International Journal of Applied Mathematics

Volume 30 No. 3 2017, 229-238
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v30i3.2

ON THE CONVERGENCE OF DIRICHLET SERIES WITH RANDOM EXPONENTS

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Abstract: For the Dirichlet series of the form $F(z,\omega) = \sum_{k=0}^{+\infty} f_k(\omega)e^{z\lambda_k(\omega)}$ $(z \in \mathbb{C},\ \omega \in \Omega)$ with pairwise independent real exponents $(\lambda_k(\omega))$ on probability space $(\Omega, \mathcal{A}, P)$ an estimates of abscissas convergence and absolutely convergence are established.

AMS Subject Classification: 30B20, 30D20
Key Words: random Dirichlet series, random exponents, abscissas of convergence

1. Introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\mathbf{\Lambda} = \big(\lambda_k(\omega)\big)_{k=0}^{+\infty}$ and $\mathbf{f} = \big(f_k(\omega)\big)_{k=0}^{+\infty}$ sequences of positive and complex-valued random variables on it, respectively. Let $\mathcal{D}$ be the class of formal random Dirichlet series of the form

$$f(z) = f(z,\omega) = \sum_{k=0}^{+\infty} f_k(\omega)e^{z\lambda_k(\omega)} \quad (z \in \mathbb{C},\ \omega \in \Omega).$$

Let $\sigma_c(f,\omega)$ and $\sigma(f,\omega)$ be the abscissa of convergence and absolute convergence of this series for fixed $\omega \in \Omega$, respectively. The simple modification of [1]–[3] one has that for the Dirichlet series $f \in \mathcal{D}$ for fixed $\omega \in \Omega$ such that

Received: March 13, 2017

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$\lambda_k(\omega) \to +\infty$ $(k \to +\infty)$

$$\sigma(f,\omega) \le \sigma_c(f,\omega) \le \alpha_0(\omega) := \varliminf_{k\to+\infty} \frac{-\ln|f_k(\omega)|}{\lambda_k(\omega)} \le \sigma(f,\omega) + \tau(\omega,\Lambda), \tag{1}$$

or in the case $-\ln|f_k(\omega)| \to +\infty$ $(k \to +\infty)$

$$(1-h)\sigma_c(f,\omega) \le (1-h)\alpha_0(\omega) \le \sigma(f,\omega), \quad h = h(\omega,\mathbf{f}), \tag{2}$$

where $\tau(\omega,\Lambda) := \varlimsup_{k\to\infty} \frac{\ln k}{\lambda_k(\omega)}$, $\quad h(\omega,\mathbf{f}) := \varlimsup_{k\to\infty} \frac{\ln k}{-\ln|f_k(\omega)|}$.

Also,

$$\sigma_c(f,\omega) = \sigma(f,\omega) = \alpha_0(\omega) \tag{3}$$

for fixed $\omega \in \Omega$ such that $\tau(\omega) = 0$ or $\ln k/(-\ln|f_k(\omega)|) \to +0$ $(k \to +\infty)$. Remark, that from condition $\tau(\omega) < +\infty$ we get $\lambda_k(\omega) \to +\infty$ $(k \to +\infty)$. In the case $\sigma_c(f,\omega) > 0$ the series of the form $\sum_{k=0}^{+\infty} f_k(\omega)$ is convergent, thus $-\ln|f_k(\omega)| \to +\infty$ $(k \to +\infty)$.

The following assertion is proved in [3, Corollary 5] (another version [2, Theorem 1]) in the case of the deterministic Dirichlet series with sequence of exponents that increase to infinity, i.e., $f_k(\omega) \equiv f_k \in \mathbb{C}$ $(k \ge 0)$ and $\lambda_k(\omega) \equiv \lambda_k$, $0 \le \lambda_k < \lambda_{k+1} \to +\infty$ $(0 \le k \to +\infty)$.

