# On the convergence of solutions of certain third-order differential equations.

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**Summary.** - *In this paper it is shown that an earlier result of Ezeilo* [1] *can be extended to the more general equation* (1.1), *and this is achieved without any extra conditions on* $h'(x)$ *and* $h''(x)$. *A result on the existence of a unique periodic solution of* (1.1) *is also obtained as an application of the convergence result.*

1. **Introduction.** - Consider the third-order differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}) \tag{1.1}$$

in which $a > 0$, $b > 0$ are constants, the functions $h(x)$, $p(t, x, \dot{x}, \ddot{x})$ are continuous for values of their respective arguments and dots denote differentiation with resprct to $t$. Under the above conditions on $h$ and $p$ solutions of equation (1.1) exist for any preassigned initial conditions.

Any two solutions $x_1(t)$, $x_2(t)$ of (1.1) are said to converge to each other if

$$x_1(t) - x_2(t) \to 0, \qquad \dot{x}_1(t) - \dot{x}_2(t) \to 0, \qquad \ddot{x}_1(t) - \ddot{x}_2(t) \to 0 \tag{1.2}$$

as $t \to \infty$. The convergence property of solutions of equation (1.1), with $p(t, x, \dot{x}, \ddot{x}) \equiv p(t)$, has been investigated by EZEILO [1; Chapter 6]. His result shows that if both $p(t)$ and $\int_0^t p(\tau)d\tau$ are bounded and if $h(x)$ satisfies the usual

$$h'(x) \leq c, \qquad ab - c > 0 \text{ for all } x,$$

tnen, subject to some additional conditions on $h'(x)$ and $h''(x)$, all ultimately bounded solutions of (1.1) converge. Our present investigation will show that it is possible to dispense with the extra conditions imposed on $h'(x)$ and $h''(x)$ in [1] even when the «forcing function» $p(t, x, \dot{x}, \ddot{x})$ in (1.1) is bounded by a function which depends explicitily on $x$, $\dot{x}$, $\ddot{x}$ as well as on $t$. As an immediate applieation of our convergence result a Theorem on the existence of a unique periodic solution of (1.1) will also be established in the case when $p(t, x, \dot{x}, \ddot{x})$ is periodic in $t$.

2. **Statement of the main result.** - It will assumed thoughout what follows that $h'(x)$ exists and is continuous for all values of $x$. Our first result is as follows:

THEOREM 1. - *Given the equation* (1.1), *suppose that*

(i) *the function* $h(x)$ *is such that*

$$h(0)=0, \qquad \frac{h(x_2)-h(x_1)}{x_2-x_1}\geq\delta>0 \quad \text{for} \quad x_1 \neq x_2, \tag{2.1}$$

$$h'(x)\leq c \quad \textit{for all } x, \textit{ where } \quad 0<c<ab, \tag{2.2}$$

(ii) *there exists a finite constant* $\Delta_* > 0$ *such that*

$$|p(t, x_2, y_2, z_2)-p(t, x_1, y_1, z_1)|\leq\Delta_*\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\}^{1/2}. \tag{2.3}$$

*Furthermore, let*

$$B=\min\left[\frac{\delta d}{16a^2(a+1)}, \frac{a\delta d^2}{4bc(a+1+d/2c)^2}\right], \qquad d=ab-c. \tag{2.4}$$

*Then, there exists a finite constant* $\varepsilon_*>0$ *such that if the constant* $\Delta_*$ *in* (2.3) *satisfies* $\Delta_*<\varepsilon_*$, *then any two distinct solutions* $x_1(t)$, $x_2(t)$ *of* (1.1) *for which*

$$[h(x_2-x_1)-\{h(x_2)-h(x_1)\}]^2\leq B(x_2-x_1)^2 \tag{2.5}$$

*for all* $t\geq t_0$ $(0<t_0<\infty)$ *necessarily converge.*

It ought to be pointed out that the constant $B$, given by (2.4), is not necessarily the best possible; an indication on how to obtain a more general, and probably better, estimate will be given at the end of § 3.

