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ON THE CONVERGENCE OF SEQUENCES OF FUNCTIONS WHICH ARE DISCONTINUOUS ON COUNTABLE SETS

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Abstract. Let (X, T_X) be a topological space and let (Y, d_Y) be a metric space. For a function $f\colon X \to Y$ denote by $C(f)$ the set of all continuity points of f and by $D(f) = X \setminus C(f)$ the set of all discontinuity points of f. Let

$$C(X,Y) = \{f\colon X \to Y;\ f \text{ is continuous}\},$$
$$H(X,Y) = \{f\colon X \to Y;\ D(f) \text{ is countable}\},$$
$$H_1(X,Y) = \{f\colon X \to Y;\ \exists_{h \in C(X,Y)}\ \{x;\ f(x) \neq h(x)\} \text{ is countable}\},$$

and $H_2(X,Y) = H(X,Y) \cap H_1(X,Y)$. In this article we investigate some convergences (pointwise, uniform, quasiuniform, discrete and transfinite) of sequences of functions from $H(X,Y)$, $H_1(X,Y)$ and $H_2(X,Y)$.

Let (X, T_X) be a topological space and let (Y, d_Y) be a metric space. For a function $f\colon X \to Y$, denote by $C(f)$ the set of all continuity points of f and by $D(f) = X \setminus C(f)$ the set of all discontinuity points of f. Let

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$C(X,Y) = \{f\colon X \to Y;\ f \text{ is continuous}\}$,
$H(X,Y) = \{f\colon X \to Y;\ D(f) \text{ is countable}\}$,
$H_1(X,Y) = \{f\colon X \to Y;\ \text{there is } h \in C(X,Y) \text{ such that the set } \{x;\ f(x) \neq h(x)\} \text{ is countable}\}$, and
$H_2(X,Y) = H(X,Y) \cap H_1(X,Y)$.

Consider the following conditions (1) and (2):

(1) Suppose that (X, T_X) is a T_1 space. Then each finite set is closed and each countable set is an F_σ-set.

Remind that a function $f\colon X \to Y$ is of Baire class 1 if there is a sequence of continuous functions $f_n\colon X \to Y$ with $f = \lim_{n\to\infty} f_n$.

(2) Assume also that each function $f\colon A \to Y$ of Baire class 1, where A is a G_δ-set in X, may be extended to a Baire 1 function $g\colon X \to Y$.

It is known that if (X, ρ_X) is a metric space and $A \subset X$ is a nonempty G_δ-set in X then every function $f\colon A \to Y$ of Baire class 1 may be extended to a Baire 1 function $g\colon X \to Y$ ([4]).

Throughout the paper we assume (1) and (2).

For a given family Φ of functions $f\colon X \to Y$ denote by $\mathcal{B}(\Phi)$ ($\mathcal{B}_u(\Phi)$) the family of all pointwise (uniform) limits of sequences of functions from Φ.

Remark 1. *Observe that if $f \in \mathcal{B}(H(X,Y))$ then there is a Baire 1 function $g\colon X \to Y$ such that the set $\{x \in X;\ f(x) \neq g(x)\}$ is countable.*

Proof. Of course, if

$$f = \lim_{n\to\infty} f_n, \quad \text{where } f_n \in H(X,Y) \text{ for } n \geq 1,$$

then the sets $D(f_n)$ are countable, and consequently the union

$$E = \bigcup_n D(f_n)$$

is also countable. The set $A = X \setminus E$ is a G_δ-set and the restricted function $f|_A$ is the limit of the sequence of continuous functions $f_n|_A$, so by (2) there is a Baire 1 function $g\colon X \to Y$ such that $g|_A = f|_A$. Since the set

$$\{x \in X;\ f(x) \neq g(x)\} \subset E$$

is countable, the proof is finished. □

Similarly we can prove the following:

Remark 2. *If $f \in \mathcal{B}(H_1(X,Y))$ then there is a Baire 1 function $g\colon X \to Y$ such that the set $\{x \in X;\ f(x) \neq g(x)\}$ is countable.*

Theorem 1. *Assume* (1) *and* (2) *and suppose that for a function* $f\colon X \to Y$ *there is a Baire* 1 *function* $g\colon X \to Y$ *such that the set* $\{x \in X;\ g(x) \neq f(x)\}$ *is countable. Then* $f \in \mathcal{B}(H_2(X,Y))$.

