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ON THE CONVERGENCE OF A MODIFIED S-ITERATION PROCESS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS IN A CAT(0) SPACE

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Abstract. In this paper, we give the sufficient condition of modified S-iteration process to converge to fixed point for asymptotically quasi-nonexpansive type mappings in the setting of CAT(0) space and also establish some strong convergence theorems of the said iteration process and mapping under suitable conditions. Our results extend and improve many known results from the existing literature.

1. Introduction

We know that a metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [2]), $\mathbb{R}$-trees (see [13]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [9]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [2].

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Fixed point theory in a CAT(0) space has been first studied by Kirk (see [15, 16]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see, *e.g.*, [2]).

The Mann iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in (0,1).

Further, the Ishikawa iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$. This iteration process reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal, O'Regan and Sahu [1] introduced the S-iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$. Note that (1.3) is independent of (1.2) (and hence (1.1)). They showed that their process independent of those of Mann and Ishikawa and converges faster than both of these (see [1, Proposition 3.1]).

Schu [23], in 1991, considered the modified Mann iteration process which is a generalization of the Mann iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

Tan and Xu [26], in 1994, studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$. This iteration process reduces to the modified Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

Recently, Agarwal, O'Regan and Sahu [1] introduced the modified S-iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). Note that (1.6) is independent of (1.5) (and hence of (1.4)). Also (1.6) reduces to (1.3) when $T^n = T$ for all $n \geq 1$.

Very recently, Şahin and Başarir [21] modified the iteration process (1.6) in a CAT(0) space as follows:

Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T\colon K \to K$ be an asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.7}$$

where and throughout the paper $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences such that $0 \leq \alpha_n, \beta_n \leq 1$ for all $n \geq 1$. They studied modified S-iteration process for asymptotically quasi-nonexpansive mappings on the CAT(0) space and established some strong convergence results under some suitable conditions which generalize some results of Khan and Abbas [11].

In this paper, we study the modified S-iteration process (1.7) for asymptotically quasi-nonexpansive type mappings and give the sufficient condition to converge to fixed point in the setting of CAT(0) space and also establish some strong convergence results under some additional conditions. Our results can be applied to an S-iteration process since the modified S-iteration process reduces to the S-iteration process when $T^n = T$ for all $n \geq 1$.

2. Preliminaries

In order to prove the main results of this paper, we need the following definitions and concepts.

Let (X,d) be a metric space and K be its nonempty subset. Let $T\colon K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$. We will also denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$.

The concept of quasi-nonexpansive mapping was introduced by Diaz and Metcalf [6] in 1967, the concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [8] in 1972. The iterative approximation problems for asymptotically quasi-nonexpansive mapping were studied by many authors in a Banach space and a CAT(0) space (see, *e.g.* [7, 12, 17, 18, 22, 24]).

Definition 2.1. Let (X,d) be a metric space and K be its subset. Then a mapping $T\colon K \to K$ said to be

(1) nonexpansive if
$$d(Tx,Ty) \le d(x,y)$$
for all $x, y \in K$;

(2) asymptotically nonexpansive if there exists a sequence $\{r_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that
$$d(T^n x, T^n y) \le (1+r_n) d(x,y)$$
for all $x, y \in K$ and $n \ge 1$;

(3) quasi-nonexpansive if
$$d(Tx,p) \le d(x,p)$$
for all $x \in K$ and $p \in F(T)$;

(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that
$$d(T^n x, p) \le (1+r_n) d(x,p)$$
for all $x \in K$, $p \in F(T)$ and $n \ge 1$;

(5) asymptotically nonexpansive type [14], if
$$\limsup_{n\to\infty} \Big\{ \sup_{x,\, y \in K} \Big(d(T^n x, T^n y) - d(x,y) \Big) \Big\} \le 0;$$

(6) asymptotically quasi-nonexpansive type [20], if $F(T) \neq \emptyset$ and
$$\limsup_{n\to\infty} \Big\{ \sup_{x \in K,\, p \in F(T)} \Big(d(T^n x, p) - d(x,p) \Big) \Big\} \le 0;$$

(7) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L\, d(x, y)$$

for all $x, y \in K$ and $n \geq 1$;

(8) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

Remark 2.2. By Definition 2.1, it is clear that the class of quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonex pansive mappings and asymptotically nonexpansive type mappings include nonexpansive mappings, whereas the class of asymptotically quasi-nonexpansive type mapping is larger than that of quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive type mappings. The reverse of these implications may not be true.

