On the Convergence of Blind Adaptive Equalizers for Constant Modulus Signals

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***Abstract*—This paper studies the behavior of the error sequence of stop-and-go variants of two adaptive blind equalizers, namely CMA2-2 and Sato's algorithm. It is shown that for transmitted signals with constant modulus γ, the equalizer output can be made to lie within the circle of radius $\gamma\sqrt{c}$ infinitely often, for some value of c that is only slightly larger than one.**

***Index Terms*—Adaptive equalization, constant modulus algorithm, feedback, l_2-stability, positive-real function, small gain theorem.**

I. INTRODUCTION

CHANNEL equalization is a relevant step in the design of reliable data communication systems. For slowly varying channels, an initial training phase is often tolerable for equalization purposes and such scenarios arise, for example, in equalizer implementations of digital cellular handsets. When the communications environment is highly nonstationary, however, it may become impractical to use training sequences [1, p. 139]. Such situations can be handled by the use of blind equalization and they arise, for example, in the operation of cordless phones.

Over the years many ingenious analyzes and modifications have been proposed in the literature with the intent of both understanding and improving the performance of blind and nonblind adaptive equalizers. Considerable progress has been made in several respects, and we may mention here, among others, the works [2]–[19]. In several of these works, the analyzes are concerned with the multimodality of the associated cost functions. For example, [15] establishes the fact that for fractionally spaced equalizers (FSE's), and under certain rank and length conditions, the associated cost function for blind (Godard) equalization has global minima at zero-forcing equalizers. While this is a reassuring conclusion, it can only guarantee that a well-designed steepest descent method can achieve a global minimum of the cost function.

But what about adaptive equalizers that are derived from steepest descent methods by resorting to instantaneous-gradient approximations? Will they perform reliably in the presence of the gradient noise that is introduced by such approximations? This issue is more complex and deserves a closer examination.

In this paper, we study two blind adaptive algorithms, viz., CMA2-2 and Sato's algorithm, for transmitted signals that satisfy a constant modulus property. We show, under some conditions on the step-sizes, that certain stop-and-go variants of these algorithms can be made to guarantee that the equalizer output will lie infinitely often within a bounded domain whose radius is only slightly larger than the assumed constant modulus (see Theorems 1 and 2). The analysis is based on a feedback framework developed in [20]–[23], and it allows us to employ some simple tools from system theory for l_2-stability analysis.

Notation: We use small boldface letters to denote vectors. The symbol "T" denotes transposition, "$*$" denotes Hermitian conjugation, and the notation $\|\mathbf{x}\|$ denotes the Euclidean norm of a vector. All vectors are column vectors except for the input data vector denoted by $\mathbf{u}_i$, which is taken to be a row vector. Moreover, since the signals we deal with are often complex, we denote by z_R the real part of a complex number z and by z_I its imaginary part. We also use the shift operator q^{-1}, defined by $q^{-1}[s(i)] = s(i-1)$, to denote the unit time delay and write $C(q^{-1})$ to denote a transfer function in q^{-1}.

II. PROBLEM FORMULATION

Fig. 1 shows an input signal $s(i)$ that is transmitted through an unknown channel $C(q^{-1})$ to an Mth-order finite-impulse response (FIR) receiver $R(q^{-1})$. The input of the receiver is denoted by $u(i)$ and its output by $z(i)$. A zero-forcing receiver is one that guarantees $z(i) = e^{j\theta}s(i-D)$, for some positive integer D and for some phase $\theta \in [0, 2\pi)$. In the sequel, we assume that such a receiver exists (with $\theta = 0$, for simplicity of notation). That is, we assume that an Mth-order equalizer $R(q^{-1})$ exists such that when used in Fig. 1 it leads to an output $z(i) = s(i-D)$. As mentioned in the introduction, this assumption is valid for FSE's under certain rank and length conditions (see, e.g., [15] and [16]). In Section V, we comment on the case of channel and equalization imperfections.

The objective of adaptive equalization is to provide estimates $\hat{R}(q^{-1})$ that approximate reasonably well the performance of $R(q^{-1})$, especially when the channel itself is unknown or even time variant. This is depicted in Fig. 2, where the weight vector of $\hat{R}(q^{-1})$ at time $(i-1)$ is denoted by $\mathbf{w}_{i-1}$. After one adaptation step, an error signal $e(i)$ is generated and the weight estimate is updated to $\mathbf{w}_i$. In blind adaptation, $e(i)$ is solely

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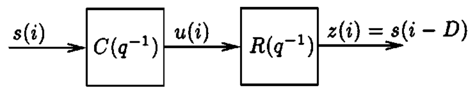

Fig. 1. Block diagram representation of zero-forcing equalization.

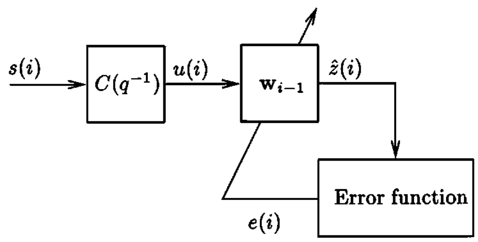

Fig. 2. Structure of the blind adaptive equalizer.

dependent on $\hat{z}(i)$. [In nonblind adaptation, a certain reference signal is used in conjunction with $\hat{z}(i)$ to produce $e(i)$.]

A. The Algorithms

Let $\mathbf{u}_i$ be a row vector that denotes the state of the adaptive FIR filter at the receiver of Fig. 2 at time i, viz.,

$$\mathbf{u}_i = [u(i) \quad u(i-1) \quad \cdots \quad u(i-M+1)]. \tag{1}$$

Then, the output of the equalizer is given by the inner product

$$\hat{z}(i) = \mathbf{u}_i \mathbf{w}_{i-1}. \tag{2}$$

Assume also that the transmitted signal $s(i)$ arises from a constant modulus constellation, say

$$|s(i)| = \gamma \tag{3}$$

for all i and for some $\gamma > 0$.