The condition (2.3) can be improved to the form

$$|p(t, x_2, y_2, z_2)-p(t, x_1, y_1, z_1)|\leq\Phi(t)[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]^{1/2} \tag{2.6}$$

where, for some constant $k_2$ in the range $1\leq k_2\leq 2$, the function $\Phi(t)\geq 0$ is such that

$$\int_{-\infty}^{\infty}\Phi^{k_2}(t)dt<\infty. \tag{2.7}$$

It will however be convenient to deal first with Theorem 1 in its present form, and then. later (see § 5), to indicate what modifications are necessary to convert our methods to the case in which $p(t, x, y, z)$ satisfies (2.6).

In the cases where the solutions $x_1(t)$, $x_2(t)$ of (1.1) are bounded, say $|x_i(t)|\leq R$ $(i=1, 2)$, it would be clear that there exists a finitite constant $H=H(R)$ such that

$$|h(x_2-x_1)-\{h(x_2)-h(x_1)\}|\leq H(R)|x_2-x_1| \quad \text{if} \quad |x_i|\leq R \quad (i=1, 2). \tag{2.8}$$

For, since $h(0)=0$ and $h'(x)$ exists and is continuous,

$$h(x_2-x_1)=(x_2-x_1)h'(\theta_1(x_2-x_1)), \qquad 0<\theta_1<1,$$
$$h(x_2)-h(x_1)=(x_2-x_1)h'(\xi), \qquad \xi=x_1+\theta_2(x_2-x_1), \qquad 0<\theta_2<1$$

by the mean — value theorem, and so

$$h(x_2-x_2)-\{h(x_2)-h(x_1)\}=(x_2-x_1)\{h'(\theta_1(x_2-x_1)-h'(\xi)\},$$

where $h'(\theta'(x_2-x_1))-h'(\xi)$ is bounded if $x_1$, $x_2$ are bounded. Thus, in cases where solutions of (1.1) are ultimately bounded, (2.5) is not an extra condition on the existence of the constant on the right-hand side of (2.5), but rather a condition on the smallness of the constant.

The following result follows immediately from Theorem 1.

THEOREM 2. - *Let the conditions of Theorem* 1 *hold and let $H$ be the function defined by* (2.8). *Then there exists a constant* $\beta$, $0<\beta<\infty$, *such that if*

$$H(\beta)\leq B^{1/2} \tag{2.9}$$

*then all solutions of* (1.1) *converge provided that* $\Delta_* < \varepsilon^*$.

Indeed, in the case $n=1$, the result [4; corollary 3] implies that under the hypoteses (i) and (ii) of Theorem 1 every solution $x(t)$ of (1.1) ultimately satisfies

$$|x(t)|\leq D_1, \qquad \dot{x}(t)|\leq D_1, \qquad |\ddot{x}(t)|\leq D_1 \tag{2.10}$$

provided $\Delta_*<\varepsilon_1^*$ $(0<\varepsilon_1^*<\infty)$. Thus, if the constant $\beta$ is chosen such that $\beta\geq D_1$ and if $\Delta_*<\varepsilon^*$, $\varepsilon^*=\min(\varepsilon_1^*, \varepsilon_*)$, then, by (2.8) and (2.9), for any two distinct solutions $x_1(t)$, $x_2(t)$ of (1.1),

$$[h(x_2-x_1)-\{h(x_2)-h(x_1)\}]^2\leq B(x_2-x_1)^2$$

for all $t\geq t_0$, and the corollary would then follow from Theorem 1.

## 3. Preliminary.

- We shall consider, instaed of (1.1), the system

$$\dot{x}=y, \qquad \dot{y}=z, \qquad \dot{z}=-az-by-h(x)+p(t, x, y, z) \tag{3.1}$$

which is obtained from (1.1) by setting $\dot{x}=y$, $\dot{y}=z$. Let $(x_1(t), y_1(t), z_1(t))$,

