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ON THE CONTROLLABILITY OF A FRACTIONAL ORDER PARABOLIC EQUATION*

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Abstract. The null-controllability property of a $1-d$ parabolic equation involving a fractional power of the Laplace operator, $(-\Delta)^\alpha$, is studied. The control is a scalar time-dependent function $g = g(t)$ acting on the system through a given space-profile $f = f(x)$ on the interior of the domain. Thus, the control g determines the intensity of the space control f applied to the system, the latter being given a priori. We show that, if $\alpha \le 1/2$ and the shape function f is, say, in L^2, no initial datum belonging to any Sobolev space of negative order may be driven to zero in any time. This is in contrast with the existing positive results for the case $\alpha > 1/2$ and, in particular, for the heat equation that corresponds to $\alpha = 1$. This negative result exhibits a new phenomenon that does not arise either for finite-dimensional systems or in the context of the heat equation.

On the contrary, if more regularity of the shape function f is assumed, then we show that there are initial data in any Sobolev space H^m that may be controlled. Once again this is precisely the opposite behavior with respect to the control properties of the heat equation in which, when increasing the regularity of the control profile, the space of controllable data decreases.

These results show that, in order for the control properties of the heat equation to be true, the dynamical system under consideration has to have a sufficiently strong smoothing effect that is critical when $\alpha = 1/2$ for the fractional powers of the Dirichlet Laplacian in $1-d$. The results we present here are, in nature and with respect to techniques of proof, similar to those on the control of the heat equation in unbounded domains in [S. Micu and E. Zuazua, *Trans. Amer. Math. Soc.*, 353 (2000), pp. 1635–1659] and [S. Micu and E. Zuazua, *Portugal. Math.*, 58 (2001), pp. 1–24].

We also discuss the hyperbolic counterpart of this problem considering a fractional order wave equation and some other models.

Key words. null controllability, parabolic equation, fractional power of the Laplace operator

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1. Introduction.

It is generally considered that, due to the strong dissipative effect on the high modes, parabolic equations behave like ordinary differential equations (i.e., finite-dimensional dynamical systems) from a control theoretical point of view. This is true for instance for the heat equation concerning the problem of null controllability, i.e., that of driving solutions from a given initial configuration to equilibrium, in several respects: (a) both finite-dimensional systems and the heat equation are controllable in an arbitrarily short time; (b) the controls may be taken to be arbitrarily smooth. In this way, for instance, the heat equation in bounded domains is controllable with L^2-controls for initial data in a Sobolev space of arbitrary negative order, in an arbitrarily short time and with controls supported in an arbitrarily small

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subdomain. Recently it was proved, however, that this is not true in unbounded domains (see [17] and [18]).

The object of this article is to further investigate to what extent this analogy is systematically true or whether it is related to the intrinsic properties of the heat equation.

To do this we consider the following *null-controllability problem: Given $T > 0$ and $f \in L^2(0,\pi)$, for any $u^0 \in L^2(0,\pi)$ find a control $g \in L^2(0,T)$ such that the solution u of the problem*

$$\begin{cases} u_t + (-\Delta)^\alpha u = g(t)f(x), & x \in (0,\pi),\ t \in (0,T), \\ u = 0, & x \in \{0,\pi\},\ t \in (0,T), \\ u(0,x) = u^0(x), & x \in (0,\pi), \end{cases} \tag{1.1}$$

satisfies

$$u(T,\cdot) = 0. \tag{1.2}$$

Here and in what follows $(-\Delta)^\alpha$ denotes the fractional power of order $\alpha > 0$ of the Dirichlet Laplacian that we shall denote by A_α. More precisely,

$$A_\alpha : D(A_\alpha) \subset L^2(\Omega) \to L^2(\Omega),$$

$$D(A_\alpha) = \left\{ u \in L^2(0,\pi) \,:\, u = \sum_{n\geq 1} a_n \sin(nx) \text{ and } \sum_{n\geq 1} |a_n|^2 n^{4\alpha} < \infty \right\}, \tag{1.3}$$

$$u(x) = \sum_{n\geq 1} a_n \sin(nx) \longrightarrow A_\alpha u(x) = \sum_{n\geq 1} a_n n^{2\alpha} \sin(nx).$$

Equation (1.1) is of parabolic type for any $\alpha > 0$. In the absence of control, solutions of (1.1) decay exponentially as $t \to \infty$ in, say, L^2. When $\alpha = 1$ we recover the classical heat equation.

When $0 < \alpha < 1$, (1.1) is a model example of parabolic dynamical system with weaker diffusivity (subdiffusion). Fractional equations of diffusion type are useful models for the description of transport processes in complex systems, slower than the Brownian diffusion. The list of systems displaying such anomalous dynamic behavior is quite extensive: charge carrier transport in amorphous semiconductors, nuclear magnetic resonance diffusometry in percolative and porous media, transport on fractal geometries, diffusion of a scalar tracer in an array of convection rolls, dynamics of a bead in a polymeric network, transport in viscoelastic materials, etc. (see [16] and [12]).

The state of system (1.1) is u and the control, which acts on its right-hand side term as an external source, is given by $g(t)f(x)$, where the shape function $f = f(x)$ is given and the intensity $g = g(t)$ is at our disposal. Such types of controls are sometimes called "lumped" or "bilinear" (see, for instance, [1] and [11]).

The null-controllability problem (1.1)–(1.2) has been considered and solved in [7] for the case $\alpha > 1/2$. The proof in [7] is based on the fact that the null-control problem may be rewritten as a *problem of moments* of the following form: *Find $g \in L^2(0,T)$ such that*

$$\int_0^T g(t)e^{\lambda_n t}dt = \beta_n \quad \forall n \geq 1, \tag{1.4}$$

where $\beta_n = -\pi a_n/2f_n$ *depend on the Fourier coefficients* $(a_n)_{n\geq 1}$ *of the initial data to be controlled and those of the control profile* $(f_n)_{n\geq 1}$.

Here λ_n is the sequence of the (real) eigenvalues of the equation under consideration: $\lambda_n = n^{2\alpha}$.

It is by now well known—and this is the second ingredient in the proof of [7]—that if

$$\lambda_n \sim cn^{\gamma} \quad \text{as} \quad n \to \infty \tag{1.5}$$

for some $\gamma > 1$ and a positive constant $c > 0$, then (1.4) has L^2-solutions if the values β_n do not increase too much.

This result may be proved by means of a careful evaluation of the norm of a biorthogonal sequence to the family of exponentials $\{e^{\lambda_n t}\}_{n\geq 1}$ and it is related to the Müntz theorem (see [20]), guaranteeing that the family of exponentials $\{e^{\lambda_n t}\}_{n\geq 1}$ is linearly independent in $L^2(0,T)$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} < \infty. \tag{1.6}$$

In the context of system (1.3), condition (1.5) and, implicitly, (1.6) are verified if and only if $\alpha > 1/2$.

According to this analysis, it was proved in [7] that, when $\alpha > 1/2$, and when the control profile f satisfies the condition

$$\lim_{n\to\infty} \left(\left| \int_0^{\pi} f(x)\sin(nx)dx \right| e^{\eta\lambda_n} \right) > 0 \quad \forall \eta > 0, \tag{1.7}$$

system (1.1) is null controllable in the sense above for an arbitrarily short time and with smooth time-dependent controls g.

It is important to note that, according to condition (1.7), the shape function f is not "too regular." In particular, its Fourier coefficients may not decay faster than a suitable exponential function. Obviously, one can find control profiles f with such a property in any Sobolev space $H^s(0,\pi)$ and, in particular, in $L^2(0,\pi)$.

The present paper deals with the case $\alpha \leq 1/2$. As we shall see, the behavior of the system from the control theoretical point of view is, surprisingly, the opposite one.

Concerning the growth condition (1.5) on the spectrum, the case $\alpha = 1/2$ is critical and the condition, clearly, does not hold when $0 < \alpha < 1/2$. The same can be said about the summability condition (1.6). In this sense, the situation we are dealing with is similar to that in [17] and [18], where the heat equation in the half-line and half-space was considered. Indeed, in [17] it was proved that when $\lambda_n = n$, the corresponding moment problem (1.4) has a solution only if the β_n grows very fast as n tends to infinity.[1] Since β_n is, essentially, the ratio between the Fourier coefficients of the initial data to be controlled and those of the control profile f, we concluded that no regular nontrivial initial data allow a L^2-solution of the moment problem, when the profile is not too smooth. Accordingly L^2-controls may not exist either. The same can be said about the control problem (1.1) in the whole range $0 < \alpha \leq 1/2$.