In this paper, we study the convergence performance of the following two blind adaptive schemes.[1]

1) **CMA2-2**: The letters CMA stand for constant modulus algorithm and the numbers 2-2 refer to the specific cost function that gave rise to it (see [4] and [10]). The algorithm is widely used in the context of adaptive blind equalization, and it offers superior performance properties in several respects when compared to other blind adaptive schemes (see, e.g., [24] and [25]). In this case, the error signal is computed as

$$e(i) = \hat{z}(i)\left[\gamma^2 - |\hat{z}(i)|^2\right] \tag{4}$$

and the weight vector is updated according to the recursion

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu(i)\mathbf{u}_i^* \hat{z}(i)[\gamma^2 - |\hat{z}(i)|^2] \tag{5}$$

for some initial weight vector $\mathbf{w}_{-1}$, and where $\mu(i)$ is a positive step-size (allowed to be time-dependent for generality).

2) **Sato's Algorithm**: For this algorithm, all signals are assumed to be real-valued and $s(i)$ arises from a 2-PSK constellation (i.e., $s(i) = \pm\gamma$). The error signal is now computed as

$$e(i) = \gamma\,\text{sign}[\hat{z}(i)] - \hat{z}(i) \tag{6}$$

and the weight vector is updated according to the recursion

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu(i)\mathbf{u}_i^T[\gamma\,\text{sign}[\hat{z}(i)] - \hat{z}(i)]. \tag{7}$$

[1]The analysis can be extended to other similar algorithms.

B. Assumptions

We shall assume that the successive regression vectors $\{\mathbf{u}_i\}$ are nonzero and also uniformly bounded from above and from below, say

$$0 < b < \|\mathbf{u}_i\| < B < \infty \tag{8}$$

for some constants $\{b, B\}$ and for all i.

Moreover, the two algorithms mentioned above are special cases of the general recursion

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu(i)\mathbf{u}_i^* e(i), \qquad i \geq 0 \tag{9}$$

for different definitions of $e(i)$ in terms of $\hat{z}(i)$. It is then clear that the weight vector will not be updated, i.e., $\mathbf{w}_i = \mathbf{w}_{i-1}$, whenever $e(i) = 0$. We shall exclude these cases from our analysis and focus only on *active* update steps, namely those for which $e(i) \neq 0$. To emphasize this fact, we shall use an alternative time index k to refer to the active update steps. In other words, in this paper, we shall study algorithms of the form

$$\mathbf{w}_k = \mathbf{w}_{k-1} + \mu(k)\mathbf{u}_k^* e(k), \qquad e(k) \neq 0;\ k = 0, 1, 2, \cdots. \tag{10}$$

In particular, we shall study the behavior of $e(k)$ as time progresses to infinity ($k \to \infty$).

III. CMA2-2 Algorithm

We start our analysis with CMA2-2, for which $e(k)$ is given by [cf. (4)]

$$e(k) = \hat{z}(k)[\gamma^2 - |\hat{z}(k)|^2].$$

Thus, let $\mathbf{w}$ denote the weight vector of the optimal receiver $R(q^{-1})$ of Fig. 1 and let $z(k) = \mathbf{u}_k\mathbf{w}$. Recall that we are assuming a zero-forcing receiver that guarantees $z(k) = s(k-D)$, for some D [so that $|z(k)| = \gamma$ by (3)].

Define further the *a priori* and *a posteriori* estimation errors

$$\begin{aligned} e_a(k) &= z(k) - \hat{z}(k) \\ &= \mathbf{u}_k \tilde{\mathbf{w}}_{k-1} \end{aligned} \tag{11}$$

$$e_p(k) = \mathbf{u}_k \tilde{\mathbf{w}}_k \tag{12}$$

where $\tilde{\mathbf{w}}_{k-1} = \mathbf{w} - \mathbf{w}_{k-1}$. Introduce also the complex-valued functions

$$f(z) = z|z|^2$$

and, for $z_1 \neq z_2$

$$h[z_1, z_2] \triangleq \frac{f(z_1) - f(z_2)}{z_1 - z_2} = \frac{z_1|z_1|^2 - z_2|z_2|^2}{z_1 - z_2}. \tag{13}$$

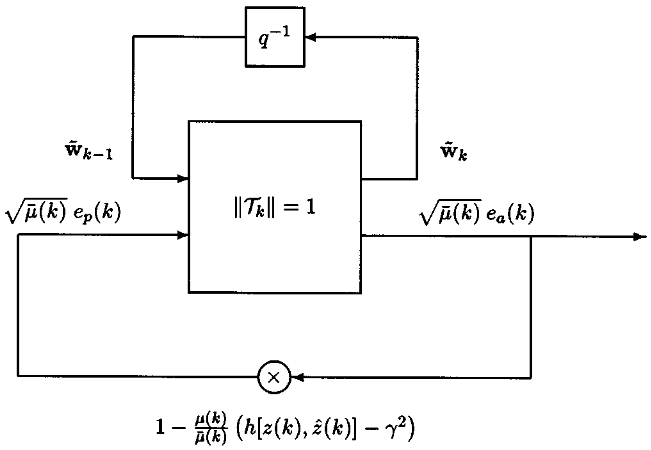

Fig. 3. A time-variant lossless mapping with gain feedback for blind operation.