Proof. Since $g\colon X \to Y$ is of Baire class 1, there is a sequence of functions $g_n \in C(X,Y)$ such that $g = \lim_{n\to\infty} g_n$. For $n \geq 1$ there are closed sets A_n such that

$$A_n \subset A_{n+1} \text{ for } n \geq 1, \quad \text{and} \quad \{x; g(x) \neq f(x)\} = \bigcup_{n\geq 1} A_n.$$

For $n \geq 1$ put

$$f_n(x) = \begin{cases} g_n(x) & \text{if } x \in X \setminus A_n \\ f(x) & \text{if } x \in A_n. \end{cases}$$

Since the sets A_n, $n \geq 1$, are closed and countable, and $g_n \in C(X,Y)$, $n \geq 1$, we obtain that $f_n \in H_2(X,Y)$, $n \geq 1$. Obviously, $f = \lim_{n\to\infty} f_n$, so the proof is completed. □

Corollary 1. *The equalites*

$$\begin{aligned} &\mathcal{B}(H(X,Y)) = \mathcal{B}(H_1(X,Y)) = \mathcal{B}(H_2(X,Y)) \\ &\quad = \{f\colon X \to Y;\ \text{there is a Baire 1 function } g\colon X \to Y \text{ such that the set} \\ &\qquad \{x \in X;\ f(x) \neq g(x)\} \text{ is countable}\} \end{aligned}$$

are true (*under asumptions* (1) *and* (2)).

Evidently, the families $H(X,Y)$, $H_1(X,Y)$ and $H_2(X,Y)$ are uniformly closed, i.e. $H = \mathcal{B}_u(H(X,Y))$, $H_1(X,Y) = \mathcal{B}_u(H_1(X,Y))$ and $H_2(X,Y) = \mathcal{B}_u(H_2(X,Y))$.

The quasi-uniform convergence, defined as follows, is a generalization of the uniform convergence. A sequence of functions $f_n\colon X \to Y$ $(n \geq 1)$ quasi-uniformly converges to a function $f\colon X \to Y$ in the sense of Predoi ([8]), if for each point $x \in X$ and each real $\varepsilon > 0$ there is a positive integer k such that for each integer $n \geq k$ there is a neighbourhood $U(x) \in T_X$ of x such that for all points $u \in U(x)$ the inequality $d_Y(f(u), f_n(u)) < \varepsilon$ is true. It is well known ([8]) that this type of convergence preserves continuity, i.e. if the functions f_n are continuous at a point x and the sequence (f_n) quasi-uniformly converges to f (in the sense of Predoi) then f is continuous at x. From this we obtain that

$$\mathcal{B}_{qu}(H(X,Y)) = H(X,Y),$$

where $\mathcal{B}_{qu}(\Phi)$ denotes the family of all functions, which are the limits of quasi-uniformly convergent sequences of functions from Φ. Of course, if a function $f\colon X \to Y$ is the limit of a quasi-uniformly convergent (in the sense of Predoi) sequence of functions $f_n \in H(X,Y)$ then the set

$$D(f) \subset \bigcup_{n\geq 1} D(f_n)$$

is countable, and consequently $f \in H(X,Y)$.

In the proof of the next theorem we will apply the following lemma:

Lemma 1. *Suppose that a metric space Y is complete. Let $A \subset X$ be a nonempty set and let $a \in X$ be an accumulation point of A. Suppose that for functions $f_n\colon (A \cup \{a\}) \to Y$ $(n \geq 1)$ there exist limits*

$$\lim_{A\ni x\to a} f_n(x) = b_n \in Y$$

and that the sequence (f_n) quasi-uniformly converges to a function $f\colon (A \cup \{a\}) \to Y$. Then there exists a limit

$$\lim_{A\ni x\to a} f(x) = \lim_{n\to\infty} b_n.$$

Proof. Fix $\varepsilon > 0$. Since the sequence (f_n) quasi-uniformly converges to f on $A \cup \{a\}$, there is a positive integer k such that for each integer $n \geq k$ there is a set $U_n \in T_X$ containing a such that for each point $x \in U_n \cap A$ we have