[1] Recently, the results of [17] and [18], and more precisely its consequences in the context of unique continuation, were generalized in [5] to the case of parabolic equations with a potential, by means of Carleman inequalities.

This negative result shows that the parabolic nature of the equation and the infinite velocity of propagation do not suffice to guarantee the controllability of the system. On the contrary, we see that, in order for the control properties of the heat equation to be true, very much like in the finite-dimensional theory, the underlying semigroup is required to have a very strong dissipative effect that fails when $\alpha \leq 1/2$.

To be more precise, we shall show that

- if the shape function f satisfies (1.7), no initial data in any negative Sobolev space may be controlled to zero;
- if this function is more regular, for instance, if it satisfies

$$\left| \int_0^\pi f(x) \sin(nx) dx \right| \leq e^{-\eta \lambda_n} \tag{1.8}$$

  for some $\eta > T$, then there are initial data in any Sobolev space $H^m(0,\pi)$ that are null controllable in time T with L^2-controls.

As we said above, and contrary to intuition, this behavior is in opposition to the control properties of the heat equation corresponding to $\alpha = 1$.

Let us mention that the property

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} > \infty \tag{1.9}$$

of the eigenvalues of the differential operator leads to a result of no spectral controllability in the case of the heat equation in multidimensional problems (see [1, Theorem IV.1.3, p. 178]). On the other hand, under hypothesis (1.9), Fattorini [6] shows that for any $T > 0$ there exist a shape function $f \in L^2(0,\pi)$ and an initial datum $u^0 \in L^2(0,\pi)$ which cannot be driven to zero in time T by means of a control of type $f(x)g(t)$. The proofs of these results are based on the fact that an entire function which vanishes at every λ_n is identically zero and are related to the methods we use in our article. However, note that, given a shape function f, we describe the space of the initial data which cannot be controlled to zero in finite time.

It is also interesting to compare the results we obtain in this paper with those that one could expect from the application of the methodology in the articles by Lebeau and Robbiano [13] and Lebeau and Zuazua [14]. In [13] and [14] an iterative method was developed to prove the null controllability of the heat equation when the control acts in an open subset of the domain where the equation holds. The same method can be used to deal with control mechanisms as in (1.1). Their main idea was to split the time interval into a sequence of decreasing consecutive subintervals. In each of these intervals an increasing finite number of Fourier components (determined by a diadic decomposition) is controlled to zero, the control being applied in two steps. In a first step (in half of the subinterval) where a nontrivial control is applied, an estimate based on Carleman inequalities guarantees that the size of the control does not grow faster than an exponential factor, in which the maximal eigenfrequency of the eigenfunctions under consideration enters. It was then shown that the dissipative property of the heat equation in the remaining half of the subinterval was able to compensate this exponential growth. A careful analysis of the method of proof in [13] and [14] shows that it works if $\alpha > 1/2$. The results of the present paper show that the results this method yields are sharp in the sense that completely opposite results hold when $\alpha \leq 1/2$. This fact confirms once more that the control theoretical results of the heat equation do hold because of its very strong dissipative properties.

The paper is organized as follows. In section 2 we present the controllability problem and some equivalent formulations. Some known results are also mentioned. In section 3 our main controllability results for the case $\alpha \leq 1/2$ are stated. They are based on two propositions concerning entire functions that are proven in section 4. In section 5 we give a negative result concerning the dual observability inequality (with respect to the control problem). In section 6 we analyze the controllability properties of a hyperbolic equation involving the same operator $(-\Delta)^\alpha$:

$$u_{tt} + (-\Delta)^\alpha u = g(t)f(x).$$

In this case the situation is even worse since the classical control properties of the $1-d$ wave equation (that correspond to the exponent $\alpha = 1$) fail for all $\alpha < 1$. Some comments and open problems are included in section 7.

2. Problem formulation and existing results. We first observe that the operator A_α in (1.3) is well defined since $(\sqrt{2}\sin(nx)/\sqrt{\pi})_{n\geq 1}$ forms an orthonormal basis in $L^2(0,\pi)$. Moreover, the operator A_α is densely defined and is self-adjoint in $L^2(0,\pi)$.

The eigenvalues of the operator A_α are given by

$$\lambda_n = n^{2\alpha} \quad \forall n \geq 1 \tag{2.1}$$

with eigenfunctions

$$\varphi_n = \sin(nx) \quad \forall n \geq 1. \tag{2.2}$$

With this notation the control problem for system (1.1) can be formulated as follows: Given $T > 0$, $f \in L^2(0,\pi)$ and $u^0 \in L^2(0,\pi)$ find $g \in L^2(0,T)$ such that the solution u of problem

$$\begin{cases} u_t + A_\alpha u = g(t)f(x), & x \in (0,\pi),\ t \in (0,T), \\ u = 0, & x \in \{0,\pi\},\ t \in (0,T), \\ u(0,x) = u^0(x), & x \in (0,\pi), \end{cases} \tag{2.3}$$

satisfies

$$u(T,\cdot) = 0. \tag{2.4}$$

An initial datum u^0 with such property is said to be *null controllable* in time T. If all initial data in $L^2(0,\pi)$ are null controllable we say that (2.3) is null controllable in $L^2(0,\pi)$.

The goal is to drive the initial datum u^0 to rest by using a control with a given shape $f(x)$ in space at each time. Then the control $g(t)$ determines the intensity of the control profile applied to the system.

Let us first give the following variational characterization of controllable initial data.

LEMMA 2.1. *The initial datum $u^0 \in L^2(0,\pi)$ is null controllable in time T with control $g \in L^2(0,T)$ if and only if the identity*

$$-\int_0^\pi u^0(x)\varphi(0,x)dx = \int_0^T g(t)\left(\int_0^\pi f(x)\varphi(t,x)dx\right)dt \tag{2.5}$$

holds for any $\varphi^T \in L^2(0,\pi)$ *with* $\varphi(t,x)$ *solution of the adjoint equation*

$$\begin{cases} -\varphi_t + A_\alpha \varphi = 0, & x \in (0,\pi),\ t \in (0,T), \\ \varphi = 0, & x \in \{0,\pi\},\ t \in (0,T), \\ \varphi(T,x) = \varphi^T(x), & x \in (0,\pi). \end{cases} \tag{2.6}$$

Proof. The proof follows immediately by multiplying (2.3) by φ, the solution of (2.6), integrating in $(0,T)\times\Omega$, and taking into account that A_α is self-adjoint. ◻

Since $(\sin(nx))_{n\geq 1}$ is complete in $L^2(0,\pi)$, considering $\varphi^T(x) = \sin(nx)$ for each $n \geq 1$ in Lemma 2.1, the following equivalent condition for the null-controllability results.

LEMMA 2.2. *An initial datum* $u^0 \in L^2(0,\pi)$ *of the form*

$$u^0(x) = \sum_{n\geq 1} a_n \sin(nx) \tag{2.7}$$

is null controllable in time T *if and only if there exists* $g \in L^2(0,T)$ *such that, for any* $n \geq 1$,

$$f_n \int_0^T g(t) e^{\lambda_n t} dt = -\frac{\pi}{2} a_n, \tag{2.8}$$

where

$$f_n = \int_0^\pi f(x) \sin(nx) dx. \tag{2.9}$$

Note that (2.8) is a *moment problem.*

Note also that, given an arbitrary initial datum u^0, a necessary condition for this moment problem to have a solution is that

$$f_n = \int_0^\pi f(x) \sin(nx) dx \neq 0 \quad \forall n \geq 1. \tag{2.10}$$

Indeed, if there exists $k \geq 1$ such that $f_k = 0$, the kth equation in (2.8) does not hold except for the case $a_k = 0$. In fact, if $f_k = 0$, it is easy to see that the kth Fourier component of the solution of the controlled problem (1.1) is invariant in time. This makes the controllability property impossible unless $a_k = 0$.

From now on we shall suppose that f verifies (2.10).

Let us now recall the following result from [7].

THEOREM 2.1. *Let* $\alpha > 1/2$ *and suppose that the Fourier coefficients of* f *satisfy* (2.10) *and the following additional condition:*

$$\liminf_{n\to\infty} |f_n| e^{\eta n^{2\alpha}} > 0 \tag{2.11}$$

for any $\eta > 0$.

Then, the initial state $u^0 = \sum_{n\geq 1} a_n \sin(nx)$ *is null controllable in time* $T > 0$ *by means of a control* $g \in L^2(0,T)$ *if, for some* $M, \eta > 0$,

$$|a_n| \leq M e^{n^{2\alpha} T} e^{-(\pi+\eta)n}, \quad n = 1, 2, \ldots. \tag{2.12}$$

Moreover, when this holds, the control g may be chosen to be in $C^m([0,T])$ for all $m \geq 1$.