Then, it is straightforward to verify that the error signals $\{e(k), e_a(k)\}$ can be related as follows:[2]

$$e(k) = (h[z(k), \hat{z}(k)] - \gamma^2)e_a(k).$$

By subtracting $\mathbf{w}$ from both sides of (10), we obtain the following recursion for the weight-error vector:

$$\tilde{\mathbf{w}}_k = \tilde{\mathbf{w}}_{k-1} - \mu(k)\mathbf{u}_k^*[(h - \gamma^2)e_a(k)] \tag{14}$$

where we are dropping the arguments $(z(k), \hat{z}(k))$ of h for compactness of notation.

Let $\bar{\mu}(k)$ denote the reciprocal of the input energy at iteration k, $\bar{\mu}(k) = 1/\|\mathbf{u}_k\|^2$. If we multiply (14) by $\mathbf{u}_k$ from the left, we conclude that $e_p(k)$ and $e_a(k)$ are related via

$$e_p(k) = \left(1 - \frac{\mu(k)}{\bar{\mu}(k)}[h - \gamma^2]\right) e_a(k) \tag{15}$$

which in turn shows that we can rewrite (14) in the equivalent form

$$\tilde{\mathbf{w}}_k = \tilde{\mathbf{w}}_{k-1} - \bar{\mu}(k)\mathbf{u}_k^*[e_a(k) - e_p(k)]. \tag{16}$$

If we square both sides of (16) and compare the resulting energies, the following equality always holds:

$$\frac{\|\tilde{\mathbf{w}}_k\|^2 + \bar{\mu}(k)|e_a(k)|^2}{\|\tilde{\mathbf{w}}_{k-1}\|^2 + \bar{\mu}(k)|e_p(k)|^2} = 1. \tag{17}$$

This relation establishes the existence of a lossless map (denoted by $\mathcal{T}_k$) from the variables $\{\tilde{\mathbf{w}}_{k-1}, \sqrt{\bar{\mu}(k)}e_p(k)\}$ to the variables $\{\tilde{\mathbf{w}}_k, \sqrt{\bar{\mu}(k)}e_a(k)\}$. Correspondingly, using (15), this analysis shows that the update (5) induces an overall feedback structure of the form shown in Fig. 3. The feedback configuration consists of a lossless feedforward map and a memoryless (yet time-variant) feedback map.

This feedback structure, along with the energy relation (17), was derived in [20]–[22] in the context of robustness analysis of adaptive filters. They have been also applied to the study of the steady-state and tracking performances of general (blind and nonblind) adaptive schemes (see, e.g., [25]–[27]). Here, we shall show how the feedback structure of Fig. 3 leads to useful insights regarding the behavior of the error sequence $\{e(k)\}$ as time progresses to infinity.

A. Small Gain Analysis

Define the quantities

$$\Delta(N) = \max_{0 \le k \le N} \left| 1 - \frac{\mu(k)}{\bar{\mu}(k)}(h[z(k), \hat{z}(k)] - \gamma^2) \right| \tag{18}$$

and

$$\kappa(N) = \max_{0 \le k \le N} \frac{\mu(k)}{\bar{\mu}(k)}. \tag{19}$$

In particular, $\Delta(N)$ is the maximum norm of the (complex-valued) gain of the feedback loop over an interval of length $(N+1)$. The following result now follows immediately from (17) (see, e.g., [20]–[22]).

Lemma 1 [CMA2-2]: If

$$\Delta(N) < 1 \tag{20}$$

then the following bound on the weighted energy of the *a priori* estimation errors holds:

$$\sqrt{\sum_{k=0}^{N} \mu(k)|e_a(k)|^2} \le \frac{\kappa^{1/2}(N)}{1 - \Delta(N)} \|\tilde{\mathbf{w}}_{-1}\|. \tag{21}$$

◇

The usefulness of this lemma will be highlighted in the sequel.

B. Selective (Stop-and-Go) Updating

Condition (20) can be met for any N by attempting to select the step-size sequence $\mu(k)$ so as to guarantee for all k

$$\left| 1 - \frac{\mu(k)}{\bar{\mu}(k)}(h[z(k), \hat{z}(k)] - \gamma^2) \right| < a < 1 \tag{22}$$

for all possible combinations of $z(k)$ and $\hat{z}(k)$, and for some positive scalar a. This motivates us to introduce the following stop-and-go variant of the CMA2-2 algorithm (5).[3] Let

$$c \triangleq \epsilon + \frac{4}{3} \tag{23}$$

for some small positive number ϵ. Now, at each time instant i, do the following.

- If $|\hat{z}(i)| \ge \gamma\sqrt{c}$, then we update $\mathbf{w}_{i-1}$ to $\mathbf{w}_i$ via
  $$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu(i)\mathbf{u}_i^*\hat{z}(i)[\gamma^2 - |\hat{z}(i)|^2]$$
  where $\mu(i)$ is chosen as explained below in Theorem 1. This corresponds to an *active* update step since the corresponding error $e(i)$ is necessarily nonzero by virtue of the conditions $|z(i)| = \gamma$ and $\sqrt{c} > 1$.
- If $|\hat{z}(i)| < \gamma\sqrt{c}$, then $\mathbf{w}_i = \mathbf{w}_{i-1}$. That is, we do not update the weight vector.

[2]The denominator of $h[z(k), \hat{z}(k)]$ is nonzero since $z(k) \ne \hat{z}(k)$. This can be seen as follows. If $\hat{z}(k) = z(k)$, then $|\hat{z}(k)| = \gamma$ (in view of the fact that $|z(k)| = \gamma$). It would then follow that $e(k) = 0$, which contradicts the assumption of active updates for which $e(k) \ne 0$.

[3]We first describe the modified algorithm in terms of the time index i used for the original recursion (5). Then, we extract the active update steps and use the index k for these, as in (24).

