On the Convergence of the Coordinate Descent Method for Convex Differentiable Minimization*

by

Zhi-Quan Luo† and Paul Tseng‡

Abstract

The coordinate descent method enjoys a long history in convex differentiable minimization. It is simple, has a good rate of convergence, and, in certain cases, is well suited for parallel computation. Thus, it is somewhat surprising that very little is known about the convergence of the iterates generated by this method. Convergence typically require restrictive assumptions such as that the cost function has bounded level sets and is in some sense strictly convex. In a recent work, Luo and Tseng showed that the iterates are convergent for the symmetric linear complementarity problem, for which the cost function is convex quadratic but not necessarily strictly convex and does not necessarily have bounded level sets. In this paper we extend their results to problems for which the cost function is the composition of an affine mapping with a strictly convex function which is twice differentiable in its effective domain. As a consequence of this result, we obtain, for the first time, that the (dual) iterates generated by a number of existing methods for matrix balancing and entropy optimization are convergent.

KEY WORDS: Coordinate descent, convex differentiable optimization, symmetric linear complementarity problem.

1. Introduction

A very important problem in optimization is that of minimizing a convex function of the Legendre type (i.e., a function that is strictly convex differentiable on an open convex set and whose gradient tends to infinity in norm at the boundary points), subject to linear constraints. As an example, when the cost function is quadratic, this problem has applications in linear programming [Man84], [MaD88], image reconstruction [HeL78], and the solution of boundary value problems [CoG78], [CGS78], [DeT84]. When the cost function is the "$x \log(x)$" entropy function, this problem has applications in information theory [Ari72], [Bla72], matrix balancing [Kru37], [LaS81], image reconstruction [Cen88], [Len77], [Pow88], speech processing [Fri75], [Jay82], [JoS84], and statistical inference [DaR72]. As a final example, when the cost function is the "$-\log(x)$" entropy function, this problem reduces to the analytic centering problem which plays a key role in many new algorithms for linear programming [Fre88], [Kar84], [Hua67], [Son88].

A popular approach to solving the above problem is to dualize the linear constraints to obtain a dual problem of the form

$$
\begin{aligned}
&\text{Minimize} \quad g(E^T x) + \langle b, x \rangle \\
&\text{subject to} \quad x \geq 0,
\end{aligned}
\tag{1.1}
$$

where g is a strictly convex essentially smooth function, E is a matrix and b is a vector (see Section 5); and then use a coordinate descent method to solve this problem whereby, at each iteration, one of the coordinates of x is adjusted in order to minimize the cost function (while the other coordinates are held fixed). Such a method is simple, uses little storage, and, in certain cases, is highly parallelizable. Methods that follow this approach include a method of Hildreth [Hil57] (also see [CoG78], [CoP82], [Cry71], [HeL78], [Man84]) for quadratic programming, a method of Kruithof [Kru37] (also see [BaK79], [Bre67], [LaS81], [ScZ87], [Zel88] and references cited in [LaS81]) for matrix balancing, as well as a number of related methods for entropy optimization [CeL87], [Fri88; p. 236], [Len77].

An outstanding question concerns the convergence of the iterates generated by the above coordinate descent scheme. Typically, convergence requires the cost function to have bounded level sets and to be strictly convex in some sense (see for example [Aus76], [BeT89; Chap. 3.3.5], [D'Es59], [Glo84], [Lue73], [Pol71], [Pow73], [SaS73], [Zan69]), neither of which, unfortunately, holds for the cost function of (1.1) (e.g. when E has redundant rows). For (1.1), it was known, under mild restrictions on the order of coordinate relaxation, that the gradient of the cost function, evaluated at the iterates, converge [Tse88], [Tse89], [TsB87b] (also see [Bre67a], [Bre67b], [CeL87], [Hil57], [Pan84]), but it was not known if the iterates themselves converge or if they are even bounded. The only nontrivial special cases for which the iterates are known to converge, without assuming uniqueness of the optimal solution, are (i) when g is separable and E is the node-arc incidence matrix for a digraph [BHT87], and (ii) when g is a strictly convex quadratic function [LuT89].

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In this paper, we give the first result on the convergence of the iterates generated by the above coordinate descent scheme (for solving (1)). In particular, we show that the iterates converge to an optimal solution of (1.1), provided that g has a positive definite Hessian and tends to infinity at the boundary of its effective domain, and that the coordinates are relaxed in a cyclic manner. This result is rather remarkable since the optimal solution set may be unbounded and the function g may have a very complicated form. As a corollary, we establish, for the first time, the convergence of the dual iterates generated by a method of Kruithof [Kru37] and by many other methods (see Section 5). Our results are, to a certain degree, based on those given in [LuT89] for the quadratic cost case. In particular, we prove our results by approximating the cost function by its quadratic expansion at an optimal solution and then applying the proof technique in [LuT89] to the approximate problem. However, the extension is by no means simple, as it requires making an accurate estimate of the approximation error (see the proof of Lemma 9), as well as other new proof techniques.

2. Algorithm Description and a Convergence Result

Consider the following problem [compare with (1.1)]

$$\text{Minimize} \quad f(x)$$
$$\text{subject to} \quad l \leq x \leq c, \tag{P}$$

where $f : \Re^n \to (-\infty, \infty]$ is a convex function of the form

$$f(x) = g(E^T x) + \langle b, x \rangle, \tag{2.1}$$

b, l and c are n-vectors with l (c) possibly having components of the extended value $-\infty$ (∞), E is an $n \times m$ matrix having no zero row, and $g : \Re^m \to (-\infty, \infty]$ is some given convex function. In our notation, all vectors are column vectors, $\Re^n$ denotes the n-dimensional Euclidean space, $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product, and superscript T denotes transpose.

For any vector x, we will denote by x_i the i-th coordinate of x and by $[x]^+$ the orthogonal projection of x onto the feasible set

$$\mathcal{X} = \{ \ x \mid l \leq x \leq c \ \} = [l_1, c_1] \times \cdots \times [l_n, c_n],$$

i.e., $[x]^+$ is the n-vector whose i-th coordinate is $[x_i]_i^+$, where we let $[x_i]_i^+ = \max\{l_i, \min\{c_i, x_i\}\}$. For any function $h : R^k \to (-\infty, \infty]$ we will denote by $\text{dom}(h)$ the effective domain of h, i.e. $\text{dom}(h) = \{x \mid h(x) < \infty\}$ and by C_h the interior of $\text{dom}(h)$.

We make the following standing assumptions about g and (P):

<u>Assumption A</u>.

(a) $C_g \neq \emptyset$, g is strictly convex twice continuously differentiable on C_g, and $g(t) \to \infty$ as t approaches any boundary point of C_g.

(b) The set of optimal solutions for (P), denoted by $\mathcal{X}^*$ (i.e. $\mathcal{X}^* = \{x^* \in \mathcal{X} \mid f(x^*) < \infty, f(x^*) \leq f(x) \ \forall x \in \mathcal{X}\}$), is nonempty.

Part (b) of Assumption A is clearly necessary. Part (a) of Assumption A implies that (C_g, g) is, in the terminology of Rockafellar [Roc70], a <u>convex function of the Legendre type</u>. Such a function has a number of nice properties (for example, its conjugate function is also a convex function of the Legendre type). Notice that the strict convexity of g implies that the function $x \to E^T x$ is invariant over $\mathcal{X}^*$, i.e., there exists a $t^* \in \Re^m$ such that

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$$E^T x^* = t^*, \ \forall x^* \in \mathcal{X}^*. \tag{2.2}$$

To see this, note that, for any $x^* \in \mathcal{X}^*$ and $y^* \in \mathcal{X}^*$, we have by the convexity of $\mathcal{X}^*$ that $\frac{x^*+y^*}{2} \in \mathcal{X}^*$. Then, $f(x^*) = f(y^*) = f(\frac{x^*+y^*}{2})$, so that (using (2.1)) $g(\frac{E^T x^* + E^T y^*}{2}) = \frac{g(E^T x^*) + g(E^T y^*)}{2}$. Since both $g(E^T x^*)$ and $g(E^T y^*)$ are finite, so that $E^T x^* \in C_g$ and $E^T y^* \in C_g$, this together with the strict convexity of g on C_g yields $E^T x^* = E^T y^*$.

Notice that since g is differentiable on C_g, then so is f on C_f. In what follows, we will denote by $d(x)$ the gradient of f at an $x \in C_f$ and by $d_i(x)$ the i-th coordinate of $d(x)$. Then, by (2.1) and the chain rule for differentiation, we have

$$d(x) = \nabla f(x) = E \nabla g(E^T x) + b. \tag{2.3}$$

From the Kuhn–Tucker conditions for (P) it is easily seen that an x belongs to $\mathcal{X}^*$ if and only if the orthogonal projection of $x - d(x)$ onto the feasible set $\mathcal{X}$ is x itself, i.e.

$$x = [x - d(x)]^+. \tag{2.4}$$

Consider the following coordinate descent method for solving (P), whereby given an n-vector $x^r \in \mathcal{X}$ at the r-th iteration ($r = 0, 1, \ldots$ and x^0 is given), a new n-vector $x^{r+1} \in \mathcal{X}$ is generated according to the iteration:

Cyclic Coordinate Descent Iteration

For $i = 1, 2, \ldots, n$, compute x_i^{r+1} as a solution of

$$x_i^{r+1} = [x_i^{r+1} - d_i(x_1^{r+1}, \ldots, x_i^{r+1}, x_{i+1}^r, \ldots, x_n^r)]_i^+. \tag{2.5}$$

The above iteration can be seen to be a Gauss–Seidel iteration whereby the cost function f is successively minimized with respect to the coordinate x_i over $[l_i, c_i]$ (with the other coordinates held fixed) for $i = 1, 2, \ldots, n$, that is,

$$x_i^{r+1} = \arg \min_{l_i \leq x_i \leq c_i} f(x_1^{r+1}, \ldots, x_{i-1}^{r+1}, x_i, x_{i+1}^r, \ldots, x_n^r). \tag{2.6}$$

General discussions of Gauss–Seidel iterations can be found in, for example, [Aus76], [BeT89], [Lue73], [OrR70].