REMARK 2.1. *The right-hand side term in* (2.12) *tends to infinity as $n \to \infty$. Thus, Theorem* 2.1 *implies, for instance, that any initial data in $L^2(0,\pi)$ are null controllable in any time $T > 0$. This result is in contrast with which we shall prove for the case $\alpha \leq 1/2$ that no initial data in a negative Sobolev space may be driven to zero in finite time with an L^2-control g.*

REMARK 2.2. *Condition* (2.11) *requires the shape function $f = f(x)$ to be not "too regular." Obviously, one can find control profiles f with such a property in any Sobolev space $H^s(0,\pi)$, but a too fast exponential decay rate of the Fourier coefficients of f is incompatible with* (2.11). *In particular, when f is a Gaussian function,* (2.11) *fails for $\alpha = 1$, i.e., for the classical heat equation.*

3. Controllability results in the case $\boldsymbol{\alpha \leq 1/2}$. Let us now address the case $0 < \alpha \leq 1/2$. Throughout this section we will assume that $0 < \alpha \leq 1/2$. However, some of the results we present here are valid for all $\alpha > 0$. This will be indicated explicitly when it is the case.

3.1. The main negative result. The following result is completely different from that obtained in Theorem 2.1.

THEOREM 3.1. *Let $0 < \alpha \leq 1/2$ and suppose that the Fourier coefficients of f satisfy* (2.11). *Then any nontrivial initial state $u^0 = \sum_{n\geq 1} a_n \sin(nx)$ with the property that for any $\mu > 0$ there exists a constant $C_\mu > 0$ such that*

$$|a_n| \leq C_\mu e^{\mu n^{2\alpha}} \quad \forall n \geq 1 \tag{3.1}$$

cannot be driven to zero in time $T > 0$ by means of a control $g \in L^2(0,T)$, whatever $T > 0$ is.

REMARK 3.1. *The right-hand side term in* (3.1) *grows exponentially as $n \to \infty$. Thus, Theorem* 3.1 *implies that there is no initial datum in any Sobolev space of negative order that might be null controllable in any time $T > 0$ with controls g in $L^2(0,T)$.*

Consequently, this result is in opposition to the positive one in Theorem 2.1 *for the case $\alpha > 1/2$.*

In particular, Theorem 3.1 *means that choosing quite irregular control profiles f, as one is required to do when $\alpha > 1/2$ according to* (2.11), *is a very bad choice when $\alpha \leq 1/2$.*

REMARK 3.2. *From* (3.1) *it seems that, as α increases, the class of data for which the null-controllability property fails increases as well. However, a careful analysis of the proof of the theorem and Proposition* 3.2 *shows the contrary. Indeed, for the null-controllability property to fail, not all, but only part, of the Fourier coefficients of the initial datum must satisfy* (3.1). *Indeed, instead of* (3.1) *it is sufficient to have*

$$|a_{n_k}| \leq C_\mu e^{\mu n_k^{2\alpha}} \quad \forall k \geq 1 \tag{3.2}$$

for a suitable subsequence $(n_k)_{k\geq 1}$ (see Lemma 4.2 *and Remark* 4.2) *satisfying*

$$|n_{k+1} - n_k| \geq \frac{1}{2\alpha} k^{\frac{1}{2\alpha}-1} - 2 \quad \forall k \geq 1. \tag{3.3}$$

Note that (3.3) *shows that the distance between two consecutive terms of the sequence $(n_k)_{k\geq 1}$ decreases when α increases. Hence, the same happens to the class of* *data for which the null-controllability property fails. This agrees with the first intuition that suggests that, as the dissipativity of the system increases, its controllability properties improve.*

A dramatic change in the controllability properties arises when $\alpha = 1/2$. For $\alpha > 1/2$ the control problem is very well behaved (see Theorem 2.1). On the contrary, the controllability properties are very poor when $\alpha \leq 1/2$. Note that the same occurs with the spectral property (1.6). Something similar happens in (3.3) where, when $\alpha > 1/2$, the gap condition is fulfilled for all indices k without extracting subsequences.

3.2. Proof of the negative result. According to Lemma 2.2, the property of null controllability of $u^0 = \sum_{n\geq 1} a_n \sin(nx)$ is equivalent to the existence of a function $g \in L^2(0,T)$ such that, for any $n \geq 1$, (2.8) is verified.

Before getting into the proof of Theorem 3.1 let us first give an equivalent condition for the existence of such a control function g.

PROPOSITION 3.1. *The following assertions are equivalent:*

(a) *There exists $g \in L^2(0,T)$ such that the following holds:*

$$\int_0^T g(s) e^{n^{2\alpha} s} ds = \alpha_n \quad \forall n \geq 1. \tag{3.4}$$

(b) *There exists an entire function F of exponential type $\leq T/2$, with*

$$\int_{-\infty}^{\infty} |F(iy)|^2 \, dy < \infty \tag{3.5}$$

and such that

$$F(n^{2\alpha}) = \alpha_n e^{-n^{2\alpha} T/2} \quad \forall n \geq 1. \tag{3.6}$$

Recall that an entire function is said to be of exponential type $\leq B$ if there exists a positive constant $A > 0$ such that (see [21])

$$|F(z)| \leq A e^{B|z|} \quad \forall z \in \mathbb{C}. \tag{3.7}$$

REMARK 3.3. *Several remarks are in order:*

- *Proposition 3.1 is a very general result in which the explicit values of the coefficients α_n and the eigenvalues $\lambda_n = n^{2\alpha}$ do not matter.*
- *The proof of Proposition 3.1 uses the Fourier transform and the Paley–Wiener theorem and will be given in the next section.*
- *As the proof of this proposition shows, the function F in (b) is uniformly bounded along the imaginary axis.*
- *From Proposition 3.1, in order to characterize the null-controllable initial data it is necessary and sufficient to characterize the sequences $\{F(n^{2\alpha})\}_{n\geq 1}$ that may be obtained by means of entire functions F of exponential type $\leq T/2$ satisfying (3.5).*

The following proposition provides significant information on the rate of growth of $F(n^{2\alpha})$ for functions F as above.

PROPOSITION 3.2. *Let $F : \mathbb{C} \to \mathbb{C}$ be a function satisfying the following properties:*

(i) *F is an entire function of exponential type $\leq T/2$;*

(ii) *$\int_{-\infty}^{\infty} |F(iy)|^2 \, dy < \infty$;*

(iii) *for any $\delta > 0$ there exists $C_\delta > 0$ such that*

$$|F(n^{2\alpha})| \leq C_\delta e^{\delta n^{2\alpha}} e^{-n^{2\alpha}T/2} \quad \forall n \geq 1.$$

Then, necessarily, $F \equiv 0$.

The proof of Proposition 3.2 is based on a result of Duffin and Schaeffer (see [4] and also [3, p. 191]) which gives conditions for the boundedness of an analytic function in a sector of the complex plane if its boundedness on a sequence of complex numbers is assumed. In our case, the information we have on the behavior of $F(n^{2\alpha})$ allows us to construct an analytic function in the right half-plane which is bounded on a sequence of complex numbers close to n and to apply the mentioned result. The complete proof of Proposition 3.2 will be given in the next section.

Let us now show how Theorem 3.1 follows from Propositions 3.1 and 3.2.

If u^0 is null controllable, then the existence of a function F as in Proposition 3.1 is ensured with $\lambda_n = n^{2\alpha}$ and $\alpha_n = -a_n/(2f_n)$. Hence, F satisfies conditions (i)–(ii) in Proposition 3.2. Then, condition (3.1) on the Fourier coefficients of u^0 and condition (2.11) on the shape function f imply that

$$|F(n^{2\alpha})| \leq Ce^{(\mu+\eta)n^{2\alpha}} e^{-n^{2\alpha}T/2}. \tag{3.8}$$

Since μ and η are arbitrary, the function F also satisfies property (iii) from Proposition 3.2.

It follows that $F \equiv 0$ and, consequently, under the growth condition (3.1) and with control profiles satisfying (2.11), the only controllable initial datum is the trivial one. □

3.3. Other controllability properties. As we have said before, condition (2.11) indicates that the shape function f is not "too regular." Let us now show that assuming more regularity on f may increase the space of controllable data. This fact is also in opposition to the behavior of the system in the case $\alpha > 1/2$, in which increasing the regularity of the profile f reduces the space of controllable data.