In other words, the corresponding active update steps [cf. (10)] of this algorithm have the form

$$\begin{cases} \mathbf{w}_k = \mathbf{w}_{k-1} + \mu(k)\mathbf{u}_k^* e(k), & |\hat{z}(k)| \geq \gamma\sqrt{c} \\ e(k) = \hat{z}(k)[\gamma^2 - |\hat{z}(k)|^2]. \end{cases} \tag{24}$$

Assume we run the above stop-and-go CMA2-2 algorithm infinitely often (i.e., $i \to \infty$), and let K denote the maximum number of active updates that occurred in the process. We now prove that, by properly designing the step-size sequence, K can be made *finite*, which in turn means that the condition $|\hat{z}(i)| < \gamma\sqrt{c}$ will hold infinitely often.

Theorem 1 [Stop-and-Go CMA2-2]: Assume $\hat{z}(k)$ stays uniformly bounded from above for all k, say

$$\gamma\sqrt{c} \leq |\hat{z}(k)| \leq P\gamma < \infty$$

for some $P \geq \sqrt{c} > 1$.[4] Choose a positive number β^o in the interval

$$\frac{64P^4 - \epsilon^2}{64P^4 + \epsilon^2} < \beta^o < 1$$

and compute an α^o via

$$\alpha^o = 8(1 - \beta^o)\frac{P^2}{\epsilon}.$$

Choose further the step-size sequence $\mu(k)$ for the active updates from within the interval

$$(1 - \beta^o)\frac{1}{\|\mathbf{u}_k\|^2}\frac{4}{3\epsilon\gamma^2} < \mu(k) < \frac{\alpha^o}{\|\mathbf{u}_k\|^2}\frac{1}{3P^2\gamma^2}.$$

(A constant step-size can also be chosen as in (54) for different values of $\{\alpha^o, \beta^o\}$.) It then holds that $K < \infty$. That is, $|\hat{z}(i)| < \gamma\sqrt{c}$ holds infinitely often.

Proof: The corresponding function $h[z(k), \hat{z}(k)]$ only needs to be defined for the case $|z(k)| = \gamma$ and $|\hat{z}(k)| \geq \gamma\sqrt{c}$

$$h[z(k), \hat{z}(k)] = \frac{z(k)|z(k)|^2 - \hat{z}(k)|\hat{z}(k)|^2}{z(k) - \hat{z}(k)}. \tag{25}$$

In this case we know from the analysis in Appendix A [expression (42)] that

$$\text{Real}\{h[z(k), \hat{z}(k)]\} \geq \gamma^2(1 + 0.75\epsilon).$$

We also know from Appendix B [expression (43)] that

$$|h[z(k), \hat{z}(k)]| \leq 3P^2\gamma^2.$$

We use these two bounds on $|h|$ and $\text{Real}[h]$ in Appendix C to show that the choice for $\mu(k)$ in the statement of the theorem above guarantees (22) for all k, viz.,

$$\left[1 - \frac{\mu(k)}{\bar{\mu}(k)}(h_R(k) - \gamma)\right]^2 + \frac{\mu^2(k)}{\bar{\mu}^2(k)}h_I^2(k) < a^2 < 1$$

where h_R and h_I denote the real and imaginary parts of h. Hence, according to Lemma 1, we must have

$$\sum_{k=0}^{K}\mu(k)|e_a(k)|^2 \leq \frac{\kappa(K)\|\tilde{\mathbf{w}}_{-1}\|^2}{(1 - \Delta(K))^2}.$$

We want to show that $K < \infty$. Assume to the contrary that K is infinite. Then, the above inequality would imply that

$$\sum_{k=0}^{\infty}\mu(k)|e_a(k)|^2 < \infty$$

from which we conclude that $\sqrt{\mu(k)}e_a(k) \to 0$, or equivalently $e_a(k) \to 0$, since $\mu(k)$ is bounded from below and from above. We thus conclude that $\hat{z}(k) \to z(k)$, which means that there exists a finite L large enough, such that for any positive number ϵ' and for all $k > L$

$$|\hat{z}(k)| < |z(k)| + \epsilon' = \gamma + \epsilon'.$$

We are free to choose ϵ' so let ϵ' be any positive number that satisfies

$$\epsilon' < \gamma(\sqrt{c} - 1).$$

It then follows that

$$|\hat{z}(k)| < \gamma\sqrt{c}.$$

This contradicts the fact that the updates occurred with $\hat{z}(k)$ such that $|\hat{z}(k)| \geq \gamma\sqrt{c}$. We thus conclude that only a finite number of updates could have occurred, i.e., $K < \infty$. ◇

Remark 1: Although the interval that defines the choice of $\mu(k)$ in the statement of Theorem 1 can generally be small, the theorem nevertheless shows that there exists a selection of step-sizes for which $|\hat{z}(k)| < \gamma\sqrt{c}$ occurs infinitely often. (In other words, we could interpret this result as essentially saying that, for suitably chosen step-sizes, the stop-and-go CMA2-2 algorithm produces a sequence of estimates $\hat{z}(i)$ that lies inside the circle of radius $\gamma\sqrt{c}$ with probability one.)

Remark 2: The analysis can be extended to a general recursion of the form

$$\mathbf{w}_k = \mathbf{w}_{k-1} + \mu(k)\mathbf{u}_k^*\hat{z}(k)[\gamma^q - |\hat{z}(k)|^q]$$

for $q = 3, 4, 5, \cdots$.

