

ON THE CONVOLUTION TRANSFORM

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1. Introduction.

In this paper we shall study the inversion theory for the class of convolution transforms

$$(1) \quad f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t) dt$$

for which the kernel $G(t)$ is of the form

$$(2) \quad G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{F(s)} e^{st} ds.$$

Here

$$(3) \quad F(s) = \prod_{k=1}^{\infty} \frac{(1-s^2/a_k^2)}{(1-s^2/c_k^2)},$$

where $\{a_k\}_1^{\infty}$ and $\{c_k\}_1^{\infty}$ are positive constants such that

$$(4) \quad 0 < a_1 \leq a_2 \leq \cdots; \quad c_1 \leq c_2 \leq \cdots; \quad a_k \leq c_k,$$

$$\lim_{n \rightarrow \infty} \frac{n}{a_n} = \Omega > \Omega' = \lim_{n \rightarrow \infty} \frac{n}{c_n}.$$

We agree that from certain point on, all c_k may $= \infty$. In fact, the case was extensively studied by Hirschman and Widder [1] Chapter IX. We shall follow after their arguments to consider the generalization.

If we set $a_k = (2k-1)/2$, $c_k = \infty$ ($k = 1, 2, 3, \dots$), Theorem 7 and Theorem 8 below will give known results for the Stieltjes transform [1].

2. Properties of the kernel.

We suppose that

$$(1) \quad E(s) = \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{a_k^2}\right)$$

where

$$(2) \quad 0 < a_1 \leq a_2 \leq \cdots, \quad \lim_{n \rightarrow \infty} \frac{n}{a_n} = \Omega.$$

LEMMA 1. *If $E(s)$ is defined by equations (1) and (2), then*

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$$\lim_{r \rightarrow \infty} r^{-1} \log |E(re^{i\theta})| = \pi \Omega |\sin \theta|$$

uniformly for θ in any closed interval not containing an integral multiple of π .

This is known; see [1] p. 213.

LEMMA 2. If $F(s)$ is defined by equations (3) and (4) of §1, then

$$\lim_{r \rightarrow \infty} r^{-1} \log |F(re^{i\theta})| = \pi(\Omega - \Omega') |\sin \theta|$$

uniformly for θ in any closed interval not containing an integral multiple of π .

This is an immediate consequence of Lemma 1.

We define

$$h_k(t) = \left(1 - \frac{a_k^2}{c_k^2}\right) \frac{1}{2} a_k \int_{-\infty}^t e^{-a_k|u|} du + \frac{a_k^2}{c_k^2} j(t)$$

where $j(t)$ is the standard jump function, that is, $j(t) = 0$ for $t < 0$, $1/2$ for $t = 0$ and 1 for $t > 0$. It is easily verified that $h_k(t)$ is a distribution function with mean 0 and variance $2(a_k^{-2} - c_k^{-2})$ and that

$$(3) \quad \int_{-\infty}^{\infty} e^{-st} dh_k(t) = \frac{1 - s^2/c_k^2}{1 - s^2/a_k^2},$$

the bilateral Laplace transform converging absolutely for $-a_k < \Re s < a_k$.

THEOREM 1. If

1. $F(s)$ is defined by (3) and (4) of §1,
2. μ denotes the multiplicity of a_1 as a zero of $F(s)$,

and

$$3. \quad G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{F(s)} e^{st} ds \quad (-\infty < t < \infty),$$

then

A. $G(t)$ is a frequency function with mean 0 and variance $2(\sum_{i=1}^{\infty} a_i^{-2} - \sum_{i=1}^{\infty} c_i^{-2})$,

B. $\int_{-\infty}^{\infty} G(t) e^{-st} dt = 1/F(s)$, the bilateral Laplace transform converging absolutely in the strip $-a_1 < \Re s < a_1$,

C. $G(t) \in C^\infty$,

D. $G(t) = p(t)e^{-a_1 t} + R_+(t)$, $G(t) = p(-t)e^{a_1 t} + R_-(t)$

where $p(t)$ is a real polynomial of degree $\mu - 1$ and

$$\left(\frac{d}{dt}\right)^n R_+(t) = O(e^{-(a_1 + \epsilon)t}) \quad \text{as } t \rightarrow \infty \quad (n = 0, 1, 2, \dots),$$

$$\left(\frac{d}{dt}\right)^n R_-(t) = O(e^{(a_1 + \epsilon)t}) \quad \text{as } t \rightarrow -\infty \quad (n = 0, 1, 2, \dots)$$

for some $\varepsilon > 0$.

