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ON THE CONVERGENCE OF POISSON BINOMIAL TO POISSON DISTRIBUTIONS

BY LOUIS H. Y. CHEN
Simon Fraser University

It is well known that under certain conditions, binomial distributions converge to Poisson distributions. Simons and Johnson (1971) showed that the convergence is actually much stronger than in the usual sense. In this note the author shows that the result of Simons and Johnson is also true for Poisson binomial distributions which include binomial distributions as special cases.

1. Introduction and assertion. Let $X_{1n}, X_{2n}, \cdots, X_{nn}$, $n = 1, 2, 3, \cdots$, be a triangular array of independent Bernoulli random variables with $P(X_{in} = 1) = 1 - P(X_{in} = 0) = p_{in}$ and let $\tilde{p}_n = \max_{1 \leq i \leq n} p_{in} \to 0$ and $\sum_{i=1}^n p_{in} = \lambda$ (fixed) as $n \to \infty$. It is well known that for $r = 0, 1, 2, \cdots$,

$$\lim_{n\to\infty} P(W_n = r) = p(r; \lambda)\,, \tag{1}$$

where $W_n = \sum_{i=1}^n X_{in}$ and $p(r; \lambda) = e^{-\lambda}\lambda^r/r!$. Stronger versions of (1) which are also known are

$$\lim_{n\to\infty} \sum_{r=0}^\infty |P(W_n = r) - p(r; \lambda)| = 0 \tag{2}$$

and

$$\sum_{r=0}^\infty |P(W_n = r) - p(r; \lambda)| \leqq (16/\lambda) \sum_{i=1}^n p_{in}^2 \qquad (\tilde{p}_n \leqq \tfrac{1}{4}) \tag{3}$$

(Le Cam (1960)).

Simons and Johnson (1971) showed that when the p_{in}'s are equal, (2) can be strengthened to

$$\lim_{n\to\infty} \sum_{r=0}^\infty h(r)|P(W_n = r) - p(r; \lambda)| = 0 \tag{4}$$

where $h(r) \geqq 0$ for $r = 0, 1, 2, \cdots$, and

$$\sum_{r=0}^\infty h(r)p(r; \lambda) < \infty\,. \tag{5}$$

The aim of this note is to show that (4) is indeed true even without assuming the equality of the p_{in}'s. The main idea of the proof here is similar to that in [3]. However, because the distribution of W_n (which is sometimes referred to as the Poisson binomial distribution) does not in general assume a form which is easy to work with, in order to show that it exhibits a certain monotone property in relation to the Poisson distribution, the author uses an identity. This identity was first used by Charles Stein in lectures at Stanford University. Generalizations of it are contained in [1].

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2. Proof. Let $W_n^{(i)} = \sum_{j=1, j\neq i}^n X_{jn}$. By using the fact that each X_{in} takes on values 0 and 1 and the independence of the X_{in}'s, we have, for every bounded function f,

$$EW_n f(W_n) = \sum_{i=1}^n p_{in} Ef(W_n^{(i)} + 1) . \tag{6}$$

By choosing $f(w)$ to be 1 if $w = r + 1$ and 0 if $w \neq r + 1$, where $0 \leqq r \leqq n$, (6) yields

$$(r + 1)P(W_n = r + 1) = \sum_{i=1}^n p_{in} P(W_n^{(i)} = r) . \tag{7}$$

Now $1 \geqq P(W_n^{(i)} = r \mid W_n = r) = (1 - p_{in})P(W_n^{(i)} = r)/P(W_n = r)$. This together with (7) implies

$$P(W_n = r + 1)/P(W_n = r) \leqq \lambda/(r + 1)(1 - \tilde{p}_n) . \tag{8}$$

Using again the fact that each X_{in} takes on values 0 and 1 and the independence of the X_{in}'s, we obtain from (6),

$$\begin{aligned} EW_n f(W_n) &= \lambda Ef(W_n + 1) + \sum_{i=1}^n p_{in} E[f(W_n^{(i)} + 1) - f(W_n + 1)] \\ &= \lambda Ef(W_n + 1) + \sum_{i=1}^n p_{in} E\{X_{in}[f(W_n^{(i)} + 1) - f(W_n^{(i)} + 2)]\} \\ &= \lambda Ef(W_n + 1) + \sum_{i=1}^n p_{in}^2 E[f(W_n^{(i)} + 1) - f(W_n^{(i)} + 2)] . \end{aligned} \tag{9}$$

This yields

$$\begin{aligned} &\frac{P(W_n = r + 1)}{p(r + 1; \lambda)} \Big/ \frac{P(W_n = r)}{p(r; \lambda)} \\ &\quad = \frac{(r + 1)P(W_n = r + 1)}{\lambda P(W_n = r)} \\ &\quad = 1 + \frac{1}{\lambda P(W_n = r)} \sum_{i=1}^n p_{in}^2 \\ &\qquad \times [P(W_n^{(i)} = r) - P(W_n^{(i)} = r - 1)] . \end{aligned} \tag{10}$$

By (8), $P(W_n^{(i)} = r)/P(W_n^{(i)} = r - 1) \leqq (\lambda - p_{in})/r(1 - \max_{j\neq i} p_{jn}) \leqq \lambda/r(1 - \tilde{p}_n)$ which is $\leqq 1$ for $n \geqq r \geqq [\lambda] + 1$ and $n \geqq n_0$ where $[\lambda]$ is the greatest integer $\leqq \lambda$ and n_0 is independent of r. Thus for $n \geqq r \geqq [\lambda] + 1$ and $n \geqq n_0$, $P(W_n^{(i)} = r) - P(W_n^{(i)} = r - 1) \leqq 0$ for $i = 1, 2, \cdots, n$, which together with (10) in turn implies

$$\frac{P(W_n = r + 1)}{p(r + 1; \lambda)} \Big/ \frac{P(W_n = r)}{p(r; \lambda)} \leqq 1 .$$

Hence noting also that $P(W_n = r) = 0$ if $r \geqq n + 1$, we have for $r \geqq [\lambda] + 1$ and $n \geqq n_0$,

$$P(W_n = r)/p(r; \lambda) \leqq p(W_n = [\lambda] + 1)/p([\lambda] + 1; \lambda) \leqq S < \infty \tag{11}$$

where S is some constant by virtue of (1). Finally, (4) follows, from (1), (5), (11) and the dominated convergence theorem, as in Simons and Johnson (1971). The assertion is proved.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SINGAPORE
SINGAPORE 10
REPUBLIC OF SINGAPORE