$(x_2(t),\ y_2(t),\ z_2(t))$ be two distinct solutions of (3.1) such that

$$[h(x_2 - x_1) - \{h(x_2) - h(x_1)\}]^2 \leq B(x_2 - x_1)^2 \tag{3.2}$$

for all $t \geq t_0$ $(0 < t_0 < \infty)$, where $B$ is given by (2.4). Then, in view of definition (1.2), it is enough, in order to prove Theorem 1, to show that

$$x_2(t) - x_1(t) \to 0, \qquad y_2(t) - y_1(t) \to 0, \qquad z_2(t) - z_1(t) \to 0 \tag{3.3}$$

as $t \to \infty$. Our proof of this will rest mainly on the properties of a certain function $W(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$ which is derived from the function $W(x,\ y,\ z)$ [2; § 2] by replacing $x,\ y,\ z$ with $x_2 - x_1,\ y_2 - y_1$ and $z_2 - z_1$ respectively. The function $W = W(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ is defined as follows:

$$\begin{aligned} 2W = {} & 2(a+1)\int_0^{x_2-x_1} h(\xi)d\xi + 2(\delta_1 + 1)(y_2 - y_1)h(x_2 - x_1) + b\delta_2(x_2 - x_1)^2 + \\ & + \{a + \delta_1 b + a^2 + b - \delta_2\}(y_2 - y_1)^2 + (\delta_1 + 1)(z_2 - z_1)^2 + \\ & + 2a\delta_2(x_2 - x_1)(y_2 - y_1) + 2(a+1)(y_2 - y_1)(z_2 - z_1) + 2\delta_2(x_2 - x_1)(z_2 - z_1), \end{aligned} \tag{3.4}$$

where $\delta_1,\ \delta_2$ are two fixed constants such that

$$\frac{1}{a} < \delta_1 < \frac{b}{c}, \qquad ab - c > a\delta_2 > 0. \tag{3.5}$$

Note that subject to the hypothesis (1) of Theorem 1, the function $W$ is positive definite in $x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1$. Indeed by [2; Lemma 1] the function $W(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$ satisfies

$$W(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1) \geq D_2[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] \tag{3.6}$$

where $D_2 = D_2(a,\ b,\ c,\ \delta_1,\ \delta_2) > 0$ is a finite constant. Our efforts will now be directed to the proof of the following result: for any two distinct solutions $(x_1(t),\ y_1(t),\ z_1(t))$, $(x_2(t),\ y(t),\ z_2(t))$ of (3.1) such that (3.2) holds, the function $W(t) \equiv W(x_2(t) - x_1(t),\ y_2(t) - y_1(t),\ z_2(t) - z_1(t))$ satisfies

$$W(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.7}$$

After this had been proved, the result (3.3) would follow since $W(t)$ is positive definite. The proof of (3.7) will be based on the following properties of the function $W$.

LEMMA 1. - *Let the conditions of Theorem* 1 *hold and let* $(x_1(t), y_1(t), z_1(t))$, $(x_2(t), y_2(t), z_2(t))$ *be any two distinct solutions of* (3.1) *such that*

$$[h(x_2 - x_1) - \{h(x_2) - h(x_1)\}]^2 \leq B(x_2 - x_1)^2 \tag{3.8}$$

*for all* $t \geq t_0$ $(0 < t_0 < \infty)$, *where* $B$ *is given by* (2.4). *Then, there exist constants* $\varepsilon_* > 0$, $D_3$ *such that if the constant* $\Delta_*$ *in* (2.3) *satisfies* $\Delta_* < \varepsilon_*$, *then the function* $W(t) \equiv W(x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t))$ *satisfies*

$$\dot{W}(t) + D_3 W(t) \leq 0 \tag{3.9}$$

*for all* $t \geq t_0$.