PROPOSITION 3.3. *Let $\alpha \leq 1/2$ and suppose that there exists $\eta > T$ such that*

$$|f_n| \leq e^{-\eta n^{2\alpha}} \quad \forall n \geq 1. \tag{3.9}$$

Then there are initial data in any Sobolev space $H^m(0,\pi)$ which are null controllable by means of a control function $g \in L^2(0,T)$.

REMARK 3.4. *It is important to note that the result in Proposition* 3.3 *holds for $\alpha > 1/2$ as well. However, in this case, as mentioned above, one can prove much better results guaranteeing that all initial data in $L^2(0,\pi)$ are controllable even if condition* (3.9) *is not satisfied.*

Proof. From Lemma 2.2 and Proposition 3.1, it follows that an initial datum whose Fourier coefficients are given by

$$a_n = -2f_n F(n^{2\alpha}) e^{\frac{T}{2}n^{2\alpha}},$$

where

$$F(z) = \frac{\sin\left(\frac{T}{2}zi\right)}{\frac{T}{2}zi},$$

is null controllable in time T.

The initial datum with these Fourier coefficients belongs to any Sobolev space $H^m(0,\pi)$ with $m \geq 0$. Indeed,

$$\sum_{n\geq 1} |a_n|^2 n^{2m} \leq 4 \sum_{n\geq 1} |f_n|^2 |F(n^{2\alpha})|^2 n^{2m} e^{Tn^{2\alpha}}.$$

We now use in an essential way that F is of exponential type $\leq T/2$. This is obvious in this case in view of the explicit form of F. It follows that

$$\sum_{n\geq 1} |a_n|^2 n^{2m} \leq 4 \sum_{n\geq 1} e^{-2\eta n^{2\alpha}} \frac{e^{Tn^{2\alpha}}}{\left(\frac{Tn^{2\alpha}}{2}\right)^2} e^{Tn^{2\alpha}} n^{2m} < \infty. \qquad \square$$

As we mentioned above, when $\alpha > 1/2$, if the regularity of the shape function increases, the space of controllable initial data diminishes. As we have just proved, this is no longer true if $\alpha \leq 1/2$. In this case, some regular initial data may be controlled only if more regularity is assumed for the shape function f.

REMARK 3.5. *There exists an alternative proof for the above proposition which allows us to construct an explicit null-controllable initial datum under hypothesis* (3.9). *Indeed, let $g \in L^2(0,T)$ such that the solution u_1 of the ordinary differential equation*

$$\begin{cases} u_1' + u_1 = g(t) f_1, & t \in (0,T), \\ u_1(0) = 1 \end{cases} \tag{3.10}$$

satisfies $u_1(T) = 0$.

Now, for each $n \geq 2$, solve the following backward ordinary differential equation:

$$\begin{cases} u_n' + n^{2\alpha} u_n = g(t) f_n, & t \in (0,T), \\ u_n(T) = 0. \end{cases} \tag{3.11}$$

It is easy to see that

$$|u_n(0)| \leq \sqrt{T} |f_n| e^{Tn^{2\alpha}} ||g||_{L^2}.$$

Under hypothesis (3.9) *the initial datum*

$$u^0 = \sin(x) + \sum_{n\geq 2} u_n(0) \sin(nx)$$

belongs to $H^m(0,\pi)$ for any $m \geq 0$ and it is null controllable.

This example can easily be generalized by choosing first the control corresponding to a finite number of Fourier components, and then determining the other Fourier components of the controllable initial datum from the final equilibrium condition in terms of this control.

More precisely, fix a finite $N \geq 1$ and an arbitrary choice of the first N Fourier components of the initial datum to be controlled: $a_1, \dots, a_N$. Let $g = g(t)$ be such that each of the solutions of

$$\begin{cases} u_n' + n^{2\alpha} u_n = g(t) f_n, & t \in (0,T), \\ u_n(0) = a_n \end{cases} \tag{3.12}$$

satisfies $u_n(T) = 0$ for all $n = 1, \dots, N$. The existence of this control g is guaranteed. Indeed, system (3.12) *is controllable since the classical Kalman rank condition is satisfied.*

Once this is done for each $n \geq N+1$ we solve the backward problem

$$\begin{cases} u_n' + n^{2\alpha} u_n = g(t) f_n, & t \in (0,T), \\ u_n(T) = 0. \end{cases} \tag{3.13}$$

Under assumption (3.9) *the controlled initial datum*

$$u^0 = \sum_{n=1}^{N} a_n \sin(nx) + \sum_{n \geq N+1} u_n(0) \sin(nx)$$

belongs to $H^m(0,\pi)$ for any $m \geq 0$.

3.4. Partial controllability. In order to better explain the previous result it is convenient to introduce the following definition.

DEFINITION 3.1. *The initial datum $u^0 \in L^2(0,\pi)$ is N-partially controllable in time $T > 0$ if there exists $g = g_N \in L^2(0,T)$ such that the solution u of* (2.3) *verifies*

$$\Pi_N(u(T,\cdot)) = 0, \tag{3.14}$$

where Π_N is the orthogonal projection over the space generated by the first N eigenfunctions $(\sqrt{2}\sin(nx)/\sqrt{\pi})_{1 \leq n \leq N}$.

Arguing as in Lemma 2.2 we can show that the N-partial controllability problem is equivalent to a finite moment problem and more precisely to the existence of $g_N \in L^2(0,T)$ such that

$$f_n \int_0^T g_N(t) e^{\lambda_n t} dt = -\frac{\pi}{2} a_n \quad \text{for any } 1 \leq n \leq N. \tag{3.15}$$

A function g_N with property (3.15) will be called N-partial control. Its existence is easy to prove since, as mentioned above, the Kalman rank condition is satisfied. The lack of controllability properties proved above on the case $\alpha \leq 1/2$ suggests that the controls g_N should diverge as $N \to \infty$. Let us check this fact in a simple but illustrative example.

The system is N-partially controllable if and only if, for all $k \geq 1$, there exists $g_{k,N} \in L^2(0,T)$ such that

$$\int_0^T g_{k,N}(t) e^{\lambda_n t} dt = \delta_{kn} \text{ for any } 1 \leq n \leq N. \tag{3.16}$$

If $(g_{k,N})_{N \geq 1}$ is bounded in $L^2(0,T)$, there exists a subsequence which weakly converges as $N \to \infty$ to $g_k \in L^2(0,T)$ and

$$\int_0^T g_k(t) e^{\lambda_n t} dt = \delta_{kn} \quad \forall n \geq 1. \tag{3.17}$$

But relation (3.17) cannot hold since $(e^{\lambda_n t})_{n \geq 1, n \neq k}$ is complete in $L^2(0,T)$ (the divergence property (1.6) still holds if one exponent is eliminated). Hence, $(g_{k,N})_{N \geq 1}$ may not be bounded in $L^2(0,T)$.

A sequence $(g_k)_{k \geq 1}$ with property (3.17) is called biorthogonal to $(e^{\lambda_n t})_{n \geq 1}$. When $\alpha \leq 1/2$ such a biorthogonal sequence does not exist. From the controllability point of view the fact that, for k fixed, $g_{k,N}$ diverges as $N \to \infty$ means that it is impossible to control to zero one Fourier mode of the initial datum. This is in agreement with the positive result in Proposition 3.3 indicating that taking a more smooth control profile may increase the space of controllable data. Note that, as the Fourier components of the control profile f decay faster, the impact of the controls on the high frequencies decreases. This does help in building smooth data that are controllable, as the construction of Remark 3.5 shows.

In fact, according to Theorem 3.1, if the Fourier coefficients of the initial datum are not large enough, the sequence of N-partial controls $(g_N)_{N\geq 1}$ diverges and no control exists. $\square$

The previous notion of N-partial controllability can be extended as follows: Given a subset $I \subset \mathbb{N}$ of indices we introduce the subspace H_I of $L^2(0,\pi)$ spanned by the eigenfunctions of the Laplacian with indices in I. More precisely,

$$H_I = \left\{ \varphi \in L^2(0,\pi) \,:\, \varphi(x) = \sum_{j\in I} a_j \sin(jx),\ \sum_{j\in I} |a_j|^2 < \infty \right\}. \tag{3.18}$$

We then introduce the orthogonal projection Π_I from $L^2(0,\pi)$ into H_I.

DEFINITION 3.2. *System* (2.3) *is H_I-partially controllable in time $T > 0$ if for every initial datum $u^0 \in L^2(0,\pi)$ there exists a control $g \in L^2(0,T)$ such that the solution u of* (2.3) *verifies*

$$\Pi_I(u(T,\cdot)) = 0. \tag{3.19}$$

This control property is also equivalent to finding $g_I \in L^2(0,T)$ such that

$$f_j \int_0^T g_I(t) e^{\lambda_j t} dt = -\frac{\pi}{2} a_j \quad \forall j \in I. \tag{3.20}$$

Obviously, this generalizes the N-partial controllability problem that corresponds to the case where $I = \{1, 2, \dots, N\}$.