Proposition 1. *Let $f \in \mathcal{D}$. Then $\sigma_a(f,\omega) \le \sigma_c(f,\omega) \le \alpha_0(\omega)$ $(\forall \omega \in \Omega)$, and*

$$\sigma_a(f,\omega) \ge \gamma(\omega)\alpha_0(\omega) - \delta(\omega) \ge \gamma(\omega)\sigma_c(f,\omega) - \delta(\omega) \tag{4}$$

for arbitrary real random variables γ, δ and for all $\omega \in \Omega$ such that $\gamma(\omega) > 0$ and

$$\sum_{k=0}^{+\infty} |f_k(\omega)|^{1-\gamma(\omega)} e^{-\delta(\omega)\lambda_k(\omega)} < +\infty. \tag{5}$$

Remark 2. Condition (5) implies, that $(\gamma(\omega)-1)\ln|f_k(\omega)| + \delta(\omega)\lambda_k(\omega) \to +\infty$ $(k \to +\infty)$ for such ω. But, in general, from this condition don't follows neither $\lambda_k(\omega) \to +\infty$ nor $\ln|f_k(\omega)| \to \infty$ $(k \to +\infty)$.

Proof of Proposition 1. It is obvious that $\sigma(f,\omega) \le \sigma_c(f,\omega)$.

We prove now that $\sigma_c(f,\omega) \le \alpha_0(\omega)$. Indeed, assume first that $\alpha_0(\omega) \ne \infty$ and put $x_0 = \alpha_0(\omega) + \varepsilon$, where $\varepsilon > 0$ is arbitrary. Then, $|f_k(\omega)|e^{x_0\lambda_k(\omega)} = \exp\{\lambda_k(\omega)(\ln|f_k(\omega)|/\lambda_k(\omega) + x_0)\}$. But by definition of $\alpha_0(\omega)$ there exists a sequence $k_j \to +\infty$ $(j \to +\infty)$ such that $\ln|f_k(\omega)|/\lambda_k(\omega) > -(\alpha_0(\omega) + \varepsilon/2)$ $(k = k_j,\ j \geq 1)$. Thus, $\ln|f_k(\omega)|/\lambda_k(\omega) + x_0 > \varepsilon/2 \quad (k = k_j,\ j \geq 1)$, and

$$|f_k(\omega)|e^{x_0\lambda_k(\omega)} \geq e^{\lambda_k(\omega)\varepsilon/2} \geq 1 \quad (k = k_j,\ j \geq 1),$$

therefore $\sigma_c(f, \omega) \leq \alpha_0(\omega) + \varepsilon$, but $\varepsilon > 0$ is arbitrary.

The case $\alpha_0(\omega) = +\infty$ is trivial. In the case $\alpha_0(\omega) = -\infty$ for every $E > 0$ and for some sequence $k_j \to +\infty$ $(j \to +\infty)$ by definition $\alpha_0(\omega)$ we obtain $\ln|f_k(\omega)|/\lambda_k > E \quad (k = k_j,\ j \geq 1)$. Therefore $|f_k(\omega)|\exp\{-E\lambda_k\} > 1$ $(k = k_j,\ j \geq 1)$, i.e. the Dirichlet series diverges at the point $z = -E$, but $E > 0$ is arbitrary. Thus, $\sigma_c = -\infty$.

Let now $x_0 = \gamma(\omega)(\alpha_0(\omega) - \varepsilon) - \delta(\omega)$ for arbitrary $\varepsilon > 0$. Then,

$$|f_k(\omega)|e^{x_0\lambda_k(\omega)} = |f_k(\omega)|^{1-\gamma(\omega)}e^{-\delta(\omega)\lambda_k(\omega)}\Big(|f_k(\omega)|e^{(\alpha_0(\omega)-\varepsilon)\lambda_k(\omega)}\Big)^{\gamma(\omega)}. \qquad (6)$$

By definition of $\alpha_0(\omega)$, we obtain $\alpha_0(\omega) < \frac{-\ln f_k(\omega)}{\lambda_k(\omega)} + \varepsilon/2$ for $k \geq k_0(\omega)$, and thus $|f_k(\omega)|e^{(\alpha_0(\omega)-\varepsilon)\lambda_k(\omega)} < \exp\{-\lambda_k\varepsilon/2\} \leq 1$ $(k \geq k_0(\omega))$. Hence by (6) one has $|f_k(\omega)|e^{x_0\lambda_k(\omega)} \leq |f_k(\omega)|^{1-\gamma(\omega)}e^{-\delta(\omega)\lambda_k(\omega)}$ and by condition (5) we obtain $\sigma(f, \omega) \geq x_0 = \gamma(\omega)(\alpha_0(\omega) - \varepsilon) - \delta(\omega)$. But, $\varepsilon > 0$ is arbitrary. □

From Proposition 1 it simply follows such a statement.