PROOF OF LEMMA 1. - Let $(x_2(t), y_2(t), z_2(t))$, $(x_1(t), y_1(t), z_1(t))$ be any two distinct solutions of (3.1), and consider the function $W(t) \equiv W(x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t))$. On differntiating this with respect to $t$ and using

$$\dot{x}_i = y_i, \quad \dot{y}_i = z_i, \quad \dot{z}_i = -az_i - by_i - h(x_i) + p(t, x_i, y_i, z_i). \quad (i = 1, 2)$$

we have, after further simplification, that

$$\dot{W}(t) - U + [\delta_2(x_2 - x_1) + (a + 1)(y_2 - y_1) + (1 + \delta_1)(z_2 - z_1)]\theta, \tag{3.10}$$

where

$$\theta = p(t, x_2, y_2, z_2) - p(t, x_1, y_1, z_1); \tag{3.11}$$

$$U = (a\delta_1 - 1)(z_2 - z_1)^2 + \{ab - a\delta_2 + b - (1 + \delta_1)h'(x_2 - x_1)\}(y_2 - y_1)^2 +$$
$$+ \delta_2(x_2 - x_1)\{h(x_2) - h(x_1)\} - \{(a+1)(y_2 - y_1) + (1 + \delta_1)(z_2 - z_1)\}[h(x_2 - x_1) - \{h(x_2) - h(x_1)\}].$$

Clearly, by (2.1) and (2.2),

$$h(x_2) - h(x_1) \geq \delta(x_2 - x_1),$$
$$ab - a\delta_2 + b - (1 + \delta_1)h'(x_2 - x_1) \geq ab - a\delta_2 + b - (1 + \delta_1)c,$$

and thus,

$$U \geq D_4(x_2 - x_1)^2 + D_5(y_2 - y_1)^2 + D_6(z_2 - z_1)^2 - \{D_7(y_2 - y_1) + D_8(z_2 - z_1)\} H(x_2, x_1), \tag{3.12}$$

where

$$H(x_2, x_1) = h(x_2 - x_1) - \{h(x_2) - h(x_1)\}$$

and

$$D_4 \equiv \delta\delta_2 > 0, \qquad D_5 \equiv (ab - c - a\delta_2) + (b - \delta_1 c) > 0, \tag{3.13}$$
$$D_6 \equiv a\delta_1 - 1 > 0, \qquad D_7 \equiv a + 1 > 0, \qquad D_8 \equiv 1 + \delta_1 > 0$$

by (3.5). The inequality (3.12) can be reset in the form

$$U \geq \frac{1}{2} D_4(x_2 - x_1)^2 + \frac{3}{4}\{D_5(y_2 - y_1)^2 + D_6(z_2 - z_1)^2\} + U_1 + U_2, \tag{3.14}$$

where

$$U_1 = \frac{1}{4}\{D_4(x_2 - x_1)^2 + D_5(y_2 - y_1)^2\} - D_7(y_2 - y_1)H(x_2, x_1),$$

$$U_2 = \frac{1}{4}\{D_4(x_2 - x_1)^2 + D_6(z_2 - z_1)^2\} - D_8(z_2 - z_1)H(x_2, x_1).$$

By a simple rearrangement of terms, it is clear that

$$U_1 = \frac{1}{4} D_4(x_2 - x_1)^2 + \frac{D_5}{4}\left\{(y_2 - y_1) - \frac{2D_7}{D_5} H(x_2, x_1)\right\}^2 - \frac{D_7^2}{D_5} H^2(x_2, x_1)$$
$$\geq \frac{1}{4} D_4(x_2 - x_1)^2 - \frac{D_7^2}{D_5} H^2(x_2, x_1),$$

and also that

$$U_2 \geq \frac{1}{4} D_4(x_2 - x_1)^2 - \frac{D_8^2}{D_6} H^2(x_2, x_1).$$

Hence $U_1 \geq 0$, $U_2 \geq 0$ provided that

$$H^2(x_2 - x_1) \leq \min\left[\frac{D_4 D_5}{4D_7^2}, \frac{D_4 D_6}{4D_8^2}\right](x_2 - x_1)^2, \tag{3.15}$$

and thus, from (3.14), we have that

$$U \geq \frac{1}{4} D_4(x_2 - x_1)^2 + \frac{3}{4}\{D_5(y_2 - y_1)^2 + D_6(z_2 - z_1)^2\} \tag{3.16}$$

provided (3.14), holds. On combining this with (3.9), we have, in view of (3.10) and (2.3), that

$$\dot{W}(t) \leq -\{D_8 - D_9\Delta_*\}\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}$$

provided $H(x_2, x_1)$ satisfies (3.14), where $D_8 = \frac{1}{2}\min(D_4, D_5, D_6)$ and $D_9 = 3^{1/2}\max(\delta_2, (a+1), 1+\delta_1)$. Now let

$$\varepsilon_* = \frac{D_8}{D_9}. \tag{3.17}$$

Then provided $\Delta_* < \varepsilon_*$, there exists a constant $D_{10}$ such that if (3.15) is satisfied, then

$$\dot{W}(t) \leq -D_{10}\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}$$

and, in view of (3.6), this implies

$$\dot{W}(t) \leq -D_{11} W(t) \tag{3.18}$$

for some constant $D_{11}$.

