On the Convection of Heat from Small Cylinders in a Stream of Fluid: Determination of the Convection Constants of Small Platinum Wires, with Applications to Hot-Wire Anemometry.

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(Abstract.)

Part I.

Sections 1 *and* 2.—Until comparatively recently, the problem of solving the equations of heat conduction in the case of a solid cooled by a stream of fluid had received little attention, although the general problem was formulated by Fourier* himself as long ago as 1820. In 1901 the problem was taken up by Boussinesq,† and many cases were dealt with in his memoir of 1905. By means of an extremely elegant transformation Boussinesq was able to express the general equation for the two-dimensional problem in a linear form: by transforming the equation to the set of orthogonal curvilinear co-ordinates determined by the stream-lines and equipotentials of the hydrodynamical problem of the flow of a uniform stream of velocity V past the cylindrical obstacle, the equation for the temperature θ at any point of the fluid takes the form

$$\partial^2\theta/\partial\alpha^2 + \partial^2\theta/\partial\beta^2 = 2n\partial\theta/\partial\beta, \quad (1)$$

where the curves $\alpha = constant$ represent the stream-lines and $\beta = constant$ the equipotentials. The constant n is given by the relation $2n = c\mathrm{V}/\kappa = s\sigma\mathrm{V}/\kappa$, where c is the specific heat of the fluid per unit volume, s that per unit mass, σ its density, and κ its thermal conductivity. If the surface of the cylinder be the particular stream-line $\alpha = 0$, and the critical equipotentials be the curves $\beta = 0$ and $\beta = \beta_0$, the heat-flux per unit length of the cylinder is given by

$$\mathrm{H} = -\int_0^{\beta_0} \kappa (\partial\theta/\partial\alpha)_0 \, d\beta, \quad (2)$$

where the integral is taken to include the two branches of the stream-line $\alpha = 0$.

* Fourier, 'Mémoires de l'Académie,' vol. 12, p. 507 (1820).

† Boussinesq, 'Comptes Rendus,' vol. 133, p. 257; also 'Journ. de Mathématiques,' vol. 1, pp. 285–332 (1905).

The transformation just described reduces the problem for any cylinder to that of calculating the temperature distribution in a uniform stream flowing parallel to the axis of $x\,(\alpha = 0)$ when the temperature or the heat-flux is prescribed over the interval $x = 0$ to $x = \beta_0$. Boussinesq obtains an approximate expression for the heat-loss by neglecting the term $\partial^2\theta/\partial\beta^2$ in equation (1), and deriving a simple Fourier solution corresponding to the condition that the temperature be constant over the cylindrical boundary.*

Section 3.—It was found by the writer that Boussinesq's result did not meet the requirements of the experiments on the convection of heat from small platinum wires cooled by a stream of air. The complete formulation of the possible boundary conditions is most conveniently obtained by expressing the problem in terms of an integral equation. Use is made of H. A. Wilson's solution† for the temperature at any point (x, y) due to a line source of strength Q at the origin in a stream of fluid flowing parallel to the x-axis in the form

$$\theta = (\mathrm{Q}/2\pi\kappa)\, e^{nx}\mathrm{K}_0(nr), \tag{3}$$

where $\mathrm{K}_0(z)$ is that solution of Bessel's equation most conveniently defined by the definite integral

$$\mathrm{K}_0(z) = \int_0^\infty e^{-z\cosh\phi}\, d\phi. \tag{4}$$

If $u(\xi)\,d\xi$ represent the total flux of heat from an elementary portion $d\xi$ of the x-axis between $x = 0$ and $x = \beta_0$, the boundary condition over the axis $y = 0$ is expressed by the relation

$$2\pi\kappa\,\theta(x) = \int_0^{\beta_0} u(\xi)\, e^{n(x-\xi)}\mathrm{K}_0\,|\,n(x-\xi)\,|\,d\xi, \tag{5}$$