As mentioned in the introduction, the solvability of (3.19) and/or (3.20) depends on the summability condition

$$\sum_{j\in I} \frac{1}{|\lambda_j|} < \infty. \tag{3.21}$$

In the case under consideration, $\lambda_j = j^{2\alpha}$. Therefore, we see that (3.21) is satisfied under the following conditions:

1. When $\alpha > 1/2$ for $I = \mathbb{N}$. In this case partial controllability turns out to be complete null controllability.
2. When $0 < \alpha \leq 1/2$ for a suitable subsequence I_α of $\mathbb{N}$. It is obvious that one needs to consider a strict subsequence I_α of $\mathbb{N}$. Moreover, as α decreases, the subsequence I_α becomes more and more sparse in $\mathbb{N}$ and, therefore, the property of partial controllability weaker and weaker. This result agrees with a first intuition suggesting that an increase of diffusivity enhances the null-controllability properties of the system.

4. Proofs of some technical results.

4.1. Proof of Proposition 3.1. First of all we observe that

$$\int_0^T g(s)e^{n^{2\alpha}s}ds = \int_{-T/2}^{T/2} g(s+T/2)e^{n^{2\alpha}(s+T/2)}ds$$
$$= e^{n^{2\alpha}T/2}\int_{-T/2}^{T/2} g(s+T/2)e^{n^{2\alpha}s}ds = e^{n^{2\alpha}T/2}\int_{-T/2}^{T/2} h(s)e^{n^{2\alpha}s}ds$$

with $h(s) = g(s+T/2)$.

Hence, statement (a) of Proposition 3.1 is equivalent to the following one:

(4.1)

$$\text{(a}'\text{)}\quad \exists h \in L^2(-T/2, T/2) \text{ such that } \int_{-T/2}^{T/2} h(s)e^{n^{2\alpha}s}ds = e^{-n^{2\alpha}T/2}\alpha_n \quad \forall n \geq 1.$$

We now prove that (a′) and (b) are equivalent.

- (a′) $\Rightarrow$ (b).
  Let H be the Fourier transform of $h(s)1_{(-T/2,T/2)}$, i.e.,
  $$H(z) = \int_{-T/2}^{T/2} h(s)e^{-izs}ds,$$
  and let $F(z) = H(iz)$. According to the Paley–Wiener theorem (see, for instance, [3] or [21]), we know that $H : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $\leq T/2$ and such that $\int_{-\infty}^{\infty} \mid H(x) \mid^2 dx < \infty$. Consequently, F is also an entire function of exponential type $\leq T/2$ such that $\int_{-\infty}^{\infty} \mid F(ix) \mid^2 dx < \infty$.
  Moreover, in view of (4.1),
  $$F(n^{2\alpha}) = H(in^{2\alpha}) = \int_{-T/2}^{T/2} h(s)e^{n^{2\alpha}s}ds = e^{-n^{2\alpha}T/2}\alpha_n.$$
  This shows that (b) holds.
- (b) $\Rightarrow$ (a′).
  Let F be an entire function of exponential type $\leq T/2$, with $\int_{-\infty}^{\infty} \mid F(ix) \mid^2 dx < \infty$ and such that (3.6) holds.
  We then set $H(z) = F(-iz)$, which is also an entire function of exponential type $\leq T/2$ with $\int_{-\infty}^{\infty} \mid H(x) \mid^2 dx < \infty$.
  From the Paley–Wiener theorem we deduce that there exists $h \in L^2(-\frac{T}{2}, \frac{T}{2})$ such that
  $$H(z) = \int_{-T/2}^{T/2} h(s)e^{-izs}ds.$$
  We have that
  $$\int_{-T/2}^{T/2} h(s)e^{n^{2\alpha}s}ds = H(in^{2\alpha}) = F(n^{2\alpha}) = \alpha_n e^{-n^{2\alpha}T/2}$$
  and (a′) is verified.

This completes the proof of Proposition 3.1. $\square$

4.2. Properties of the eigenvalues. Let us recall that the operator A_α we are dealing with has a sequence of eigenvalues $\lambda_n = n^{2\alpha}$, $n \geq 1$. Recall also that we are dealing with the case $0 < \alpha \leq 1/2$. In this section we deduce some properties of these eigenvalues.

LEMMA 4.1. *The sequence* $(\lambda_n)_{n\geq 1}$ *has the following properties:*
1. *It is strictly increasing and* $\lim_{n\to\infty} \lambda_n = \infty$.
2. *For any* $n \geq 1$,

$$\lambda_{n+1} - \lambda_n \leq \frac{2\alpha}{n^{1-2\alpha}}. \tag{4.2}$$

Proof. The first part is obvious. For the second one let us note that

$$\lambda_{n+1} - \lambda_n = (n+1)^{2\alpha} - n^{2\alpha} = n^{2\alpha}\left[\left(\frac{1}{n}+1\right)^{2\alpha} - 1\right]$$

and use the fact that for any $x > 0$ there exists ξ in $[0, x]$ such that

$$(x+1)^{2\alpha} = 1 + 2\alpha x + 2\alpha(2\alpha-1)\frac{x^2}{2}(\xi+1)^{2\alpha-2} \leq 1 + 2\alpha x.$$

It follows that

$$\lambda_{n+1} - \lambda_n = n^{2\alpha}\left[\left(\frac{1}{n}+1\right)^{2\alpha} - 1\right] \leq n^{2\alpha}\left[\left(1+2\alpha\frac{1}{n}\right) - 1\right] = \frac{2\alpha}{n^{1-2\alpha}}. \qquad \square$$

Concerning the distribution of the sequence $(\lambda_n)_{n\geq 1}$ the following can be said.

LEMMA 4.2. *There exists an increasing sequence* $(n_k)_{k\in\mathbb{N}^*}$ *in* $\mathbb{N}^*$ *such that*
1. *there exists* $\beta > 0$ *such that* $0 < \beta < n_{k+1}^{2\alpha} - n_k^{2\alpha}$, *for any* $k \geq 1$;
2. *for any* $k \geq 1$, $|k - n_k^{2\alpha}| \leq \alpha$.

Proof. If $\alpha = 1/2$, we may take $n_k = k$ and both properties are verified.

Consider now the case $\alpha < 1/2$. If $k = 1$, we take $n_k = 1$. Suppose that $k \geq 2$. Let $n'_k = \inf\{n \in \mathbb{N}^* : k \leq n^{2\alpha}\}$. We have

$$(n'_k - 1)^{2\alpha} < k \leq (n'_k)^{2\alpha}$$

and $n'_k > 1$.

Define

$$n_k = \begin{cases} n'_k - 1 & \text{if } k - (n'_k-1)^{2\alpha} \leq (n'_k)^{2\alpha} - k, \\ n'_k & \text{if } (n'_k)^{2\alpha} - k < k - (n'_k-1)^{2\alpha}. \end{cases}$$

Taking Lemma 4.1 into account we obtain that

$$\begin{aligned} |k - (n_k)^{2\alpha}| &= \min\{k - (n'_k-1)^{2\alpha},\, (n'_k)^{2\alpha} - k\} \leq \frac{1}{2}\left(k - (n'_k-1)^{2\alpha} + (n'_k)^{2\alpha} - k\right) \\ &= \frac{1}{2}\left((n'_k)^{2\alpha} - (n'_k-1)^{2\alpha}\right) = \frac{1}{2}(\lambda_{n'_k} - \lambda_{n'_k-1}) \leq \frac{\alpha}{(n'_k-1)^{1-2\alpha}} \leq \alpha \end{aligned}$$

and the second property of the statement of the Lemma is verified. On the other hand

$$\begin{aligned} |n_k^{2\alpha} - n_{k-1}^{2\alpha}| &\geq 1 - \left(|n_k^{2\alpha} - k| + |n_{k-1}^{2\alpha} - (k-1)|\right) \\ &\geq 1 - \alpha\left[\frac{1}{(n'_k-1)^{1-2\alpha}} + \frac{1}{(n'_{k-1}-1)^{1-2\alpha}}\right] \geq 1 - 2\alpha > 0 \end{aligned}$$

and the first property is verified as well. □

Remark 4.1. *Lemma* 4.2 *says that there exists a subsequence* $(\lambda_{n_k})_{k\geq 1}$ *of the sequence of eigenvalues* $(\lambda_n)_{n\geq 1}$ *such that*