$$d_Y(f(x), f_n(x)) < \frac{\varepsilon}{5}.$$

Fix integers $n, m \geq k$. Since

$$\lim_{A\ni x\to a} f_n(x) = b_n \quad \text{and} \quad \lim_{A\ni x\to a} f_m(x) = b_m,$$

there is a set $U \in T_X$ containing a such that for $x \in (U \cap A) \setminus \{a\}$ we have

$$\max(d_Y(f_n(x), b_n), d_Y(f_m(x), b_m)) < \frac{\varepsilon}{5}.$$

Fix a point $x \in (U_n \cap U_m \cap U \cap A) \setminus \{a\}$ and observe that

$$\begin{aligned} d_Y(b_n, b_m) \leq & d_Y(b_n, f_n(x)) + d_Y(f_n(x), f(x)) + d_Y(f(x), f_m(x)) \\ & + d_Y(f_m(x), b_m) < \frac{4\varepsilon}{5} < \varepsilon. \end{aligned}$$

So, the sequence (b_n) satisfies the Cauchy condition. Since (Y, d_Y) is a complete space, there is a point $b \in Y$ such that $b = \lim_{n\to\infty} b_n$. We will prove that

$$\lim_{A\ni x\to a} f(x) = b. \tag{$*$}$$

For this, fix an integer $j > k$ such that $d_Y(b, b_j) < \varepsilon/5$ and find a set $V \in T_X$ containing a such that for $x \in (V \cap A) \setminus \{a\}$ the inequality

$$d_Y(f_j(x), b_j) < \frac{\varepsilon}{5}$$

is true. Observe that for $x \in (U_j \cap V \cap A) \setminus \{a\}$ we obtain

$$d_Y(f(x), b) \le d_Y(f(x), f_j(x)) + d_Y(f_j(x), b_j) + d_Y(b_j, b) < 3\frac{\varepsilon}{5} < \varepsilon,$$

and $(*)$ is true. This completes the proof. $\square$

We will show that in the above lemma the hypothesis that f and f_n's are defined on $A \cup \{a\}$ and the sequence (f_n) is quasi-uniformly convergent (in the sense of Predoi) on $A \cup \{a\}$ is essential.

Example. Let $X = Y = (0, 1]$, $d(y_1, y_2) = |y_1 - y_2|$ for $y_1, y_2 \in (0, 1]$ and let T_X be the topology introduced by the Euclidean metric d. For $n \ge 1$ put

$$f_n(x) = \begin{cases} 0 & \text{if } x \in \left(0, \dfrac{1}{2n\pi}\right] \\ \sin\dfrac{1}{x} & \text{if } x \in \left[\dfrac{1}{2n\pi}, 1\right], \end{cases}$$

$$f(x) = \sin\frac{1}{x} \quad \text{for } x \in (0, 1].$$

Then for each integer $n \ge 1$ there exists $\lim_{x \to 0^+} f(x) = 0$, and the sequence (f_n) quasi-uniformly converges to f on $(0, 1]$, but the function f does not have a limit at 0.

Theorem 2. *Suppose that (Y, d_Y) is a complete metric space and for each countable set $A \subset X$ the complement $X \setminus A$ is dense in X. Then the equality $\mathcal{B}_{qu}(H_1(X, Y)) = H_1(X, Y)$ is true.*

Proof. Assume that a function $f \colon X \to Y$ is the limit of a quasi-uniformly convergent sequence of functions $f_n \in H_1(X, Y)$, $(n \ge 1)$. Since $f_n \in H_1(X, Y)$, there are continuous functions $g_n \colon X \to Y$ such that the sets $A_n = \{x \in X \colon f_n(x) \ne g_n(x)\}$ are countable $(n \ge 1)$. Consequently, the union

$$A = \bigcup_{n \ge 1} \{x \in X;\ f_n(x) \ne g_n(x)\}$$

is also countable, and its complement $X \setminus A$ is dense in X. The sequence of continuous restricted functions $f_n|_{(X \setminus A)} = g_n|_{(X \setminus A)}$, $(n \ge 1)$ quasi-uniformly converges to $f|_{(X\setminus A)}$, so the restricted function $f|_{(X\setminus A)}$ is also continuous. By Lemma 1, for each point $x \in X \setminus A$ there exists

$$\lim_{(X\setminus A)\ni t\to x} f(t) = h_1(x) \in Y.$$

Evidently, the function

$$h(x) = f(x) \text{ for } x \in X \setminus A \quad \text{and} \quad h(x) = h_1(x) \text{ for } x \in A$$

is continuous and the set $\{x \in X;\ f(x) \neq h(x)\} \subset A$ is countable. This completes the proof. □