where $\theta(x)$ is the temperature at the point $(x, 0)$ of the boundary. If the temperature is prescribed over the boundary, (5) constitutes an integral equation for the determination of $u(\xi)$; if the flux of heat $u(\xi)$ is prescribed over the boundary, the same equation gives the temperature of the stream in contact with the boundary. In either case the total heat-loss from the cylinder per unit length is given by

$$\mathrm{H} = \int_0^{\beta_0} u(\xi)\, d\xi. \tag{6}$$

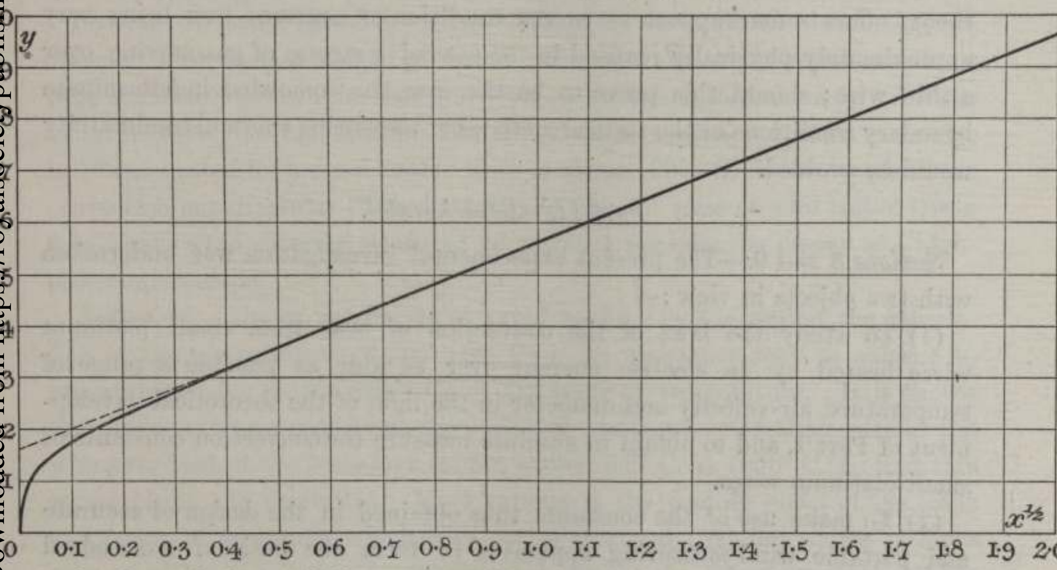
* Boussinesq's expression for the heat-loss per unit length from a cylinder of radius a is given by $\mathrm{H} = 8\sqrt{(s\sigma\kappa\mathrm{V}a/\pi)}\,\theta_0$, where θ_0 is the excess of temperature of the cylinder above that of the fluid at a distance.

† H. A. Wilson, "On the Convection of Heat," 'Camb. Phil. Soc. Proc.,' vol. 12, p. 413 (1904).

Sections 4–7.—A solution of equation (5) in which the heat-flux is assumed to be constant over the boundary gives rise to an expression for the heat-loss which is in good agreement with the results of experiment on the convection of heat from small platinum wires cooled by a stream of air In this case the heat-loss per unit length is given by the equation

$$\mathrm{H} = 2\pi\kappa\theta_0 n\beta_0 \Big/ \left[\int_0^{\beta_0} e^u \mathrm{K}_0(u)\, du\right]. \tag{7}$$

In order to test the theory, it is necessary to tabulate the function $\int_0^x e^u \mathrm{K}_0(u)\, du$. Use is made of the tabulated values of the function $\mathrm{K}_0(x)$,* and the integral is evaluated step by step by making use of Euler's formula for quadratures, the arithmetical operations being easily carried out by means of a calculating machine. The graph of the function $y = x \Big/ \left[\int_0^x e^u \mathrm{K}_0(u)\, du\right]$ plotted against $\sqrt{x}$ is shown in the accompanying figure. Over the range



of values of the variable corresponding to the interpretation of the experiments on the convection of heat, the curve lies extremely close to its asymptote

$$y = 1/(2\pi) + \sqrt{[x/(2\pi)]}, \tag{8}$$

while for small values of the variable it can easily be shown that an approximate value is given by

$$y = 1/[(1-\gamma) - \log \tfrac{1}{2}x], \tag{9}$$

* Tables due to W. S. Aldis ('Roy. Soc. Proc.,' vol. 64, p. 219 (1898)) are given by Jahnke and Emde, 'Funktionentafeln,' p. 135 (Teubners, 1909).