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We claim that (2.6) [or, equivalently, (2.5)] is well-defined. To see this, suppose the contrary, so that, for some r and i, the minimum in (2.6) is not attained. Let $\bar{x} = (x_1^{r+1}, ..., x_{i-1}^{r+1}, x_i^r, ..., x_n^r)^T$ and let e^i denote the i-th coordinate vector in $\Re^n$. Then, either (i) $l_i = -\infty$ and $f(\bar{x} - \lambda e^i)$ is monotonically decreasing with increasing λ or (ii) $c_i = \infty$ and $f(\bar{x} + \lambda e^i)$ is monotonically decreasing with increasing λ. Suppose that case (i) holds. [Case (ii) may be treated analogously.] Then, since the set $\{E^T x \mid l \leq x \leq c, f(x) \leq f(\bar{x})\}$ is bounded by Lemma A.1 in Appendix A, there holds $E^T e^i = 0$, a contradiction of the assumption that E contains no zero row.

In order to prove the convergence of the iterates x^r generated by (2.5), we make the following standing assumption (in addition to Assumption A):

<u>Assumption B.</u> $\nabla^2 g(t^*)$ is positive definite.

Assumption B states that g has a positive curvature locally around the optimal solution point t^*. This condition is guaranteed to hold if g has a positive curvature everywhere on C_g. [There are many important functions that satisfy this latter condition (in addition to Assumption A (a)), the most notable of which are the quadratic function, the exponential function, and the negative of the logarithm function. We will discuss these example functions in detail in Section 5.] Notice that if g is strongly convex and twice differentiable everywhere, then g automatically satisfies both Assumption A (a) and Assumption B.

The main result of this paper is the following:

Theorem 1. If $\{x^r\}$ is a sequence of iterates generated by (2.5), then $\{x^r\}$ converges to an element of $\mathcal{X}^*$.

The proof of Theorem 1 is quite intricate and will be given by a sequence of lemmas which we present in the following two sections.

The remainder of this paper proceeds as follows: In Section 3 we prove some preliminary convergence results for the iteration (2.5). In Section 4, we used these results to prove Theorem 1. In Section 5, we consider dual applications of Theorem 1. In Section 6 we discuss possible extensions of our work.

In what follows, $\|\cdot\|$ and $\|\cdot\|_\infty$ will denote, respectively, the L_2-norm and the L_∞-norm in some Euclidean space (i.e., $\|x\| = \sqrt{\langle x, x \rangle}$ and $\|x\|_\infty = \max_i |x_i|$). If A is a square matrix, $\|A\|$ will denote the matrix norm of A induced by the vector norm $\|\cdot\|$, i.e., $\|A\| = \max_{\|x\|=1} \|Ax\|$. For any $k \times m$ matrix A, we will denote by A_i the i-th row of A and, for any nonempty $I \subseteq \{1, ..., k\}$ and $J \subseteq \{1, ..., m\}$, by A_I the submatrix of A obtained by removing all rows $i \notin I$ of A, and by A_{IJ} the submatrix of A_I obtained by removing all columns $j \notin J$ of A_I. We will also denote by $\text{Span}(A)$ the space spanned by the columns of A. Analogously, for any k-vector x and any nonempty subset $J \subseteq \{1, ..., k\}$, we denote by x_J the vector with components x_i, $i \in J$ (with the x_i's arranged in the same order as in x). Finally, for any $J \subseteq \{1, ..., n\}$, we denote by $\bar{J}$ the complement of J with respect to $\{1, ..., n\}$.

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3. Technical Preliminaries

In this section we prove a number of useful facts about the optimal solution set and the iterates x^r generated by the cyclic coordinate descent iteration (2.5). These properties, some of which are interesting in themselves, will be used in Section 4 to prove Theorem 1.

First, since $E^T x^* = t^*$ for all $x^* \in \mathcal{X}^*$ [cf. (2.2)], we have that $d(x)$ is itself invariant over $\mathcal{X}^*$. In particular, we have from (2.3) that

$$d(x^*) = d^*, \quad \forall x^* \in \mathcal{X}^*, \tag{3.1}$$

where we let

$$d^* = E\nabla g(t^*) + b. \tag{3.2}$$

Also let

$$\mathcal{I}^* = \{i \,|\, d_i^* = 0\}.$$

Since $\nabla^2 g(t^*)$ is positive definite [cf. Assumption B], it follows from the continuity property of $\nabla^2 g$ [cf. Assumption A (a)] that $\nabla^2 g$ is positive definite in some open neighborhood of t^*. This in turn implies that g is strongly convex near t^*, i.e. there exist a positive scalar $\sigma > 0$ and an open set $\mathcal{U}^*$ containing t^* such that

$$g(z) - g(y) - \langle \nabla g(y), z - y \rangle \geq \sigma \|z - y\|^2, \quad \forall z \in \mathcal{U}^*, \ \forall y \in \mathcal{U}^*. \tag{3.3}$$

By interchanging the role of y with that of z in (3.3) and adding the resulting relation to (3.3), we also obtain

$$\langle \nabla g(z) - \nabla g(y), z - y \rangle \geq 2\sigma \|z - y\|^2, \quad \forall z \in \mathcal{U}^*, \ \forall y \in \mathcal{U}^*. \tag{3.4}$$

Next, we have the following lemma on the Lipschitz continuity of the solution of a linear system as a function of the right hand side [Hof52] (also see [Rob73], [MaS87]):

Lemma 1. Let B be any $k \times n$ matrix. Then, there exists a constant $\theta > 0$ depending on B only such that, for any $\bar{x} \in \mathcal{X}$ and any k-vector d such that the linear system $By = d, \ y \in \mathcal{X}$ is consistent, there is a point $\bar{y}$ satisfying $B\bar{y} = d, \ \bar{y} \in \mathcal{X}$, with

$$\|\bar{x} - \bar{y}\| \leq \theta \|B\bar{x} - d\|.$$

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Lemma 1 will be used extensively in the proof of Lemmas 2 and 6 to follow.

Let $\{x^r\}$ be a sequence of iterates generated by (2.5). Also let

$$x^{r,i} = (x_1^{r+1}, ..., x_i^{r+1}, x_{i+1}^r, ..., x_n^r)^T,\tag{3.5}$$

for all r and all $i \in \{0, 1, ..., n\}$. Notice that $x^{r,n} = x^{r+1}$ and $x^{r,0} = x^r$. In the remainder of this section, we will prove various convergence properties of $\{x^r\}$ and $\{x^{r,i}\}$.

First, by using Lemma 1, we have the following result on the convergence of $\{E^T x^{r,i}\}$.

Lemma 2. For every $i \in \{0, 1, ..., n\}$, there holds

$$\{E^T x^{r,i}\} \to t^*.$$

The proof of Lemma 2, which is quite technical, is given in Appendix A. The proof is a shortened version of one given in [Tse89], specialized to the iteration (2.5) (cf. proof of Proposition 1 in [Tse89]).

As a corollary of Lemma 2, we obtain [cf. (2.3), (3.2) and the continuity of ∇g at t^*]

$$\{d(x^{r,i})\} \to d^*, \quad \forall i.\tag{3.6}$$

Notice that Lemma 2 shows that if E has full row rank so that the optimal solution of (P) is unique, then $\{x^{r,i}\}$ converges to this optimal solution for all i. However, for most practical problems, the matrix E does not have full row rank, in which case, as we shall see, proving convergence is much more difficult.

In addition to Lemma 2, we have the following result that states that $\{x^r - x^{r+1}\} \to 0$ "sufficiently fast."

Lemma 3. $\sum_{r=0}^\infty \|x^r - x^{r+1}\|^2 < \infty$.

Proof: First we show that

$$f(x^r) - f(x^{r+1}) \geq \sigma \min_j\{\|E_j\|^2\}\|x^r - x^{r+1}\|^2,\tag{3.7}$$

for all r sufficiently large. By Lemma 2 we have $\{E^T x^{r,i}\} \to t^*$ for all i, so that for all r sufficiently large there holds [cf. (3.3)] $E^T x^{r,i} \in \mathcal{U}^*$ for all i. Consider any such r. For every $i \in \{1, ..., n\}$, since $x^{r,i}$ is obtained from $x^{r,i-1}$ by minimizing f along the i-th coordinate [cf. (2.6)], there holds

$$\langle d(x^{r,i}), x^{r,i} - x^{r,i-1} \rangle \leq 0.$$

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Hence, by using (2.1) and (2.3), we have

$$f(x^{r,i-1}) - f(x^{r,i}) \geq f(x^{r,i-1}) - f(x^{r,i}) + \langle d(x^{r,i}), x^{r,i} - x^{r,i-1} \rangle$$

$$= g(E^T x^{r,i-1}) - g(E^T x^{r,i}) - \langle \nabla g(E^T x^{r,i}), E^T(x^{r,i-1} - x^{r,i}) \rangle$$

$$\geq \sigma \| E^T(x^{r,i-1} - x^{r,i}) \|^2$$

$$= \sigma \| E_i \|^2 | x_i^r - x_i^{r+1} |^2$$

$$\geq \sigma \min_j \{ \| E_j \|^2 \} | x_i^r - x_i^{r+1} |^2,$$

where the second inequality follows from $E^T x^{r,i} \in \mathcal{U}^*$, $E^T x^{r,i-1} \in \mathcal{U}^*$ and (3.3); the last equality follows from the observation that $x^{r,i}$ and $x^{r,i-1}$ differ only in their i-th coordinate and this difference is exactly $x_i^{r+1} - x_i^r$. Summing the above inequality over all $i \in \{1, ...n\}$ yields (3.7).

By summing (3.7) over all r and using the fact that $\{f(x^r)\}$ is bounded from below [cf. Assumption A (b)], we obtain $\infty > \sigma \min_j \{ \| E_j \|^2 \} \sum_{r=0}^{\infty} \| x^r - x^{r+1} \|^2$. **Q.E.D.**

Lemma 3 combined with Lemma 2 yield the following result:

Lemma 4. The following hold:

(a) $\{ x^{r+1} - x^r \} \to 0$.

(b) For all r sufficiently large, there holds $x_i^r = l_i$ for all i with $d_i^* > 0$ and $x_i^r = c_i$ for all i with $d_i^* < 0$.

(b) $\{ x^r - [x^r - d(x^r)]^+ \} \to 0$.

