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On the copula for multivariate extreme value distributions

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Abstract. We show that all multivariate extreme value distributions, which are the possible weak limits of the K largest order statistics of i.i.d. samples, have the same copula, the so called K-extremal copula. This copula and its density are described through exact expressions. We also study measures of dependence, we obtain a weak convergence result and we propose a simulation algorithm for the K-extremal copula.

1 Introduction

In the study of extremes of i.i.d. sequences, a question of interest is whether or not the dependence relation among the marginals of the limit distribution of the K largest order statistics relies on the parent distribution function of the sequence. One way to evaluate nonlinear dependence between random variables is through the copula associated to them, this is already discussed in several books as the ones by Joe (1997), Nelsen (2006) and Drouet-Mari and Kotz (2001). In the present paper, we show that every multivariate extreme value distribution, which are the possible weak limits of the K largest order statistics of i.i.d. samples, have the same copula called the K-extremal copula. From the extremal types theorem, see below, extremal distributions are obtained from linear transformations of one of three basic distributions. We prove that the copula for the three basic types is the K-extremal copula, thus all K-dimensional multivariate extremal distribution have the same nonlinear dependence among its marginals. This is not remarkable since the copula for any group of order statistics of an i.i.d. sample of size n with continuous parent distribution do not depend on this distribution, see Lemma 6 in Averous, Genest and Kochar (2005). However, a proper characterization of the K-extremal copula is relevant as well as their consequences. Our result generalizes the case $K = 2$ which was considered in Mendes and Sanfins (2007).

The K-extremal copula is a K-dimensional continuous distribution function which, together with its density, will be described through exact expressions. We show that the copula of the K largest order statistics of i.i.d. sequences with continuous parent distribution converges in distribution to the K-extremal copula.

Key words and phrases. Copula, order statistics, independent random variables, extreme value distribution.

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We also study the behavior of Spearman's rho and Kendall's tau for the bivariate marginals of the K-extremal copula. As a last result, we propose a simulation algorithm to sample from the K-extremal copula.

As pointed out above, this paper deals with a multivariate copula. Most of the literature on copulas focusses on the bivariate case. In contrast, there are only few descriptions and construction schemes of higher dimensional copulas. For more on recent results and innovations concerning multivariate copulas, we suggest to the reader the papers Aas et al. (2009), Liebscher (2008) and Morillas (2005), together with the references therein.

In Section 2, we will present and discuss the results in this paper postponing all the proofs to Section 3.

2 Statements

Fix an integer $K \geq 2$. For every $n \geq K$, let $M_{1,n}, \dots, M_{K,n}$ be the K largest order statistics of an i.i.d. sample of size n with parent distribution not depending on n. The extremal types theorem, see Sections 2.2 and 2.3 in Leadbetter, Lindgreen and Rootzen (1983) and Section 4.2 in Embrechts, Kluppelberg and Mikosch (1997), states that if for some sequences of real numbers $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ the random variables $a_n M_{1,n} + b_n$ converge in distribution then the random vectors

$$(a_n M_{1,n} + b_n, \dots, a_n M_{K,n} + b_n) \tag{2.1}$$

also converge in distribution. The limit belongs to a family of distributions parametrized by $-\infty < \mu < \infty$, $\sigma > 0$ and $-\infty < \xi < \infty$. For a choice (μ, σ, ξ) of the parameters, the marginals of a limit distribution have distribution function and density functions given respectively, by

$$G_m(z) = \begin{cases} \exp\{-\Lambda(z)\} \sum_{j=0}^{m-1} \frac{\Lambda(z)^j}{j!}, & \text{if } \xi\left(\frac{z-\mu}{\sigma}\right) > -1 \text{ for } \xi \neq 0 \text{ or } \\ & \quad z \in \mathbb{R} \text{ for } \xi = 0, \\ 0, & \text{if } z < \mu - \frac{\sigma}{\xi} \text{ for } \xi > 0, \\ 1, & \text{if } z > \mu - \frac{\sigma}{\xi} \text{ for } \xi < 0. \end{cases} \tag{2.2}$$

and

$$g_m(z) = \begin{cases} \exp\{-\Lambda(z)\} \frac{\Lambda'(z)\Lambda(z)^{m-1}}{(m-1)!}, & \\ \qquad \text{if } \xi\left(\frac{z-\mu}{\sigma}\right) > -1 \text{ for } \xi \neq 0 \text{ or} & \\ \qquad z \in \mathbb{R} \text{ for } \xi = 0, & \\ 0, \quad \text{otherwise,} & \end{cases} \tag{2.3}$$

where

$$\Lambda(z) = \Lambda_{\xi,\mu,\sigma}(z) = \begin{cases} \left[1 + \xi\left(\dfrac{z-\mu}{\sigma}\right)\right]^{-1/\xi}, & \text{if } \xi \neq 0, \\ \exp\left(-\dfrac{z-\mu}{\sigma}\right), & \text{if } \xi = 0 \end{cases}$$

for $m \geq 1$. A distribution function as above is called a Generalized Extreme Value (GEV) distribution which are classified in types I, II and III according, respectively, to $\xi = 0$, $\xi > 0$ and $\xi < 0$. Note that the function Λ is strictly decreasing positive function and satisfies

$$\begin{aligned} &\lim_{z \to -\infty} \Lambda(z) = +\infty \quad \text{and} \quad \lim_{z \to \infty} \Lambda(z) = 0, && \text{if } \xi = 0, \\ &\lim_{z \downarrow (\mu - \sigma/\xi)} \Lambda(z) = +\infty \quad \text{and} \quad \lim_{z \to \infty} \Lambda(z) = 0, && \text{if } \xi > 0, \\ &\lim_{z \to -\infty} \Lambda(z) = +\infty \quad \text{and} \quad \lim_{z \uparrow (\mu - \sigma/\xi)} \Lambda(z) = 0, && \text{if } \xi < 0. \end{aligned} \tag{2.4}$$

