On the convex hull of random points in a polytope — **Source link** ↗

Rex A. Dwyer

Institutions: Carnegie Mellon University

Topics: Convex polytope, Simple polytope, Convex hull, Convex combination and Polytope

Related papers:

- Convex-bodies, economic cap coverings, random polytopes

- The convex hull of a random set of points

- über die konvexe Hülle von n zufällig gewählten Punkten

- Random approximation of convex sets

- Intrinsic volumes and f -vectors of random polytopes

1987

On the convex hull of random points in a polytope

Rex A. Dwyer
Carnegie Mellon University

On the Convex Hull of
Random Points in a Polytope

Rex A. Dwyer *

1 May 1987

CMU–CS–87–118 (2)

Abstract

It is shown that the expected number of vertices of the convex
hull of n points drawn from a uniform distribution over the interior
of a d-dimensional polytope is $O(\log^{d-1} n)$. A similar lower bound is
derived for a large class of polytopes. An algorithm is presented for
constructing the convex hull of such sets of points in linear average
time.

On the Convex Hull of
Random Points in a Polytope

Rex A. Dwyer *
Computer Science Department
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

1 May 1987

1. Introduction.

Let $S = (X_1, X_2, \ldots, X_n)$ be a set of n points in $\mathbf{R}^d$. The *convex hull* of S is the intersection of all closed half-spaces containing S. The convex hull of a finite set is a polytope; conversely every polytope is the convex hull of a finite set. The vertices of the polytope are called the *extreme points* of S, denoted ext S.

If the elements of S are independent but identically distributed random variates, one may study the expected complexity of the convex hull of S. In particular, one may investigate the expected number of vertices or facets of the convex hull. Rényi & Sulanke [12,13] studied these expectations for uniform distributions over various convex figures in the plane as well as for the two-dimensional normal distribution. Carnal [5] extended this study to circularly symmetric distributions in the plane. Raynaud [11] investigated the uniform distribution in the d-dimensional ball and the d-dimensional normal distribution. Bentley *et al.* [1] and later Devroye [6] examined d-dimensional distributions in which each coordinate of each point is selected independently of the other coordinates; the uniform distribution on the interior of a hypercube and the normal distribution are examples of this class. Buchta *et al.* [4] investigated the uniform distribution on the surface of the d-dimensional hypersphere.

This paper demonstrates that the expected number of extreme points is $O(\log^{d-1} n)$ when the points of S are drawn from a uniform distribution on the interior of any d-dimensional polytope. An $O(\log^{(d-1)\lfloor d/2 \rfloor} n)$ upper bound on the expected number of facets follows easily. We will also see that the bound on $E(|\operatorname{ext} S|)$ is tight for a large class of polytopes. (Whether it is tight for all polytopes remains an open question.) Finally, we apply this bound to the average-case analysis of a new algorithm for identifying extreme points and constructing the convex hull. We show that both of these tasks can be carried out in linear average time for these distributions.

2. Preliminaries.

If $X = (\xi_1, \xi_2, \ldots, \xi_d)$ is a point in $\mathbf{R}^d$, we will write $\Pi(X)$ for the product $\xi_1 \xi_2 \cdots \xi_d$. Now define $F(d, c)$ to be the volume of the part of the d-dimensional unit hypercube $[0, 1]^d$ that satisfies $\Pi(X) \leq c$. The function F satisfies the following recurrence:

$$
\begin{aligned}
F(1, c) &= c \\
F(d, c) &= c + \int_c^1 F(d-1, c/x)\, dx
\end{aligned}
$$

Lemma 1 $F(d, c) = c \sum_{0 \leq i < d} \log^i(1/c)/i!$

Proof. By induction on d. The basis case, $d = 1$, is trivial.