γ being Euler's constant, $\gamma = 0{\cdot}5771$. The corresponding expressions for the heat-loss per unit length for a cylindrical wire of radius a are:—

$$\text{(High velocities)} \quad \mathrm{H} = \kappa\theta_0 + 2\surd(\pi\kappa s\sigma a)\,\mathrm{V}^{\frac{1}{2}}\theta_0, \tag{10}$$

$$\text{(Low velocities)} \quad \mathrm{H} = 2\pi\kappa\theta_0/(\log b/a), \tag{11}$$

where b in (11) is given by $b = \kappa e^{1-\gamma}/(s\sigma \mathrm{V})$.

The boundary condition of constant flux gives rise to a discontinuity of temperature over the boundary; from the point of view of the kinetic theory of gases, temperature conditions cannot be strictly defined in the immediate neighbourhood of the heated cylinder, and it is only at a distance of several free paths when equi-partition is nearly complete that temperature can strictly be defined and that normal thermal conduction takes place. Equation (10) gives so good an account of the experiments that it seems possible to make use of the method of flow in a determination of the thermal conductivity of gases; the form of apparatus required lends itself easily to measurements over a wide range of temperature and pressure. The kinetic theory offers some support as to the condition of constant flux being very approximately physically realised in the case of a stream of gas moving over a thin wire; should this prove to be the case, the somewhat indeterminate boundary conditions in the statical methods of measuring thermal conductivity would be avoided.

Part II.—Experimental.

Sections 8 *and* 9.—The present experimental investigation was undertaken with two objects in view:—

(1) To study the laws of the convection of heat from small platinum wires heated by an electric current over as wide as possible a range of temperature, air-velocity and diameter in the light of the theoretical development of Part I, and to obtain in absolute measure the convection constants of small platinum wires.

(2) To make use of the constants thus obtained in the design of accurate and portable wind-measuring apparatus to form the basis of a standard system of anemometry, as well as to serve for use in a great variety of engineering and aerotechnical problems.

The general arrangement of apparatus necessary to carry out the requisite measurements of heat-losses from a series of platinum wires of diameters 1 to 6 mils consisted of a rotating arm capable of adjustment to any speed as calculated from a chronograph record. At various lengths along this arm could be clamped a light fork designed to hold the specimens of wire under test. The latter formed part of a Kelvin double bridge, electrical connection being obtained through a central mercury connecting switch and overhead wires to the remainder of the bridge. By means of a rheostat it was possible at each speed to adjust a measured current through the wire so as to bring its resistance to a value corresponding to a pre-determined temperature. In this way it was possible to vary at will the various factors of temperature, air-velocity, and heat-loss.

In order to obtain a correct measurement of velocity by the use of a rotating arm, it was found necessary to make a correction for the velocity of the vortex set up in the laboratory. This was accomplished by making use of one of the wires previously tested on the rotating arm as a hot-wire anemometer for measuring the velocity of the vortex set up by the rotating arm; for a wire fixed at any radius it was found that the velocity V relative to the air of the room is connected with the velocity V_r relative to the room itself by the relation $V = (1-s) V_r$, s being a constant for the radius employed and the disposition of the apparatus in the room. The constant s may be conveniently called the "*swirl*" and expressed in percentages of the apparent velocity; for a radius of 2·6 metres s is as much as 5 per cent.

Sections 10 *and* 11.—The wires employed were drawn down through diamond dies from a length of 6 mil pure platinum wire whose constants were accurately known for the purposes of platinum thermometry. It was found necessary to redetermine the temperature coefficients after each wire had been heated for a considerable time to about 1200° C. in the course of the convection experiments. The diameters of the 10 sizes of wire tested (from 6 to 1 mil) were directly measured to within 1 per cent. by means of a high-power microscope.