Also by the extremal types theorem, the joint density function $\tilde{g}_K$ of a limiting extreme value distribution for normalized sums of the K largest order statistics of an i.i.d. sequence, as in (2.1), is given by

$$\tilde{g}_K(z_1, \ldots, z_K) = \begin{cases} (-1)^K \exp\{-\Lambda(z_K)\} \displaystyle\prod_{j=1}^{K} \Lambda'(z_j), \\ \qquad \text{if } (z_1, \ldots, z_K) \in \Omega_\xi, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{2.5}$$

where

$$\Omega_\xi = \begin{cases} \mathbb{R}^K, & \text{if } \xi = 0, \\ \left\{(z_1, \ldots, z_K) \in \mathbb{R}^K : z_1 > \cdots > z_K > \mu - \dfrac{\sigma}{\xi}\right\}, & \text{if } \xi > 0, \\ \left\{(z_1, \ldots, z_K) \in \mathbb{R}^K : \mu - \dfrac{\sigma}{\xi} > z_1 > \cdots > z_K\right\}, & \text{if } \xi < 0. \end{cases}$$

A continuous distribution function with density as in (2.5) for parameters $-\infty < \mu < \infty$, $\sigma > 0$ and $-\infty < \xi < \infty$ is called a Multivariate Generalized Extreme Value (MGEV) distribution function.

Remark 2.1. A broader class of stationary sequences of random variables have a MGEV distribution as the assymptotic distribution of the largest maxima. These sequences should satisfy some weak dependence condition. The results can be found for instance in Embrechts, Kluppelberg and Mikosch (1997).

Our first result gives an explicitly expression for the distribution function associated to the density $\tilde{g}_K$.

Proposition 2.1. *The distribution function $\tilde{G}_K$ of a limiting extreme value distribution for a normalized vector of the K largest order statistics of i.i.d. continuous random variables has the following representation*

$$\tilde{G}_K(z_1, \dots, z_K) = H_K(z_1, \min(z_1, z_2), \min(z_1, z_2, z_3), \dots, \min(z_1, \dots, z_K))$$

for every $(z_1, \dots, z_K) \in \mathbb{R}^K$, where

$$H_K(z_1, \dots, z_K) = \exp\{-\Lambda(z_K)\} J_K(\Lambda(z_1), \dots, \Lambda(z_K))$$

for $\min(z_1, \dots, z_K) > \mu - \frac{\sigma}{\xi}$, *if* $\xi > 0$, *or for* $\min(z_1, \dots, z_K) < \mu - \frac{\sigma}{\xi}$, *if* $\xi < 0$, *or* $(z_1, \dots, z_K) \in \mathbb{R}^K$, *if* $\xi = 0$, *otherwise* $H_K(z_1, \dots, z_K) = 0$. *The function $J_K : \mathbb{R}_+^K \to \mathbb{R}_+$ is a polynomial in K variables which is defined by induction by putting $J_1 \equiv 1$ and*

$$J_m(x_1, \dots, x_m) = \sum_{j=0}^{m-1} \frac{x_m^j}{j!} - \sum_{j=1}^{m-1} \frac{x_j^j}{j!} J_{m-j}(x_{j+1}, \dots, x_m) \qquad \textit{for } m \geq 1.$$

We can now compute the density of the copula associated to the density $\tilde{g}_K$ of a MGEV distribution function, which we call the K-extremal copula and turns out to not depend on the parameters ξ, μ and σ.

Proposition 2.2. *The density of the copula of a MGEV distribution function is given by*

$$c_K(u_1, \dots, u_K) = \left(\prod_{j=1}^{K-1} \frac{d \log \psi_j}{d u_j}(u_j) \right) \frac{d\psi_K}{du_K}(u_K) \tag{2.6}$$

$$= \left(\prod_{j=1}^{K-1} (-1)^{j-1} \psi_j(u_j) \frac{(\log \psi_j(u_j))^{j-1}}{(j-1)!} \right)^{-1} \times \left(\frac{(-\log \psi_K(u_K))^{K-1}}{(K-1)!} \right)^{-1} \tag{2.7}$$

for $(u_1, \dots, u_K) \in (0, 1)^K$ such that $u_1 > \psi_2(u_2) > \cdots > \psi_K(u_K)$, where $\psi_m : (0, 1) \to (0, 1)$ is the increasing function that satisfies the following implicit equation

$$u = \psi_m(u) \sum_{j=0}^{m-1} (-1)^j \frac{(\log \psi_m(u))^j}{j!}, \tag{2.8}$$

otherwise $c_K(u_1, \dots, u_K) = 0$.

Remark 2.2. The function ψ_m which appears in the expression for the density of the K-extremal copula can be obtained from a MGEV distribution function as $\psi_m(u) = \exp\{-\Lambda(G_m^{-1}(u)\}$ for every $u \in (0, 1)$ and $m \geq 1$.

Also with the expression of the MGEV distribution function, it is straightforward to write the expression for the K-extremal copula which we present in the next result.