Corollary 2. *The equality $\mathcal{B}_{qu}(H_2(X,Y)) = H_2(X,Y)$ is true.*

In the article [2] the authors introduced the notion of discrete convergence of sequences of functions and investigated the discrete limits in different families, for example in the family $C(X,\mathbb{R})$, where X is a nonempty set. We will say that a sequence of functions $f_n\colon X \to Y$, $n = 1, 2, \ldots$, discretely converges to the limit f, ($f = d\text{-}\lim_{n\to\infty} f_n$) if for each point $x \in X$ there exists a positive integer $n(x)$ such that for all $n > n(x)$ the equality $f_n(x) = f(x)$ is true. For any family Φ of functions $f\colon X \to Y$ denote by $\mathcal{B}_d(\Phi)$ the family of all discrete limits of sequences of functions from the family Φ.

Theorem 3. *Assume that for each nonempty closed set $A \subset X$ and for each continuous function $h\colon A \to Y$ there is a continuous function $g\colon X \to Y$ such that $g|_A = h$. If $f \in \mathcal{B}_d(H_1(X,Y))$ then there is a function $g \in \mathcal{B}_d(C(X,Y))$ such that the set $\{x \in X;\ f(x) \neq g(x)\}$ is countable.*

Proof. Fix $f \in \mathcal{B}_d(H_1(X,Y))$. There are functions $f_n \in H_1(X,Y)$, $n \geq 1$, such that $f = d\text{-}\lim_{n\to\infty} f_n$. For each positive integer n there is a continuous function $g_n\colon X \to Y$ such that the set $A_n = \{x \in X;\ f_n(x) \neq g_n(x)\}$ is countable. Let

$$A = \bigcup_n A_n = \{a_1, \ldots, a_n, \ldots\} \quad \text{and} \quad G_n = \{a_k;\ k \leq n\} \quad \text{for } n \geq 1.$$

For $n \geq 1$ put

$$S_n = \{x \in X;\ f_k(x) = f(x) \text{ for } k \geq n\}$$

and observe that

$$S_n \subset S_{n+1} \text{ for } n \geq 1 \quad \text{and} \quad X = \bigcup_n S_n.$$

Without loss of generality we can suppose that $S_1 \neq \emptyset$. Let

$$B_n = \operatorname{cl}(S_n \setminus A) \cup G_n \quad \text{for } n \geq 1.$$

Observe that the sets B_n are closed for $n \geq 1$. Since for $k \geq n$ we have

$$g_k(x) = f_k(x) = f_n(x) = g_n(x) \quad \text{for } x \in S_n \setminus A,$$

so

$$g_k(x) = g_n(x) \quad \text{for } x \in \text{cl}(S_n \setminus A) \text{ and } k \geq n.$$

For $n \geq 1$ define

$$h_n(x) = \begin{cases} g_n(x) & \text{if } x \in \text{cl}(S_n \setminus A) \\ f_n(x) & \text{if } x \in G_n \setminus \text{cl}(S_n \setminus A). \end{cases}$$

Then the functions $h_n \colon B_n \to Y$ are continuous and there are continuous functions $\phi_n \colon X \to Y$ with $\phi_n | B_n = h_n$.

Observe that the sequence of (ϕ_n) discretely converges to a function $g \colon X \to Y$. Indeed, fix a point $x \in X$. If there is an index $n(x)$ such that $x \in cl(S_n \setminus A)$ for $n \geq n(x)$, then

$$\phi_n(x) = h_n(x) = g_n(x) = g_{n(x)}(x) \quad \text{for } n \geq n(x).$$

In the opposite case, for sufficiently large n we have $x \in A \setminus \text{cl}(S_n \setminus A)$ and $\phi_n(x) = f_n(x)$. Since the sequence (f_n) discretely converges to f, there is a positive integer $n(x)$ such that $f_n(x) = f(x)$ for $n \geq n(x)$. So for $n \geq n(x)$ we have

$$\phi_n(x) = f_n(x) = f(x),$$

and the sequence (ϕ_n) is discretely convergent. Denote its limit by $g \in \mathcal{B}_d(C(X, Y))$. From the construction of ϕ_n it follows that if $x \in X \setminus A = \bigcup_n S_n \setminus A$ then for sufficiently large n the equalities $\phi_n(x) = f_n(x) = f(x)$ are true, and consequently, the set $\{x \in X;\ f(x) \neq g(x)\}$ is countable. This completes the proof. □