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) a *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denoted by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [2]).

CAT(0) space

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \quad \leq \quad d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [10]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \;\leq\; \frac{1}{2}\, d^2(x, y_1) + \frac{1}{2}\, d^2(x, y_2) - \frac{1}{4}\, d^2(y_1, y_2). \tag{2.2}$$

The inequality (2.2) is the (CN) inequality of Bruhat and Tits [4]. The above inequality has been extended in [5] as

$$d^2(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d^2(z, x) + (1-\alpha) d^2(z, y) - \alpha(1-\alpha) d^2(x, y) \tag{2.3}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [2, page 163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \;\leq\; \alpha d(z, x) + (1-\alpha) d(z, y), \tag{2.4}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\}$.

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

In the sequel we need the following useful lemmas.

Lemma 2.3. (See [19]) *Let X be a CAT(0) space.*

(i) *For $x,\, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x,\, y]$ such that*

$$d(x,\, z) = t\, d(x,\, y) \quad \textit{and} \quad d(y,\, z) = (1-t)\, d(x,\, y). \tag{A}$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) *For $x,\, y \in X$ and $t \in [0, 1]$, we have*

$$d((1-t)x \oplus ty, z) \leq (1-t) d(x, z) + t d(y, z).$$

Lemma 2.4. (See [25]) *Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that*

$$a_{n+1} \leq a_n + b_n$$

for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists.

3. Main results

In this section, we establish some strong convergence results of modified S-iteration scheme (1.7) to converge to a fixed point for asymptotically quasi-nonexpansive type mapping in the setting of CAT(0) space.

Theorem 3.1. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be an asymptotically quasi-nonexpansive type mapping with $F(T) \neq \emptyset$ is closed. Suppose that $\{x_n\}$ is defined by the iteration process* (1.7)*. Put*

$$\mu_n = \max\Big\{0, \sup_{p\in F(T),\, n\geq 1}\Big(d(T^n x, p) - d(x, p)\Big)\Big\} \tag{3.1}$$

such that $\sum_{n=1}^{\infty}\mu_n < \infty$. If

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0 \quad or \quad \limsup_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) = \inf_{p\in F(T)} d(x, p)$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $p \in F(T)$. From (1.7), (3.1) and Lemma 2.3(ii), we have

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)x_n \oplus \beta_n T^n x_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(T^n x_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n[d(x_n, p) + \mu_n] \\ &\leq d(x_n, p) + \mu_n. \end{aligned} \tag{3.2}$$

Again using (1.7), (3.1), (3.2) and Lemma 2.3(ii), we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, p) \\ &\leq (1-\alpha_n)d(T^n x_n, p) + \alpha_n d(T^n y_n, p) \\ &\leq (1-\alpha_n)[d(x_n, p) + \mu_n] + \alpha_n[d(y_n, p) + \mu_n] \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p) + \mu_n \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n[d(x_n, p) + \mu_n] + \mu_n \\ &= d(x_n, p) + (1+\alpha_n)\mu_n \\ &\leq d(x_n, p) + 2\mu_n. \end{aligned} \tag{3.3}$$

This implies that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + 2\mu_n. \tag{3.4}$$

Since by hypothesis $\sum_{n=1}^{\infty}\mu_n < \infty$, by Lemma 2.4 and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(T)) = 0$ gives that

$$\lim_{n\to\infty} d(x_n, F(T)) = 0. \tag{3.5}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in K. From (3.3), we have

$$\begin{aligned} d(x_{n+m},p) &\leq d(x_{n+m-1},p)+2\mu_{n+m-1} \\ &\leq d(x_{n+m-2},p)+2\mu_{n+m-2}+2\mu_{n+m-1} \\ &\leq \cdots \\ &\leq \cdots \\ &\leq d(x_n,p)+2\sum_{k=n}^{n+m-1}\mu_k, \end{aligned} \tag{3.6}$$

for the natural numbers m,n and $p \in F(T)$. Since $\lim_{n\to\infty} d(x_n,F(T))=0$, therefore for any $\varepsilon>0$, there exists a natural number n_0 such that

$$d(x_n,F(T))<\varepsilon/8$$

and

$$\sum_{k=n}^{n+m-1}\mu_k<\varepsilon/8$$

for all $n \geq n_0$. So, we can find $p^* \in F(T)$ such that

$$d(x_{n_0},p^*)<\varepsilon/4.$$

Hence, for all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned} d(x_{n+m},x_n) &\leq d(x_{n+m},p^*)+d(x_n,p^*) \\ &\leq d(x_{n_0},p^*)+2\sum_{k=n_0}^{\infty}\mu_k+d(x_{n_0},p^*)+2\sum_{k=n_0}^{\infty}\mu_k \\ &= 2d(x_{n_0},p^*)+4\sum_{k=n_0}^{\infty}\mu_k \\ &< 2\Big(\frac{\varepsilon}{4}\Big)+4\Big(\frac{\varepsilon}{8}\Big) \\ &= \varepsilon. \end{aligned} \tag{3.7}$$