Proposition 2.3. *The copula of a MGEV distribution is given by*

$$C_K(u_1,\dots,u_K) = \mathcal{H}_K(u_1, r_1(u_1,u_2), r_2(u_1,u_2,u_3),\dots,r_{K-1}(u_1,\dots,u_K))$$

for every $(u_1,\dots,u_K)\in[0,1]^K$, *where*

$$r_{m-1}(u_1,\dots,u_m) = \psi_m^{-1}(\psi_l(u_l)) = \psi_l(u_l)\sum_{j=0}^{m-1}(-1)^j\frac{(\log\psi_l(u_l))^j}{j!},$$

if $\psi_l(u_l) = \min(\psi_1(u_1),\dots,\psi_m(u_m))$ *and for every* $(u_1,\dots,u_K)$ *such that* $u_1 = \psi_1(u_1)\geq\psi_2(u_2)\geq\cdots\geq\psi_K(u_K)$

$$\begin{aligned}
&\mathcal{H}_K(u_1,\dots,u_K)\\
&\quad= \psi_K(u_K)J_K(-\log u_1, -\log\psi_2(u_2),\dots,-\log\psi_K(u_K))\\
&\quad= u_K - \psi_K(u_K)\sum_{j=1}^{K-1}\frac{(-\log\psi_j(u_j))^j}{j!}\\
&\qquad\qquad\times J_{K-j}(-\log\psi_{j+1}(u_{j+1}),\dots,-\log\psi_K(u_K))
\end{aligned}$$

with J_m *defined in the statement of Proposition* 2.1.

By a simple generalization of Lemma 6 in Averous, Genest and Kochar (2005), we have that the multivariate copula of the K largest order statistics of an i.i.d. sample of size n do not depend on the continuous parent distribution of the sample. This copula will be denoted by $\tilde{C}_K^{(n)}$, where n denotes the size of the sample. The next proposition is a convergence result for copulas that has the consequence that for continuous distributions the nonlinear dependence structure of the K-largest order statistics of large i.i.d. samples is approximately captured by the K-extremal copula.

Proposition 2.4. *The copula* $\tilde{C}_K^{(n)}$ *converges in distribution to* C_K *as* $n\to\infty$.

From the K-extremal copula, we can obtain the copula between the m largest and the l largest limiting order statistics for every choice of m and l, or between any two marginals of a MGEV distribution. Then we can use these bivariate copulas to obtain measures of dependence as the Spearman's rho and Kendall's tau. For a copula C, the Spearman's rho is defined by

$$12\int_0^1\int_0^1 C(u,v)\,du\,dv - 3 = 12\int_0^1\int_0^1 uv\,dC(u,v) - 3$$

and Kendall's tau by

$$4\int_0^1\int_0^1 C(u,v)\,dC(u,v)-1.$$

We are going to study here the behavior of Spearman's rho and Kendall's tau for the mth and the lth marginals of the K-extremal, for $1 \le m < l \le K$ and $K \ge 2$. We denote these measures respectively by $\rho_{m,l}$ and $\tau_{m,l}$. We point out that $\rho_{1,2} = 2/3$ and $\tau_{1,2} = 1/2$ have been obtained in Mendes and Sanfins (2007). For more on measures of dependence of order statistics see Averous, Genest and Kochar (2005) and Chen (2007).

As a first result, we show that $\rho_{1,K}$ and $\tau_{1,K}$ converge to zero as $K \to \infty$. Using the convergence result in Proposition 2.4, this characterizes the behavior of these measures for the first and the Kth largest order statistics of large samples with continuous parent distribution.

Proposition 2.5. *Both sequences* $(\rho_{1,K})$ *and* $(\tau_{1,K})$ *converges to zero as* $K \to \infty$.

Remark 2.3. In the proof of Proposition 2.5, we obtain an explicit expression for $\rho_{1,K}$ which is

$$\rho_{1,K} = 12\left((-1)^{K-1}\sum_{l=0}^{K-1}\sum_{j=K-1}^{\infty}(-1)^j\binom{l+j}{l}\frac{1}{2^{l+j+1}}\right)-3. \tag{2.9}$$

From this formula, we can verify that $\rho_{1,2} = 2/3$ and show that $\rho_{1,3} = 19/36$. For the sake of completeness, we have included the straightforward computation of $\rho_{1,2}$ and $\rho_{1,3}$ in Section 3, just after the proof of Proposition 2.5. For $\tau_{1,K}$, we were not able to obtain an explicit expression.

Still concerning measures of dependence, it is reasonable to expect that both $\rho_{m,l}$ and $\tau_{m,l}$ decrease as l increases and increase as m increases. In Tables 1

Table 1 *Estimates for Spearman's rho*, $\rho_{m,l}$, *obtained from the bivariate marginals of the* 10-*extremal copula using a sample of size* 10,000

$\rho_{m,l}$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$	$l=7$	$l=8$	$l=9$	$l=10$
$m=1$	1.000	0.664	0.525	0.446	0.395	0.358	0.330	0.308	0.289	0.274
$m=2$		1.000	0.794	0.675	0.597	0.541	0.498	0.464	0.436	0.412
$m=3$			1.000	0.850	0.751	0.680	0.626	0.582	0.547	0.517
$m=4$				1.000	0.882	0.798	0.734	0.682	0.641	0.605
$m=5$					1.000	0.903	0.829	0.771	0.723	0.683
$m=6$						1.000	0.917	0.852	0.798	0.754
$m=7$							1.000	0.927	0.869	0.820
$m=8$								1.000	0.935	0.882
$m=9$									1.000	0.942
$m=10$										1.000

Table 2 *Estimates for Kendall's tau, $\tau_{m,l}$, obtained from the bivariate marginals of the* 10*-extremal copula using a sample of size* 10,000