Proof: Part (a) follows from Lemma 3. To see part (b), note from (3.6) that if $d_i^* > 0$, then for all r sufficiently large there holds $d_i(x^{r,i}) > 0$, which together with the fact $x_i^{r+1} = [x_i^{r+1} - d_i(x^{r,i})]_i^+$ [cf. (2.5), (3.5)] yields $x_i^{r+1} = l_i > -\infty$. A symmetric argument shows that if $d_i^* < 0$, then $x_i^{r+1} = c_i < \infty$ for all r sufficiently large. To see part (c), we note from part (b) and the fact $\{d(x^r)\} \to d^*$ that if $d_i^* > 0$, then, for all r sufficiently large, there holds $d_i(x^r) > 0$ and $x_i^r = l_i$, so that $[x_i^r - d_i(x^r)]_i^+ = l_i = x_i^r$. A symmetric argument proves the same relation for the case when $d_i^* < 0$. If $d_i^* = 0$, then $\{d_i(x^r)\} \to 0$ and we obtain $|x_i^r - [x_i^r - d_i(x^r)]_i^+| \leq |d_i(x^r)| \to 0$, where the inequality follows from the nonexpansive property of the projection operator $[\cdot]_i^+$. **Q.E.D.**

For each $x \in \Re^n$, let $\phi(x)$ denote the distance from x to $\mathcal{X}^*$, i.e.

$$\phi(x) = \min_{x^* \in \mathcal{X}^*} \| x - x^* \|.$$

The next lemma, which shows that $\{x^r\}$ approaches $\mathcal{X}^*$, follows as a consequence of Lemmas 2 and 4 (b).

Lemma 5. $\{ \phi(x^r) \} \to 0$.

Proof: By Lemma 4 (b), for all r sufficiently large, x^r satisfies

$$x_i^r = l_i \text{ if } d_i^* > 0, \quad x_i^r = c_i \text{ if } d_i^* < 0, \quad x^r \in \mathcal{X}.$$

From (2.2), (2.4), (3.1) and (3.2) we also have that $\mathcal{X}^*$ is the solution set of the nonlinear system $E^T x = t^*, x = [x - d^*]^+$ or, equivalently, the solution set of the <u>linear</u> system

$$E^T x = t^*, \quad x_i = l_i \text{ if } d_i^* > 0, \quad x_i = c_i \text{ if } d_i^* < 0, \quad x \in \mathcal{X}. \tag{3.8}$$

By applying Lemma 1 to the above linear system and x^r, we obtain that there exists, for each r sufficiently large, a $y^r \in \mathcal{X}^*$ satisfying

$$\|x^r - y^r\| \le \theta \|E^T x^r - t^*\|,$$

where θ is some constant that depends on E only. Since $y^r \in \mathcal{X}^*$, this in turn implies $\phi(x^r) \le \theta \|E^T x^r - t^*\|$. Our claim then follows from Lemma 2. **Q.E.D.**

Lemma 5 shows that the iterates x^r approach the solution set $\mathcal{X}^*$. [This however does not imply that $\{x^r\}$ converges or is even bounded since $\mathcal{X}^*$ may be unbounded.] The following result, based on the local strong convexity property of g [cf. (3.3)], shows that the distance from x^r to $\mathcal{X}^*$ is upper bounded by some constant times $\|x^{r+1} - x^r\|$. This is a crucial step which enables us to bound the error in approximating g by its quadratic expansion around t^* in Section 4.

Lemma 6. There exists a constant $\omega > 0$ such that $\|E^T x^r - t^*\| \le \omega \|x^{r+1} - x^r\|$ for all r.

The proof of Lemma 6, which is quite technical, is given in Appendix B.

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4. Convergence Analysis

In this section we use the results developed in Section 3 to prove Theorem 1, i.e., the iterates x^r generated by (2.5) converge to an optimal solution of (P). The proof is based on taking a quadratic approximation of g at t^* and then applying the proof technique developed in [LuT89] for the quadratic cost case to this approximate problem. We show that the error in taking the quadratic approximation is small enough so that it does not affect the convergence of the iterates.

Let M denote the Hessian of f evaluated at any $x^* \in \mathcal{X}^*$. By (2.1) and (2.2), we see that

$$M = \nabla^2 f(x^*) = E\nabla^2 g(t^*)E^T, \tag{4.1}$$

so that M is independent of x^*. We will denote by m_{ij} the (i,j)-th entry of M. Notice that since M is the Hessian of a convex function, M is symmetric positive semi-definite. Moreover, since E has no zero row and $\nabla^2 g(t^*)$ is positive definite (cf. Assumption B), there holds $m_{ii} > 0$ for all i.

By exploiting the symmetric positive semi-definite property of M, we have the following lemma based on Lemma 5 in [LuT89]:

Lemma 7. For any $J \subseteq \mathcal{I}^*$, there holds $\mathrm{Span}(M_{Jj}) \subseteq \mathrm{Span}(M_{JJ})$.

Proof: For each $i \notin J$, consider the convex quadratic program

$$\text{Minimize} \quad \langle x, Mx \rangle$$

$$\text{subject to} \quad x_i = 1, x_j = 0 \quad \forall j \notin J \text{ with } j \neq i.$$

This problem is clearly feasible and, since M is positive semi-definite, its optimal value is finite. Then, since we are dealing with a convex quadratic program, we have that the Kuhn–Tucker conditions hold, from which we find $M_{Ji} \in \mathrm{Span}(M_{JJ})$. Since the choice of $i \notin J$ was arbitrary, this implies $\mathrm{Span}(M_{Jj}) \subseteq \mathrm{Span}(M_{JJ})$. **Q.E.D.**

Let B denote the lower triangular portion of M (i.e., the (i,j)-th entry of B is m_{ij} if $i \geq j$ and is 0 otherwise) and let $C = M - B$ (so that C is the strictly upper triangular portion of M). We claim that $B - C$ is positive <u>definite</u>. [To see this, note that since M is symmetric, we have $B = D + C^T$, where D is the $n \times n$ diagonal matrix whose i-th diagonal entry is $m_{ii} > 0$, so that $\langle x, (B - C)x \rangle = \langle x, Dx \rangle > 0$ for all $x \neq 0$.] Then, since (B, C) is a <u>splitting</u> of M (see [OrR70]), i.e.,

$$M = B + C,$$

we conclude from Lemma 4 in [LuT89] that B and C have the following contraction properties:

Lemma 8. The following hold:

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(a) For any nonempty $J \subseteq \{1, ..., n\}$, there exist $\rho_J \in (0, 1)$ and $\tau_J > 0$ such that

$$\|(I - M_{JJ}(B_{JJ})^{-1})^k z\| \leq \tau_J (\rho_J)^k \|z\|, \quad \forall k \geq 1, \ \forall z \in \text{Span}(M_{JJ}).$$

(b) There exists a $\Delta \geq 1$ such that, for any nonempty $J \subseteq \{1, ..., n\}$,

$$\|(I - (B_{JJ})^{-1} M_{JJ})^k z\| \leq \Delta \|z\|, \quad \forall k \geq 1, \ \forall z.$$

Let

$$\beta = \max_{J \subseteq \mathcal{I}^*} \sqrt{\text{Card}(\tilde{J})} \{ (\tau_J \|(B_{JJ})^{-1}\| \|M_{JJ}\| / (1 - \rho_J) + \Delta + 1) \|(B_{JJ})^{-1} B_{J\tilde{J}}\|$$
$$+ \tau_J \|(B_{JJ})^{-1}\| \|M_{J\tilde{J}}\| / (1 - \rho_J) \},$$

where $\text{Card}(\tilde{J})$ denotes the cardinality of $\tilde{J}$. The following lemma, based on Lemmas 6, 7 and 8, shows that those coordinates of x^r that stay away from the boundary are influenced by the remaining coordinates only through the distance, scaled by β, of these remaining coordinates from the boundary. This result allows us to separate the effect of these two sets of coordinates on each other.

Lemma 9. Consider any $J \subseteq \mathcal{I}^*$. If for some two integers $s \geq t \geq 0$ we have $l_i < x_i^r < c_i$ for all $r = t + 1, t + 2, ..., s$, then, for any $x^* \in \mathcal{X}^*$, there holds

$$\|x_J^s - x_J^*\| \leq \Delta \|x_J^t - x_J^*\| + \beta \max_{r \in \{t, ..., s\}} \|x_{\tilde{J}}^r - x_{\tilde{J}}^*\|_\infty + \mu \sum_{r=t}^{s-1} \|x^r - x^{r+1}\|^2,$$

where μ is some positive constant which is independent of s and t.

Proof: The claim clearly holds if $s = t$ (since $\Delta \geq 1$). Suppose that $s > t$. Fix any $r \in \{t, ..., s-1\}$ and any $i \in J$. Since $l_i < x_i^{r+1} < c_i$, it follows from the fact [cf. (2.5), (3.5)] $x_i^{r+1} = [x_i^{r+1} - d_i(x^{r,i})]_i^+$ that $d_i(x^{r,i}) = 0$. Since $i \in \mathcal{I}^*$ so that $d_i(x^*) = 0$, this implies

$$0 = d_i(x^{r,i}) - d_i(x^*)$$
$$= E_i \left(\nabla g(E^T x^{r,i}) - \nabla g(E^T x^*) \right)$$
$$= E_i \left(\nabla^2 g(E^T x^*)(E^T x^{r,i} - E^T x^*) \right) + O(\|E^T x^{r,i} - E^T x^*\|^2).$$

Using the triangle inequality $\|E^T(x^{r,i} - x^*)\| \leq \|E^T(x^{r,i} - x^r)\| + \|E^T(x^r - x^*)\|$ and the fact [cf. Lemma 6 and (2.2)] that $\|E^T(x^r - x^*)\| \leq \omega \|x^r - x^{r+1}\|$, we see that the last term in the above relation is of the order $\|x^r - x^{r+1}\|^2$. Thus, we obtain, by using the fact that the j-th component of $E_i \nabla^2 g(E^T x^*) E^T$ is m_{ij} [cf. (4.1)], that the above expression can be written as

$$0 = \sum_{j \leq i} m_{ij}(x_j^{r+1} - x_j^*) + \sum_{j > i} m_{ij}(x_j^r - x_j^*) + O(\|x^r - x^{r+1}\|^2).$$

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Since our choice of i was arbitrary, the above holds for all $i \in J$, so that in matrix form they can be expressed as

$$0 = B_J(x^{r+1} - x^*) + C_J(x^r - x^*) + w_J^r \|x^r - x^{r+1}\|^2,$$

where w_J^r is some vector whose norm is upper bounded by a constant which is independent of r.