- $|\lambda_{n_k} - k| \leq \alpha$ *for all* $k \geq 1$*;*
- $|\lambda_{n_k} - \lambda_{n_{k-1}}| > \beta > 0$ *for all* $k \geq 2$*.*

This subsequence (λ_{n_k}) *will be used to prove Proposition* 3.2*.*

Remark 4.2. *The subsequence* $(n_k)_{k\geq 1}$ *constructed in the proof of Lemma* 4.2 *satisfies*

$$|n_{k+1} - n_k| \geq \frac{1}{2\alpha} k^{\frac{1}{2\alpha}-1} - 2 \quad \forall k \geq 1. \tag{4.3}$$

Indeed,

$$n_{k+1} - n_k \geq n'_{k+1} - n'_k - 1 \geq (k+1)^{\frac{1}{2\alpha}} - k^{\frac{1}{2\alpha}} - 2 \geq \frac{1}{2\alpha} k^{\frac{1}{2\alpha}-1} - 2.$$

4.3. Proof of Proposition 3.2. We introduce the function $G : \mathbb{C} \to \mathbb{C}$:

$$G(z) = e^{Tz/2} F(z). \tag{4.4}$$

In view of properties (i)–(ii) of F it is immediate that

$$G \text{ is an entire function of exponential type } \leq T; \tag{4.5}$$

$$\int_{-\infty}^{\infty} | G(iy) |^2 \, dy < \infty; \tag{4.6}$$

$$\forall \delta > 0 :| G(n^{2\alpha}) |\leq C_\delta e^{\delta n^{2\alpha}} \quad \forall n \geq 1. \tag{4.7}$$

Moreover, G is bounded on the negative semiaxis, i.e.,

$$\exists L > 0 :| G(-x) |\leq L \quad \forall x \geq 0. \tag{4.8}$$

Property (4.8) is an immediate consequence of the fact that F is of exponential type $\leq T/2$.

We now introduce

$$G_1(z) = G\left(-ze^{i\pi/4}\right) \tag{4.9}$$

and apply the Phragmén–Lindelöf theorem to G_1 in the sector $| \arg z |< \pi/4$ to deduce that there exists $M_1 > 0$ such that

$$| G_1(z) |\leq M_1 \quad \forall z \in \mathbb{C} :| \arg z |\leq \pi/4. \tag{4.10}$$

This is possible since

$$G_1 \text{ is analytic on } \mathbb{C}; \tag{4.11}$$

$$G_1 \text{ is bounded when } \arg z = \pm\pi/4; \tag{4.12}$$

$$| G_1(z) |= O\left(e^{|z|^\beta}\right) \text{ for some } \beta < 2, \text{ as } | z |\to \infty. \tag{4.13}$$

Note that (4.12) holds because G is bounded along the imaginary axis by (4.6) and on the negative semiaxis by (4.8). On the other hand, (4.13) holds for any $\beta > 1$ since $| G_1(z) |= O\left(e^{T|z|}\right)$, due to (4.5).

As a consequence of (4.10) we deduce that

$$| G(z) | \leq M_1 \tag{4.14}$$

for all $z \in \mathbb{C}$ with $\arg(z) \in [\pi/2, \pi]$.

In a similar way we may prove the existence of $M_2 > 0$ such that

$$| G(z) | \leq M_2 \tag{4.15}$$

for all $z \in \mathbb{C}$ with $\arg(z) \in [\pi, 3\pi/2]$. Hence, G is bounded in the half complex plane $\operatorname{Re} z \leq 0$.

Let us now consider the function

$$H_\delta(z) = G(z)e^{-\delta z} = e^{Tz/2}e^{-\delta z}F(z) \tag{4.16}$$

defined on the half-plane $\operatorname{Re} z \geq 0$. It is easy to see that H_δ satisfies the following properties:

(4.17) H_δ is analytic on the closed half-plane $\operatorname{Re} z \geq 0$;

(4.18) H_δ is of exponential type;

(4.19) $\exists C_\delta > 0 : | H_\delta(n^{2\alpha}) | \leq C_\delta \quad \forall n \geq 1$;

(4.20) H_δ is bounded on the imaginary axis.

We now introduce the indicator function

$$h_{H_\delta}(\theta) = \limsup_{r \to \infty} \left[\frac{1}{r} \log \left| H_\delta\left(re^{i\theta}\right) \right| \right] \quad \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]. \tag{4.21}$$

Lemma 4.3. *For any $\delta < T$, there exists a positive constant $A > 0$ such that*

$$h_{H_\delta}(\theta) \leq A \cos\theta \quad \forall \theta \in [-\pi/2, \pi/2]. \tag{4.22}$$

Proof of Lemma 4.3. We have

$$\begin{aligned} \log \left| H_\delta\left(re^{i\theta}\right) \right| &= \log \left| e^{(T-2\delta)re^{i\theta}/2} F\left(re^{i\theta}\right) \right| \\ &= \log \left| e^{(T-2\delta)re^{i\theta}/2} \right| + \log \left| F\left(re^{i\theta}\right) \right| = \frac{(T-2\delta) r \cos\theta}{2} + \log \left| F\left(re^{i\theta}\right) \right|. \end{aligned} \tag{4.23}$$

On the other hand, arguing as in the proof of Proposition 3.1, we deduce from the Paley–Wiener theorem the existence of a function $\psi \in L^2(-T/2, T/2)$ such that

$$F(z) = \int_{-T/2}^{T/2} \psi(s) e^{zs} ds.$$

Therefore

$$\left| F\left(re^{i\theta}\right) \right| \leq \int_{-T/2}^{T/2} | \psi(s) | \, e^{sr\cos\theta} ds \leq e^{Tr|\cos\theta|/2} \int_{-T/2}^{T/2} | \psi(s) | \, ds. \tag{4.24}$$

Combining (4.22) and (4.24) we deduce that

$$\log \left| H_\delta\left(re^{i\theta}\right) \right| \leq (T-\delta) r \, | \cos\theta | + \log \| \psi \|_{L^1(-T/2, T/2)} . \tag{4.25}$$

From (4.25) we easily deduce that (4.21) holds with

$$A = T - \delta. \qquad \square \tag{4.26}$$

Let us now return to the proof of Proposition 3.2. By a result of Duffin and Schaeffer [4] (see also [3, p. 191]) we have the following theorem.

THEOREM 4.1. *Let f be analytic in $\mid \arg(z) \mid \leq \gamma \leq \pi/2$ and suppose that its indicator function h_f satisfies*

$$\mid h_f(\theta) \mid \leq a \mid \cos\theta \mid + b \mid \sin\theta \mid \quad \forall \mid \theta \mid \leq \gamma \tag{4.27}$$

with $a, b > 0$ and $b < \pi$.

If $(\nu_k)_{k\geq 1}$ is an increasing sequence of real numbers such that

$$\nu_{k+1} - \nu_k \geq \beta > 0 \quad \forall k \geq 1, \tag{4.28}$$

$$\mid \nu_k - k \mid \leq L \quad \forall k \geq 1, \tag{4.29}$$

and $f(\nu_k)$ is bounded, then $f(x)$ is bounded for all $x > 0$.

We apply Theorem 4.1 to the function H_δ with $\nu_k = \lambda_{n_k} = n_k^{2\alpha}$, where n_k are given by Lemma 4.2. The sequence $(\nu_k)_{k\geq 1}$ satisfies the hypotheses of Theorem 4.1. Moreover,

$$|H_\delta(\nu_k)| = |G(\nu_k)|e^{-\delta\nu_k} = |G(\lambda_{n_k})|e^{-\delta n_k^{2\alpha}} \leq C_\delta.$$

We deduce from Theorem 4.1 that H_δ is bounded on the positive real axis. Since, by (4.20), H_δ is also bounded on the imaginary axis, we deduce, by the Phragmén–Lindelöf theorem, that H_δ is bounded in the half-plane $\operatorname{Re} z \geq 0$ for all $0 < \delta < S$.

Consequently,

- G is bounded on the half-plane $\operatorname{Re} z \leq 0$;
- $\mid G(z) \mid \leq C(\delta)e^{\delta|z|}$ on the half-plane $\operatorname{Re} z \geq 0$ for all $0 < \delta < S$;
- G is entire;
- $\int_{-\infty}^{\infty} \mid G(iy) \mid^2 dy < \infty$.

According to the Paley–Wiener theorem, these properties are sufficient to guarantee that $G \equiv 0$. $\square$

5. On the lack of observability estimates. A natural approach to the problem of null controllability of heat equations consists in dealing with the dual observability problem for the adjoint system (see, for instance, [8], [22], and [23]).