Proof. If we set

$$H_n(t) = h_1(t) \# h_2(t) \# \cdots \# h_n(t)$$

where operation $\#$ denotes the Stieltjes convolution for distribution functions, that is, $h \# k$ means

$$\int_{-\infty}^{\infty} h(t-u) dk(u),$$

then by the convolution theorem [2] $H_n(t)$ is a distribution function with the bilateral Laplace transform

$$\int_{-\infty}^{\infty} e^{-st} dH_n(t) = \prod_{k=1}^n \frac{1-s^2/c_k^2}{1-s^2/a_k^2}.$$

We have

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{1-s^2/c_k^2}{1-s^2/a_k^2} = \frac{1}{F(s)}$$

uniformly for s in any compact set of the s -plane punctured at $\pm a_1, \pm a_2, \dots$. Thus $1/F(i\tau)$ is the characteristic function of a distribution function $H(t) = \lim_{n \rightarrow \infty} H_n(t)$,

$$\int_{-\infty}^{\infty} e^{-i\tau t} dH(t) = \frac{1}{F(i\tau)}.$$

Further, by Lévy's theorem

$$H(t_1) - H(t_2) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st_1} - e^{st_2}}{sF(s)} ds.$$

Since by Lemma 2

$$(4) \quad \log |F(i\tau)| \sim \pi(\Omega - \Omega')|\tau| \quad \text{as } \tau \rightarrow \pm\infty,$$

it follows that $H(t)$ is infinitely differentiable. If $G(t) = dH(t)/dt$, then $G(t)$ is a frequency function, and

$$(5) \quad G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{F(s)} ds.$$

From this the conclusion C follows.

To demonstrate the conclusion D, let us choose $\varepsilon > 0$ so small that no a_k ($k=2, 3, \dots$) lies in the interval $-a_1 - \varepsilon \leq \sigma < a_1$ ($s = \sigma + i\tau$). Integrating about the rectangular contour with vertices at $\pm iT$, $-a_1 - \varepsilon \pm iT$ and letting T increase without limit, we obtain

$$G(t) = p(t)e^{-a_1 t} + R_+(t) \quad R_+(t) = \frac{1}{2\pi i} \int_{-a_1 - \varepsilon - i\infty}^{-a_1 - \varepsilon + i\infty} \frac{e^{st}}{F(s)} ds.$$

Again by Lemma 2 if $\eta > 0$ then

$$\left| \frac{1}{F(s)} \right| = O(e^{-\pi(\Omega - \Omega' - \eta)|\tau|}) \quad \text{as } \tau \rightarrow \pm\infty$$

uniformly for σ in any finite interval. From this it is easily seen that

$$\left(\frac{d}{dt}\right)^n R_+(t) = O(e^{-(a_1+\varepsilon)t}) \quad \text{as } t \rightarrow \infty.$$

The second part of conclusion D will be established similarly.

From D we see that

$$\int_{-\infty}^{\infty} e^{st} G(t) dt$$

converges absolutely for $|\Re s| < a_1$ and defines in this strip an analytic function. Since

$$\int_{-\infty}^{\infty} e^{-i\tau t} G(t) dt = \frac{1}{F(i\tau)},$$

we have demonstrated the conclusion B; that is, for $|\Re s| < a_1$

$$(6) \quad \int_{-\infty}^{\infty} e^{-st} G(t) dt = \frac{1}{F(s)}.$$

From this equation the conclusion A follows by the the straightforward computations.