IV. Sato's Algorithm

We now extend the results to Sato's algorithm (7). In this case, we introduce the functions

$$f(z) = \text{sign}[z]$$

and, for $z_1 \neq z_2$,

$$h[z_1, z_2] \triangleq \frac{f(z_1) - f(z_2)}{z_1 - z_2} = \frac{\text{sign}[z_1] - \text{sign}[z_2]}{z_1 - z_2}. \tag{26}$$

Then, it is straightforward to verify that the error signals $\{e(k), e_a(k)\}$ are now related via

$$e(k) = (1 - \gamma h[z(k), \hat{z}(k)])e_a(k).$$

[4] The lower bound is automatically satisfied by the stop-and-go nature of the algorithm. The upper bound is expressed conveniently in terms of γ; any upper bound can be expressed in this form for some P.

Moreover, it further holds that

$$e_p(k) = \left(1 - \frac{\mu(k)}{\bar{\mu}(k)}[1-\gamma h]\right) e_a(k)$$

so that the same feedback structure of Fig. 1 will still hold with the gain in the feedback path replaced by

$$\left(1 - \frac{\mu(k)}{\bar{\mu}(k)}[1-\gamma h]\right).$$

The same conclusion of Theorem 1 will also hold with $\Delta(N)$ now defined as the maximum of the above feedback gain over an interval of length $(N+1)$

$$\Delta(N) = \max_{0\le k\le N}\left|1 - \frac{\mu(k)}{\bar{\mu}(k)}(1-\gamma h[z(k),\hat{z}(k)])\right|. \qquad (27)$$

A. Selective (Stop-and-Go) Updating

We thus see that we now need to select the step-size sequence $\mu(k)$ so as to guarantee

$$\left|1 - \frac{\mu(k)}{\bar{\mu}(k)}(1-\gamma h[z(k),\hat{z}(k)])\right| < a < 1 \qquad (28)$$

for all k and for some positive scalar a.

This again motivates us to consider a stop-and-go variant of Sato's algorithm.[5] Select a positive real number a satisfying $a<1$, and a positive real number λ such that

$$\frac{1-a}{1+a} < \lambda < 1. \qquad (29)$$

Then, define

$$\sqrt{c} \triangleq \frac{1+\lambda}{1-\lambda}.$$

Now at each time instant i, do the following.

- If $|\hat{z}(i)| \ge \gamma\sqrt{c}$, then we update $\mathbf{w}_{i-1}$ to $\mathbf{w}_i$ via

  $$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu(i)\mathbf{u}_i^*[\gamma\,\mathrm{sign}[\hat{z}(i)] - \hat{z}(i)]$$

  where $\mu(i)$ is chosen as explained below. This corresponds to an *active* update step since the corresponding error $e(i)$ is necessarily nonzero in view of the condition $|z(i)| = \gamma$.
- If $|\hat{z}(i)| < \gamma\sqrt{c}$, then $\mathbf{w}_i = \mathbf{w}_{i-1}$. That is, we do not update the weight vector.

In other words, the corresponding active update steps [cf. (10)] of this algorithm have the form

$$\begin{cases} \mathbf{w}_k = \mathbf{w}_{k-1} + \mu(k)\mathbf{u}_k^T e(k), & |\hat{z}(k)| \ge \gamma\sqrt{c} \\ e(k) = \gamma\mathrm{sign}[\hat{z}(k)] - \hat{z}(k). \end{cases} \qquad (30)$$

Assume we run the stop-and-go Sato's algorithm infinitely often (i.e., $i\to\infty$), and let K denote the maximum number of active updates that occurred in the process. We can also establish that, by properly designing the step-size sequence, K can be made *finite*, which in turn means that the condition $|\hat{z}(i)| < \gamma\sqrt{c}$ will hold infinitely often.

[5]Once more, we describe the modified algorithm in terms of the time index i used for the original recursion (7). Then, we extract the active update steps and use the index k for these, as in (30).

Theorem 2 [Stop-and-Go Sato]: Assume $\hat{z}(k)$ stays uniformly bounded from above for all k, say

$$\gamma\sqrt{c} \le |\hat{z}(k)| \le P\gamma < \infty$$

for some $P \ge \sqrt{c} > 1$. Choose also the step-size sequence $\mu(k)$ for the active updates from within the interval

$$\frac{1-a}{\lambda\|\mathbf{u}_k\|^2} < \mu(k) < \frac{1+a}{\|\mathbf{u}_k\|^2}.$$

(A constant step-size can also be chosen by using the bounds (8) on $\|\mathbf{u}_k\|$.) It then holds that $K<\infty$. That is, $|\hat{z}(i)| < \gamma\sqrt{c}$ holds infinitely often.

Proof: The corresponding (real-valued) function $h[z(k),\hat{z}(k)]$ only needs to be defined for the case $|z(k)|=\gamma$ and $|\hat{z}(k)| \ge \gamma\sqrt{c}$. In this case, we have that

$$h[z(k),\hat{z}(k)] = \begin{cases} \dfrac{2}{|z(k)|+|\hat{z}(k)|}, & \mathrm{sign}[z(k)] \ne \mathrm{sign}[\hat{z}(k)] \\ 0, & \text{otherwise.} \end{cases}$$

Then, it holds that

$$0 \le h[z(k),\hat{z}(k)] < \frac{2}{(1+\sqrt{c})\gamma}$$

so that

$$\lambda < 1-\gamma h < 1. \qquad (31)$$

In order to meet (28), we need to choose the step-size according to

$$\frac{1-a}{\|\mathbf{u}_k\|^2} < \mu(k)(1-\gamma h) < \frac{1+a}{\|\mathbf{u}_k\|^2}$$

which in view of the bounds on $(1-\gamma h)$ in (31) can be achieved by selecting $\mu(k)$ according to

$$\frac{1-a}{\lambda\|\mathbf{u}_k\|^2} < \mu(k) < \frac{1+a}{\|\mathbf{u}_k\|^2}.$$

We still need to guarantee that the lower bound in the above inequality is smaller than the upper bound. This requires that λ be such that

$$\frac{1-a}{\lambda} < 1+a$$

or, equivalently, as in (29).