More precisely, the null controllability of system (2.3) in $L^2(0,\pi)$ with controls in $L^2(0,T)$ is equivalent to the existence of a positive constant $C > 0$ such that

$$\| \varphi(0) \|^2_{L^2(0,\pi)} \leq C \int_0^T \left| \int_0^\pi \varphi(t,x) f(x) dx \right|^2 dt \quad \forall \varphi^T \in L^2(0,\pi), \tag{5.1}$$

where φ is the solution of (2.6).

As we have shown in Theorem 3.1, when $0 < \alpha \leq 1/2$, the null-controllability result is false and therefore (5.1) does not hold. In fact, according to the statement of Theorem 3.1, it turns out that all the possible weaker versions of (5.1) in which the L^2-norm of the left-hand side is replaced by an $H^{-\sigma}$-norm for any $\sigma > 0$ are false as well.

In this section we describe how the lack of observability inequalities of form (5.1) may be proved directly.

In view of the Fourier series expansion of the solution φ of (2.6) we have

$$\varphi(t,x) = \sum_{n\geq 1} a_n e^{-n^{2\alpha}(T-t)} \sin(nx).$$

Thus (5.1) is equivalent to

$$\sum_{n\geq 1} |a_n|^2 e^{-2n^{2\alpha}T} \leq C \int_0^T \left| \sum_{n\geq 1} a_n f_n e^{-n^{2\alpha}t} \right|^2 dt. \tag{5.2}$$

Inequalities of form (5.2) are well known to be true when $\alpha > 1/2$ (see, for instance, [7] and [19]). But they fail when $\alpha \leq 1/2$ since the series $\sum_{n\geq 1} 1/n^{2\alpha}$ diverges in that case (see [17]). More precisely, the following negative result holds.

PROPOSITION 5.1. *When $0 < \alpha \leq 1/2$ there is no sequence $(\rho_n)_{n\geq 1}$ of positive weights, i.e., $\rho_n > 0$ for all $n \geq 1$, such that*

$$\sum_{n\geq 1} \rho_n |b_n|^2 \leq \int_0^T \left| \sum_{n\geq 1} b_n e^{-n^{2\alpha}t} \right|^2 dt \tag{5.3}$$

for all finite sequence $(b_n)_{n\geq 1}$.

This result excludes inequality (5.2) and any other weaker version of it. Observe that an inequality like (5.2) is equivalent to the null controllability in time T of all initial data in the class

$$H = \left\{ u^0 = \sum_{n\geq 1} a_n \sin(n\pi) : \sum_{n\geq 1} |a_n|^2/\rho_n < \infty \right\},$$

and, according to the result of Theorem 3.1, we know that this is false for all sequences of weights $(\rho_n)_{n\geq 1}$.

Proposition 5.1 is an immediate consequence of the following one.

PROPOSITION 5.2. *Let $(\nu_n\}_{n\geq 1}$ be an increasing sequence of positive real numbers. Assume that there exists a sequence of positive weights $(\rho_n)_{n\geq 1}$ such that*

$$\sum_{n\geq 1} \rho_n |a_n|^2 \leq \int_0^T \left| \sum_{n\geq 1} a_n e^{-\nu_n t} \right|^2 dt \tag{5.4}$$

for all finite sequence $(a_n)_{n\geq 1}$. Then, necessarily,

$$\sum_{n\geq 1} \frac{1}{\nu_n} < \infty. \tag{5.5}$$

We refer to Proposition 3.5 in [17] for a proof.

The proof of Proposition 5.2 provides in fact a stronger result. Namely, it shows that if the sequence $(\nu_n)_{n\geq 1}$ is such that for some n_0 and $\rho > 0$ we have

$$\rho |a_{n_0}|^2 \leq \int_0^1 \left| \sum_{n\geq 0} a_n e^{-\nu_n t} \right|^2 dt \tag{5.6}$$

for all finite sequence $(a_n)_{n\geq 1}$, then, necessarily,

$$\sum_{n\geq 1} \frac{1}{\nu_n} < \infty. \tag{5.7}$$

Inequalities of form (5.6) are related to the so-called spectral controllability problem, which consists in analyzing whether all the eigenfunctions may be driven to zero in finite time. Theorem 3.1 provides a negative answer.[2] Proposition 5.1 provides a second proof of this negative result in which the effect of the divergence of the series $\sum_{n\geq 1} 1/\nu_n$ is clearly seen.

Note that spectral controllability also implies that all finite combinations of eigenfunctions are controllable and also show the controllability property in the infinite-dimensional space generated by the eigenfunctions, with suitable weights as the frequency increases.

The results on partial controllability of section 3.4 can also be understood in terms of observability inequalities. Indeed, the H_I-partial controllability property is equivalent to the observability property (5.1) in the subspace of solutions of the adjoint system (2.6) with initial data φ^T in H_I, i.e., of solutions φ of (2.6) involving only the Fourier coefficients with indices $j \in I$. This turns out to be equivalent to an inequality of the form

$$\sum_{j\in I} |a_j|^2 e^{-2\lambda_j T} \leq C \int_0^T \left| \sum_{j\in I} a_j e^{-\lambda_j t} \right|^2 dt \tag{5.8}$$

for all finite sequence $(a_n)_{n\geq 1}$.

Inequality (5.8) holds provided the subsequence $(\lambda_j)_{j\in I}$ fulfills a gap condition and the summability condition

$$\sum_{j\in I} \frac{1}{|\lambda_j|} < \infty. \tag{5.9}$$

As indicated in section 3.4 these conditions are satisfied provided the sequence I is sparse enough.

6. A hyperbolic problem.

In this section we consider a hyperbolic system involving the operator A_α and address the corresponding control problem: Given $T > 0$, $f \in L^2(0,\pi)$, and initial data (u^0, u^1), find $g \in L^2(0,T)$ such that the solution u of the problem

$$\begin{cases} u_{tt} + A_\alpha u = g(t)f(x), & x \in (0,\pi),\ t \in (0,T), \\ u = 0, & x \in \{0,\pi\},\ t \in (0,T), \\ u(0,x) = u^0(x), \quad u_t(0,x) = u^1(x), & x \in (0,\pi), \end{cases} \tag{6.1}$$

satisfies

$$u(T,\cdot) = u_t(T,\cdot) = 0. \tag{6.2}$$

[2] In fact the lack of (5.6) for any index n_0 shows that there is no single eigenfunction that may be driven to zero in final time with $L^2(0,T)$ controls.

Note that system (6.1) is a generalization of the wave equation

$$u_{tt} - u_{xx} = g(t)f(x)$$

that corresponds to the case $\alpha = 1$. In the absence of control (i.e., when $g = 0$) system (6.1) is conservative and generates a group of isometries in the corresponding energy space.

The eigenvalues corresponding to (6.1) are given by $i\nu_n$, $n \in \mathbb{Z}^*$, where

$$\nu_n = \operatorname{sgn}(n)|n|^\alpha \quad \forall n \neq 0.$$

When $\alpha \geq 1$ system (6.1) is controllable provided the control profile is such that all the Fourier components do not vanish. In particular that is the case for the wave equation in time $T = 2\pi$. As we shall see, the situation is even better when $\alpha > 1$, in which case the control property holds for an arbitrarily short time $T > 0$.

More precisely, the following holds.

THEOREM 6.1. *Let $\alpha \geq 1$ and f_k given by* (2.9) *satisfying* (2.10). *Any initial state in the space*

$$H = \left\{ (u^0, u^1) = \sum_{k \in \mathbb{Z}^*} a_k (\frac{1}{k^\alpha\, i}, -1) \sin(kx) : \sum_{k \in \mathbb{Z}^*} \frac{|a_k|^2}{|f_k|^2} < \infty \right\}$$

is controllable in time $T \geq 2\pi$ if $\alpha = 1$ and any time $T > 0$ if $\alpha > 1$, by means of a control $g \in L^2(0,T)$.

Proof. We first claim that the controllability of all initial data from H is equivalent to the inequality

$$C \sum_{n \in \mathbb{Z}^*} |c_n|^2 \leq \int_0^T \left| \sum_{n \in \mathbb{Z}^*} c_n e^{i\nu_n t} \right|^2 \tag{6.3}$$

for every sequence $(c_n)_{n \in \mathbb{Z}^*} \in \ell^2$.