THEOREM 2. *If $G(t)$ is defined as in Theorem 1, then*

$$\operatorname{sgn} \frac{dG(t)}{dt} = -\operatorname{sgn} t.$$

This follows from the fact that the functions $h_k(t)$ are convex distribution functions.

3. Properties of the transform.

THEOREM 3. *If*

1. $G(t)$ is defined as in Theorem 1,
2. $\alpha(t)$ is of bounded variation in every finite interval,

and

3. $\int_{-\infty}^{\infty} G(x_0 - t) d\alpha(t)$ converges,

then

$$\int_{-\infty}^{\infty} G(x - t) d\alpha(t)$$

converges uniformly for x in any finite interval.

Proof. It is enough to show that

$$(1) \quad \lim_{A, B \rightarrow +\infty} \int_A^B G(x - t) d\alpha(t) = 0,$$

$$(1)' \quad \lim_{A, B \rightarrow +\infty} \int_A^B G(x - t) d\alpha(t) = 0,$$

uniformly for x in any finite interval. By Theorem 1 we have

$$(2) \quad \frac{G(x-t)}{G(x_0-t)} = O(1) \quad \text{and} \quad \frac{d}{dt} \left[\frac{G(x-t)}{G(x_0-t)} \right] = O\left(\frac{1}{t^2}\right) \quad \text{as } t \rightarrow \infty,$$

uniformly for x in any finite interval. If we set

$$L(t) = \int_t^\infty G(x_0-t) d\alpha(t),$$

then

$$(3) \quad L(t) = o(1) \quad \text{as } t \rightarrow +\infty.$$

We have

$$\begin{aligned} \int_A^B G(x-t) d\alpha(t) &= \int_A^B \frac{G(x-t)}{G(x_0-t)} G(x_0-t) d\alpha(t) \\ &= \left[-\frac{G(x-t)}{G(x_0-t)} L(t) \right]_A^B + \int_A^B \left(\frac{d}{dt} \frac{G(x-t)}{G(x_0-t)} \right) L(t) dt. \end{aligned}$$

Using equations (2) and (3) we see that equation (1) holds uniformly for x . We can establish (1)' similarly.

4. Operational calculus.

Denote by D the operation of differentiation. We define the operation $(1-D/a_k)^{-1}$ after Hirschman and Widder [1] by the following equation:

$$(1-D/a_k)^{-1}\varphi(x) = \int_{-\infty}^{\infty} e^{-yD/a_k} \varphi(x) h(y) dy,$$

where

$$h(y) = \begin{cases} e^y & (-\infty, 0), \\ 0 & (0, \infty), \end{cases}$$

that is, by the equations

$$(1-D/a_k)^{-1}\varphi(x) = \begin{cases} a_k e^{a_k x} \int_x^\infty \varphi(y) e^{-a_k y} dy & \text{if } a_k > 0, \\ -a_k e^{a_k x} \int_{-\infty}^x \varphi(y) e^{-a_k y} dy & \text{if } a_k < 0. \end{cases}$$

For example, if $a_k > 0$ then

$$(1-D/a_k)^{-1}e^{st} = \frac{e^{st}}{1-s/a_k} \quad \text{for } \Re s < a_k.$$

Therefore

$$(1-D/a_k)^{-1}G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{F(s)(1-s/a_k)} ds,$$

the integral converging absolutely by Lemma 2.

Let us define

$$(1) \quad F_n(s) = \prod_{k=n+1}^{\infty} \frac{1 - s^2/a_k^2}{1 - s^2/c_k^2},$$

$$(2) \quad F_n^*(s) = \prod_{k=1}^n \frac{1 - s^2/a_k^2}{1 - s^2/c_k^2},$$

$$(3) \quad G_n(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{F_n(s)} ds.$$

THEOREM 4. *If*

1. $F_n(s)$ is defined by equation (1),
2. $G_n(t)$ is defined by equation (3),

then

A. $G_n(t)$ is a frequency function of mean 0 and variance $2(\sum_{n+1}^{\infty} a_k^{-2} - \sum_{n+1}^{\infty} c_k^{-2})$.