Hence, according to Lemma 1, we must have

$$\sum_{k=0}^{K}\mu(k)|e_a(k)|^2 \le \frac{\kappa(K)\|\tilde{\mathbf{w}}_{-1}\|^2}{(1-\Delta(K))^2}.$$

We want to show that $K<\infty$. Assume to the contrary that K is infinite. Then, $\hat{z}(k)\to z(k)$, which means that there exists a finite L large enough, such that for any positive number ϵ' and for all $k>L$

$$|\hat{z}(k)| < |z(k)| + \epsilon' = \gamma + \epsilon'.$$

We are free to choose ϵ' so let ϵ' be any positive number that satisfies

$$\gamma + \epsilon' < \gamma\sqrt{c}.$$

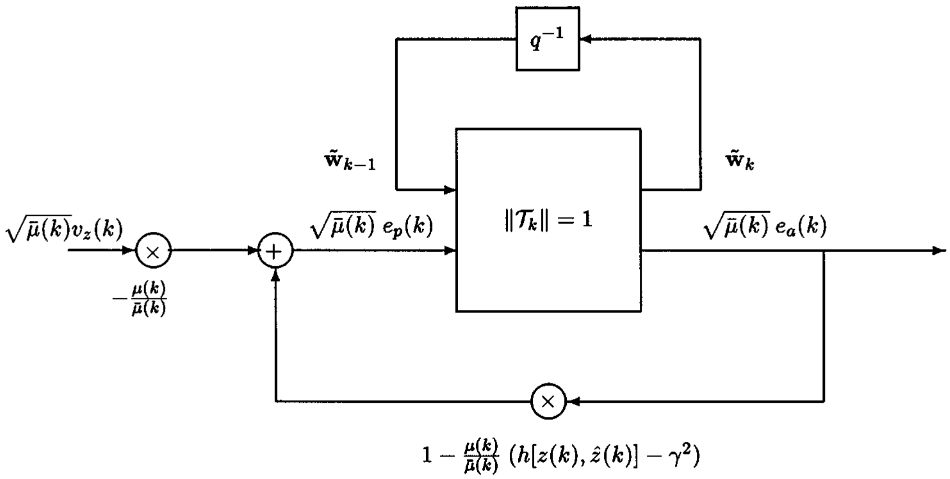

Fig. 4. A feedback structure for the case of channel imperfections.

It then follows that $|\hat{z}(k)| < \gamma\sqrt{c}$, which contradicts the fact that the active updates occurred with $\hat{z}(k)$ such that $|\hat{z}(k)| \geq \gamma\sqrt{c}$. We thus conclude that only a finite number of updates could have occurred, i.e., $K < \infty$. ◇

V. Channel Imperfections

So far in the paper we have assumed that there were no noise distortions or that the channel could be perfectly equalized. Under channel and equalization imperfections, the output $z(k)$ of the optimal equalizer in Fig. 1 will not be constant modulus anymore. However, assuming that these imperfections lead to disturbances that are small enough compared to the amplitude γ, we may assume that $z(k)$ in the CMA2-2 case is such that

$$z(k)|z(k)|^2 = z(k)\gamma^2 - v_z(k) \tag{32}$$

for some additive noise component $v_z(k)$ that explains the mismatch between $z(k)|z(k)|^2$ and $\gamma^2 z(k)$ in the nonideal case. Then, it is easy to verify that the error signal $e(k) = \hat{z}(k)[\gamma^2 - |\hat{z}(k)|^2]$ is now related to $e_a(k)$ via

$$e(k) = (h[z(k), \hat{z}(k)] - \gamma^2)e_a(k) + v_z(k) \tag{33}$$

where

$$h[z(k), \hat{z}(k)] \triangleq \begin{cases} \dfrac{z(k)|z(k)|^2 - \hat{z}(k)|\hat{z}(k)|^2}{z(k) - \hat{z}(k)}, & z(k) \neq \hat{z}(k) \\ v_z(k), & z(k) = \hat{z}(k). \end{cases}$$

Moreover, relation (15) between the *a priori* error $e_a(k)$ and the *a posteriori* error $e_p(k)$ becomes

$$e_p(k) = \left(1 - \frac{\mu(k)}{\bar{\mu}(k)}[h - \gamma^2]\right) e_a(k) - \frac{\mu(k)}{\bar{\mu}(k)} v_z(k) \tag{34}$$

while equality (17) still holds. This means that the feedback structure of Fig. 3 should now be replaced by the one shown in Fig. 4.

The following result now follows from the same arguments that led to Lemma 1 (see, e.g., [20]–[22]).

Lemma 2 [CMA2-2 with Channel Imperfections]: With the same definitions of $\Delta(N)$ and $\kappa(N)$ prior to Lemma 1, if

$$\Delta(N) < 1 \tag{35}$$

then the following bound on the weighted energy of the *a priori* estimation errors holds:

$$\sqrt{\sum_{k=0}^{N} \mu(k)|e_a(k)|^2} \leq \frac{\kappa^{1/2}(N)}{1 - \Delta(N)} \left[\|\tilde{\mathbf{w}}_{-1}\| + \kappa^{1/2}(N) \sqrt{\sum_{k=0}^{N} \mu(k)|v_z(k)|^2} \right]. \tag{36}$$

◇

In a similar vein to what was done in the earlier sections, the result of the above lemma could be used to establish conclusions of the following form for $N \to \infty$. If $\kappa(N)$ is uniformly bounded, $\|\tilde{\mathbf{w}}_{-1}\|$ is bounded, $\Delta(N)$ is uniformly bounded by one, say

$$\Delta(N) < a < 1, \qquad \text{for all } N \tag{37}$$

for some a, and if the distortion $v_z(i)$ has weighted finite energy,

$$\sum_{k=0}^{\infty} \mu(k)|v_z(k)|^2 < \infty \tag{38}$$

then it would follow that

$$\sum_{k=0}^{\infty} \mu(k)|e_a(k)|^2 < \infty.$$

This in turn implies that $\{\sqrt{\mu(k)}e_a(k)\}$ is a Cauchy sequence and that it tends to zero. In particular, if the step-size were constant or even for step-sizes that are bounded from below, this fact would lead to the conclusion that $e_a(k) \to 0$.