So far we have not used the condition (3.8). To facilitate the use of this, it is convenient to reset the right-hand side of (3.15) differently.
Let $d \equiv ab - c$, then by (3.5), there are constants $\alpha_1 > 0$, $\alpha_2 > 0$ such that

$$\begin{aligned} a\delta_1 &= 1 + \alpha_1, \qquad 0 < \alpha_1 < d/c \\ \delta_2 &= \frac{d}{a} - \alpha_2, \qquad 0 < \alpha_2 < d/a. \end{aligned} \tag{3.19}$$

From these and (3.13), it is readily seen that

$$\frac{D_4 D_6}{4D_6^2} = \frac{a\delta\alpha_1(d - a\alpha_2)}{4(1 + a + \alpha_1)^2} \qquad \frac{D_4 D_5}{4D_7^2} = \frac{\delta(d - a\alpha_2)(d + a^2\alpha_2 - c\alpha_1)}{4a^2(a+1)^2}$$

and the right-hand side of (3.15) now becomes

$$B^* = \min\left[\frac{\delta(d - a\alpha_2)(d + a^2\alpha_2 - c\alpha_1)}{4a^2(a+1)^2};\ \frac{a\delta\alpha_1(d - a\alpha_2)}{4(1 + a + \alpha_1)^2}\right].$$

For the special values of $\alpha_1$, $\alpha_2$ chosen, in view of (3.19), such that

$$\alpha_1 = d/2c, \qquad \alpha_2 = d/2a,$$

$B^*$ reduces to (2.4) and so, by (3.8), $H(x_2, x_1)$ satisfies (3.15) for all $t \geq t_0$. Thus, the inequality (3.18) holds for all $t \geq t_0$, and this completes the proof of the lemma with $\varepsilon_*$ given by (3.17).

The estimate (2.4) can be considerably improved by choosing $\alpha_1$, $\alpha_2$ to make $B^*$ a maximum.

4. **Completion of the proof of Theorem 1.** - Let $\varepsilon^* > 0$ and let $t_0$ $(0 < t_0 < 0)$ be fixed as in Lemma 1. Then, by Lemma 1, for any two distinct solutions $(x_1(t), y_1(t), z_1(t))$, $(x_2(t), y_2(t), z_2(t))$ of (3.1) for which (3.8) holds, the function $W(t) \equiv W(x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t))$ satisfies

$$\dot{W}(t) + D_3 W(t) \leq 0$$

for all $t \geq t_0$ provided $\Delta_* < \varepsilon_*$. On integrating this between $t_0$ and $t$, we have that

$$W(t) \leq W(t_0) e^{-D_3(t-t_0)}, \qquad t \geq t_0,$$

which implies

$$W(t) \to 0 \text{ as } t \to \infty$$

since $W(t_0)$ is bounded and the function $W$ is positive definite. In view of the preceding remarks in § 3, this proves (3.3) and thus the theorem is proved with $\varepsilon_*$ given by (3.17).