Indeed, as in Lemma 2.2, it is easy to show that the controllability of

$$(u^0, u^1) = \sum_{k \in \mathbb{Z}^*} a_k \left(\frac{1}{k^\alpha\, i}, -1 \right) \sin(kx)$$

is equivalent to the following moment problem: Find $g \in L^2(0,T)$ such that

$$f_k \int_0^T g(t) e^{i\nu_k t} dt = a_k \quad \forall k \in \mathbb{Z}^*. \tag{6.4}$$

The moment problem (6.4) has a solution for any $(a_n/f_n)_{n \in \mathbb{Z}^*} \in \ell^2$ if and only if[3] the sequence $(e^{i\nu_n t})_{n \in \mathbb{Z}^*}$ is a Riesz–Fischer sequence in $L^2(0,T)$.

On the other hand, from the characterization of the Riesz–Fischer sequences (see [21, Theorem 3, p. 155]), it follows that the sequence $(e^{i\nu_n t})_{n \in \mathbb{Z}^*}$ is a Riesz–Fischer sequence in $L^2(0,T)$ if and only if (6.3) holds. This proves the claim.

We deduce that the moment problem (6.4) has a solution for any $(a_n/f_n)_{n \in \mathbb{Z}^*} \in \ell^2$ if and only if (6.3) holds.

[3]This is an immediate consequence of the definition of *Riesz–Fischer sequence*. Recall that a sequence of vectors $(x_n)_{n \in \mathbb{Z}^*}$ belonging to a Hilbert space H is said to be a Riesz–Fischer sequence if the moment problem $(x, x_n) = c_n$ for all $n \in \mathbb{Z}^*$ has a solution $x \in H$ for any $(c_n)_{n \in \mathbb{Z}^*} \in \ell^2$.

This can also be seen by the so-called HUM method by Lions [15] (see Remark 6.1).

Let us now show that, in the case $\alpha \geq 1$, (6.3) holds under the restrictions on T in the statement of the theorem. Indeed, from [2] (see also [9] and [10]), it follows that (6.3) holds for any $T > 2\pi/\gamma_\infty$ if

$$\liminf_{|n|\to\infty} |\nu_{n+1} - \nu_n| \geq \gamma_\infty > 0. \tag{6.5}$$

Since $|\nu_{n+1} - \nu_n| = (n+1)^\alpha - n^\alpha$, it follows that property (6.5) holds for any $T > 2\pi$ if $\alpha = 1$ (since $\gamma_\infty = 1$) and for any $T > 0$ if $\alpha > 1$ (since $\gamma_\infty = \infty$). Moreover, when $\alpha = 1$, in view of the time-orthogonality of the complex exponentials involved in the Fourier series development of solutions, property (6.5) holds for $T = 2\pi$ as well.

This completes the proof of the theorem. □

REMARK 6.1. *The controllability of* (6.1) *with initial data in H and controls in $L^2(0,T)$ is equivalent to the existence of a positive constant $C > 0$ such that*

$$||(\varphi^0,\varphi^1)||^2_{H'} \leq C \int_0^T \left| \int_0^\pi \varphi(t,x) f(x)dx \right|^2 dt,$$

$$\forall(\varphi^0,\varphi^1) \in H' = \left\{ (\varphi^0,\varphi^1) = \sum_{k\in\mathbb{Z}^*} a_k \left(\frac{1}{k^\alpha\, i}, -1 \right) \sin(kx) : \sum_{k\in\mathbb{Z}^*} |a_k|^2 |f_k|^2 < \infty \right\}, \tag{6.6}$$

where (φ, φ_t) is the solution of

$$\begin{cases} \varphi_{tt} + A_\alpha \varphi = 0, & x \in (0,\pi),\ t \in (0,T), \\ \varphi = 0, & x \in \{0,\pi\},\ t \in (0,T), \\ \varphi(0,x) = \varphi^0(x), \quad \varphi_t(0,x) = \varphi^1(x), & x \in (0,\pi). \end{cases} \tag{6.7}$$

Inequality (6.6) *is usually called the* observation inequality*.*

Using the Fourier expansion of the solutions of (6.7) *it is easy to see that* (6.6) *may be written as* (6.3)*. Inequality* (6.3) *may be proved by means of the classical Ingham inequality* (*see* [21])*.*

Once inequality (6.6) *is known to hold, the control $g = g(t)$ can be built by minimizing the quadratic functional*

$$J(\varphi^0,\varphi^1) = \frac{1}{2} \int_0^T \left| \int_0^\pi \varphi(t,x) f(x)dx \right|^2 dt + \langle (u^1, -u^0), (\varphi^0,\varphi^1) \rangle \tag{6.8}$$

in the space H'. Indeed, under the assumption that $(u^1, -u^0) \in H$ (the dual of H') the functional J is continuous, convex, and coercive in the Hilbert space H'. Thus its minimum exists. It is then easy to see that the control g we are looking for is $g(t) = \int_0^\pi \hat{\varphi}(t,x) f(x)dx$, where $\hat{\varphi}$ is the solution of (6.7) *with the minimizer of J as initial datum.*

The proof of Theorem 6.1 is based on inequality (6.3), which holds in the case $\alpha \geq 1$. Nevertheless, if $\alpha < 1$, there exists no uniform gap between two consecutive eigenvalues and (6.3) does not hold. The controllability properties are very different in this case.

In fact, when $0 < \alpha < 1$ system (6.1) is very badly controllable. Even the *spectral control property* fails to hold. We recall that system is said to be spectrally controllable if all initial data consisting in a single eigenfunction of the system may be controlled.

THEOREM 6.2. *If $\alpha < 1$, equation (6.1) is not spectrally controllable in any time $T > 0$.*

Proof. Suppose that (6.1) is spectrally controllable. Hence, every eigenfunction of system (6.1) may be driven to zero by using a control in $L^2(0,T)$.

But an initial datum of the form $(u^0, u^1) = (1/k^\alpha\, i, -1)\sin(kx)$ is controllable if and only if there exists $g \in L^2(0,T)$ such that

$$\int_0^T g(t)e^{i\nu_n t}dt = \delta_{nk}/f_k \quad \forall n \neq 0. \tag{6.9}$$

From the Paley–Wiener theorem we obtain that (6.9) implies the existence of an entire function G of exponential type $T/2$, such that $\int_{-\infty}^{\infty} |G(x)|^2 dx < \infty$ and $G(\nu_n) = 0$ for all $n \neq 0, k$.

Let $n_G(r)$ denote the number of zeros of the function G which belong to the ball of center zero and radius r,

$$n_G(r) = \#\{z \in \mathbb{C} : G(z) = 0 \text{ and } |z| \leq r\}.$$

We have

$$n_G(r) = 2\#\{n \in \mathbb{N}^* : n^\alpha \leq r\} = 2\left[r^{\frac{1}{\alpha}}\right].$$

Since $\alpha < 1$ it follows that

$$\lim_{r\to\infty} n_G(r)/r = \infty. \tag{6.10}$$

We need now the following result, which is a consequence of the well-known Jensen formula (see [21, Theorems 2 and 3, pp. 59–61]): *if f is an entire nontrivial function of exponential type, then $n_f(r)/r$ remains bounded as r tends to infinity.*

From (6.10) and the previous theorem it follows that $G \equiv 0$, which contradicts (6.9). $\square$

Our results show that for the hyperbolic equation (6.1) the critical exponent becomes $\alpha = 1$, instead of the exponent $\alpha = 1/2$ we have obtained for the parabolic equations (2.3).

7. Comments.

7.1. More general $1 - d$ problems. In this article we have considered the problem of controllability of a parabolic equation involving the fractional power of the Laplace operator. The control has a fixed shape, given by the function f. The problems of distributed control of the form $v(t,x)1_\omega$, with ω a subinterval of $(0,\pi)$, or of boundary control $v(t)$ may also be considered and will be treated elsewhere by similar techniques. The $1 - d$ analysis on the wave equation in section 6 may also be carried out for the Schrödinger and the beam equations.

7.2. Multidimensional problems. In several space dimensions, $N \geq 2$, similar problems can be analyzed. Consider the Dirichlet problem,

$$\begin{cases} u_t + (-\Delta)^\alpha u = g(t)f(x) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0,x) = u^0(x) & \text{in } \Omega. \end{cases} \tag{7.1}$$

Here Ω is a bounded domain of $\mathbb{R}^N$.

By Weyl's theorem, the spectrum of the Laplacian grows as the frequency increases in the following way: $\lambda_n \sim c(\Omega)n^{2/N}$.

According to this, the spectrum of the α-power of the Laplacian, $(-\Delta)^\alpha$, grows at a rate $n^{2\alpha/N}$ as $n \to \infty$. The critical case is then $\alpha = N/2$. One can then expect to obtain positive results for $\alpha > N/2$ and negative ones, as those presented here, for $\alpha \leq N/2$. An analysis of this multidimensional problem is also to be done.

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