B. $\int_{-\infty}^{\infty} G_n(t)e^{-st} dt = 1/F_n(s)$, the bilateral Laplace transform converging absolutely in the strip $-a_{n+1} < \Re s < a_{n+1}$,

C. $G_n(t) \in C^\infty$, $-\infty < t < \infty$,

and

D. $G_n(t) = p_n(t)e^{-a_{n+1}t} + R_{n,+}(t)$, $G_n(t) = p_n(-t)e^{a_{n+1}t} + R_{n,-}(t)$,

where $p_n(t)$ is a polynomial of degree $\mu_n - 1$, μ_n denoting the multiplicity of $s = a_{n+1}$ as a zero of $F_n(s)$, and

$$\left(\frac{d}{dt}\right)^m R_{n,+}(t) = O(e^{-(a_{n+1}+\varepsilon)t}) \quad \text{as } t \rightarrow +\infty \quad (m = 0, 1, 2, \dots),$$

$$\left(\frac{d}{dt}\right)^m R_{n,-}(t) = O(e^{(a_{n+1}+\varepsilon)t}) \quad \text{as } t \rightarrow -\infty \quad (m = 0, 1, 2, \dots)$$

for some $\varepsilon > 0$.

This is an immediate consequence of Theorem 1.

From this theorem and Theorem 1 we have

$$(4) \quad F_n^*(D)G(t) = G_n(t).$$

5. Inversion theorem.

THEOREM 5. *If*

1. $G(t)$ is defined as in Theorem 1,
2. $F_n^*(D)$ and $G_n(t)$ are defined by (2) and (3) of §4,
3. $\alpha(t)$ is of bounded variation in every finite interval,

and

4. $f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t)$ converges,

then

$$F_n^*(D)f(x) = \int_{-\infty}^{\infty} G_n(x-t) d\alpha(t),$$

the integral converging uniformly for x in any finite interval.

Proof. From the relation (4) of §4 it is enough to show that the integral

$$(1) \quad \int_{-\infty}^{\infty} G_n(x-t) d\alpha(t)$$

converges uniformly for x in any finite interval. By Theorem 1 and Theorem 4 the integral

$$(2) \quad \int_{-\infty}^{\infty} \frac{d}{dt} \frac{G_n(x-t)}{G(x-t)} dt$$

converges uniformly for x in any finite interval and we have

$$\lim_{t \rightarrow \pm\infty} \frac{G_n(x-t)}{G(x-t)} < \infty,$$

uniformly for x . For any x ($-\infty < x < \infty$) we set

$$L(t) = \int_0^t G(x-t) d\alpha(t),$$

then by Theorem 4, $L(t)$ is bounded and $L(+\infty)$, $L(-\infty)$ exist. For arbitrary T_1, T_2 we have

$$\begin{aligned} \int_{T_1}^{T_2} G_n(x-t) d\alpha(t) &= \int_{T_1}^{T_2} \frac{G_n(x-t)}{G(x-t)} dL(t) \\ &= \left[\frac{G_n(x-t)}{G(x-t)} L(t) \right]_{T_1}^{T_2} - \int_{T_1}^{T_2} \left[\frac{d}{dt} \frac{G_n(x-t)}{G(x-t)} \right] L(t) dt. \end{aligned}$$

The last two terms converge as $T_1 \rightarrow -\infty$, $T_2 \rightarrow +\infty$, uniformly for x in any finite interval.

COROLLARY 5. 1. $G(t)$ is defined as in Theorem 1,

2. $F_n^*(D)$, $G_n(t)$ are defined by (2) and (3) of §4,

3. $\varphi(t)$ is integrable on every finite interval,

4. $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t) dt$ converges,

then

$$F_n^*(D)f(x) = \int_{-\infty}^{\infty} G_n(x-t)\varphi(t) dt,$$

the integral converging absolutely for x in any finite interval.