Theorem 4. *Suppose that $f \colon X \to Y$. If there is a function $g \in \mathcal{B}_d(C(X, Y)))$ such that the set $\{x \in X;\ g(x) \neq f(x)\}$ is countable then $f \in \mathcal{B}_d(H_2(X, Y))$.*

Proof. Assume that there is a function $g \in \mathcal{B}_d(C(X, Y))$ such that $A = \{x \in X;\ f(x) \neq g(x)\}$ is countable, i.e. $A = \{a_n;\ n \geq 1\}$. There are continuous functions $g_n \colon X \to Y$ with $g = d\text{-}\lim_{n\to\infty} g_n$. For $n \geq 1$ define

$$f_n(x) = \begin{cases} f(x) & \text{for } x \in \{a_k;\ k \leq n\} \\ g_n(x) & \text{otherwise on } X. \end{cases}$$

Then $f_n \in H_2(X, Y)$ for $n \geq 1$ and $f = d\text{-}\lim_{n\to\infty} f_n$. So, $f \in \mathcal{B}_d(H_2(X, Y))$ and the proof is completed. □

Theorem 5. *Let (X, T_X) be an uncountable space and let $f\colon X \to Y$. If $f \in \mathcal{B}_d(H(X,Y))$, there is a countable set $A \subset X$ such that $f|_{(X \setminus A)} \in \mathcal{B}_d(C(X \setminus A, Y))$.*

Proof. Since $f \in \mathcal{B}_d(H(X,Y))$, there are functions $f_n \in H(X,Y)$ $(n \geq 1)$ such that $f = d\text{-}\lim_{n\to\infty} f_n$. The sets $D(f_n)$ are countable. Let

$$A = \bigcup_{n=1}^{\infty} D(f_n).$$

Then A is countable and the sequence of continuous functions $f_n|_{(X\setminus A)}$ discretely converges to $f|_{(X\setminus A)}$. □

Theorem 6. *Suppose that for each countable set $A \subset X$, the complement $X \setminus A$ is dense in X and that (Y, d_Y) is a compact metric space. Let $f\colon X \to Y$. If there is a countable set $E \subset X$ such that $f|_{(X\setminus E)} \in \mathcal{B}_d(C(X \setminus E, Y))$ then $f \in \mathcal{B}_d(H(X,Y))$.*

Proof. Since $f|_{(X\setminus E)} \in \mathcal{B}_d(C(X \setminus E, Y))$, there are continuous functions $g_n\colon (X \setminus E) \to Y$ such that

$$f|_{(X\setminus E)} = d - \lim_{n\to\infty} g_n.$$

Since the set $X \setminus E$ is dense in X and (Y, d_Y) is compact, for each integer $n \geq 1$ and each point $x \in E$ there is an $a_n(x) \in Y$ such that for each $\varepsilon > 0$ and each set $U \in T_X$ containing x there is a point $u_{n,U} \in U \setminus E$ such that $d_Y(g_n(u_{n,U}), a_n(x)) < \varepsilon$. Such a point $a_n(x)$ may be chosen in the product of all closures $\mathrm{cl}(g_n(U \setminus \{x\}))$, where $U \in T_X$ are arbitrary open sets containing x.

Let

$$E = \{b_n;\ n \geq 1\} \quad \text{and} \quad E_n = \{b_k;\ k \leq n\} \quad \text{for } n \geq 1.$$

For $n \geq 1$ define

$$h_n(x) = \begin{cases} g_n(x) & \text{if } x \in X \setminus E \\ a_n(x) & \text{if } x \in E, \end{cases}$$

and

$$f_n(x) = \begin{cases} h_n(x) & \text{if } x \in X \setminus E_n \\ f(x) & \text{if } x \in E_n. \end{cases}$$

Then

$$D(f_n) \subset D(h_n) \cup E_n \subset E \text{ for } n \geq 1,$$

and consequently, $f_n \in H(X,Y)$. Evidently $f = d\text{-}\lim_{n\to\infty} f_n$, so the proof is completed. □

Problem. Is Theorem 6 true for a σ-compact metric spaces (Y, ρ_Y)?