Proposition 3. *Let $f \in \mathcal{D}$. Then equalities* (3) *hold for all $\omega \in \Omega$ such, that*

$$\ln k = o(\ln|f_k(\omega)|) \quad (k \to +\infty). \qquad (7)$$

Remark 4. If the sequences Λ and $\mathbf{f}$ such that $\big(|f_k(\omega)|e^{x\lambda_k(\omega)}\big)$ are the sequences of independent random variables for every $x \in \mathbb{R}$, then by Kolmogorov's Zero-One Law ([4]) random variable $\sigma(f, \omega)$ is almost surely (a.s.) constant. That is, $\sigma(f, \omega) = \sigma \in [-\infty, +\infty]$ a.s. In the book [4] it is written when Λ monotonic increasing to infinity sequence $\lambda_k(\omega) \equiv \lambda_k$. The same we get when $\big(\frac{-\ln|f_k(\omega)|}{\lambda_k(\omega)}\big)$ is the sequence of independent random variables, and $\tau(\omega, \Lambda) = 0$ or $h(\omega, \mathbf{f}) = 0$. It follows from Proposition 3 and equalities (3).

In the papers [5]–[10] considered question about abscissas of convergence random Dirichlet series from the class $\mathcal{D}$ in case, when $\Lambda_+ = (\lambda_k)$ is increasing sequence of positive numbers, i.e., $0 = \lambda_0 < \lambda_k < \lambda_{k+1} \to +\infty$ $(1 \leq k \to +\infty)$ and $\tau(\omega, \Lambda) \equiv \tau(\Lambda) < +\infty$.

We have such elementary assertion.

Proposition 5. *Let $f \in \mathcal{D}(\Lambda)$ be a Dirichlet series of the form $f(z) = f(z,\omega) = \sum_{k=0}^{+\infty} a_k Z_k(\omega) e^{z\lambda_k(\omega)}$, where $(Z_k(\omega))$ is a sequence of random complex-valued variables.*

1^0. *If the condition $\tau(\omega,\Lambda) = 0$ holds and*

$$\lim_{k\to+\infty} \frac{-\ln |Z_k(\omega)|}{\lambda_k(\omega)} = 0 \quad \text{a.s.}, \tag{8}$$

then $\sigma_c(f,\omega) = \sigma(f,\omega) = \alpha_0^*(\omega) := \varliminf_{k\to+\infty} -\ln |a_k|/\lambda_k(\omega) \quad$ a.s.

2^0. *If $\alpha_0(\omega) = +\infty$ and the conditions $\tau(\omega,\Lambda) < +\infty$,*

$$\varliminf_{k\to+\infty} \frac{-\ln |Z_k(\omega)|}{\lambda_k(\omega)} > -\infty \ \ \text{a.s.} \tag{9}$$

hold, then $\sigma(f,\omega) = +\infty$ a.s.

We obtain Proposition 5 immediately from inequalities (1).

In the paper [6], it is considered only 1^0 for the case of the Dirichlet series $f \in \mathcal{D}(\Lambda_+)$ of the form $f(z) = f(z,\omega) = \sum_{k=0}^{+\infty} a_k Z_k(\omega) e^{z\lambda_k}$.

From Proposition 5, in particular, they follow Theorem 1 (when $\alpha_0 := \alpha_0^* = +\infty$) and Theorem 3 (when $\alpha_0 = 0$) from [6], which are proved under such condition for expectation:

$$(\exists \alpha > 0, \beta > 0)\colon \ \sup\{\mathbf{E}|Z_k|^{\alpha}, \mathbf{E}|Z_k|^{-\beta}\colon \ k \geq 0\} < +\infty. \tag{10}$$

By the Bienayme-Chebyshev inequality ([11, 12]) and the Borel-Cantelli Lemma $\big($[4], also about refined Second Borel-Cantelli lemma see [13]$\big)$ from condition (10) it easy follows, that a.s. for all enough large k inequalities $k^{-\gamma} \leq |Z_k(\omega)| < k^{\gamma}$ with $\gamma = \max\{2/\alpha, 2/\beta\}$ hold, and if $\tau(\Lambda) = 0$, then and condition (8). Similarly, if $\tau(\Lambda) < +\infty$, then from condition $(\exists \beta > 0)\colon \ \sup\{\mathbf{E}|Z_k|^{-\beta}\colon \ k \geq 0\} < +\infty$ follows condition (9).