Since $B - C$ is positive definite, it follows from $2B = M + (B - C)$ (cf. $M = B + C$) and the positive semi-definite property of M that B is also positive definite. Hence, B_{JJ} is invertible and, by rearranging terms in the above expression and using $C = M - B$ (also see the proof of Lemma 8 in [LuT89]), we obtain

$$x_J^{r+1} - x_J^* = (I - (B_{JJ})^{-1} M_{JJ})(x_J^r - x_J^*) - (B_{JJ})^{-1} M_{J\bar{J}}(x_{\bar{J}}^r - x_{\bar{J}}^*)$$
$$- (B_{JJ})^{-1} B_{J\bar{J}}(x_{\bar{J}}^{r+1} - x_{\bar{J}}^r) - (B_{JJ})^{-1} w_J^r \|x^r - x^{r+1}\|^2.$$

Since the choice of r was arbitrary, the above relation holds for all $r \in \{t, ..., s-1\}$, so that, by using Lemmas 7 and 8 and an argument analogous to that used in the proof of Lemma 8 in [LuT89], we obtain

$$\|x_J^s - x_J^*\| \le \Delta \|x_J^t - x_J^*\| + \beta \max_{r \in \{t,...,s\}} \|x_{\bar{J}}^r - x_{\bar{J}}^*\|_\infty + \sum_{r=t}^{s-1} \|(I - (B_{JJ})^{-1} M_{JJ})^{s-r-1} (B_{JJ})^{-1} w_J^r\| \|x^r - x^{r+1}\|^2.$$

Since $\|(I - (B_{JJ})^{-1} M_{JJ})^k (B_{JJ})^{-1} w_J^r\| \le \Delta \|(B_{JJ})^{-1} w_J^r\| \le \Delta \|(B_{JJ})^{-1}\| \cdot \|w_J^r\|$ for all $k \ge 1$ [cf. Lemma 8 (b)], this, together with the fact that $\|w_J^r\|$ is bounded from above by a constant for all r, proves our claim. **Q.E.D.**

By using Lemmas 3, 4, 5 and 9, we can now prove Theorem 1, the convergence of $\{x^r\}$. The idea of the proof is identical to that used in the proof of Lemma 9 in [LuT89], that is, to show that those coordinates of x^r that are bounded sufficiently far away from the boundary are essentially unaffected by the rest. This then allows us to treat these coordinates as if they are unconstrained and, by using a certain contraction property of the algorithmic mapping [cf. Lemma 8 (a)], to conclude convergence for these coordinates. Some modifications, albeit small, need to be made to the original proof in [LuT89] to account for the extra error term $\sum_{r=t}^{\infty} \|x^r - x^{r+1}\|^2$ (compare Lemma 9 with Lemma 8 in [LuT89]). We define the following scalars for the subsequent analysis:

$$\begin{aligned} \sigma_0 &= 1, \\ \sigma_k &= \Delta + 3 + \beta + (\beta + 1)\sigma_{k-1} + \mu, \quad k = 1, 2, ..., n. \end{aligned} \tag{4.2}$$

(Notice that $\sigma_k \ge 1$ for all k and is monotonically increasing with k.)

Lemma 10. For any $\delta > 0$, there exists an $x^* \in \mathcal{X}^*$ and an $\hat{r} > 0$ such that

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$$\|x^r - x^*\| \leq \sigma_n \delta + \delta, \quad \forall r \geq \hat{r}.$$

Proof: To simplify the proof, we will assume that $c_i = \infty$ for all i. The case where $c_i < \infty$ for some i can be handled by making a symmetric argument. Furthermore, by using Lemmas 3, 4 (a) and 5, we will, by taking r sufficiently large if necessary, assume that

$$\phi(x^r) \leq \delta, \quad \forall r, \tag{4.3a}$$

$$\|x^{r+1} - x^r\| \leq \delta, \quad \forall r, \tag{4.3b}$$

$$\sum_{k=r}^{\infty} \|x^k - x^{k+1}\|^2 \leq \delta, \quad \forall r. \tag{4.3c}$$

We first have the following lemma which states that Lemma 10 holds in the special case where the coordinates that start near the boundary of $\mathcal{X}$ remain near the boundary (also assuming that the remaining coordinates start far from the boundary).

Lemma 11. Fix any $k \in \{1, ..., n\}$. If for some nonempty $J \subseteq \mathcal{I}^*$ and some two integers $t' > t$ we have

$$x_i^t > l_i + \sigma_k \delta, \quad \forall i \in J, \tag{4.4}$$

$$x_i^r \leq l_i + \sigma_{k-1} \delta, \quad \forall i \notin J, \quad \forall r = t, t+1, \ldots, t'-1, \tag{4.5}$$

then the following hold:

(a) $x_i^{t'} > l_i + \sigma_{k-1} \delta, \forall i \in J$.

(b) There exists an $x^* \in \mathcal{X}^*$ such that

$$\|x^r - x^*\|_\infty \leq \sigma_k \delta, \quad \forall r = t, t+1, \ldots, t'-1.$$

Proof: let x^* be any element of $\mathcal{X}^*$ satisfying $\phi(x^t) = \|x^t - x^*\|$. Then, we have from (4.3a) that

$$\|x^t - x^*\| \leq \delta. \tag{4.6}$$

Also, we have from (4.5) that, for all $i \notin J$, $x_i^* \leq x_i^t + \|x^t - x^*\| \leq l_i + \sigma_{k-1} \delta + \|x^t - x^*\|$, which together with (4.6) and the fact $x^* \geq l$ implies that $l_i \leq x_i^* \leq l_i + \sigma_{k-1} \delta + \delta$. Since $l_i \leq x_i^r \leq l_i + \sigma_{k-1} \delta$ for all $r = t, t+1, \ldots, t'-1$ [cf. (4.5)], this in turn implies that

$$|x_i^r - x_i^*| \leq \sigma_{k-1} \delta + \delta, \quad \forall i \notin J, \forall r = t, t+1, \ldots, t'-1. \tag{4.7}$$

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Next we prove by induction that, for $r = t, t+1, ..., t'-1$, there holds

$$x_i^r > l_i + \sigma_{k-1}\delta + \delta, \qquad \forall i \in J. \tag{4.8}$$

Eq. (4.8) clearly holds for $r = t$ [cf. (4.4) and $\sigma_k \geq \sigma_{k-1} + 1$]. Suppose that (4.8) holds for $r = t, t+1, ..., s$, for some $s \in \{t, t+1, ..., t'-2\}$. We will prove that it also holds for $r = s+1$. Since $l_i < x_i^r < c_i$ for all $i \in J$ and all $r = t+1, ..., s$ [cf. (4.8) and $c_i = \infty$ for all i], we have from Lemma 9 that

$$\|x_J^s - x_J^*\| \leq \Delta\|x_J^t - x_J^*\| + \beta \max_{r \in \{t,...,s\}} \|x_J^r - x_J^*\|_\infty + \mu \sum_{r=t}^{s-1} \|x^r - x^{r+1}\|^2,$$

which together with (4.3c), (4.6) and (4.7) implies

$$\|x_J^s - x_J^*\| \leq \Delta\delta + \beta(\sigma_{k-1}\delta + \delta) + \mu\delta. \tag{4.9}$$

Then, we have that, for any $i \in J$,

$$\begin{aligned}
x_i^{s+1} &\geq x_i^t - \|x_J^t - x_J^{s+1}\| \\
&\geq x_i^t - (\|x_J^t - x_J^*\| + \|x_J^* - x_J^s\| + \|x_J^s - x_J^{s+1}\|) \\
&> l_i + \sigma_k\delta - (\delta + \|x_J^* - x_J^s\| + \delta) \\
&\geq l_i + \sigma_k\delta - (\delta + (\Delta\delta + \beta\sigma_{k-1}\delta + \beta\delta + \mu\delta) + \delta) \\
&= l_i + \sigma_{k-1}\delta + \delta,
\end{aligned}$$

where the strict inequality follows from (4.3b), (4.4), (4.6), and the equality follows from (4.2). This completes the induction and proves that (4.8) holds for $r = t, t+1, ..., t'-1$. Since (4.8) holds for $r = t, t+1, ..., t'-1$, it can be seen from the arguments above that (4.9) holds for $s = t, t+1, ..., t'-1$, which when combined with (4.7) (and using the facts $\beta > 1$ and $\|z\|_\infty \leq \|z\|$ for all z) yields

$$\|x^r - x^*\|_\infty \leq (\Delta + \beta\sigma_{k-1} + \beta + \mu)\delta, \quad \forall r = t, t+1, ..., t'-1.$$

Since $\Delta + \mu + \beta\sigma_{k-1} + \beta \leq \sigma_k$ [cf. (4.2)], this proves part (b). From (4.8) with $r = t'-1$ we have that, for all $i \in J$,

$$\begin{aligned}
x_i^{t'} &\geq x_i^{t'-1} - \|x^{t'-1} - x^{t'}\| \\
&> l_i + \sigma_{k-1}\delta + \delta - \|x^{t'-1} - x^{t'}\|.
\end{aligned}$$

Since $\|x^{t'-1} - x^{t'}\| \leq \delta$ [cf. (4.3b)], this proves part (a). **Q.E.D.**

The remainder of the proof follows the proof of Lemma 9 in [LuT89] (cf. Lemma 11 in [LuT89] and the argument that follows it) and for brevity is omitted here. [The proof in [LuT89] considers the special case where $l_i = 0$ for all i, but the argument used therein readily extends to arbitrary l_i's.] **Q.E.D.**

Now we are coming to the end of our proof of Theorem 1. By Lemma 10, for any $\epsilon > 0$, there exist an $x^* \in \mathcal{X}^*$ and an $\hat{r} > 0$ such that

$$\|x^r - x^*\|_\infty < \epsilon/2, \quad \forall r \geq \hat{r}.$$

Hence, for all r_1, $r_2 > \hat{r}$, there holds

$$\|x^{r_1} - x^{r_2}\|_\infty \leq \|x^{r_1} - x^*\|_\infty + \|x^* - x^{r_2}\|_\infty$$
$$\leq \epsilon/2 + \epsilon/2 = \epsilon.$$

This implies that $\{x^r\}$ is a Cauchy sequence so that it converges. By Lemma 5, it converges to an element of $\mathcal{X}^*$. This completes the proof of Theorem 1.

5. Dual Applications to Quadratic Programming and Entropy Optimization

As we noted in Section 1, an important application of the coordinate descent method is to the solution of problems with strictly convex costs and linear constraints (see for example [BaK79], [BHT87], [Bre67b], [Cen88], [CoP82], [DaR72], [Fri75], [LaS81], [LiP87], [ScZ87], [Tse88], [TsB87a]). In this section, we consider a number of such problems, including those that arise in matrix balancing and, more generally, in entropy optimization. By using Theorem 1, we establish, for the first time, the convergence of the (dual) iterates generated by a number of known methods for solving these problems.