VI. Concluding Remarks

The paper studied the behavior of the error sequence of stop-and-go variants of two blind adaptive schemes, viz., CMA2-2 and Sato's algorithm. For both algorithms, the main conclusion was that for transmitted signals with constant modulus γ, the equalizer output can be made to lie within the circle of radius $\gamma\sqrt{c}$ infinitely often, for some value of c that is only slightly larger than one. The analysis can be extended to other classes of algorithms, e.g., CMA1-1 and with greater detail to cases with channel imperfections. These extensions will be pursued elsewhere.

Appendix A

We prove that the real part of the function $h[z,\hat{z}]$ defined by (25) is positive and bounded from below (we are omitting the indices k for simplicity of notation). Recall that $|z| = \gamma$ and $|\hat{z}| \geq \gamma\sqrt{c}$.

We start by writing $z/\hat{z} = re^{j\phi}$ for some $r < 1$ and for some $\phi \in [0, 2\pi)$. Then, expression (25) leads to

$$\begin{aligned} h[z,\hat{z}] &= |\hat{z}|^2 \left(1 + \left[1 - \frac{|z|^2}{|\hat{z}|^2}\right] \frac{z}{\hat{z} - z}\right) \\ &= |\hat{z}|^2 \left(1 + \left[1 - r^2\right] \frac{re^{j\phi}}{1 - re^{j\phi}}\right). \end{aligned} \tag{39}$$

Since $|\hat{z}|^2 > 0$, it is enough for our purposes to verify whether the real part of

$$\left(1 + [1 - r^2]\frac{re^{j\phi}}{1 - re^{j\phi}}\right) \tag{40}$$

is positive. The term $(1 - r^2)$ in (40) is positive since $r \in (0,1)$. Hence, we need to focus on the values that can be assumed by the term $re^{j\phi}/(1 - re^{j\phi})$. For any fixed value of r, if we allow the angle ϕ to vary from zero to 2π, then the term $re^{j\phi}/(1 - re^{j\phi})$ describes a circle in the complex plane whose most negative value is $-r/(1+r)$, obtained for $\phi = \pi$, and whose most positive value is $r/1 - r$, obtained for $\phi = 0$. This shows that for $r \in (0,1)$, the real part of the function $h[z,\hat{z}]/|\hat{z}|^2$ lies in the interval

$$1 - (1 - r^2)\frac{r}{1 + r} \leq \mathrm{Real}\left\{\frac{h[z,\hat{z}]}{|\hat{z}|^2}\right\} \leq 1 + (1 - r^2)\frac{r}{1 - r}.$$

The lower and upper bounds can be equivalently rewritten as

$$\frac{1 + r^3}{1 + r} \leq \mathrm{Real}\left\{\frac{h[z,\hat{z}]}{|\hat{z}|^2}\right\} \leq \frac{1 - r^3}{1 - r}. \tag{41}$$

Using the lower bound $|\hat{z}| \geq \gamma\sqrt{c}$, we get that

$$\mathrm{Real}\{h[z,\hat{z}]\} \geq c\gamma^2 \min_{r \in (0,1)} \left(\frac{1 + r^3}{1 + r}\right).$$

It can be easily verified that the rational function $(1+r^3)/(1+r)$ has a unique minimum in the interval $(0,1)$: it occurs at $r_o = 1/2$ and the minimum value is $3/4$ so that

$$\mathrm{Real}\{h[z,\hat{z}]\} \geq \frac{3}{4}c\gamma^2.$$

Using the definition of c from (23), we conclude that

$$\mathrm{Real}\{h[z,\hat{z}]\} \geq \gamma^2(1 + 0.75\epsilon). \tag{42}$$

Appendix B

We know from Appendix A that the function $h[z,\hat{z}]/|\hat{z}|^2$ assumes values that lie inside a circle in the right-half plane; its center is on the real axis and its right-most intersection with the real axis is at

$$1 + (1 - r^2)\frac{r}{1 - r} = \frac{1 - r^3}{1 - r}.$$

Hence, using $|\hat{z}| \leq P\gamma$, we get

$$|h[z,\hat{z}]| < P^2\gamma^2\frac{1 - r^3}{1 - r}.$$

Now, since

$$\frac{1 - r^3}{1 - r} = \sum_{n=0}^{2} r^n < 3$$

we obtain

$$|h[z,\hat{z}]| < 3P^2\gamma^2. \tag{43}$$

Appendix C

In this appendix, we establish the bounds on $\mu(k)$ in Theorem 1. Thus, let h_R and h_I denote the real and imaginary parts of h. Let also α and β be any two positive numbers satisfying

$$\alpha^2 + \beta^2 < 1. \tag{44}$$

(We shall be more specific about $\{\alpha, \beta\}$ further ahead.) If we can find a $\mu(k)$ that satisfies

$$\left|\frac{\mu(k)}{\bar{\mu}(k)}h_I[z(k), \hat{z}(k)]\right| < \alpha \tag{45}$$

and

$$\left|1 - \frac{\mu(k)}{\bar{\mu}(k)}(h_R[z(k), \hat{z}(k)] - \gamma^2)\right| < \beta \tag{46}$$

then $\mu(k)$ will also satisfy (22) for all k, as desired.