It should be noted, that condition (8) follows from such condition (see [10]) on sequence of distribution functions of random variables $(|Z_k(\omega)|)$,

$$(\forall \varepsilon > 0)\colon \sum_{k=0}^{+\infty} \big(1 - F_k^*(e^{\varepsilon\lambda_k}) + F_k^*(e^{-\varepsilon\lambda_k})\big) < +\infty,$$

where $F_k^*(x) := P\{\omega\colon |Z_k(\omega)| < x\}$. In particular, from this condition it follows $\lim_{k\to+\infty} F_k^*(+0) = 0$.

In the papers [7]–[9] in the case of independent random variables $\mathbf{f} = (f_k)$, besides, generalized on class $\mathcal{D}(\Lambda)$ assertion of known Blackwell's conjecture on power series with random coefficients, proved in [14] (see also [4]).

In the general case, for Dirichlet series from the class $\mathcal{D}(\Lambda_+)$ in [10] (see also similar results for random gap power series in [15]–[18]) two theorems are proved. In particular, we find ([10]) the following theorem.

Theorem 6 ([10]). *Let $f \in \mathcal{D}(\Lambda_+)$ and $\mathbf{f} = \big(f_k(\omega)\big)$ be a sequence such that $\big(|f_k(\omega)|\big)$ is the sequence of pairwise independent random variables with functions of distribution $F_k(x) := P\{\omega : |f_k(\omega)| < x\},\ x \in \mathbb{R},\ k \geq 0$. The following assertions are true:*

a) *If $\sigma(\omega) = \sigma(f, \omega) \geq \rho \in (-\infty, +\infty)$ a.s., then*

$$(\forall \varepsilon > 0)\colon \ \sum_{k=0}^{+\infty}(1 - F_k((e^{-\rho} + \varepsilon)^{\lambda_k})) < \infty.$$

b) *If there exists a sequence $(\delta_k)\colon\ \delta_k > -\infty\ (k \geq 0),\ \underline{\lim}_{k\to+\infty} \delta_k = e^{-\rho},\ \rho \in (-\infty, +\infty]$, and $\sum_{k=0}^{+\infty}(1 - F_k(\delta_k^{\lambda_k})) = +\infty$, then $\sigma(f, \omega) \leq \rho$ a.s.*

Another theorem in [10] contains the converse statements.

In this paper we prove similar theorems for Dirichlet series with random exponents $(\lambda_k(\omega))$ and deterministic coefficients $\mathbf{f} = (f_k),\ f_k \in \mathbb{C},\ k \geq 0$. Note that in paper [19] a power series of the form $\sum_{k=0}^{+\infty} z^{X_k(\omega)}$ is studied, where $(X_k(\omega))$ is a strictly increasing integer-valued stochastic process.

2. The Main Results: Series with Random Exponents

In this section we assume that $f_k(\omega) \equiv f_k \in \mathbb{C}\ (k \geq 0)$ and condition $\ln k = o(\ln |f_k|)\quad (k \to +\infty)$ holds, that condition (7) is satisfied for all $\omega \in \Omega$, therefore by Proposition 3 equalities (3) for every $\omega \in \Omega$ hold.