Theorem 7. *Suppose that for each countable set $A \subset X$, the complement $X \setminus A$ is dense in X and that $Y = \mathbb{R}$ is endowed with the metric $d_Y(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. Let $f\colon X \to Y$ be a function. If there is a countable set $E \subset X$ such that $f|_{(X \setminus E)} \in \mathcal{B}_d(C(X \setminus E, Y))$ then $f \in \mathcal{B}_d(H(X, Y))$.*

Proof. Since $f \in \mathcal{B}_d(C(X \setminus E, Y))$, there are continuous functions $\phi_n\colon (X \setminus E) \to Y$ such that

$$f|_{(X\setminus E)} = d - \lim_{n\to\infty} \phi_n.$$

For $n \geq 1$ put

$$g_n(x) = \begin{cases} \phi_n(x) & \text{if } |\phi_n(x)| \leq n \\ n & \text{if } \phi_n(x) \geq n \\ -n & \text{if } \phi_n(x) \leq -n. \end{cases}$$

Then evidently $g_n\colon (X \setminus E) \to [-n, n]$ are continuous for $n \geq 1$, and

$$f|_{(X\setminus E)} \;=\; d - \lim_{n\to\infty} g_n.$$

Since the set $X \setminus E$ is dense in X and $([-n, n], d_Y)$ is compact, for each integer $n \geq 1$ and each point $x \in E$ there is an $a_n(x) \in Y$ such that for each $\varepsilon > 0$ and each set $U \in T_X$ containing x there is a point $u_{n,U} \in U \setminus E$ such that $|g_n(u_{n,U}) - a_n(x)| < \varepsilon$. Then we define the sets E, E_n $(n \geq 1)$ and functions h_n, f_n $(n \geq 1)$ as in the proof of Theorem 6. Consequently, $f_n \in H(X, Y)$ $(n \geq 1)$ and

$$f = d - \lim_{n\to\infty} f_n,$$

which completes the proof. □

Remark 3. *The inclusion $\mathcal{B}_d(H_1(X, Y)) \subset \mathcal{B}_d(H(X, Y))$ holds.*

Proof. Of course, if $f \in \mathcal{B}_d(H_1(X, Y))$ then there are functions $f_n \in H_1(X, Y)$, $(n \geq 1)$ such that $f = d\text{-}\lim_{n\to\infty} f_n$. For each integer $n \geq 1$ there is a continuous function $g_n\colon X \to Y$ such that $A_n = \{x \in X;\ g_n(x) \neq f_n(x)\}$ is countable. The set

$$A = \bigcup_{n=1}^{\infty} A_n \text{ is also countable, so let } A = \{a_1, a_2, \dots\}.$$

For $n \geq 1$ put

$$A_n = \{a_k; k \leq n\}$$

and

$$h_n = \begin{cases} f(x) & \text{if } x \in A_n \\ g_n(x) & \text{if } x \in X \setminus A_n. \end{cases}$$

Then $D(h_n) \subset A_n$, and $h_n \in H(X,Y)$ and $f = d\text{-}\lim_{n\to\infty} h_n$, so the proof is finished. □

Observe that if $f \in \mathcal{B}_d(H_1(X,Y))$ then there are continuous functions $f_n\colon X \to Y$, $(n \geq 1)$ whose graphs $Gr(f_n) = \{(x, f_n(x));\ x \in X\}$ cover the graph $Gr(f)$ of the function f, i.e.

$$Gr(f) \subset \bigcup_{n=1}^{\infty} Gr(f_n).$$

Indeed, if $f \in \mathcal{B}_d(H_1(X,Y))$ then there are functions $g_n \in H_1(X,Y)$ such that $f = d\text{-}\lim_{n\to\infty} g_n$. For each integer $n \geq 1$ there is a continuous function $h_n\colon X \to Y$ such that $A_n = \{x;\ g_n(x) \neq h_n(x)\}$ is countable and so, the union

$$A = \bigcup_n A_n$$

is countable. Then

$$Gr(f|_{(X\setminus A)}) \subset \bigcup_n Gr(h_n),$$

and the graph $Gr(f|_A)$ may be covered by a countable family of the graphs of constant functions.