$\tau_{m,l}$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$	$l=7$	$l=8$	$l=9$	$l=10$
$m=1$	1.000	0.500	0.375	0.313	0.274	0.247	0.226	0.210	0.196	0.185
$m=2$		1.000	0.625	0.500	0.430	0.383	0.349	0.322	0.301	0.283
$m=3$			1.000	0.688	0.570	0.500	0.451	0.414	0.385	0.362
$m=4$				1.000	0.726	0.617	0.549	0.500	0.462	0.432
$m=5$					1.000	0.754	0.651	0.585	0.537	0.500
$m=6$						1.000	0.774	0.678	0.614	0.567
$m=7$							1.000	0.790	0.699	0.637
$m=8$								1.000	0.804	0.716
$m=9$									1.000	0.814
$m=10$										1.000

and 2, just below, we present a numerical evidence of this fact listing all the values of $\rho_{m,l}$ and $\tau_{m,l}$ estimated from a sample of the 10-Extremal copula. The procedure to sample from the copula is discussed later in this section.

We here present a formal proof that $\rho_{m,l}$ increases as m increases.

Proposition 2.6. *For every $0 < m < l$, $\rho_{m,l} \le \rho_{m+1,l}$.*

We now describe a simulation algorithm to generate samples from the K-extremal copula. The method is based on a technique of conditional sampling to sample from multivariate copulas, see, for instance, Cherubini, Luciano and Vecchiato's book Cherubini, Luciano and Vecchiato (2004). We can resume the procedure with the following steps:

(i) Put $C_m(u_1, u_2, \dots, u_m) = C(u_1, u_2, \dots, u_m, 1, \dots, 1)$ for $m = 2, \dots, K$;
(ii) Sample u_1 from the uniform distribution in (0, 1);
(iii) Sample u_m from the conditional distribution $C_m(\cdot|u_1, \dots, u_{m-1})$ for $m = 2, \dots, K$;

We now are going to focus on how to sample u_k from the conditional distribution $C_k(\cdot|u_1, \dots, u_{k-1})$. To sample u_m from $C_m(\cdot|u_1, \dots, u_{m-1})$, we sample q from $U(0, 1)$ and we put $u_m = C_m^{-1}(q|u_1, \dots, u_{m-1})$. Therefore, we should know explicitly $C_m(\cdot|u_1, \dots, u_{m-1})$. We compute it in the following lemma.

Lemma 2.7. *The conditional distribution function of $U_m|(U_1, U_2, \dots, U_{m-1})$ when $(U_1, \dots, U_K)$ has distribution function given by the K-extremal copula is given by*

$$C_m(u_m|u_1, \dots, u_{m-1}) = \frac{\psi_m(u_m)}{\psi_{m-1}(u_{m-1})}. \tag{2.10}$$

If we now put $q = C_m(u_m|u_1, \ldots, u_{m-1})$, we have that:

$$u_m = C_m^{-1}(q|u_1, \ldots, u_{m-1}) = \psi_m^{-1}(q \cdot \psi_{m-1}(u_{m-1})).$$

From definition (2.8), we get

$$u_m = \psi_m(q \cdot \psi_{m-1}(u_{m-1})) \sum_{j=0}^{m-1} (-1)^j \frac{(\log \psi_m(q \cdot \psi_{m-1}(u_{m-1})))^j}{j!}.$$

Therefore, we solve numerically $\psi_{m-1}(u_{m-1})$ and then $\psi_m(q \cdot \psi_{m-1}(u_{m-1}))$ to obtain u_m.

In Figure 1, we plot the bivariate marginals taken from a sample of size 200 of the 4-extremal copula.

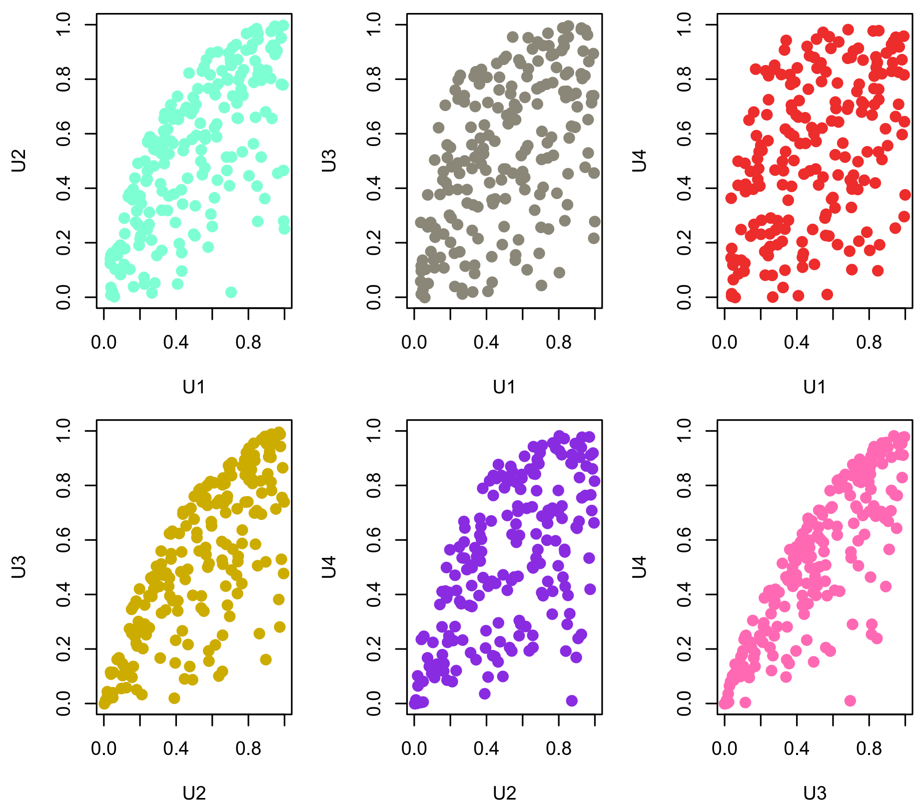

Figure 1 *The six bivariate samples taken from a unique sample of size* 200 *of the* 4*-extremal copula whose marginals are denoted by* $U_1, \ldots, U_4$.