Section 12.—Under conditions of rapid cooling by convection, the calculation of the temperatures of the wire from its resistance may be subject to uncertainties due to the existence of gradients of temperature. It is shown that under extreme conditions the excess of temperature of the centre of the wire over that of the boundary cannot exceed 0·6° C. at 1000° C., so that this source of error is negligible. More serious is the cooling effect of the leads and potential terminals, which must be so arranged that this source of error may be within the limits of experimental errors; the effect is calculated out in detail and a numerical table is drawn up showing that with the disposition of apparatus of the convection measurements the error due to the leads and potential terminals may be neglected. The possibility of error becomes more serious in the design of hot-wire anemometers with short wires, and in any particular case may be kept within small limits by a reference to the above-mentioned Table. The importance of keeping the anemometer wire from vibration is shown by a mathematical investigation of the error involved.

Section 13.—Observations on the heat-loss per unit length from a series of ten platinum wires of diameters 6 to 1 mil under varying conditions of temperature and wind-velocity are analysed in detail. For each velocity (corrected for "*swirl*") the currents required to heat the wire to a pre-determined series of resistances (from which the temperatures were calculated) were measured. The corresponding heat-loss in watts per unit length was calculated for each temperature: the theory of Part I suggests that the results be examined in the light of the formula $W = B\sqrt{V} + C$, where B and C are functions of the temperature and of the dimensions of the wire. When W is plotted against $\sqrt{V}$ for each temperature, a family of straight lines is obtained and by determining the line of closest fit to the observed points, the constants B and C can be found for each wire.

It is found by plotting B against $\theta - \theta_0$, the excess of temperature of the wire above the surrounding air, that the resulting graph approximates very closely to a straight line for a range of temperature attaining to as high as 1200° C.: this result may be expressed by the relation $B = \beta(\theta - \theta_0)$, where β shows the existence of a small temperature coefficient represented by $\beta = \beta_0[1 + b(\theta - \theta_0)]$, b having the value $b = 0{\cdot}00008$. Finally theory requires that β_0 be proportional to $\sqrt{a_0}$, a_0 being the radius of the wire. A graph of β_0^2 against a_0 shows this condition to be satisfied, leading to the final result

$$\beta_0/\sqrt{a_0} = 1{\cdot}432 \times 10^{-3} \quad \text{(experimental).} \tag{12}$$

The theoretical formula (10) requires

$$\beta_0/\sqrt{a_0} = 2\sqrt{(\pi s_0 \sigma_0 \kappa_0)} = 1{\cdot}66 \times 10^{-3} \quad \text{(theoretical),} \tag{13}$$

taking $\sigma_0 = 0{\cdot}001293$, $\kappa_0 = 5{\cdot}66 \times 10^{-5}$ calorie and $s_0 = 0{\cdot}171$ calorie. The agreement of (12) and (13) must be considered fair in view of the uncertainty attached to the value of the thermal conductivity for air, and also in the fact that the theoretical investigation does not take into account the variation of this and other factors with the temperature gradient in the neighbourhood of the wire.

In order to interpret the constant C in terms of formula (10), it is necessary to calculate the contribution to the term C due to radiation. It is shown from the observations of Lummer and Kurlbaum* that the radiation loss from polished platinum at absolute temperature Θ° C. is given in watts per cm.^{2} by the relation

$$e = 0{\cdot}514\,(\Theta/1000)^{5{\cdot}2}. \tag{14}$$

Having calculated the radiation loss per centimetre of the wire from the formula $E = 2\pi a\, e$, we obtain the true convection loss $C_0 = C - E$: it is found for each wire that C_0 is very nearly proportional to the temperature difference

* Lummer and Kurlbaum, 'Verh. Deut. Phys. Ges., Berlin,' vol. 17, p. 106 (1898).

