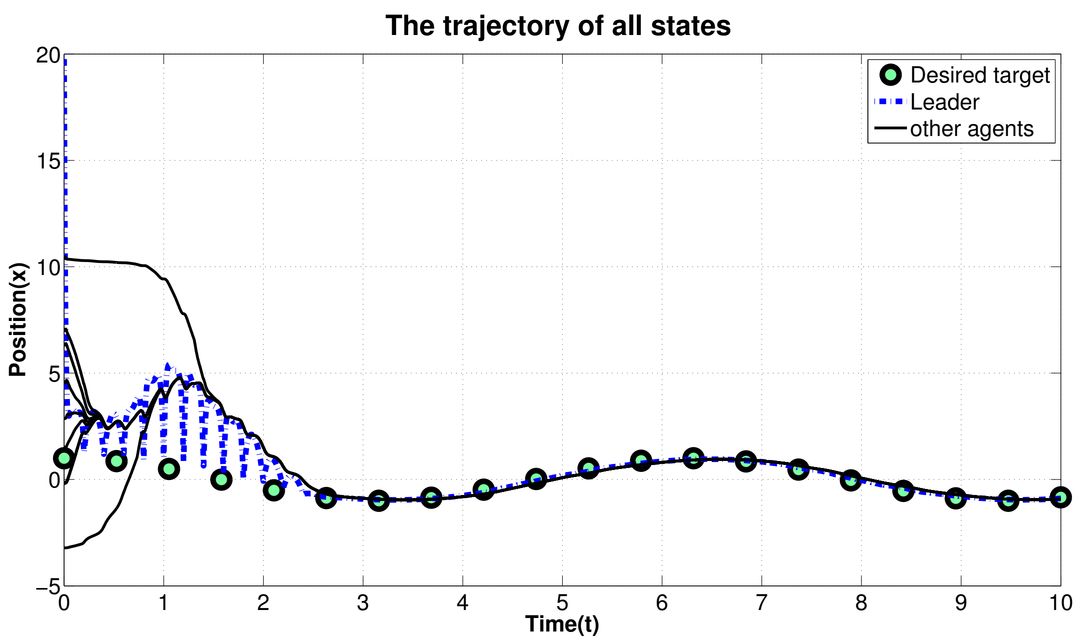


(b) t=[0,10]

Figure 5: Optimal control. Simulation with $N + 1 = 10$ agents. Figures (a) and (b) represent the optimal results of the HK with leadership for $\mu = 0$. The objective is to force all agents to reach the consensus $\mathbf{x}^*$. The opinion evaluation of each agent is denoted by a continuous line and the dashed line represents the opinion of the leader.



(a) t=[0,4]



(b) t=[0,10]

Figure 6: Optimal control. Simulation with $N+1 = 10$ agents. In figures (a) and (b) the optimal results of HK with leadership for $\mu = 1$. The objective is to force all agents to reach the desired position $\mathbf{x}_{des} = \cos([0, 10])$. The opinion evaluation of each agent is denoted by a continuous line and the dashed line represents the opinion of the leader.