In the previous theorem, set

$$\int_0^t \varphi(t) dt = \alpha(t).$$

Then the result follows immediately.

LEMMA 3. [1] *Let $\varphi(t)$ be continuous and $\alpha(t)$ of bounded variation in every finite subinterval of $a \leq t < \infty$. If*

1. $\varphi(t)$ is positive and monotonic,
and

2. $\int_a^\infty \varphi(t) d\alpha(t)$ converges,

then $\lim_{t \rightarrow \infty} \varphi(t) = 0$ implies that

$$\alpha(t) = o\left(\frac{1}{\varphi(t)}\right) \text{ as } t \rightarrow +\infty.$$

THEOREM 6. *If*

1. $G(t)$ is defined as in Theorem 1,

2. $F_n^*(D)$ and $G_n(t)$ are defined by (2) and (3) of §4,

3. $\alpha(t)$ is of bounded variation in any finite interval,

and

4. $f(x) = \int_{-\infty}^\infty G(x-t) d\alpha(t)$ converges,

then for n sufficiently large

$$\int_{x_1}^{x_2} F_n^*(D)f(x) dx = \int_{-\infty}^\infty G_n(x_2-t)\alpha(t) dt - \int_{-\infty}^\infty G_n(x_1-t)\alpha(t) dt.$$

Proof. By Theorem 2 and Lemma 3 we have

$$(1) \quad \alpha(t) = o[G(x-t)]^{-1} \text{ as } t \rightarrow \pm\infty.$$

By Theorem 5

$$F_n^*(D)f(x) = \int_{-\infty}^\infty G_n(x-t) d\alpha(t),$$

the integral converging uniformly for x in any finite interval. Integrating by parts, we obtain

$$F_n^*(D)f(x) = \left[G_n(x-t)\alpha(t) \right]_{-\infty}^\infty - \int_{-\infty}^\infty \left[\frac{\partial}{\partial t} G_n(x-t) \right] \alpha(t) dt.$$

Theorem 1, Theorem 4 and the estimation (1) show that the integrated parts vanishes uniformly for $x_1 \leq x \leq x_2$. Thus

$$F_n^*(D)f(x) = - \int_{-\infty}^\infty \left[\frac{\partial}{\partial t} G_n(x-t) \right] \alpha(t) dt = \int_{-\infty}^\infty \left[\frac{\partial}{\partial x} G_n(x-t) \right] \alpha(t) dt,$$

the integral converging uniformly for $x_1 \leq x \leq x_2$. We have

$$\int_{x_1}^{x_2} F_n^*(D)f(x) dx = \int_{x_1}^{x_2} dx \int_{-\infty}^\infty \left[\frac{\partial}{\partial x} G_n(x-t) \right] \alpha(t) dt.$$

Because of the uniform convergence of the inner integral we may invert the

order of integration and we have

$$(2) \quad \int_{x_1}^{x_2} F_n^*(D)f(x)dx = \int_{-\infty}^{\infty} [G_n(x_2-t) - G_n(x_1-t)]\alpha(t)dt.$$

Using Theorem 1, Theorem 4 and the estimate (1), we see that if n is sufficiently large, the integral (2) will converge absolutely.

LEMMA 4. *If $G_n(t)$ is defined as in (3) of §4, then*

$$\lim_{n \rightarrow \infty} G_n(t) = 0 \quad (0 < |t| < \infty).$$

Proof. Let t_0 be an arbitrary number different from zero. Then we have

$$\begin{aligned} \left| \int_{t_0}^{t_0/2} G_n(t) dt \right| &\leq \int_{|t| > |t_0|/2} G_n(t) dt \leq \frac{|t_0|^2}{4} \int_{-\infty}^{\infty} t^2 G_n(t) dt \\ &= \frac{|t_0|^2}{2} \left(\sum_{n+1}^{\infty} a_k^{-2} - \sum_{n+1}^{\infty} c_k^2 \right). \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \left| \int_{t_0}^{t_0/2} G_n(t) dt \right| = 0.$$

But by Theorem 2 $G_n(t)$ is monotonic over the range of this integral and takes its smallest value at t_0 ; i.e.

$$G_n(t_0) \frac{|t_0|}{2} \leq \left| \int_{t_0}^{t_0/2} G_n(t) dt \right|.$$

From this inequality the result follows immediately.