$$
\begin{aligned}
F(d, c) &= c + \int_c^1 F(d-1, c/x)\, dx \\
&= c + \int_c^1 \frac{c}{x} \sum_{i=0}^{d-2} \frac{\log^i(x/c)}{i!}\, dx \\
&= c \left[1 + \sum_{i=0}^{d-2} \frac{1}{i!} \int_c^1 \frac{(\log x - \log c)^i}{x}\, dx \right] \\
&= c \left[1 + \sum_{i=0}^{d-2} \frac{1}{i!} \sum_{j=0}^i \binom{i}{j} \int_c^1 \frac{(\log x)^j}{x}\, dx (-\log c)^{i-j} \right] \\
&= c \left[1 + \sum_{i=0}^{d-2} \frac{1}{i!} \sum_{j=0}^i \binom{i}{j} \frac{-1}{j+1} (\log c)^{j+1} (-\log c)^{i-j} \right] \\
&= c \left[1 + \sum_{i=0}^{d-2} \frac{1}{i!} \sum_{j=0}^i \binom{i}{j} \frac{(-1)^{i-j+1}}{j+1} (\log c)^{i+1} \right] \\
&= c \left[1 + \sum_{i=0}^{d-2} (-\log c)^{i+1} \sum_{j=0}^i \binom{i}{j} \frac{(-1)^j}{i!(j+1)} \right] \\
&\stackrel{(*)}{=} c \left[1 + \sum_{i=0}^{d-2} \frac{(-\log c)^{i+1}}{(i+1)!} \right] \\
&= c \left[1 + \sum_{i=1}^{d-1} \frac{(-\log c)^i}{i!} \right] \\
&= c \sum_{i=0}^{d-1} \frac{\log^i(1/c)}{i!}
\end{aligned}
$$

The equality $(*)$ holds because

$$
\sum_{j=0}^i \binom{i}{j} \frac{(-1)^j}{i!(j+1)} = \sum_{j=0}^i \binom{i+1}{j+1} \frac{(-1)^j}{i!(i+1)}
$$

$$= \frac{-1}{(i+1)!} \sum_{j=1}^{i+1} \binom{i+1}{j} (-1)^j$$

$$= \frac{-1}{(i+1)!} \left(\sum_{j=0}^{i+1} \binom{i+1}{j} (-1)^j - \binom{i+1}{0} (-1)^0 \right)$$

$$= \frac{1}{(i+1)!}. \quad \square$$

We will write M_n for random variate whose value is the number of extreme points among n points randomly chosen from a polytope. The following lemma of Devroye [7] bounding the moments of M_n will be useful in bounding the number of facets and in bounding the running time of algorithms.

Lemma 2 *The pth moment of M_n grows as the pth power of the mean of M_n, i.e.,*

$$E(M_n^p) = \Theta((EM_n)^p) \text{ for } p \geq 1.$$

The implicit constant in this lemma depends on p.

3. Upper Bounds.

Theorem 3 *Let $S = \{X_1, X_2, \ldots, X_n\}$ be a set of n points chosen independently from a uniform distribution on the interior of a regular simplex of d dimensions. The expected cardinality of ext S is $O(\log^{d-1} n)$.*

Proof. Partition the simplex into $d+1$ cells $C_0, C_1, \ldots, C_d$ corresponding to $d+1$ vertices $V_0, V_1, \ldots, V_d$ of the d-simplex. A point X lies in C_i if $d(X, V_i) \leq d(X, V_j)$ for $0 \leq j \leq d$. C_i is itself a polytope; its vertices are the centers of each of the faces of the simplex to which V_i belongs. Thus each cell has $\sum_{0 \leq i \leq d} \binom{d}{i} = 2^d$ vertices.

It will be more convenient to work with the simplex defined by the origin and the points $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$. Since such transformations do not affect the combinatorial properties of the convex hull, we will apply an affine transformation that maps V_0 to the origin and $V_1, \ldots, V_n$ to the other points.

It is easily verified that the vertices of C_0 that are centers of k-faces are mapped to points with k coordinates of $1/(k+1)$ and $d - k$ zero coordinates.

Now suppose that $X_1 = (\xi_1, \xi_2, \ldots, \xi_d)$ lies inside the cell. The hyperplanes $x_i = \xi_i$ for $1 \leq i \leq d$ partition the simplex into 2^d pieces. If the point X_1 is to be an extreme point, one of these pieces must be empty.

Claim. The smallest of the 2^d pieces has volume at least $(1/d!)\Pi(X_1)$.

Proof. By induction on d.

3

The basis case, $d = 1$, is easy. The volume of one piece is ξ_1; the volume of the other is $1 - \xi_1 \geq 1/2 \geq \xi_1$.