Theorem 7. *Let $f \in \mathcal{D}(\Lambda)$ and $\Lambda = \big(\lambda_k(\omega)\big)$ be a sequence of pairwise independent random variables with distribution functions $F_k(x) := P\{\ \omega : \lambda_k(\omega) < x\},\ x \in \mathbb{R},\ k \geq 0$. The following assertions hold:*

i) *If $\sigma(\omega) = \sigma(f, \omega) \geq \rho \in (0, +\infty)$ a.s. then*

$$(\forall \varepsilon \in (0, \rho))\colon\ \sum_{k=0}^{+\infty} \big(1 - F_k(\ln |f_k|/(-\rho + \varepsilon))\big) < \infty.$$

ii) *If $0 \geq \sigma(\omega) = \sigma(f, \omega) \geq \rho \in (-\infty, 0]$ a.s. then*

$$(\forall \varepsilon > 0)\colon\ \sum_{k=0}^{+\infty} F_k(\ln |f_k|/(-\rho + \varepsilon)) < \infty.$$

Proof of Theorem 7. **i**) If $\sigma(f, \omega) \geq \rho \in (0, +\infty)$ a.s., then from (3) we have $(\exists B \in \mathcal{A}, P(B) = 1)(\forall \omega \in B)\colon\ \underline{\lim}_{k\to+\infty} -\ln |f_k|/\lambda_k(\omega) \geq \rho$, and by definition of

$\underline{\lim}$,

$$(\forall\omega\in B)(\forall\varepsilon\in(0,\rho))(\exists k^*(\omega)\in\mathbb{N})(\forall k\geqslant k^*(\omega))\colon\\ \lambda_k(\omega)<\ln|f_k|/(-\rho+\varepsilon). \tag{11}$$

We denote

$$A_k:=\Big\{\omega\colon \lambda_k(\omega)\geq\frac{\ln|f_k|}{(-\rho+\varepsilon)}\Big\}.$$

It is clear, that $B\subset\overline{C}:=\bigcup_{N=0}^{\infty}\bigcap_{k=N}^{\infty}\overline{A}_k$, hence $P(\overline{C})=1$, and $C=\bigcap_{N=0}^{\infty}\bigcup_{k=N}^{\infty}A_k$ is the event "(A_k) infinitely often", i.e. $\overline{C}$ is the event "(A_k) finitely often". From pairwise independence of random variables $(\lambda_k(\omega))$ follows pairwise independence of events (A_k). Therefore, by refined Second Borel-Cantelli Lemma ([13, p.84])

$$\sum\nolimits_{k=0}^{+\infty}\big(1-F_k\big(\ln|f_k|/(-\rho+\varepsilon)\big)\big)=\sum\nolimits_{k=0}^{+\infty}P(A_k)<+\infty.$$

ii) If $0\geq\sigma(\omega,f)\geq\rho\in(-\infty,0]$ a.s., then instead of (11) we obtain

$$(\exists B,P(B)=1)(\forall\omega\in B)(\forall\varepsilon>0)(\exists k^*(\omega)\in\mathbb{N})(\forall k\geq k^*(\omega))\colon\\ \lambda_k(\omega)>\ln|f_k|/(-\rho+\varepsilon).$$

Therefore, for $A_k:=\Big\{\omega\colon\lambda_k(\omega)\leq\ln|f_k|/(-\rho+\varepsilon)\Big\}$ by the refined Second Borel-Cantelli lemma we obtain again

$$\sum\nolimits_{k=0}^{+\infty}F_k\big(\ln|f_k|/(-\rho+\varepsilon)\big)=\sum\nolimits_{k=0}^{+\infty}P(A_k)<+\infty.$$

This completes the proof of Theorem 7. □

Remark 8. If $\sigma(f,\omega)>\rho\in[0,+\infty)$ a.s., then from (3) by definition of $\underline{\lim}$ we have $(\forall\omega\in B)(\exists\varepsilon^*=\varepsilon^*(\omega)>0)(\exists k^*(\omega)\in\mathbb{N})(\forall k\geqslant k^*(\omega))\colon\ \lambda_k(\omega)<\ln|f_k|/-(\rho+\varepsilon^*)$, and similarly as in proof of **i**) we obtain

$$\sum_{k=0}^{+\infty}\big(1-F_k(-\ln|f_k|/\rho)\big)<+\infty$$

in the case $\rho>0$ and in the case $\rho=0$ one has

$$\sum_{k=0}^{+\infty}\big(1-F_k(+0)\big)<+\infty,$$

i.e., in particular, $\lim_{k\to+\infty}F_k(+0)=1$. Namely, if $\underline{\lim}\limits_{k\to+\infty}F_k(+0)<1$, then $\sigma(f,\omega)\leq 0$ a.s.