3 Proofs

Proof of Proposition 2.1. We show that $\tilde{G}_K$ is a K-dimensional distribution function with density given by $\tilde{g}_K$. By the definition of $\tilde{g}_K$, the multiple integral

$$\int_{-\infty}^{z_1} \cdots \int_{-\infty}^{z_K} \tilde{g}_K(y_1, \dots, y_K)\, dy_1 \cdots dy_K$$

is equal to

$$\int_{-\infty}^{z_1} \int_{-\infty}^{\min(z_1,z_2)} \cdots \int_{-\infty}^{\min(z_1,\dots,z_K)} \tilde{g}_K(y_1, \dots, y_K)\, dy_1 \cdots dy_K.$$

Therefore, $\tilde{G}_K(z_1, \dots, z_K) = \tilde{G}_K(z_1, \min(z_1, z_2), \dots, \min(z_1, \dots, z_K))$. From now on, we suppose that $z_1 > z_2 > \cdots > z_K$. Then, from (2.5), $\tilde{G}_K(z_1, \dots, z_K)$ is equal to

$$(-1)^K \int_{A_\xi}^{z_K} \int_{y_K}^{z_{K-1}} \cdots \int_{y_3}^{z_2} \int_{y_2}^{z_1} \exp\{-\Lambda(y_K)\} \prod_{j=1}^{K} \Lambda'(y_j)\, dy_1 \cdots dy_K,$$

where $A_\xi = \mu - \frac{\sigma}{\xi}$, if $\xi > 0$, and $A_\xi = -\infty$ otherwise. Considering the following change of variables in the last integral, $x_j = \Lambda(y_j)$, for $1 \le j \le K$, we get the following integral

$$I_K(w_1, \dots, w_K) := (-1)^K \int_{w_K}^{+\infty} \int_{w_{K-1}}^{x_K} \cdots \int_{w_2}^{x_3} \int_{w_1}^{x_2} e^{-x_K} dx_1 \cdots dx_K,$$

where $w_j = \Lambda(z_j)$. To complete the proof, we show by induction that

$$I_K(w_1, \dots, w_K) = e^{-w_K} J_K(w_1, \dots, w_K).$$

For $K = 1$, a simple verification shows that the result holds. Now suppose that it holds for $1 \le K \le L - 1$. For $K = L$, we perform the first iterated integral in the expression for $I_K(w_1, \dots, w_K)$ to obtain that it is equal to

$$(-1)^K \int_{w_K}^{+\infty} \int_{w_{K-1}}^{x_K} \cdots \int_{w_2}^{x_3} x_2 e^{-x_K} dx_2 \cdots dx_K - w_1 I_{K-1}(w_2, \dots, w_K).$$

Then perform the first iterated integral in the first term of the previous expression to obtain

$$(-1)^K \int_{w_K}^{+\infty} \int_{w_{K-1}}^{x_K} \cdots \int_{w_3}^{x_4} \frac{x_3}{2} e^{-x_K}\, dx_3 \cdots dx_K$$
$$- \frac{w_2}{2} I_{K-2}(w_3, \dots, w_K) - w_1 I_{K-1}(w_1, \dots, w_K).$$

Following recursively this procedure, we get

$$I_K(w_1, \dots, w_K) = e^{-w_K} \sum_{j=0}^{m-1} \frac{w_K^j}{j!} - \sum_{j=1}^{m-1} \frac{w_j^j}{j!} I_{K-j}(w_{j+1}, \dots, w_K).$$

By the definition of J_K and the induction hypotheses, we complete the proof. □

Proof of Proposition 2.2. Let us fix a limiting extreme value distribution function $\tilde{G}_K$. We have that

$$c_K(u_1, \dots, u_K) = \frac{\tilde{g}_K(G_1^{-1}(u_1), \dots, G_K^{-1}(u_K))}{\prod_{j=1}^K g_j(G_j^{-1}(u_j))}.$$

Therefore, we just apply formulas (2.3) and (2.5) to obtain that $c_K(u_1, \dots, u_K)$ is equal to

$$\left(\prod_{j=1}^{K-1} \exp\{-\Lambda(G_j^{-1}(u_j))\} \frac{\Lambda(G_j^{-1}(u_j))^{j-1}}{(j-1)!}\right)^{-1} \left(\frac{\Lambda(G_K^{-1}(u_K))^{K-1}}{(K-1)!}\right)^{-1}.$$

From this formula, if we put $\psi_m(u) = \exp\{-\Lambda(G_m^{-1}(u)\}$ we get (2.7) in the statement. Now (2.8) is a direct consequence of the explicit formulas for the distribution function G_m given in (2.2).