The piece satisfying $x_i < \xi_i$ for all i has volume $\Pi(X_1)$. Every other piece satisfies $x_i > \xi_i$ for some i. The cross-section of the simplex on the hyperplane $x_i = \xi_i$ is a $(d-1)$-dimensional simplex divided into 2^{d-1} pieces by the $(d-1)$ hyperplanes $x_j = \xi_j$ for $j \neq i$. By the induction hypothesis (modified by a scaling factor), the $(d-1)$-volume of each of these pieces exceeds $(1/(d-1)!)(\Pi(X_1)/\xi_i)$. Each of the d-dimensional pieces satisfying $x_i > \xi_i$ contains the intersection of the line $x_j = \xi_j$ for $j \neq i$ with the hyperplane $\sum x_j = 1$. This point is $(\xi_1, \ldots, \xi_{i-1}, 1 - \sum_{j \neq i} \xi_j, \xi_{i+1}, \ldots, \xi_d)$. Thus each d-dimensional piece contains a pyramid with base volume exceeding $(1/(d-1)!)(\Pi(X_1)/\xi_i)$, height $1 - \sum_{1 \leq i \leq d} \xi_i$, and total volume at least $(1/d)(1/(d-1)!)(\Pi(X_1)/\xi_i)\left(1 - \sum_{1 \leq i \leq d} \xi_i\right)$. The result follows if $1 - \sum_{1 \leq j \leq d} \xi_j \geq \xi_i$ for all points lying in the cell. The condition surely holds for the vertices of the cell: the center of a k-face has $\sum_{1 \leq j \leq d} \xi_j = k/(k+1)$ and $\xi_i \leq 1/(k+1)$. Since the condition is linear, this guarantees that it holds for the entire cell.

Claim. If point X_1 lies inside cell $\mathcal{C}_0$, it is an extreme point with probability less than $(2^d + 2)\exp(-n\Pi(X_1))$.

Proof. By the previous claim, all 2^d pieces formed at X_1 have volume at least $(1/d!)\Pi(X_1)$ and thus probability content $\Pi(X_1)$. At least one of these pieces must be empty if X_1 is an extreme point. The probability that a particular piece is empty is $(1 - \Pi(X_1))^{n-1} \leq (1 - (d+1)^{-d})^{-1}\exp(-n\Pi(X_1))$. The probability that at least one is empty is at most 2^d times as great, or less than $(2^d + 2)\exp(-n\Pi(X_1))$.

Now

$$\Pr\{X_1 \in \text{ext}\, S\}$$
$$= \Pr\{X_1 \in \text{ext}\, S \mid X_1 \in \mathcal{C}_0\}$$
$$= \int_0^{(d+1)^{-d}} \Pr\{X_1 \in \text{ext}\, S \mid \Pi(X_1) = y\} \cdot \Pr\{\Pi(X_1) = y \mid X_1 \in \mathcal{C}_0\}\, dy$$
$$< \int_0^{(d+1)^{-d}} (2^d + 2)e^{-ny} \Pr\{\Pi(X_1) = y \mid X_1 \in \mathcal{C}_0\}\, dy$$

We now apply the integration-by-parts formula $\int u\, dv = uv - \int v\, du$ with $u = e^{-ny}$, $dv = \Pr\{\Pi(X_1) = y \mid X_1 \in \mathcal{C}_0\}\, dy$, $du = -ne^{-ny}\, dy$, and $v = \Pr\{\Pi(X_1) \leq y \mid X_1 \in \mathcal{C}_0\}$. At the same time, we observe that, since $\mathcal{C}_0 \subset [0,1]^d$,

$$v = \frac{\text{vol}\{x \in \mathcal{C}_0 \mid \Pi(x) \leq y\}}{\text{vol}\, \mathcal{C}_0}$$
$$\leq \frac{\text{vol}\{x \in [0,1]^d \mid \Pi(x) \leq y\}}{\text{vol}\, \mathcal{C}_0}$$
$$= (d+1)! F(d, y)$$
$$= (d+1)! \sum_{0 \leq i < d} \frac{y \log^i(1/y)}{i!}$$