$\theta-\theta_0$, and may be represented by the formula $C_0 = \gamma_0(\theta-\theta_0)[1+c(\theta-\theta_0)]$, γ_0 being nearly independent of the diameter of the wire and having the value

$$\gamma_0 = 2{\cdot}50 \times 10^{-4}(1+70a) \quad \text{(experimental).} \qquad (15)$$

According to the theoretical equation (10), $C_0 = \kappa_0(\theta-\theta_0)$, giving

$$\gamma_0 = \kappa_0 = 2{\cdot}37 \times 10^{-4} \text{ watts} \quad \text{(theoretical),} \qquad (16)$$

in excellent agreement with the observed value (15). The coefficient c has the value $c = 0{\cdot}00114$, which may be considered to represent in large measure the variation of the heat conductivity with the temperature.

Section 14.—It was found that the constant β_0 varied in a marked manner with the inclination of the wire to the direction of the stream, an effect which can be utilised in practical anemometry in determining the direction of the resultant flow in a complicated distribution of air-velocity.

Section 15.—Formula (11) for small velocities agrees in form with the empirical formula proposed by Langmuir* to represent the results of his experiments on the free convection of heat from small platinum wires. The corresponding mathematical problem has not yet been solved completely, but may be dealt with in the light of the present investigation by supposing that the wire is cooled by a current of effective velocity V due to the ascent of heated air over the surface of the cylinder. Making use of the constants obtained from the present experiments on forced convection, Langmuir's observations can be interpreted and the velocity of the "effective" convection current estimated; these results are of some importance in hot-wire anemometry as the "effective" velocity sets a lower limit to value of the air-velocities which it is possible to measure by this means.

Part III.

Sections 16–18.—The special type of portable hot-wire anemometer developed by the writer may be called a linear anemometer in contra-distinction to several forms of integrating instruments which have already been described.† Detailed specifications are given for the construction of such anemometers. The Kelvin bridge connections are retained, and this makes it possible to make use of previously calibrated anemometer wires. A 3-mil platinum wire, heated to about 1000° C., was found to be the most convenient in practice; by means of a portable galvanometer the current required to bring the resistance of the anemometer wire to about

* Langmuir, "Conduction and Convection of Heat in Gases," 'Physical Review,' vol. 34, p. 415 (1912).

† Bordoni, U., 'Nuovo Cimento,' Series VI, vol. 3, pp. 241–283, April, 1912; Morris, J. T., 'Engineer,' September 27, 1912; 'Electrician,' October 4, 1912, p. 1056; Gerdien, H., 'Ver. Deut. Phys. Ges.,' No. 20, 1913.

four times its resistance at room temperature is measured. For the same wire the current i required to bring the wire to the prescribed resistance and temperature in a current of air of velocity V is given by a relation of the form

$$i^2 = i_0^2 + k\sqrt{V}, \tag{17}$$

i_0^2 and k being the constants of the instrument determined either by calculation or by direct calibration. Experiments carried out by Prof. A. M. Gray* and the writer indicate that consistent measurements of turbulent flow and of sharp gradients may be obtained independently of the diameter of the wire. It was found possible to resolve a gradient in which the velocity changed by 5 cm./sec. over a distance of 1/10 mm.

The advantages of the anemometer designs and connections described may be briefly stated as follows:—

(i) The use of the Kelvin bridge connections makes it possible to standardise and calibrate anemometer wires at a central laboratory independently of the remainder of the apparatus with which the wire is to be employed.

(ii) These connections also enable the wire to be heated to a high temperature, with the result that the determinations of velocity are practically independent of ordinary variations of room temperature.

(iii) The use of the linear anemometer makes it possible to establish a consistent measure of turbulent flow.

(iv) The linear anemometer makes it possible to analyse sharp gradients of velocities without disturbing the flow at the point of measurement.

(v) The properties of the Kelvin bridge connections make it possible to connect an anemometer through low-resistance slip rings, and provide a means of attacking such problems as the analysis of velocities in the neighbourhood of rapidly revolving aeroplane propeller blades, or between the blades of centrifugal fans.

An instrument of the type described could easily be constructed to give a continuous graphical record, thereby greatly increasing its usefulness in the analysis of complicated velocity distributions.

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