Consider the following convex program

$$\text{Minimize} \quad h(y)$$
$$\text{subject to} \quad Ey \geq b,$$

$$(5.1)$$

where $h: \Re^m \to (-\infty, \infty]$ is a convex function, E is an $n \times m$ matrix having no zero row, and b is an n–vector. (We remark that our results easily extend to problems with both linear equality and inequality constraints.) We make the following standing assumptions about h and (5.1):

Assumption C.

(a) The conjugate function of h [Roc70] given by

$$h^*(t) = \sup_y \{\langle t, y \rangle - h(y)\}$$

satisfies (i) $C_{h^*} \neq \emptyset$, (ii) h^* is strictly convex twice continuously differentiable on C_{h^*}, (iii) $h^*(t) \to \infty$ as t approaches a boundary point of C_{h^*}, and (iv) $\nabla^2 h^*(t)$ is positive definite for all t in C_{h^*}.

(b) (5.1) has an optimal solution.

(c) C_h intersects $\{y | Ey \geq b\}$.

Notice that part (a) of Assumption C implies that (C_{h^*}, h^*) is a convex function of the Legendre type, so that, by Theorem 26.5 in [Roc70], (C_h, h) must also be a convex function of the Legendre type. Part (c) of Assumption C is a constraint qualification condition that ensures the existence of an optimal Lagrange multiplier vector associated with the constraints $Ey \geq b$.

By attaching a non-negative Lagrange multiplier vector p to the constraints $Ey \geq b$ in (5.1), we obtain the following dual functional

$$q(p) = \min_y \{h(y) + \langle p, b - Ey \rangle\}$$
$$= -h^*(E^T p) + \langle b, p \rangle.$$

The dual problem is then to maximize $q(p)$ subject to $p \geq 0$ (see [Roc70, Chap. 28]) or, equivalently,

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$$\text{Minimize} \quad h^*(E^T p) - \langle b, p \rangle$$
$$\text{subject to} \quad p \geq 0.$$

(5.2)

The problem (5.2) is clearly of the form (P). Moreover, since (5.1) has an optimal solution and C_h intersects $\{y | Ey \geq b\}$ [cf. parts (b) and (c) of Assumption C], it follows from Theorem 28.2 in [Roc70] that there exists a Kuhn–Tucker vector associated with the constraints $Ey \geq b$. By Corollary 28.4.1 in [Roc70], this vector is also an optimal solution of the dual problem (5.2) and therefore (5.2) has an optimal solution. This together with part (a) of Assumption C implies that h^* and (5.2) satisfy Assumptions A and B, so that, by Theorem 1, the iteration (2.5) applied to (5.2) generates iterates that converge to an optimal solution of (5.2). The optimal solution of (5.1), which is unique since h is strictly convex, can be recovered by using the fact $\nabla h^*(t) = \arg\max_y \{\langle t, y \rangle - h(y)\}$ for all t (see [Roc70, Theorem 23.5]) so that, for any optimal solution p^* of (5.2), $\nabla h^*(E^T p^*)$ is the optimal solution of (5.1).

Below we apply the above convergence result to a number of known methods based on solving the dual program (5.2) using cyclic coordinate descent.

First, consider the special case of (5.1) where h is a strictly convex quadratic function, i.e.,

$$h(y) = \langle y, Qy \rangle / 2 + \langle q, y \rangle,$$

where Q is an $m \times m$ symmetric positive definite matrix and q is an m–vector. For the above h, its conjugate function can be verified to be $h^*(t) = \langle t - q, Q^{-1}(t - q) \rangle / 2$, which clearly satisfies conditions (i)–(iv) of Assumption C (a). If in addition the set $\{y | Ey \geq b\}$ is nonempty, then parts (b) and (c) of Assumption C also hold [(5.1) has an optimal solution since it is feasible and h, being in fact strongly convex, has bounded level sets] and we can conclude that the iterates generated by applying (2.5) to solve this special case of (5.2) converge to an optimal solution of the problem. (This special case has been treated earlier in [LuT89] in the more general setting of a matrix splitting algorithm.)

Next, consider the special case of (5.1) where h is the "$-\log(y)$" entropy function, i.e.,

$$h(y) = \begin{cases} -\sum_{j=1}^{m} \log(y_j) & \text{if } y > 0; \\ \infty & \text{otherwise.} \end{cases}$$

(We can also allow positive weights on the "$\log(y_j)$" terms.) In this case, the conjugate function of h can be verified to be

$$h^*(t) = \begin{cases} -\sum_{j=1}^{m} \log(-t_j) & \text{if } t < 0; \\ \infty & \text{otherwise,} \end{cases}$$

which clearly satisfies conditions (i)–(iv) of Assumption C (a). If in addition, the set $\{y | Ey \geq b, y > 0\}$ is nonempty and bounded[1], then it can be shown that parts (b) and (c) of Assumption C also hold and,

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once again, we can conclude that the iterates generated by applying (2.5) to solve this special case of (5.2) converge to an optimal solution of the problem. This result significantly improves upon that obtained in [CeL87] which only considered the equality constrained case and only showed that the iterates, multiplied by E^T, converge (also see Lemma 2).

Finally, consider the special case of (5.1) where h is the "$y \log(y)$" entropy function, i.e.,

$$h(y) = \begin{cases} \sum_{j=1}^{m} y_j \log(y_j) & \text{if } y \geq 0; \\ \infty & \text{otherwise.} \end{cases}$$

In this case, the conjugate function of h can be verified to be the exponential function $h^*(t) = \sum_{j=1}^{m} e^{t_j - 1}$ which clearly satisfies conditions (i)-(iv) of Assumption C (a). If in addition the set $\{y | Ey \geq b, y > 0\}$ is nonempty, then since h has bounded level sets, it can be seen that parts (b) and (c) of Assumption C also hold and, once again, we can conclude that the iterates generated by applying (2.5) to solve this special case of (5.2) converge to an optimal solution of the problem. As a corollary, we obtain the convergence of the dual iterates generated by the very popular matrix balancing method of Kruithof [Kru37] (also see [BaK79], [BeT89, p. 408], [Bre67a], [ScZ87], [ZeI88]), which is effectively (2.5) applied to the special case of (5.2) where the non-negativity constraints $p \geq 0$ are removed (i.e., the primal problem is an equality constrained problem) and E is the node-arc incidence matrix for a bipartite graph. Our convergence result for this method significantly improves upon those obtained previously (see [Bre67a], [LaS81]), which only showed that the iterates, multiplied by E^T, converge.

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[1] It is easily seen that the boundedness of the set $\{y | Ey \geq b, y > 0\}$ is necessary to ensure that the primal problem has a finite optimal value.

6. Extensions

We remark that Theorem 1 still holds if the condition "$g(t) \to \infty$ as t approaches any boundary point of C'_g" in Assumption A (a) is replaced by the weaker condition that

$$\left\{ \frac{g(\bar{t}) - g(t)}{\|\bar{t} - t\|} \right\} \to \infty, \tag{6.1}$$

as t approaches any boundary point $\bar{t}$ of C_g from inside C_g. (It can be verified that all of our arguments go through under this weaker assumption except for the proof of Lemma 2. That Lemma 2 also holds is a consequence of Proposition 1 in [Tse89].) Functions that satisfy this weaker version of Assumption A (a) include all <u>separable</u> convex functions of the Legendre type, i.e., functions g of the form

$$g(t) = \sum_{j=1}^{m} g_j(t_j), \tag{6.2}$$

where $g_j : \Re \to (-\infty, \infty]$ and (C_{g_j}, g_j) is a convex function of the Legendre type. Hence, our results apply to all functions g of the form (6.2) where each $g_j : \Re \to (-\infty, \infty]$ is strictly convex, twice differentiable on C'_{g_j} and satisfies $\nabla^2 g_j(t_j) > 0$ for all $t_j \in C'_{g_j}$, $|\nabla g_j(t_j)| \to \infty$ as t_j approaches a boundary point of C'_{g_j}. A concrete example of such a g is the negative square root function

$$g(t) = \begin{cases} -\sum_{j=1}^{m} \sqrt{t_j} & \text{if } t \geq 0; \\ \infty & \text{otherwise}, \end{cases}$$

which arises in the dual of certain routing problems [BeG87, Chap. 5] and of certain resource allocation problems [MSTW88]. Another example is the "$y \log(y)$" entropy function discussed in Section 5. (Notice that both these example functions are finite at the boundary of their respective effective domains, so that neither satisfies Assumption A (a).)

It can also be verified that Theorem 1 holds if each x_i comprises, instead of a single coordinate, a <u>block</u> of coordinates, provided that the rows of E corresponding to the coordinates in each block are independent.

Finally, we remark that an extension of the iteration (2.5) to allow under/over-relaxation of the coordinates is also possible. In particular, consider the following iteration

$$x^{r+1} = \omega^r x^r + (1 - \omega^r) \tilde{x}^r,$$

where $\tilde{x}^r$ is the n-vector obtained by applying (2.5) to x^r, i.e.,

$$\tilde{x}_i^r = \left[\tilde{x}_i^r - d_i(\tilde{x}_1^r, \ldots, \tilde{x}_i^r, x_{i+1}^r, \ldots, x_n^r) \right]_i^+, \qquad i = 1, \ldots, n,$$

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and $\omega^r > 0$ is some relaxation factor. [If $\omega^r = 1$, then the above iteration reduces to (2.5).] Under suitable restrictions on the ω^r's so that a "sufficient" decrease in the objective value is achieved at every iteration, it can be shown that the iterates generated according to the above iteration still converge to an optimal solution of (P) (cf. Theorem 1).

Appendix A. Proof of Lemma 2

First, from the strict convexity of g, we can prove the following boundedness property of the level sets of f:

Lemma A.1. $\{E^T x | x \in \mathcal{X}, f(x) \leq \zeta\}$ is bounded for all $\zeta \in \Re$.