Theorem 9. *Let* $\Lambda = \big(\lambda_k(\omega)\big)$ *be a sequence of random variables with distribution functions* $F_k(x) := P\{\ \omega : \lambda_k(\omega) < x\}$, $x \in \mathbb{R}$, $k \geq 0$, *and* $f \in \mathcal{D}(\Lambda)$. *The following assertions hold:*

i) *If there exist* $\rho \in (0, +\infty)$ *and a sequence* (ε_k) *such that* $\varepsilon_k \to +0$ $(k \to +\infty)$ *and* $\sum_{k=0}^{+\infty} \big(1 - F_k(\frac{\ln|f_k|}{-\rho+\varepsilon_k})\big) < +\infty$, *then* $\sigma(f, \omega) \geq \rho$ *a.s.*

ii) *If there exist* $\rho \in (-\infty, 0]$ *and a sequence* (ε_k) *such that* $\varepsilon_k \to +0$ $(k \to +\infty)$ *and* $\sum_{k=0}^{+\infty} F_k(\frac{\ln|f_k|}{-\rho+\varepsilon_k}) < +\infty$, *then* $\sigma(f, \omega) \geq \rho$ *a.s.*

Proof of Theorem 9. **i**) We note $1 - F_k(\ln|f_k|/(-\rho + \varepsilon_k)) = P(A_k)$, where

$$A_k := \big\{\omega\colon \lambda_k(\omega) \geq \ln|f_k|/(-\rho + \varepsilon_k)\big\}.$$

Therefore, from condition one has $\sum_{k=0}^{+\infty} P(A_k) < \infty$. Thus, by the first part of Borel-Cantelli Lemma $P(\overline{C}) = 1$, $C := \bigcap_{N=0}^{\infty} \bigcup_{k=N}^{\infty} A_k$. Since, $\overline{C} = \bigcup_{N=0}^{\infty} \bigcap_{k=N}^{\infty} \overline{A}_k$, then for all $\omega \in \overline{C}$ there exists $k = k^*(\omega)$ such that $\omega \in \overline{A}_k$ and $-\rho + \varepsilon_k < 0$ for all $k \geq k^*(\omega)$. Here, $(\forall k \geq k^*(\omega))\colon \lambda_k(\omega) < \frac{\ln|f_k|}{-\rho+\varepsilon_k}$. Using $\frac{-\ln|f_k|}{\lambda_k(\omega)} > \rho - \varepsilon_k$, we get

$$\sigma(f, \omega) = \varliminf_{k\to+\infty} \frac{-\ln|f_k|}{\lambda_k(\omega)} \geq \varliminf_{k\to+\infty} (\rho - \varepsilon_k) = \rho \quad \text{a.s.} \tag{12}$$

ii) By the condition $\sum_{k=0}^{+\infty} P(A_k) < +\infty$, where

$$A_k := \big\{\omega\colon \lambda_k(\omega) < \ln|f_k|/(-\rho + \varepsilon_k)\big\}.$$

Since, by the first part of Borel-Cantelli Lemma

$$P(\overline{C}) = 1, \quad C := \bigcap_{N=0}^{\infty} \bigcup_{k=N}^{\infty} A_k.$$

Where, as above for every $\omega \in \overline{C} = \bigcup_{N=0}^{\infty} \bigcap_{k=N}^{\infty} \overline{A}_k$ there exists $k = k^*(\omega)$ such that $\omega \in \overline{A}_k$ and $-\rho+\varepsilon_k > 0$ for all $k \geq k^*(\omega)$, such hat, $(\forall k \geq k^*(\omega))\colon \lambda_k(\omega) \geq \frac{\ln|f_k|}{-\rho+\varepsilon_k}$. Hence, $\frac{-\ln|f_k|}{\lambda_k(\omega)} > \rho - \varepsilon_k$ and, therefore, we have again the "chain" of relations (12).