Thus

$$\Pr\{X_1 \in \text{ext}\, S\}$$

4

$$< (2^d + 2)(d+1)! \left[e^{-ny} \sum_{0 \le i < d} \frac{y \log^i(1/y)}{i!} \right]_{y=0}^{y=(d+1)^{-d}}$$

$$+ n(2^d + 2)(d+1)! \int_0^{(d+1)^{-d}} e^{-ny} \sum_{0 \le i < d} \frac{y \log^i(1/y)}{i!} \, dy.$$

An examinination of first derivatives shows that the functions $y \log^i(1/y)$ are all finite and increasing in the interval $(0, (d+1)^{-d}]$. Thus

$$\Pr\{X_1 \in \operatorname{ext} S\}$$

$$< O(1) \cdot \exp(-n(d+1)^{-d})$$

$$+ n(2^d + 2)(d+1)!(1/n) \sum_{0 \le i < d} \frac{\log^i n}{i!} \int_0^{1/n} e^{-ny} \, dy$$

$$+ n(2^d + 2)(d+1)! \sum_{0 \le i < d} \frac{\log^i n}{i!} \int_{1/n}^{(d+1)^{-d}} y e^{-ny} \, dy$$

$$< O(1) \cdot \exp(-n(d+1)^{-d})$$

$$+ n(2^d + 2)(d+1)!(1/n) \sum_{0 \le i < d} \frac{\log^i n}{i!} \left[\frac{-e^{-ny}}{n} \right]_0^{1/n}$$

$$+ n(2^d + 2)(d+1)! \sum_{0 \le i < d} \frac{\log^i n}{i!} \left[\frac{-(ny+1)e^{-ny}}{n^2} \right]_{1/n}^{(d+1)^{-d}}$$

$$\le o(n^{-1})$$

$$+ n^{-1}(1 - e^{-1})(2^d + 2)(d+1)d \log^{d-1} n + O(n^{-1} \log^{d-2} n)$$

$$+ n^{-1}(2e^{-1})(2^d + 2)(d+1)d \log^{d-1} n + O(n^{-1} \log^{d-2} n)$$

$$= n^{-1}(1 + e^{-1})(2^d + 2)(d+1)d \log^{d-1} n + O(n^{-1} \log^{d-2} n).$$

Since the points $X_1, X_2, \ldots, X_n$ are identically distributed, the expected number of extreme points is less than

$$(1 + e^{-1})(2^d + 2)(d+1)d \log^{d-1} n + O(\log^{d-2} n). \quad \square$$

Corollary 4 *Let $S = \{X_1, X_2, \ldots, X_n\}$ be a set of n points chosen independently from a uniform distribution on the interior of a d-dimensional polytope P. The expected cardinality of $\operatorname{ext} S$ is $O(\log^{d-1} n)$.*

Proof. The polytope P can be partitioned into some finite number of simplices $P_1, P_2, \ldots, P_k$. Let $S_i = S \cap P_i$ for $1 \le i \le k$. By the preceding theorem, $E(|\operatorname{ext} S_i|) = O(\log^{d-1} n)$. Since $\operatorname{ext} S \subset \bigcup_{1 \le i \le k} \operatorname{ext} S_i$, it follows immediately that $E(|\operatorname{ext} S|) = O(\log^{d-1} n)$. $\square$

Corollary 5 *Let $S = \{X_1, X_2, \ldots, X_n\}$ be a set of n points chosen independently from a uniform distribution on the interior of a d-dimensional polytope P. The expected number of facets of the convex hull of S is $O(\log^{(d-1)\lfloor d/2 \rfloor} n)$.*

Proof. The result follows immediately from the preceding corollary, Lemma 2, and the Upper Bound Theorem[3], which states that the number of facets is $O(|\operatorname{ext} S|^{\lfloor d/2 \rfloor})$. $\square$

4. Lower Bounds.

Theorem 6 *Let P be a d-dimensional polytope, and suppose that $S = \{X_1, X_2, \ldots, X_n\}$ is a set of n points drawn independently from the uniform distribution over the interior of P. If P has at least one vertex that lies in exactly d facets, then the expected number of extreme points among the n points is $\Omega(\log^{d-1} n)$. (Every vertex lies in* at least *d facets; if every vertex lies in* exactly *d facets, the polytope is called* simple.*)*

Proof. Let V_0 be a vertex lying in exactly d facets. V_0 also lies in exactly d edges. Call the other endpoints of these edges $V_1, V_2, \ldots, V_d$. These $d+1$ points define a d-simplex. Now apply an affine transformation to P which maps V_0 to the origin and V_i to the point $d\epsilon_i$ for $1 \leq i \leq d$, where ϵ_i is the ith standard basis vector for $\mathbf{R}^d$. Let P' be the image of P. The unit hypercube $[0, 1]^d$ is contained by the image of the simplex $V_1 V_2 \ldots V_d$ and by P'.