Proof: We will argue by contradiction. Suppose that the claim does not hold so that the convex set $\{(t, x, \zeta) | t = E^T x, x \in \mathcal{X}, f(x) \leq \zeta\}$ in $\Re^{m+n+1}$ has a direction of recession $(v, u, 0)$ satisfying $v \neq 0$ (see [Roc70, Theorem 8.3]). Then, $v = E^T u$ and, for any $x \in \mathcal{X}$, there holds $x + \lambda u \in \mathcal{X}$ and $f(x + \lambda u) \leq f(x)$ for all $\lambda \geq 0$. Choose x to be an element of $\mathcal{X}^*$. Then, there furthermore holds $f(x + \lambda u) = f(x)$ for all $\lambda \geq 0$, so that $g(E^T x + \lambda v) + \langle b, x + \lambda u \rangle = g(E^T x) + \langle b, x \rangle$ or, equivalently, $g(E^T x + \lambda v) = g(E^T x) - \lambda \langle b, u \rangle$ for all $\lambda \geq 0$. Also, since x is by choice an optimal solution of (P), then $E^T x$ is in C_g. Since $v \neq 0$, we see that the relation $g(E^T x + \lambda v) = g(E^T x) - \lambda \langle b, u \rangle$ contradicts the strict convexity of g on C_g [cf. Assumption A (a)]. **Q.E.D.**

Let $\{x^r\}$ be a sequence of iterates generated by (2.5). Let $x^{r,i}$ be given by (3.5) and let

$$t^{r,i} = E^T x^{r,i},$$

for all r and all $i \in \{0, 1, ..., n\}$. Our goal then is to show that $\{t^{r,i}\} \to t^*$ for all i.

Since $f(x^{r,i}) \leq f(x^{r,i-1})$ for all r and all i [cf. (2.6)], we have from the observation $x^{r+1} = x^{r,n}, x^r = x^{r,0}$ [cf. (3.5)] that

$$f(x^{r+1}) \;=\; f(x^{r,n}) \;\leq\; f(x^{r,n-1}) \;\leq\; \cdots \;\leq\; f(x^{r,0}) \;=\; f(x^r), \quad \forall r. \tag{A.1}$$

Hence, $\{f(x^{r,i})\}$ is bounded from above for all i, so that, by Lemma A.1, there holds

$$\{t^{r,i}\} \text{ is bounded for all } i. \tag{A.2}$$

The next lemma strengthens (A.2) by showing that both $\{g(t^{r,i})\}$ and $\{\langle b, x^{r,i}\rangle\}$ are bounded for all i.

Lemma A.2. Let $\{w^r\}$ be any infinite sequence of n-vectors $\in \mathcal{X}$ such that $\{f(w^r)\}$ is bounded. Then, both $\{g(E^T w^r)\}$ and $\{\langle b, w^r\rangle\}$ are bounded for all i.

Proof: Since $\{E^T w^r\}$ is bounded by Lemma A.1, then $\{g(E^T w^r)\}$ is bounded from below. Therefore, if $\{g(E^T w^r)\}$ is not bounded, then there must exist a subsequence $\mathcal{R}$ of $\{0, 1, ...\}$ such that $\{g(E^T w^r)\}_\mathcal{R} \to \infty$. This in turn implies (since $f(w^r) = g(E^T w^r) + \langle b, w^r \rangle$ is bounded) that

$$\{\langle b, w^r\rangle\}_\mathcal{R} \to -\infty. \tag{A.3}$$

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Let us (by further passing into a subsequence if necessary) assume that, for each $i \in \{1, ..., n\}$, either $\{w_i^r\}_{\mathcal{R}}$ is bounded or $\{|w_i^r|\} \to \infty$. Let $\mathcal{I}$ denote the set of i's such that $\{w_i^r\}_{\mathcal{R}}$ is bounded and let $\bar{x}$ be any point in $\mathcal{X}$. For each $r \in \mathcal{R}$, consider the linear system

$$E^T y = E^T w^r, \quad y_i = w_i^r \quad \forall i \in \mathcal{I}, \quad y \in \mathcal{X}.$$

This system is clearly consistent since w^r is a solution. By Lemma 1, there exists a solution y^r of this system satisfying $\|\bar{x} - y^r\| \leq \theta(\|E^T \bar{x} - E^T w^r\| + \sum_{i \in \mathcal{I}} |\bar{x}_i - w_i^r|)$, where θ is a constant depending on E only. Since the right hand side of the above expression is bounded for all $r \in \mathcal{R}$, it follows that $\{y^r\}_{\mathcal{R}}$ is also bounded. Let $z^r = w^r - y^r$. Then, $E^T z^r = 0$, $z_{\mathcal{I}}^r = 0$ for all $r \in \mathcal{R}$, and [cf. (A.3)] $\langle b, z^r \rangle < 0$ for all $r \in \mathcal{R}$ sufficiently large. Moreover, for each $i \notin \mathcal{I}$, we have from $\{|w_i^r|\}_{\mathcal{R}} \to \infty$ that either (i) $c_i = \infty$ and $z_i^r > 0$ for all $r \in \mathcal{R}$ sufficiently large, or (ii) $l_i = -\infty$ and $z_i^r < 0$ for all $r \in \mathcal{R}$ sufficiently large. Hence, for any $r \in \mathcal{R}$ sufficiently large, z^r is a feasible direction of unbounded cost (i.e., for any x satisfying $l \leq x \leq c$ and $f(x) < \infty$, we have $l \leq x + \lambda z^r \leq c$ for all $\lambda > 0$ and $f(x + \lambda z^r) = g(E^T x) + \langle b, x + \lambda z^r \rangle = f(x) + \lambda \langle b, z^r \rangle \to -\infty$ as $\lambda \to \infty$). This contradicts the hypothesis [cf. Assumption A (b)] that (P) has an optimal solution. **Q.E.D.**

Since $\{f(x^{r,i})\}$ is bounded, Lemma A.2 yields that $\{g(t^{r,i})\}$ is bounded for all i. Hence, if t^∞ is any limit point of $\{t^{r,i}\}$, then we have from the lower semicontinuity property of g [Roc70] that $g(t^\infty) \leq \limsup_r \{g(t^{r,i})\} < \infty$. Since g is finite only on C_g [cf. Assumption A (a)], this implies

$$\text{every limit point of } \{t^{r,i}\} \text{ is in } C_g, \tag{A.4}$$

for all i.

By using (A.4), we can prove the following lemma:

Lemma A.3. $\{x^{r+1} - x^r\} \to 0$.

Proof: We will argue by contradiction. If the claim does not hold, then there exist an $\epsilon > 0$, an $i \in \{1, ..., n\}$, and a subsequence $\mathcal{R} \subseteq \{0, 1, ...\}$ such that $|x_i^{r+1} - x_i^r| \geq \epsilon$ for all $r \in \mathcal{R}$. Then, $\|t^{r,i} - t^{r,i-1}\| = \|E_i\| \cdot |x_i^{r+1} - x_i^r| \geq \|E_i\|\epsilon$ for all $r \in \mathcal{R}$ [cf. (3.5)]. Since both $\{t^{r,i}\}$ and $\{t^{r,i-1}\}$ are bounded by (A.2), we will (by further passing into a subsequence if necessary) assume that $\{t^{r,i-1}\}_{\mathcal{R}}$ and $\{t^{r,i}\}_{\mathcal{R}}$ converge to, say, t' and t'' respectively. Then, $t' \neq t''$ and, by (A.4), both t' and t'' are in C_g.

Since t' and t'' are in C_g and g is continuous on C_g (see [Roc70, Theorem 10.1]), we have that $\{g(t^{r,i})\}_{\mathcal{R}} \to g(t')$ and $\{g(t^{r,i})\}_{\mathcal{R}} \to g(t'')$ or, equivalently (since $f(x) = g(E^T x) + \langle b, x \rangle$ for all x),

$$\{\langle b, x^{r,i-1} \rangle\}_{\mathcal{R}} \to f^\infty - g(t'), \quad \{\langle b, x^{r,i} \rangle\}_{\mathcal{R}} \to f^\infty - g(t''), \tag{A.5}$$

where $f^\infty = \lim_{r\to\infty} f(x^r)$ (so that $\{f(x^{r,i})\} \to f^\infty$ for all i by $(A.1)$). Also, for each $r \in \mathcal{R}$, since $x^{r,i}$ is obtained from $x^{r,i-1}$ by performing a line minimization along the i-th coordinate direction in $\Re^n$ [cf. (2.6)], the convexity of f then yields

$$f(x^{r,i}) \leq f((x^{r,i} + x^{r,i-1})/2) = g((t^{r,i} + t^{r,i-1})/2) + \langle b, x^{r,i} + x^{r,i-1}\rangle/2 \leq f(x^{r,i-1}), \quad \forall r \in \mathcal{R}.$$

Upon passing into the limit as $r \to \infty$, $r \in \mathcal{R}$, and using $(A.5)$ and the continuity of g on C_g, we obtain

$$f^\infty \leq g((t'' + t')/2) + f^\infty - \frac{g(t'') + g(t')}{2} \leq f^\infty,$$

a contradiction of the strict convexity of g on C_g, i.e., $g((t' + t'')/2) < (g(t') + g(t''))/2$. **Q.E.D.**

Fix any $i \in \{0, 1, ..., n\}$. Since $\{t^{r,i}\}$ is bounded by $(A.2)$, it has a limit point. Let t^∞ be any such limit point and let $\mathcal{R}$ be a subsequence of $\{0, 1, ...\}$ such that $\{t^{r,i}\}_\mathcal{R}$ converges to t^∞. By $(A.4)$, $t^\infty \in C_g$, so that g is continuously differentiable in an open set around t^∞. We show below that t^∞ is equal to the t^*, which, since the choice of t^∞ was arbitrary, would then complete our proof.