Meanwhile in [2] the authors observed that if $f\colon \mathbb{R} \to \mathbb{R}$ is a stricly increasing function whose $D(f)$ of discontinuity points is dense (with respect to the Euclidean topology in $\mathbb{R}$) then for each countable family of continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ we have

$$Gr(f) \setminus \bigcup_{n=1}^{\infty} Gr(f_n) \neq \emptyset.$$

This example shows that in the case where $X = Y = \mathbb{R}$ and the topology T_X is introduced by the metric $d_Y(x,y) = |x-y|$ for $x, y \in \mathbb{R}$, the following is true:

$$H(X,Y) \setminus \mathcal{B}_d(H_1(X,Y)) \neq \emptyset.$$

Finishing we consider the transfinite convergence. For this let ω_1 denote the first uncountable ordinal. A transfinite sequence of functions $f_\alpha\colon X \to Y$, $\alpha < \omega_1$, converges to a function $f\colon X \to Y$ ($f = \lim_{\alpha<\omega_1} f_\alpha$) if for each point $x \in X$ there is a countable ordinal $\alpha(x)$ such that $f_\alpha(x) = f_{\alpha(x)}(x)$ for all countable ordinals $\alpha > \alpha(x)$ ([3, 11]). For given family Φ of functions $f\colon X \to Y$ denote by $\mathcal{B}_{tr}(\Phi)$ the family of all transfinite limits of transfinite sequences of functions from Φ.

Observe that, if (X, ρ_X) is a dense in itself metric space for which there exists a subset $A = \{x_\alpha;\ \alpha < \omega_1\} \subset X$ such that for all different points $x, y \in A$ we have $\rho_X(x, y) \geq 1$ holds, then the functions

$$f_\alpha(x_\beta) = 1 \text{ for } \beta \leq \alpha < \omega_1 \quad \text{and} \quad f_\alpha(x) = 0 \text{ otherwise on } X$$

belong to $H_2(X, Y)$ and the transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ converges to the function f given by

$$f(x) = 1 \text{ for } x \in A \quad \text{and} \quad f(x) = 0 \text{ otherwise on } X,$$

which does not belong to $H(X, Y) \cup H_1(X, Y)$.

However, the following theorem is true:

Theorem 8. *Suppose that (X, T_X) is a topological space and (Y, ρ_Y) is a metric space such that every nonempty subset of the product space $X \times Y$ is a topological separable space. If a transfinite sequence of functions $f_\alpha \colon X \to Y$, $\alpha < \omega_1$, belonging to $H(X, Y)$, converges to a function $f \colon X \to Y$, then $f \in H(X, Y)$.*

Proof. Assume, to a contradiction, that the set $D(f)$ is uncountable. The graph $Gr(f)$ is a separable subspace of $X \times Y$, so there is a countable set $B \subset Gr(f)$ dense in $Gr(f)$. Let $E = Pr_X(B) = \{x \in X; (x, f(x)) \in B\}$. Then the set E is countable and consequently, there is a countable ordinal β such that

$$f_\alpha(x) = f(x) \quad \text{for } x \in E \text{ and } \beta \leq \alpha < \omega_1.$$

Since the set B is dense in $Gr(f)$ and $f(x) = f_\beta(x)$ for $x \in E$, we obtain $D(f) \subset D(f_\beta)$. So the set $D(f_\beta)$ is uncountable, a contradiction with $f_\beta \in H(X, Y)$. □

If the Continuum Hypothesis CH is true and if (X, T_X) is a topological space of the cardinality ω_1 then each real function $f \colon X \to \mathbb{R}$ is the limit of a transfinite sequence of functions $f_\alpha \colon X \to \mathbb{R}$, $\alpha < \omega_1$, such that the sets $\{x \in X;\ f_\alpha(x) \neq 0\}$ are countable for $\alpha < \omega_1$. Such functions $f_\alpha \in H_1(X, \mathbb{R})$, so the last theorem does not hold in this case of the family $H_1(X, \mathbb{R})$.