We will now bound the probability that $X_1 = (\xi_1, \xi_2, \ldots, \xi_d)$ is an extreme point. The hyperplane $\sum_{1 \leq i \leq d} (x_i/\xi_i) = d$ passes through X_1. If the half-space $\sum_{1 \leq i \leq d} (x_i/\xi_i) \leq d$ contains none of the points of S, then X_1 is surely an extreme point. This hyperplane intersects the x_i-axis at $d\xi_i$, so it cuts a volume of at most $d^d \Pi(X_1)/d!$ from the polytope. Thus the probability content of the half-space is at most

$$\frac{d^d \Pi(X_1)}{d! \operatorname{vol} P'}.$$

Let $y = \Pi(X_1)$ and $C = d^d/d! \operatorname{vol} P'$. The probability that the half-space is empty is at least $f(y) = (1 - Cy)^{n-1} \geq (1 - Cy)^n \to e^{-Cny}$.

Now consider the probability that X_1 satisfies $\Pi(X_1) \leq y$. This is at least

$$G(y) = \frac{F(d, y)}{\operatorname{vol} P'} = \sum_{0 \leq i < d} \frac{y \log^i(1/y)}{i! \operatorname{vol} P'}.$$

The corresponding density function is

$$
\begin{aligned}
g(y) &= \sum_{0 \leq i < d} \left(\frac{-i \log^{i-1}(1/y)}{i! \operatorname{vol} P'} + \frac{\log^i(1/y)}{i! \operatorname{vol} P'} \right) \\
&= \sum_{0 \leq i < d-1} \frac{-\log^i(1/y)}{i! \operatorname{vol} P'} + \sum_{0 \leq i < d} \frac{\log^i(1/y)}{i! \operatorname{vol} P'} \\
&= \frac{\log^{d-1}(1/y)}{(d-1)! \operatorname{vol} P'}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\Pr&\{X_1 \in \operatorname{ext} S\} \\
&\geq \int_0^1 \Pr\{X_1 \text{ is extreme} \mid \Pi(X_1) = y\} \cdot \Pr\{\Pi(X_1) = y\} \, dy \\
&\geq \int_0^1 f(y) g(y) \, dy \\
&\geq \int_0^{1/n} e^{-Cny} \frac{\log^{d-1}(1/y)}{(d-1)! \operatorname{vol} P'} \, dy
\end{aligned}
$$

6

$$\geq \frac{\log^{d-1} n}{(d-1)! \operatorname{vol} \mathcal{P}'} \int_0^{1/n} e^{-Cny} \, dy$$

$$\geq \frac{\log^{d-1} n}{(d-1)! \operatorname{vol} \mathcal{P}'} \left[\frac{-e^{-Cny}}{Cn} \right]_0^{1/n}$$

$$\geq \frac{(1 - e^{-C}) \log^{d-1} n}{C(d-1)!(\operatorname{vol} \mathcal{P}')n}$$

$$\geq \frac{(1 - e^{-C})d! \log^{d-1} n}{d^d(d-1)!n}$$

$$\geq \frac{(1 - e^{-1}) \log^{d-1} n}{d^{d-1}n}$$

The expected number of extreme points is at least n times as large. $\square$

Corollary 7 *The expected number of extreme points among n chosen independently from the uniform distribution on a d-dimensional hypercube is at least*

$$\frac{(1 - e^{-1})2^d \log^{d-1} n}{d^{d-1}}.$$

Proof. It is convenient to consider the hypercube $[0, 2]^d$. We proceed as in the preceding proof, but omit the affine transformation. The expected number of extreme points in $[0, 1]^d$ is as before. The total expected number of extreme points is at least 2^d times as large, since $[0, 2]^d$ contains 2^d such unit hypercubes. $\square$