5. We now turn to the case mentioned in § 2 in which $p(t, x, y, z)$ satisfies (2.6). The proof of Theorem 1 in this case follows the lines indicated in § 3 - § 4 except for some minor modifications which we now outline. Assume that (2.5) holds. Then by (3.10), (3.11), (3.16) and (3.6) $\dot{W}(t)$ satisfies

$$\dot{W}(t) \leq -2D_{12} W(t) + D_{13} \{ W(t) \}^{1/2} | \Phi |, \tag{5.1}$$

where

$$\Phi = p(t, x_2, y_2, z_2) - p(t, x_1, y_1, z_1).$$

Let $k_2$ be any constant in the range $1 \leq k_2 \leq 2$. Then, by proceeding as in [3, § 2], using (2.6), it can be shown that

$$-D_{12} W(t) + D_{13} \{ W(t) \}^{1/2} | \Phi | \leq D_{14} \varphi^{k_2}(t) W(t)$$

for some constant $D_{14}$. On combining this with (5.1), we have that

$$\dot{W}(t) + D_{12} W(t) \leq D_{14} \varphi^{k_2}(t) W(t) \tag{5.2}$$

for all $t \geq t_0$. A straightforward integration of (5.2) between $t_0$ and $t$ now yelds

$$W(t) \exp\left[D_{12}t - D_{14}\int_0^t \varphi^{k_2}(\tau)d\tau\right] \leq W(t_0) \exp\left[D_{12}t_0 - D_{14}\int_0^{t_0} \varphi^{k_2}(\tau)d\tau\right], \qquad t \geq t_0,$$

which, in view (2.7), implies that

$$W(t) \leq D_{15} W(t_0) e^{-D_{12}(t-t_0)}, \qquad t \geq t_0$$

for some constant $D_{15}$ $\Delta < D_{15} < \infty$. The proof of the theorem may now be completed by proceeding as in § 4.

**6.** With the convergence result, Theorem 1, out of the way, we are now in a position to prove the following result on the existence of a unique periodic solution of (1.1).

THEOREM 3. - *Further to the condition of Theorem* 1 *let* $p(t, x, y, z)$ *be periodic in* $t$, *with pediod* $\omega$ *uniformly with respect to* $x, y, z$. *Suppose that*

(i) *for any constant* $R$ $(0 < R < \infty)$ *the function* $H = H(R)$ *is defined by* (2.9);

(ii) *the constant* $\beta$ *appearing in Corollary* 1 *is fixed such that*

$$\beta \geq D_1 \tag{6.1}$$

*where* $D_1$ *is the constant in* (2.10).

*Then, if*

$$H(\beta) \leq B^{1/2} \tag{6.2}$$

*there exists a unique periodic solution of* (1.1) *of period* $\omega$ *provided* $\Delta_* < \varepsilon_*$.

PROOF OF THEOREM 3. - First assume that the conditions of Theorem 1 hold. Then, by specializing the result [4; Theorem 2] to the case $n = 1$, there exists a constant $\varepsilon_1^*$ $(0 < \varepsilon_1^* < \infty)$ such that if $\Delta_* < \varepsilon_1^*$, then equation (1.1) has at east one periodic solution of period $\omega$. Next, assume that the hypotheses (i) and (ii) of Theorem 3 hold and that the function $H(\beta)$ satisfies (6.2). Then, by Theorem 2, all solutions of (1.1) converge provided that $\Delta_* < \varepsilon^*$, $\varepsilon^* = (\varepsilon_1^*, \varepsilon_*)$, where $\varepsilon_*$ is given by (3.17). From these, it now follows that if $\Delta_* < \varepsilon^*$ the equation (1.1) has a unique periodic solution of period $\omega$. For otherwise, (1.1) would have at least two distinct periodic salutions $x_1(t)$, $x_2(t)$, say, of period $\omega$, satisfying

$$(6.3) \qquad x_1(t) - x_2(t) \to 0, \qquad \dot{x}_1(t) - \dot{x}_2(t) \to 0, \qquad \ddot{x}_1(t) - \ddot{x}_2(t) \to 0$$

as $t \to \infty$, and by the periodicity of $x_i(t)$, $\dot{x}_i(t)$, $\ddot{x}_i(t)$ $(i = 1, 2)$, (6.3) would imply

$$x_1(t) = x_2(t), \qquad \dot{x}_1(t) = \dot{x}_2(t), \qquad \ddot{x}_1(t) = \ddot{x}_2(t)$$

for all $t$. Thus the theorem is proved with $\varepsilon^*$ given by $\varepsilon^* = \min(\varepsilon_1^*, \varepsilon_*)$.

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