It remains to verify (2.6). If we derive both sides of (2.8), we get that

$$\begin{aligned} 1 &= \left(\sum_{j=0}^{m-1} (-1)^j \frac{(\log \psi_m)^j}{(j)!} - \sum_{j=0}^{m-2} (-1)^j \frac{(\log \psi_m)^j}{(j)!}\right) \frac{d\psi_m}{du} \\ &= (-1)^{m-1} \frac{(\log \psi_m)^{m-1}}{(m-1)!} \frac{d\psi_m}{du}, \end{aligned}$$

which implies that

$$\frac{d\psi_m}{du} = (-1)^{m-1} \left(\frac{(\log \psi_m)^{m-1}}{(m-1)!}\right)^{-1} \tag{3.1}$$

and

$$\frac{d \log \psi_m}{du} = (-1)^{m-1} \left(\psi_m \frac{(\log \psi_m)^{m-1}}{(m-1)!}\right)^{-1}. \tag{3.2}$$

From (3.1), (3.2) and (2.7), we arrive at (2.6). □

Proof of Proposition 2.3. Let us fix a limiting extreme value distribution function $\tilde{G}_K$. Then the K-extremal copula is given by

$$C_K(u_1, \dots, u_K) = \tilde{G}_K(G_1^{-1}(u_1), \dots, G_K^{-1}(u_K))$$

for every $(u_1, \dots, u_K) \in [0, 1]^K$ which by Proposition 2.1 is equal to

$$H_K(G_1^{-1}(u_1), \min(G_1^{-1}(u_1), G_2^{-1}(u_2)), \dots, \min(G_1^{-1}(u_1), \dots, G_K^{-1}(u_K))).$$

By the definition of H_K, monotonicity and the expression for ψ_m in Remark 2.2, see also the proof of Proposition 2.2, the previous expression is equal to

$$\min_{1\le l\le K}(\psi_l(u_l))J_K\Big(-\log u_1, -\log\min_{l=1,2}(\psi_l(u_l)),\dots,-\log\min_{1\le l\le K}(\psi_l(u_l))\Big).$$

Using the definition of r_m in the statement, write the above expression as

$$\psi_K(r_K(u_1,\dots,u_m))J_K(-\log u_1, -\log\psi_2(r_2(u_1,u_2)),\dots,$$
$$-\log\psi_K(r_K(u_1,\dots,u_m))),$$

which completes the proof. □

Proof of Proposition 2.4. Let $M_{1,n},\dots,M_{K,n}$ be the K-largest order statistics of a sample of size n with a given continuous parent distribution function F which belongs to the domain of attraction of a GEV distribution. This means that there exists $(a_n)_{n=1}^{+\infty}$ and $(b_n)_{n=1}^{+\infty}$ sequences of real numbers such that the random vector

$$(a_nM_{1,n}+b_n,\dots,a_nM_{K,n}+b_n)$$

converges in distribution to some $\tilde{G}_K$ which is a MGEV distribution function. By invariance concerning composition with affine transformations the copula associated to $(M_{1,n},\dots,M_{K,n})$ and $(a_nM_{1,n}+b_n,\dots,a_nM_{K,n}+b_n)$ is $\tilde{C}_K^{(n)}$ independently of F.

Let $F_{j,n}$ be the distribution function of $a_nM_{j,n}+b_n$. Therefore, if we define the function $V_n(x_1,\dots,x_K)=(F_{1,n}(x_1),\dots,F_{K,n}(x_K))$, $(x_1,\dots,x_K)\in\mathbb{R}^n$ then

$$V_n(a_nM_{1,n}+b_n,\dots,a_nM_{K,n}+b_n) \tag{3.3}$$

has the distribution function equal to the copula $\tilde{C}_K^{(n)}$.

The K-extremal copula is the distribution function of $V(Y_1,\dots,Y_K)$, where $V(x_1,\dots,x_K)=(G_1(x_1),\dots,G_K(x_K))$, $(x_1,\dots,x_K)\in\mathbb{R}^n$. By Theorem 5.1 in Billingsley (1968), (3.3) converges in distribution to the K-extremal copula if V_n converges uniformly to V on compact intervals, but this is a consequence of Pólyas's theorem which implies that $F_{j,n}$ converges uniformly to G_j since the last is absolutely continuous. □

Proof of Proposition 2.5. We shall prove through estimates on exact expressions that $\rho_{1,K}\to 0$. The analogous result can be applied to $\tau_{1,K}$ since $\rho_{1,K}\ge\tau_{1,K}\ge 0$. This last assertion can be verified through Theorem 5.1 of Fredricks and Nelsen (2007). Indeed, according to their terminology, for two order statistics, the largest is always left-tail decreasing and smallest is right-tail increasing.

Applying directly the definition, we can write $(\rho_{1,K}+3)/12$ as

$$\int_0^1\int_{\psi_{K-1}^{-1}(\psi_K(u_K))}^1\cdots\int_{\psi_2^{-1}(\psi_3(u_3))}^1\int_{\psi_2(u_2)}^1 u_1u_Kc_K(u_1,\dots,u_K)\,du_1\cdots du_K \tag{3.4}$$

which we are going to show that converges to $1/4$ as $K \to \infty$ resulting in $\rho_{1,K} \to 0$. By (2.6), the previous iterated integral can be rewritten as

$$\int_0^1 \int_{\psi_{K-1}^{-1}(\psi_K(u_K))}^1 \cdots \int_{\psi_2^{-1}(\psi_3(u_3))}^1 \int_{\psi_2(u_2)}^1 u_1 u_K \times \left(\prod_{j=1}^{K-1} \frac{d \log \psi_j}{du_j}(u_j) \right) \frac{d\psi_K}{du_K}(u_K)\, du_1 \cdots du_K.$$