First, notice that since $\{x^{r,j} - x^{r,i}\} \to 0$ for all j (cf. Lemma A.3), we have $\{t^{r,j} - t^{r,i}\} \to 0$ for all j, so that

$$\{t^{r,j}\}_\mathcal{R} \to t^\infty, \quad \forall j. \tag{A.6}$$

Let $d^\infty = E\nabla g(t^\infty) + b$. Then, since g is continuously differentiable in an open set around t^∞, we obtain from $(A.6)$ (and using $d(x^{r,j}) = E\nabla g(t^{r,j}) + b$) that $\{d(x^{r,j})\}_\mathcal{R} \to d^\infty$ for all j. Since $x_j^{r+1} = [x_j^{r+1} - d_j(x^{r,j})]_j^+$ for all $r \in \mathcal{R}$ and all j [cf. (2.5), (3.5)], this implies

$$\begin{aligned}
\{x_j^{r+1}\}_\mathcal{R} \to l_j > -\infty & \quad \text{if } d_j^\infty > 0, \\
\{x_j^{r+1}\}_\mathcal{R} \to c_j < \infty & \quad \text{if } d_j^\infty < 0.
\end{aligned} \tag{A.7}$$

For each $r \in \mathcal{R}$, consider the linear system

$$E^T x = t^{r+1}, \quad x_j = x_j^{r+1} \ \forall j \text{ with } d_j^\infty \neq 0, \quad x \in \mathcal{X}.$$

This system is clearly consistent since x^{r+1} is a solution. Fix any point $\bar{x}$ in $\mathcal{X}$. By Lemma 1, for every $r \in \mathcal{R}$ there exists a solution y^r of the above system satisfying $\|\bar{x} - y^r\| \leq \theta(\|E^T\bar{x} - t^{r+1}\| + \sum_{d_j^\infty \neq 0} |\bar{x}_j - x_j^{r+1}|)$, where θ is a constant depending on E only. Since the right hand side of the above expression is bounded for all $r \in \mathcal{R}$, it follows that $\{y^r\}_\mathcal{R}$ is also bounded. Then, every limit point of $\{y^r\}_\mathcal{R}$, say y^∞, satisfies [cf. $(A.6), (A.7)$]

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$$E^T y^\infty = t^\infty, \quad y_j^\infty = l_j \quad \text{if } d_j^\infty > 0, \quad y_j^\infty = c_j \quad \text{if } d_j^\infty < 0, \quad y^\infty \in \mathcal{X}.$$

Since $E^T y^\infty = t^\infty$ so that $d(y^\infty) = E\nabla g(t^\infty) + b = d^\infty$, the above relation then yields $y^\infty = [y^\infty - d(y^\infty)]^+$. Hence, by (2.4), y^∞ is in $\mathcal{X}^*$ and we obtain from (2.2) that $t^\infty = E^T y^\infty = t^*$.

Appendix B. Proof of Lemma 6

To simplify the proof, we will assume that $c_i = \infty$ for all i. (The case where some of the c_i's are finite can be treated by making a symmetric argument). Let $\{x^r\}$ be a sequence of iterates generated by (2.5) and let $x^{r,i}$ be given by (3.5). First, we have from Lemma 2 that

$$\{E^T x^{r,i}\} \to t^*, \quad \forall i, \qquad (B.1a)$$

and from (3.6) that

$$\{d(x^{r,i})\} \to d^*, \quad \forall i. \qquad (B.1b)$$

Since $c_i = \infty$ for all i so that $d^* \geq 0$, we have from Lemma 4 (b) that there exists an $r_0 \geq 0$ such that

$$x_i^r = l_i, \quad \forall i \notin \mathcal{I}^*, \quad \forall r \geq r_0. \qquad (B.2)$$

Consider an arbitrary (possibly empty) subset $\mathcal{I}$ of $\{1, ..., n\}$ and let $\mathcal{R}$ denote the set of indices $r \geq r_0$ for which

$$\begin{aligned} d_i(x^{r,i}) &= 0 \quad \forall i \in \mathcal{I}, \\ d_i(x^{r,i}) &> 0 \quad \forall i \notin \mathcal{I}. \end{aligned} \qquad (B.3)$$

We will show that there exists a constant $\omega_{\mathcal{I}} > 0$ such that

$$\|E^T x^r - t^*\| \leq \omega_{\mathcal{I}} \|x^r - x^{r+1}\|, \quad \forall r \in \mathcal{R}. \qquad (B.4)$$

Then, since every $r \in \{r_0, r_0 + 1, ...,\}$ belongs to an $\mathcal{R}$ corresponding to some $\mathcal{I}$ and the number of distinct $\mathcal{I}$'s is finite, we would immediately have that

$$\|E^T x^r - t^*\| \leq \max_{\mathcal{I}} \omega_{\mathcal{I}} \|x^r - x^{r+1}\|, \quad \forall r \geq r_0,$$

and Lemma 6 would be proven. Hence, it remains only to show that $(B.4)$ holds for an arbitrary $\mathcal{I}$.

Fix any subset $\mathcal{I}$ of $\{1, ..., n\}$ and let $\mathcal{R}$ denote the corresponding index set (i.e., $(B.2), (B.3)$ hold for all $r \in \mathcal{R}$). If $\mathcal{R}$ is empty or finite, then $(B.4)$ holds trivially. Hence, in what follows we will assume that $\mathcal{R}$ is infinite. Then, we have from $(B.3)$ and the fact $x_i^{r+1} = [x_i^{r+1} - d_i(x^{r,i})]_i^+$ [cf. (2.5), (3.5)] that

$$x_{\bar{\mathcal{I}}}^{r+1} = l_{\bar{\mathcal{I}}}, \quad \forall r \in \mathcal{R}. \qquad (B.5)$$

We next have the following lemma.

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Lemma B.1 There exist $\kappa_1 > 0$ and $y^r \in \mathcal{X}^*$, for all $r \in \mathcal{R}$, such that

$$y_{\bar{\mathcal{I}}}^r = l_{\bar{\mathcal{I}}}, \qquad (B.6a)$$

$$\|y^r - x^r\| \le \kappa_1(\|E^T x^r - t^*\| + \|x^{r+1} - x^r\|). \qquad (B.6b)$$

Proof: First, we argue that the linear system

$$y_{\bar{\mathcal{I}}} = l_{\bar{\mathcal{I}}}, \quad y \in \mathcal{X}^*. \qquad (B.7)$$

is consistent. To see this, for every $r \in \mathcal{R}$, let ξ^r be the element in $\mathcal{X}^*$ attaining $\|x^r - \xi^r\| = \phi(x^r)$. By $(B.5)$, we have $x_{\bar{\mathcal{I}}}^{r+1} = l_{\bar{\mathcal{I}}}$ so that $\xi_{\bar{\mathcal{I}}}^r = (\xi_{\bar{\mathcal{I}}}^r - x_{\bar{\mathcal{I}}}^r) + (x_{\bar{\mathcal{I}}}^r - x_{\bar{\mathcal{I}}}^{r+1}) + l_{\bar{\mathcal{I}}}$. Since $\{\xi^r - x^r\}_\mathcal{R} \to 0$ (cf. Lemma 5) and $\{x^r - x^{r+1}\} \to 0$ [cf. Lemma 4 (a)], this yields $\{\xi_{\bar{\mathcal{I}}}^r\}_\mathcal{R} \to l_{\bar{\mathcal{I}}}$ so that there exist elements of $\mathcal{X}^*$ that come arbitrarily close to the affine space $\{y \mid y_{\bar{\mathcal{I}}} = l_{\bar{\mathcal{I}}}\}$. Since both $\mathcal{X}^*$ and $\{y \mid y_{\bar{\mathcal{I}}} = l_{\bar{\mathcal{I}}}\}$ are polyhedral sets [$\mathcal{X}^*$ is polyhedral by (3.8)], this shows that they must make a nonempty intersection. In other words, the linear system $(B.7)$ has a nonempty solution set.

It can be seen from (2.1) and (2.2) that, for each $r \in \mathcal{R}$, $(B.7)$ has the same solution set as the following linear system

$$y_{\bar{\mathcal{I}}} = l_{\bar{\mathcal{I}}}, \quad E^T y = t^*, \quad \langle b, y \rangle = v^*, \quad y \in \mathcal{X}, \qquad (B.7')$$

where $v^* = ($ optimal value of $(P)) - g(t^*)$. Since $x^r \in \mathcal{X}$, then, by Lemma 1, there exists a solution y^r to $(B.7')$ (i.e., $y_{\bar{\mathcal{I}}}^r = l_{\bar{\mathcal{I}}}$, $y^r \in \mathcal{X}^*$) satisfying

$$\|x^r - y^r\| \le \theta(\|x_{\bar{\mathcal{I}}}^r - l_{\bar{\mathcal{I}}}\| + \|E^T x^r - t^*\| + |\langle b, x^r \rangle - v^*|)$$

$$= \theta(\|x_{\bar{\mathcal{I}}}^r - x_{\bar{\mathcal{I}}}^{r+1}\| + \|E^T x^r - t^*\| + |\langle b, x^r - y^r \rangle|),$$

where θ is a constant depending on E and b only and the equality follows from $(B.5)$. Hence, to complete our proof, it suffices to show that $|\langle b, x^r - y^r \rangle|$ is upper bounded by some constant times $\|E^T x^r - t^*\|$. Now, we have from $(B.2)$ that $x_i^r = l_i$ for all $i \notin \mathcal{I}^*$, so that (also using $y_i^r = l_i$ for all $i \notin \mathcal{I}^*$) $x_i^r - y_i^r = 0$ for all $i \notin \mathcal{I}^*$. Also, we have from (3.2) that $0 = d_{\mathcal{I}^*}^* = E_{\mathcal{I}^*} \nabla g(t^*) + b_{\mathcal{I}^*}$. Combining these two facts yields

$$\langle b, x^r - y^r \rangle = \langle b_{\mathcal{I}^*}, x_{\mathcal{I}^*}^r - y_{\mathcal{I}^*}^r \rangle$$

$$= -\langle E_{\mathcal{I}^*} \nabla g(t^*), x_{\mathcal{I}^*}^r - y_{\mathcal{I}^*}^r \rangle$$

$$= -\langle E \nabla g(t^*), x^r - y^r \rangle$$

$$= -\langle \nabla g(t^*), E^T x^r - t^* \rangle.$$

Hence, $|\langle b, x^r - y^r \rangle| \le \|\nabla g(t^*)\| \|E^T x^r - t^*\|$. **Q.E.D.**

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Corollary B.1 Let y^r be given by Lemma B.1. Then,

$$\|y_{\mathcal{I}}^r - x_{\mathcal{I}}^r\| \leq \kappa_1 \|E\| \left(\|x^{r+1} - x^r\| + \|(E_{\mathcal{I}})^T (x_{\mathcal{I}}^r - y_{\mathcal{I}}^r)\| \right), \quad \forall r \in \mathcal{R}.$$

Proof: Simply note that $\|y_{\bar{\mathcal{I}}}^r - x_{\bar{\mathcal{I}}}^r\| \leq \|y^r - x^r\|$ and that [cf. $(B.5)$, $(B.6a)$] $\|(E_{\bar{\mathcal{I}}})^T(x_{\bar{\mathcal{I}}}^r - y_{\bar{\mathcal{I}}}^r)\| = \|(E_{\bar{\mathcal{I}}})^T(x_{\bar{\mathcal{I}}}^r - x_{\bar{\mathcal{I}}}^{r+1})\| \leq \|E\|\|x^r - x^{r+1}\|$. **Q.E.D.**

In addition to Corollary B.1, we have the following technical lemma:

Lemma B.2 Let y^r be given by Lemma B.1. Then, there exist constants $\kappa_2 > 0$ and r_1 such that

$$\|d_{\mathcal{I}}(x^r)\| \geq \|E_{\mathcal{I}}\nabla g(E^T x^r) - E_{\mathcal{I}}\nabla g\left((E_{\mathcal{I}})^T y_I^r + (E_{\bar{\mathcal{I}}})^T x_{\bar{\mathcal{I}}}^r\right)\| - \kappa_2\|E\|^2\|x_{\bar{\mathcal{I}}}^r - y_{\bar{\mathcal{I}}}^r\|, \qquad (B.8)$$

for all $r \in \mathcal{R}$, $r \geq r_1$.