Now, we showed in Appendix B that

$$|h[z(k), \hat{z}(k)]| < 3P^2\gamma^2$$

and we can use this same bound for both $|h_R|$ and $|h_I|$ so that we also have

$$|h_I[z(k), \hat{z}(k)]| < 3P^2\gamma^2.$$

We can then satisfy (45) by selecting $\mu(k)$ such that

$$\mu(k) < \frac{\alpha}{\|\mathbf{u}_k\|^2}\frac{1}{3P^2\gamma^2}. \tag{47}$$

Likewise, we showed in Appendix A that

$$\mathrm{Real}\{h[z(k), \hat{z}(k)]\} \geq \gamma^2(1 + 0.75\epsilon).$$

Therefore, by using

$$\frac{3}{4}\epsilon\gamma^2 \leq h_R[z(k), \hat{z}(k)] - \gamma^2 < (3P^2 - 1)\gamma^2$$

we see that we can satisfy (46) by selecting $\mu(k)$ such that

$$\frac{(1-\beta)}{\|\mathbf{u}_k\|^2}\frac{4}{3\epsilon\gamma^2} < \mu(k) < \frac{(1+\beta)}{\|\mathbf{u}_k\|^2}\frac{1}{(3P^2-1)\gamma^2}.$$

Since we also want to satisfy (47), and since $(1+\beta) > \alpha$, we select a $\mu(k)$ that satisfies

$$\frac{(1-\beta)}{\|\mathbf{u}_k\|^2}\frac{4}{3\epsilon\gamma^2} < \mu(k) < \frac{\alpha}{\|\mathbf{u}_k\|^2}\frac{1}{3P^2\gamma^2}. \tag{48}$$

We of course need to guarantee that the upper bound in the above expression is larger than the lower bound. This can be achieved by choosing $\{\alpha, \beta\}$ properly so that

$$\frac{\alpha}{3P^2\gamma^2} > \frac{4(1-\beta)}{3\epsilon\gamma^2}$$

which is equivalent to requiring

$$\frac{4(1-\beta)}{\alpha} < \frac{\epsilon}{P^2}. \tag{49}$$

We now show that (44) and (49) can be satisfied simultaneously for some $\{\alpha, \beta\}$. Indeed let, for example, $\{\alpha^o, \beta^o\}$ be such that

$$1 - \beta^o = \frac{1}{8}\alpha^o\frac{\epsilon}{P^2}. \tag{50}$$

Then, $\{\alpha^o, \beta^o\}$ satisfy (49). Substituting into (44), we see that β^o must be such that

$$\left(8(1-\beta^o)\frac{P^2}{\epsilon}\right)^2 + (\beta^o)^2 < 1$$

which reduces to the following quadratic inequality in β^o:

$$\left(1 + \frac{64P^4}{\epsilon^2}\right)(\beta^o)^2 - \frac{128P^4}{\epsilon^2}\beta^o + \frac{64P^4}{\epsilon^2} - 1 < 0.$$

If we can find a β^o that satisfies this inequality, then a pair $\{\alpha^o, \beta^o\}$ satisfying (44) and (49) exists. So consider the quadratic function

$$g(\beta) = \left(1 + \frac{64P^4}{\epsilon^2}\right)\beta^2 - \frac{128P^4}{\epsilon^2}\beta + \frac{64P^4}{\epsilon^2} - 1.$$

It has a negative minimum and it crosses the real axis at the positive roots

$$\beta^{(1)} = \frac{64P^4 - \epsilon^2}{64P^4 + \epsilon^2} < 1, \qquad \beta^{(2)} = 1.$$

This means that there exist many values of β (between the roots) at which the quadratic function in β evaluates to negative values. Hence, β^o can be chosen as any value in the interval

$$\frac{64P^4 - \epsilon^2}{64P^4 + \epsilon^2} < \beta^o < 1. \tag{51}$$

The bounds on $\mu(k)$ in Theorem 1 are thus justified.

In the constant step-size case, we can show that there exists a μ that satisfies for all k

$$\frac{(1-\beta)}{\|\mathbf{u}_k\|^2}\frac{4}{3\epsilon\gamma^2} < \mu < \frac{\alpha}{\|\mathbf{u}_k\|^2}\frac{1}{3P^2\gamma^2}.$$

Indeed, since $b < \|\mathbf{u}_k\| < B$ for all k, we should choose a μ that satisfies

$$(1-\beta)\frac{1}{b^2}\frac{4}{3\epsilon\gamma^2} < \mu < \frac{\alpha}{B^2}\frac{1}{3P^2\gamma^2}.$$

By following the above arguments, it is easy to verify that the same construction for an $\{\alpha, \beta\}$ will hold if we replace throughout ϵ by $b^2\epsilon$ and P by BP, i.e., we choose $\{\alpha^o, \beta^o\}$ as follows:

$$\alpha^o = 8(1-\beta^o)\frac{B^2P^2}{b^2\epsilon} \tag{52}$$

and

$$\frac{64B^4P^4 - b^4\epsilon^2}{64B^4P^4 + b^4\epsilon^2} < \beta^o < 1 \tag{53}$$

so that μ can be selected from the interval

$$(1-\beta^o)\frac{1}{b^2}\frac{4}{3\epsilon\gamma^2} < \mu < \frac{\alpha^o}{B^2}\frac{1}{3P^2\gamma^2}. \tag{54}$$

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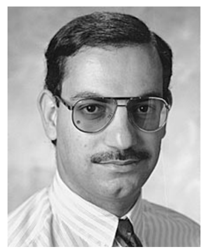

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