By induction in $1 \leq m \leq K-1$, we show that

$$\int_{\psi_m^{-1}(\psi_{m+1}(u_{m+1}))}^1 \cdots \int_{\psi_2^{-1}(\psi_3(u_3))}^1 \int_{\psi_2(u_2)}^1 u_1 \prod_{j=1}^{m} \frac{d \log \psi_j}{du_j}(u_j)\, du_1 \cdots du_m$$

is equal to

$$(-1)^m \left[\psi_{m+1}(u_{m+1}) - \sum_{j=0}^{m-1} \frac{(\log \psi_{m+1}(u_{m+1}))^j}{j!} \right]. \tag{3.5}$$

Indeed, ψ_1 is the identity function in $(0, 1)$ and therefore

$$\int_{\psi_2(u_2)}^1 u_1 \frac{d \log \psi_1}{du_1}(u_1)\, du_1 = (-1)[\psi_2(u_2) - 1].$$

Now suppose that (3.5) holds for some $1 \leq l \leq K-2$ then

$$(-1)^l \left[\psi_{l+1}(u_{l+1}) - \sum_{j=0}^{l-1} \frac{(\log \psi_{l+1}(u_{l+1}))^j}{j!} \right] \frac{d \log \psi_{l+1}}{du_{l+1}}(u_{l+1})$$

is equal to

$$(-1)^l \frac{d}{du_{l+1}} \left(\psi_{l+1}(u_{l+1}) - \sum_{j=1}^{l} \frac{(\log \psi_{l+1}(u_{l+1}))^j}{j!} \right)$$

and, since $\psi_{l+1}(1) = 1$, integrating on the variable u_{l+1} over the interval $(\psi_{l+1}^{-1}(\psi_{l+2}(u_{l+2})), 1)$, we obtain that (3.5) holds for $m = l + 1$.

Therefore, the integral in (3.4) is equal to

$$\int_0^1 u \frac{d\psi_K}{du}(u)(-1)^{K-1} \left[\psi_K(u) - \sum_{j=0}^{K-2} \frac{(\log \psi_K(u))^j}{j!} \right] du.$$

Put $v = \psi_K(u)$, $u \in (0, 1)$ and uses the power series expansion

$$v = \sum_{j=0}^{\infty} \frac{\log(v)^j}{j!}$$

to write the previous integral as

$$(-1)^{K-1}\int_0^1 \psi_K^{-1}(v)\left(\sum_{j=K-1}^{\infty}\frac{\log(v)^j}{j!}\right)dv.$$

Another change of variables and (2.8) allows us to write the integral in (3.4) as

$$(-1)^{K-1}\sum_{l=0}^{K-1}\sum_{j=K-1}^{\infty}\frac{(-1)^j}{j!l!}\int_0^{+\infty} y^{l+j}e^{-2y}\,dy$$

which, since

$$\int_0^{+\infty} y^{l+j}e^{-2y}\,dy=\frac{(l+j)!}{2^{l+j+1}},$$

can be rewritten as

$$(-1)^{K-1}\sum_{l=0}^{K-1}\sum_{j=K-1}^{\infty}(-1)^j\binom{l+j}{l}\frac{1}{2^{l+j+1}}.$$

We finish the proof showing that

$$\lim_{K\to\infty}\left\{(-1)^{K-1}\sum_{l=0}^{K-1}\sum_{j=K-1}^{\infty}(-1)^j\binom{l+j}{l}\frac{1}{2^{l+j}}\right\}=\frac{1}{2}.$$

From this point, we suppose that K is odd, for K even the proof is similar with few sign changes. The left-hand side term in the previous convergence statement is equal to

$$\sum_{l=0}^{K-1}\sum_{j=K-1}^{\infty}\binom{l+j}{l}\frac{1}{2^{l+j}}-\sum_{l=0}^{K-1}\sum_{j=(K-1)/2}^{\infty}\binom{l+2j+1}{l}\frac{1}{2^{l+2j}}. \tag{3.6}$$

Now apply the identities

$$\binom{l+2j}{0}=1 \quad\text{and}\quad \binom{l+2j+1}{l}=\binom{l+2j}{l-1}+\binom{l+2j}{l}\qquad \text{for } l\geq 1,$$

to write the second term in (3.6) as

$$\sum_{j=K-1}^{\infty}\binom{2j+1}{K-1}\frac{1}{2^{2j+1}}-\sum_{l=0}^{K-1}\sum_{j=K-1}^{\infty}\binom{l+j}{l}\frac{1}{2^{l+j}}.$$

Therefore, (3.6) is equal to

$$\sum_{j=K-1}^{\infty}\binom{2j+1}{K-1}\frac{1}{2^{2j+1}}$$

which is

$$\sum_{j=2K}^{\infty}\binom{j-1}{K-1}\frac{1}{2^j}+\sum_{j=K-1}^{\infty}\binom{2j+1}{K-1}\Big(1-\frac{2j+2}{2(2j-K+3)}\Big)\frac{1}{2^{2j+2}}.$$

Let Y be a random variable with negative binomial distribution with parameters K and $1/2$. Then the second term in the sum above is equal to

$$\mathbb{E}\Big[\Big(1-\frac{Y}{2(Y-K+1)}\Big)\mathrm{I}\{Y \text{ even}, Y\geq 2K\}\Big],$$

which is bounded above by

$$\begin{aligned}\mathbb{E}\Big[\Big(1-&\frac{Y}{2(Y-K+1)}\Big)\mathrm{I}\{2K\leq Y\geq 2K+K^{3/4}\}\Big]+\mathbb{P}(Y\geq 2K+K^{3/4})\\ &\leq\Big(1-\frac{2+K^{-1/4}}{2+2K^{-1/4}+2K^{-1}}\Big)+\mathbb{P}\Big(\frac{Y-\mathbb{E}[Y]}{\sqrt{2K}}\geq\frac{K^{1/4}}{\sqrt{2}}\Big)\end{aligned}$$

that goes to zero as $K\to\infty$ by the central limit theorem.