Devroye [6] computed the upper bound $(2^d \log^{d-1} n/(d-1)!) + O(\log^{d-2} n)$ for the case of the hypercube. Asymptotically, these two bounds differ by a factor of

$$\frac{d^{d-1}}{(1 - e^{-1})(d-1)!} \approx O(1) \cdot e^d.$$

5. Applications to Algorithms.

Suppose that n points are chosen as before, and that it is required to construct the convex hull. We may begin by identifying the extreme points. A point is extreme if and only if there is a hyperplane passing through it such that all the other points lie on one side. Thus we can determine whether a given point is extreme by solving a system of n linear inequalities in d unknowns in $O(n)$ time using the linear programming algorithm of Megiddo[9], and all extreme points can be identified in $O(n^2)$ time in the worst case by solving n such systems.

This running time can be improved to $O(n)$ in the average case by applying the randomizing divide-and-conquer technique of Bentley & Shamos. [2] We randomly divide the n points into two sets of $n/2$ points, apply the technique recursively to the subproblems to find the extreme points of each of the two subsets, then merge the two sets of extreme points by solving systems of linear inequalities as before. In the expected case, the subsets will have few extreme points, so there will be only a few small systems of inequalities in the merge step.

The average-case running time of this algorithm satisfies

$$T(n) \leq 2T(n/2) + E(M_n^2),$$

or, by Lemma 2,

$$T(n) \leq 2T(n/2) + O(\log n)^{2(d-1)},$$

or $T(n) = O(n)$. The convex hull can then be constructed from the extreme points in $O(n)$ expected time using any polynomial-time algorithm. It is easily verified that the worst-case running time is increased by only a small constant factor.

An alternative method to achieve linear average time is to identify the extreme points by applying the Bentley-Shamos technique with *any* polynomial-time convex-hull algorithm.

If $O(n)$ processors are available, the two subproblems can be attacked independently by different processors. Also, the merging step can be carried out in $O((\log n)^{d-1})$ expected time, since a separate processor can be assigned to each of the expected $O((\log n)^{d-1})$ systems of $O((\log n)^{d-1})$ inequalities. In this case, the expected running time satisfies the recurrence

$$T(n) \leq T(n/2) + O((\log n)^{d-1})$$

or $T(n) \leq O((\log n)^d)$.

6. References.

[1] J. L. Bentley, H. T. Kung, M. Schkolnick, and C. D. Thompson. On the average number of maxima in a set of vectors. *J. ACM*, 25:536–543, 1978.

[2] J. L. Bentley and M. I. Shamos. Divide and conquer for linear expected time. *IPL*, 7(2):87–91, Feb. 1978.

[3] A. Brønsted. *An Introduction to Convex Polytopes*. Springer-Verlag, New York, 1983.

[4] C. Buchta, F. Müller, and R. F. Tichy. Stochastical approximation of convex bodies. *Math. Annal.*, 271(2):225–235, Apr. 1985.

[5] H. Carnal. Die konvexe Hülle von n rotationssymmetrisch verteilten Punkten. *Z. Wahrsch.*, 15:168–176, 1970.

[6] L. Devroye. A note on finding convex hulls via maximal vectors. *IPL*, 11(1):53–56, Aug. 1980.

[7] L. P. Devroye. Moment inequalities for random variables in computational geometry. *Computing*, 30:111–119, 1983.

[8] B. Efron. The convex hull of a random set of points. *Biometrika*, 52:331–342, 1965.

[9] N. Megiddo. Linear programming in linear time when the dimension is fixed. *J. ACM*, 31(1):114–127, Jan. 1984.

[10] F. P. Preparata and M. I. Shamos. *Computational Geometry*. Springer-Verlag, 1985.

[11] H. Raynaud. Sur l'envellope convexe des nuages des points aléatoires dans $\mathbf{R}^n$, I. *J. Appl. Prob.*, 7:35–48, 1970.

[12] A. Rényi and R. Sulanke. Über die konvexe Hülle von n zufällig gewählten Punkten. *Z. Wahrsch.*, 2:75–84, 1963.

[13] A. Rényi and R. Sulanke. Über die konvexe Hülle von n zufällig gewählten Punkten, II. *Z. Wahrsch.*, 3:138–147, 1964.