Proof: Since $\{d(x^{r,i})\} \to d^*$ [cf. $(B.1b)$], we have from $(B.3)$ that $d_i^* = 0$ for all $i \in \mathcal{I}$. Then, $d_i(y^r) = 0$ for all $i \in \mathcal{I}$ [cf. (3.1)], and we have from (2.3) and the triangle inequality that

$$
\begin{aligned}
\|d_{\mathcal{I}}(x^r)\| &= \|d_{\mathcal{I}}(x^r) - d_{\mathcal{I}}(y^r)\| \\
&= \|E_{\mathcal{I}}\nabla g(E^T x^r) - E_{\mathcal{I}}\nabla g(E^T y^r)\| \\
&\geq \|E_{\mathcal{I}}\nabla g(E^T x^r) - E_{\mathcal{I}}\nabla g\left((E_{\mathcal{I}})^T y_I^r + (E_{\bar{\mathcal{I}}})^T x_{\bar{\mathcal{I}}}^r\right)\| \\
&\quad - \|E_{\mathcal{I}}\nabla g\left((E_{\mathcal{I}})^T y_I^r + (E_{\bar{\mathcal{I}}})^T x_{\bar{\mathcal{I}}}^r\right) - E_{\mathcal{I}}\nabla g(E^T y^r)\|.
\end{aligned}
\qquad (B.9)
$$

Let $\mathcal{U}^*$ be the neighborhood around t^* given in (3.3). Then, g is twice differentiable on $\mathcal{U}^*$ [cf. Assumption A (a)] so that ∇g is Lipschitz continuous on $\mathcal{U}^*$, i.e., there exists a constant $\kappa_2 > 0$ such that

$$\|\nabla g(z) - \nabla g(y)\| \leq \kappa_2\|z - y\|, \quad \forall z \in \mathcal{U}^*, \quad \forall y \in \mathcal{U}^*. \qquad (B.10)$$

Now, since $E^T y^r = t^*$ for all $r \in \mathcal{R}$, then clearly $E^T y^r \in \mathcal{U}^*$ for all $r \in \mathcal{R}$. Also, since $x_{\bar{\mathcal{I}}}^{r+1} = y_{\bar{\mathcal{I}}}^r$ [cf. $(B.5)$, $(B.6a)$] so that $((E_{\mathcal{I}})^T y_{\mathcal{I}}^r + (E_{\bar{\mathcal{I}}})^T x_{\bar{\mathcal{I}}}^r) - E^T y^r = (E_{\bar{\mathcal{I}}})^T(x_{\bar{\mathcal{I}}}^r - x_{\bar{\mathcal{I}}}^{r+1}) \to 0$ as $r \to \infty$, $r \in \mathcal{R}$ [cf. Lemma 4 (a)], we have that $(E_{\mathcal{I}})^T y_I^r + (E_{\bar{\mathcal{I}}})^T x_{\bar{\mathcal{I}}}^r \in \mathcal{U}^*$ for all $r \in \mathcal{R}$, $r \geq$ some r_1. Hence, for any $r \in \mathcal{R}$ such that $r \geq r_1$, $(B.10)$ holds with $z = (E_{\mathcal{I}})^T y_I^r + (E_{\bar{\mathcal{I}}})^T x_{\bar{\mathcal{I}}}^r$ and $y = E^T y^r$. Using this to bound the last term in $(B.9)$ then yields $(B.8)$. **Q.E.D.**

By combining Corollary B.1 with Lemma B.2, we can now prove that $(B.4)$ holds.

Lemma B.3 There exists an $\omega_{\mathcal{I}} > 0$ such that $\|E^T x^r - t^*\| \leq \omega_{\mathcal{I}}\|x^{r+1} - x^r\|$ for all $r \in \mathcal{R}$.

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Proof: Let r_1 be given by Lemma B.2. Since $\{E^T x^{r,i}\} \to t^*$ for all i [cf. $(B.1a)$], there exists an $r_2 \geq r_1$ such that $E^T x^{r,i} \in \mathcal{U}^*$ for all i and all $r \in \mathcal{R}$ with $r \geq r_2$. Fix any $r \in \mathcal{R}$ with $r \geq r_2$. Then, for any $i \in \mathcal{I}$, we have from $(B.3)$, (2.3), $(B.10)$ and $E^T x^{r,i} \in \mathcal{U}^*$ that

$$
\begin{aligned}
|d_i(x^r)| &= |d_i(x^r) - d_i(x^{r,i})| \\
&= |E_i \nabla g(E^T x^r) - E_i \nabla g(E^T x^{r,i})| \\
&\leq \kappa_2 \|E\|^2 \|x^r - x^{r,i}\| \\
&\leq \kappa_2 \|E\|^2 \|x^r - x^{r+1}\|,
\end{aligned}
$$

so that

$$
\|d_{\mathcal{I}}(x^r)\| \leq \kappa_2 n \|E\|^2 \|x^{r+1} - x^r\|.
$$

Let y^r be given by Lemma B.1. Then, the above relation together with Lemma B.2 yields

$$
\kappa_2 n \|E\|^2 \|x^{r+1} - x^r\| \geq \|E_{\mathcal{I}} \nabla g(E^T x^r) - E_{\mathcal{I}} \nabla g((E_{\mathcal{I}})^T y_{\mathcal{I}}^r + (E_{\bar{\mathcal{I}}})^T x_{\bar{\mathcal{I}}}^r)\| - \kappa_2 \|E\|^2 \|x_{\bar{\mathcal{I}}}^r - y_{\bar{\mathcal{I}}}^r\|. \tag{B.11}
$$

Since $x^r \in \mathcal{U}^*$ and $y^r \in \mathcal{U}^*$, by the strong convexity of g on $\mathcal{U}^*$ [cf. (3.4)], we have

$$
\begin{aligned}
2\sigma \|(E_{\mathcal{I}})^T (x_{\mathcal{I}}^r - y_{\mathcal{I}}^r)\|^2 &\leq \langle (E_{\mathcal{I}})^T (x_{\mathcal{I}}^r - y_{\mathcal{I}}^r), \nabla g(E^T x^r) - \nabla g((E_{\mathcal{I}})^T y_{\mathcal{I}}^r + (E_{\bar{\mathcal{I}}})^T x_{\bar{\mathcal{I}}}^r) \rangle, \\
&\leq \|x_{\mathcal{I}}^r - y_{\mathcal{I}}^r\| \cdot \|E_{\mathcal{I}} \nabla g(E^T x^r) - E_{\mathcal{I}} \nabla g((E_{\mathcal{I}})^T y_{\mathcal{I}}^r + (E_{\bar{\mathcal{I}}})^T x_{\bar{\mathcal{I}}}^r)\|,
\end{aligned}
$$

where the second inequality follows from the Cauchy-Schwartz inequality. By using $(B.11)$ to bound the right hand side of the above expression, we then obtain

$$
\begin{aligned}
2\sigma \|(E_{\mathcal{I}})^T (x_{\mathcal{I}}^r - y_{\mathcal{I}}^r)\|^2 &\leq \|x_{\mathcal{I}}^r - y_{\mathcal{I}}^r\| \left(\kappa_2 n \|E\|^2 \|x^{r+1} - x^r\| + \kappa_2 \|E\|^2 \|x_{\bar{\mathcal{I}}}^r - y_{\bar{\mathcal{I}}}^r\| \right) \\
&\leq \|x_{\mathcal{I}}^r - y_{\mathcal{I}}^r\| \left(\kappa_2 n \|E\|^2 \|x^{r+1} - x^r\| + \kappa_2 \|E\|^2 \|x^{r+1} - x^r\| \right) \\
&\leq \kappa_1 \kappa_2 (n+1) \|E\|^3 \left(\|x^{r+1} - x^r\| + \|(E_{\mathcal{I}})^T (x_{\mathcal{I}}^r - y_{\mathcal{I}}^r)\| \right) \cdot \|x^{r+1} - x^r\|,
\end{aligned}
$$

where the second inequality follows from $y_{\bar{\mathcal{I}}}^r = x_{\bar{\mathcal{I}}}^{r+1}$ [cf. $(B.5)$, $(B.6a)$] and the last inequality follows from Corollary B.1. Thus, when we view the above expression as a quadratic inequality in the variable $\|(E_{\mathcal{I}})^T (x_{\mathcal{I}}^r - y_{\mathcal{I}}^r)\|/\|x^{r+1} - x^r\|$, we see that

$$
\|(E_{\mathcal{I}})^T (x_{\mathcal{I}}^r - y_{\mathcal{I}}^r)\| \leq \kappa_3 \|x^{r+1} - x^r\|, \tag{B.12}
$$

for some constant $\kappa_3 > 0$ independent of r.

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Finally, we notice from the fact $y_{\tilde{\mathcal{I}}}^r = x_{\tilde{\mathcal{I}}}^{r+1}$ [cf. $(B.5)$, $(B.6a)$] that

$$\|(E_{\tilde{\mathcal{I}}})^T (x_{\tilde{\mathcal{I}}}^r - y_{\tilde{\mathcal{I}}}^r)\| \quad = \|(E_{\tilde{\mathcal{I}}})^T (x_{\tilde{\mathcal{I}}}^r - x_{\tilde{\mathcal{I}}}^{r+1})\| \quad \leq \|E\|\|x^{r+1} - x^r\|.$$

Combining the above relation with $(B.12)$ yields

$$\|E^T x^r - t^*\| \quad = \|E^T (x^r - y^r)\| \quad \leq (\kappa_3 + \|E\|)\|x^{r+1} - x^r\|.$$

Since the choice of $r \in \mathcal{R}$, $r \geq r_2$ was arbitrary, this completes our proof. **Q.E.D.**

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