This proves that $\{x_n\}$ is a Cauchy sequence in K. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n=z$. Since K is closed, therefore $z \in K$. Next, we show that $z \in F(T)$. Since $\lim_{n\to\infty} d(x_n,F(T))=0$, we get $d(z,F(T))=0$. From closedness of $F(T)$ gives that $z \in F(T)$. This completes the proof. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be an asymptotically quasi-nonexpansive type mapping with $F(T) \neq \emptyset$ is closed. Suppose that $\{x_n\}$ is defined by the iteration process* (1.7) *and μ_n is taken as in Theorem* 3.1. *If T satisfies the following conditions:*

(i) $\lim_{n\to\infty} d(x_n, Tx_n) = 0$;
(ii) *If the sequence* $\{z_n\}$ *in* K *satisfies* $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, *then*

$$\liminf_{n\to\infty} d(z_n, F(T)) = 0 \quad \text{or} \quad \limsup_{n\to\infty} d(z_n, F(T)) = 0,$$

then the sequence $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. It follows from the hypothesis that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. From (ii),

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0$$

or

$$\limsup_{n\to\infty} d(x_n, F(T)) = 0.$$

Therefore, the sequence $\{x_n\}$ must converges strongly to a fixed point of T from Theorem 3.1. This completes the proof. □

Theorem 3.3. *Let* K *be a nonempty closed convex subset of a complete* $CAT(0)$ *space* X *and let* $T: K \to K$ *be a uniformly* L*-Lipschitzian and asymptotically quasi-nonexpansive type mapping with* $F(T) \neq \emptyset$ *is closed. Suppose that* $\{x_n\}$ *is defined by the iteration process* (1.7) *and* μ_n *is taken as in Theorem* 3.1. *If* T *is semi-compact and*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

then the sequence $\{x_n\}$ *converges to a fixed point of* T.

Proof. From the hypothesis, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Also, since T is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in K$. Hence, we have

$$\begin{aligned} d(p, Tp) &\leq d(p, x_{n_j}) + d(x_{n_j}, Tx_{n_j}) + d(Tx_{n_j}, Tp) \\ &\leq (1 + L)d(p, x_{n_j}) + d(x_{n_j}, Tx_{n_j}) \to 0. \end{aligned}$$

Thus $p \in F(T)$. By (3.3),

$$d(x_{n+1}, p) \leq d(x_n, p) + 2\mu_n.$$

Since by hypothesis $\sum_{n=1}^{\infty} \mu_n < \infty$, by Lemma 2.4, $\lim_{n\to\infty} d(x_n, p)$ exists and $x_{n_j} \to p \in F(T)$ gives that $x_n \to p \in F(T)$. This shows that $\{x_n\}$ converges to a fixed point of T. This completes the proof. □

As an application of Theorem 3.1, we establish another strong convergence result as follows.

Theorem 3.4. *Let* K *be a nonempty closed convex subset of a complete* $CAT(0)$ *space* X *and let* $T: K \to K$ *be an asymptotically quasi-nonexpansive type mapping with* $F(T) \neq \emptyset$ *is closed. Suppose that* $\{x_n\}$ *is defined by the* *iteration process* (1.7) *and* μ_n *is taken as in Theorem* 3.1. *If* T *satisfies the following conditions:*

(i) $\lim_{n\to\infty} d(x_n, Tx_n) = 0$;
(ii) *There exists a constant* $A > 0$ *such that*

$$d(x_n, Tx_n) \geq A\, d(x_n, F(T)),$$

then the sequence $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. From conditions (i) and (ii), we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$, it follows as in the proof of Theorem 3.1, that $\{x_n\}$ must converges strongly to a fixed point of T. This completes the proof. □

4. Conclusion

The class of quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive type mappings include nonexpansive mappings, whereas the class of asymptotically quasi-nonexpansive type mapping is larger than that of quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive type mappings. Thus the results presented in this article extend and generalize some previous works for a CAT(0) space from the existing literature.

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