THEOREM 7. *If*

1. $G(t)$ is defined as in Theorem 1,
2. $\varphi(t)$ is integrable on every finite interval,
3. $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$ converges,
4. $F_n^*(D)$ is defined by (2) of §4,

and

5. $\varphi(t)$ is continuous at x ,

then

$$\lim_{n \rightarrow \infty} F_n^*(D)f(x) = \varphi(x).$$

Proof. By Corollary 5 we have

$$F_n^*(D)f(x) = \int_{-\infty}^{\infty} G_n(x-t)\varphi(t)dt.$$

Since $G_n(t)$ is a frequency function

$$F_n^*(D)f(x) - \varphi(x) = \int_{-\infty}^{\infty} G_n(x-t)[\varphi(t) - \varphi(x)] dt$$

By the condition 5, for an arbitrary $\varepsilon > 0$ we may choose $\delta > 0$ so small that

$$|\varphi(t) - \varphi(x)| \leq \varepsilon \quad |t - x| \leq \delta.$$

Put

$$\int_{-\infty}^{\infty} G_n(x-t)[\varphi(t) - \varphi(x)] dt = \int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty} = I_1 + I_2 + I_3, \text{ say.}$$

We have

$$|I_2| \leq \varepsilon \int_{x-\delta}^{x+\delta} G_n(x-t) dt \leq \varepsilon \int_{-\infty}^{\infty} G_n(t) dt = \varepsilon.$$

$$I_3 = \int_{x+\delta}^{\infty} G_n(x-t)[\varphi(t) - \varphi(x)] dt = \int_{-\infty}^{\infty} \left[\frac{G_n(x-t)}{G(x-t)} \right] [G(x-t)\{\varphi(t) - \varphi(x)\}] dt.$$

By Theorem 1 and Theorem 4, for $\varepsilon > 0$, there exists T_0 such that for sufficiently large n

$$\left| \frac{G_n(x-t)}{G(x-t)} \right| < \varepsilon \quad (t > T_0).$$

Thus

$$\left| \int_{T_0}^{\infty} \left[\frac{G_n(x-t)}{G(x-t)} \right] [G(x-t)\{\varphi(t) - \varphi(x)\}] dt \right| < \varepsilon O(1).$$

Furthermore by Lemma 4 we have

$$\lim_{n \rightarrow \infty} \left| \int_{x+\delta}^{T_0} G_n(x-t)\{\varphi(t) - \varphi(x)\} dt \right| = 0.$$

Hence

$$\overline{\lim}_{n \rightarrow \infty} |I_3| \leq \varepsilon(1 + O(1)),$$

and similarly

$$\overline{\lim}_{n \rightarrow \infty} |I_1| \leq \varepsilon(1 + O(1)).$$

Thus we get

$$\overline{\lim}_{n \rightarrow \infty} \left| \int_{-\infty}^{\infty} G_n(x-t)[\varphi(t) - \varphi(x)] dt \right| \leq \varepsilon O(1),$$

Since ε is arbitrary our theorem is proved.

THEOREM 8. *If*

1. $G(t)$ is defined as in Theorem 1,
2. $\alpha(t)$ is of bounded variation in every finite interval,
3. $f(x) = \int_{-\infty}^{\infty} G(x-t) d\alpha(t)$ converges,
4. $F_n^*(D)$ is defined by (2) of §4,

and

5. $\alpha(t)$ is continuous at x_1 and x_2 ,

then

$$\lim_{n \rightarrow \infty} \int_{x_1}^{x_2} F_n^*(D)f(x)dx = \alpha(x_2) - \alpha(x_1).$$

This is an immediate consequence of Theorem 6 and Theorem 7.

REFERENCES

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