The proof of Theorem 9 is complete. □

3. Some Corollaries

Corollary 10. *Let $f \in \mathcal{D}(\Lambda)$ and $\Lambda = (\lambda_k(\omega))$ be a sequence of pairwise independent random variables with distribution functions $F_k(x)$, $k \geq 0$. If $\varliminf\limits_{k\to+\infty} F_k(+0) < 1$ and $f_k \to 0$ $(k \to +\infty)$, then $\sigma(f,\omega) = 0$ a.s.*

Proof of Corollary 10. By Remark 8, $\sigma(f,\omega) \leq 0$ a.s. It is remains to prove that $\sigma(f,\omega) \geq 0$ a.s. Indeed, $\lambda_k(\omega) \geq 0$, therefore $F_k(0) = P\{\omega\colon \lambda_k(\omega) < 0\} = 0$. Hence, $\sum_{k=k_0}^{+\infty} F_k(\ln|f_k|/\varepsilon_k) < +\infty$ because $\ln|f_k|/\varepsilon_k < 0$ $(k \geq k_0)$. Thus, by Theorem 9 **ii**), $\sigma(f,\omega) \geq 0$ a.s. □

Corollary 10 implies immediately the statement of Corollary 11.

Corollary 11. *Let $f \in \mathcal{D}(\Lambda)$ and $\Lambda = (\lambda_k(\omega))$ be a sequence of pairwise independent random variables with distribution functions $F_k(x)$, $k \geq 0$. If there exists a positive random variable $a(\omega)$ such that $(\forall x \geq 0)(\forall k \in \mathbb{Z}_+)$: $F_k(x) \leq F_a(x) := P\{\omega\colon a(\omega) < x\}$ and $F_a(+0) < 1$ and $f_k \to 0$ $(k \to +\infty)$, then $\sigma(f,\omega) = 0$ a.s.*

Corollary 12. *Let $f \in \mathcal{D}(\Lambda)$ and $\Lambda = (\lambda_k(\omega))$ be a sequence of random variables with distribution functions $F_k(x)$, $k \geq 0$. If $f_k \to 0$ $(k \to +\infty)$ and there exist a positive random variable $b(\omega)$ and $\rho > 0$ such that $(\forall x \geq 0)(\forall k \in \mathbb{Z}_+)$: $F_k(x) \geq F_b(x) := P\{\omega\colon b(\omega) < x\}$, $\int_0^{+\infty} n_\mu(t\rho)\, dF_b(t) < +\infty$, where $n_\mu(t) = \sum_{\mu_k \leq t} 1$ is the counting function of a sequence $\mu_k = -\ln|f_k|$, then $\sigma(f,\omega) \geq \rho$ a.s.*

Proof of Corollary 12. We remark that

$$\begin{aligned}\sum_{k=k_0}^{n} \big(1 - F_k\big(\frac{\ln|f_k|}{-\rho+\varepsilon_k}\big)\big) &\leq \int_{\mu_{k_0}}^{\mu_n} \big(1 - F_k\big(t/\rho\big)\big) dn_\mu(t) \\ &\leq \int_{\mu_{k_0}}^{\mu_n} (1 - F_b(t/\rho)) dn_\mu(t) + O(1) \\ &= \int_{\mu_{k_0}/\rho}^{\mu_n/\rho} n_\mu(t\rho)\, dF_b(t) + O(1),\end{aligned}$$

$(n \to +\infty)$, because $-\ln\,|f_k| > 0$ $(k \geq k_0)$ and $\rho - \varepsilon_k < \rho$ for all $k \geq 0$. Therefore, the series $\sum_{k=k_0}^{+\infty} \big(1 - F_k\big(\frac{\ln|f_k|}{-\rho+\varepsilon_k}\big)\big)$ converges. Hence by Theorem 9 **ii**) we complete the proof. □

Corollary 13. *Let $\Lambda = (\lambda_k(\omega))$ be a increasing (a.s.) sequence of pairwise independent random variables and $f \in \mathcal{D}(\Lambda)$. If $F_0(+0) < 1$, where F_0 is distribution function of $\lambda_0(\omega)$, and $f_k \to 0$ $(k \to +\infty)$, then $\sigma(f, \omega) = 0$ a.s.*

Proof of Corollary 13. We remark that $F_{k+1}(x) \leq F_k(x)$, because $\lambda_k(\omega) \leq \lambda_{k+1}(\omega)$ $(k \geq 0)$ a.s. Therefore, by Corollary 11 we obtain the conclusion of Corollary 13. □

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