In [3] Lipiński proved that CH implies that each Baire 1 function $f \colon \mathbb{R} \to \mathbb{R}$ is the limit of a transfinite sequence of approximately continuous functions $f_\alpha \colon \mathbb{R} \to \mathbb{R}$, $\alpha < \omega_1$. Since there are Baire 1 functions $g \colon \mathbb{R} \to \mathbb{R}$ approximately discontinuous at all points of some uncountable subsets, this shows that in this case

$$\mathcal{B}_{tr}(A(\mathbb{R}, \mathbb{R})) \setminus H_1(\mathbb{R}, \mathbb{R}) \neq \emptyset,$$

where $A(\mathbb{R}, \mathbb{R})$ denotes the family of all approximately continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 9. *Let (X, T_X) be a topological space and let (Y, ρ_Y) be a metric space. Suppose that*

(a) *each nonempty subset $A \subset X \times Y$ with the restricted topology is a separable topological subspace of $X \times Y$, and*
(b) *each nonempty set $U \in T_X$ is uncountable.*

Then $\mathcal{B}_{tr}(H_2(X, Y)) = H_2(X, Y)$.

Proof. The inclusion $H_2(X, Y) \subset \mathcal{B}_{tr}(H_2(X, Y))$ is evident. From Theorem 8 it follows that $\mathcal{B}_{tr}(H_2(X, Y)) \subset H(X, Y)$. Let $f_\alpha \colon X \to Y$, $\alpha < \omega_1$, be a transfinite sequence of functions from $H_2(X, Y)$ convergent to a function $f \colon X \to Y$. Since $f \in \mathcal{B}_{tr}(H_2(X, Y)) \subset H(X, Y)$, the set $D(f)$ is countable. Let $\mathbb{N}$ be the set of all positive integers and let

$$A = \{(x_n, f(x_n));\ n \in \mathbb{N}_0 \subset \mathbb{N}\} \subset Gr(f)$$

be a countable set dense in $Gr(f)$ such that $D(f) \subset \{x_n;\ n \in \mathbb{N}_0\}$. But $\lim_{\alpha<\omega_1} f_\alpha = f$, so there is a countable ordinal β such that

$$f_\alpha(x_n) = f(x_n) \quad \text{for } n \in \mathbb{N}_0 \text{ and } \omega_1 > \alpha > \beta.$$

For each countable ordinal $\alpha > \beta$ there is a continuous function $g_\alpha \colon X \to Y$ such that the set

$$B_\alpha = \{x \in X;\ f_\alpha(x) \neq g_\alpha(x)\}$$

is countable. Observe that for a countable ordinal $\alpha > \beta$ we have

$$f_\alpha(x) = g_\alpha(x) = f(x) \quad \text{for } x \in C(f_\alpha) \cap C(f).$$

So, for countable ordinals $\alpha_1 > \alpha_2 > \beta$, the equality $g_{\alpha_1} = g_{\alpha_2}$ is true. Indeed, if there is a point $u \in X$ with $g_{\alpha_1}(u) \neq g_{\alpha_2}(u)$ then there is a set $U \in T_X$ containing u such that $g_{\alpha_1}(v) \neq g_{\alpha_2}(v)$ for all $v \in U$. But the set U is uncountable, so there is a point $w \in U \setminus (B_{\alpha_1} \cup B_{\alpha_2} \cup D(f_{\alpha_1}) \cup D(f_{\alpha_2}) \cup D(f))$. Consequently, $f(w) = f_{\alpha_1}(w) = g_{\alpha_1}(w) \neq g_{\alpha_2}(w) = f_{\alpha_2}(w) = f(w)$ and this contradiction proves that $g_{\alpha_1} = g_{\alpha_2}$.

Fix a countable ordinal $\alpha > \beta$. Since

$$\{x \in X; f(x) \neq g_\alpha(x)\} \subset B_\alpha \cup D(f) \cup D(f_\alpha),$$

we obtain $f \in H_1(X, Y)$. This finishes the proof. □

Remark 4. *Evidently, if the space X is countable then each function $f \colon X \to Y$ belongs to $H_2(X, Y)$ and $\mathcal{B}_{tr}(H_2(X, Y)) = H_2(X, Y)$.*

Problem. Is Theorem 9 true without the hypothesis (b)?

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