Therefore, the limit of (3.6) as $K\to\infty$ is the same as the limit of

$$\sum_{j=2K}^{\infty}\binom{j-1}{K-1}\frac{1}{2^j}$$

which is the probability that a negative binomial distribution with parameters K and $1/2$ takes a value greater or equal to $2K$. This probability converges to $1/2$ again by the central limit theorem. □

Computation of $\rho_{1,2}$ and $\rho_{1,3}$. From Formula (2.9), we have that

$$\begin{aligned}\rho_{1,2}&=12\sum_{l=0}^{1}\sum_{j=1}^{+\infty}(-1)^{j+1}\binom{l+j}{l}\frac{1}{2^{l+j+1}}-3\\ &=12\Big(\frac{3}{2}\sum_{j=1}^{+\infty}\frac{(-1)^{j+1}}{2^{j+1}}+\sum_{j=1}^{+\infty}\frac{(-1)^{j+1}j}{2^{j+2}}\Big)-3\end{aligned}$$

and

$$\begin{aligned}\rho_{1,3}&=12\sum_{l=0}^{2}\sum_{j=2}^{+\infty}(-1)^{j}\binom{l+j}{l}\frac{1}{2^{l+j+1}}-3\\ &=12\Big(\frac{7}{4}\sum_{j=2}^{+\infty}\frac{(-1)^{j}}{2^{j+1}}+\frac{7}{4}\sum_{j=2}^{+\infty}\frac{(-1)^{j}j}{2^{j+2}}+\frac{1}{2}\sum_{j=2}^{+\infty}\frac{(-1)^{j}j^2}{2^{j+3}}\Big)-3.\end{aligned}$$

Just to simplify the calculus of the above series, let Y be a geometric distribution with parameter $3/4$, that is, $P(Y=k)=\frac{3}{4^k}$, for $k=1,2,3,\ldots$. We have the following equalities

$$\sum_{j=1}^{+\infty}\frac{(-1)^{j+1}}{2^{j+1}}=\sum_{k=1}^{+\infty}\frac{1}{4^k}-\frac{1}{2}\sum_{k=1}^{+\infty}\frac{1}{4^k}=\frac{1}{2}\sum_{k=1}^{+\infty}\frac{1}{4^k}=\frac{1}{6},$$

$$\begin{aligned}\sum_{j=1}^{+\infty}\frac{(-1)^{j+1}j}{2^{j+2}}&=\sum_{k=0}^{+\infty}\Big(\frac{2k+1}{2^{2k+3}}-\frac{2k+2}{2^{2k+4}}\Big)\\&=\sum_{k=0}^{+\infty}\frac{k}{2^{2k+3}}=\frac{1}{8}\sum_{k=0}^{+\infty}\frac{k}{4^k}=\frac{E[X-1]}{6}=\frac{1}{18}\end{aligned}$$

and

$$\begin{aligned}\sum_{j=2}^{+\infty}\frac{(-1)^{j}j^2}{2^{j+3}}&=\sum_{k=1}^{+\infty}\Big(\frac{4k^2}{2^{2k+3}}-\frac{(2k+1)^2}{2^{2k+4}}\Big)\\&=\frac{1}{2}\sum_{k=1}^{+\infty}\frac{k^2}{4^k}-\frac{1}{16}\sum_{k=1}^{+\infty}\frac{(2k+1)^2}{4^k}\\&=\frac{E[X^2]}{6}-\frac{E[(2X+1)^2]}{48}=\frac{8}{48}\frac{20}{9}-\frac{1}{48}\frac{137}{9}=\frac{23}{48\times 9}.\end{aligned}$$

Therefore,

$$\rho_{1,2}=12\Big(\frac{3}{2}\frac{1}{6}+\frac{1}{18}\Big)-3=\frac{2}{3}$$

and

$$\rho_{1,3}=12\Big(\frac{7}{4}\Big(\frac{1}{4}-\frac{1}{6}\Big)+\frac{7}{4}\Big(\frac{1}{8}-\frac{1}{18}\Big)+\frac{1}{2}\frac{23}{48\times 9}\Big)-3=\frac{254}{72}-3=\frac{19}{36}.$$

Proof of Proposition 2.6. Fix $K>m$. Write $(\rho_{m,l}+3)/12$ as

$$\int_0^1\int_{\psi_{K-1}^{-1}(\psi_K(u_K))}^1\cdots\int_{\psi_2^{-1}(\psi_3(u_3))}^1\int_{\psi_2(u_2)}^1 u_m u_l \times c_K(u_1,\dots,u_K)\,du_1\cdots du_K. \tag{3.7}$$

By induction, we can show that for $2\le m\le K-1$

$$\begin{aligned}&\int_{\psi_{m-1}^{-1}(\psi_m(u_m))}^1\cdots\int_{\psi_2(u_2)}^1 c_K(u_1,\dots,u_K)\,du_1\cdots du_{m-1}\\&=\frac{(-\log\psi_m(u_m))^{m-1}}{(m-1)!}\frac{d\log\psi_m}{du_m}(u_m)\Bigg(\prod_{j=m+1}^{K-1}\frac{d\log\psi_j}{du_j}(u_j)\Bigg)\frac{d\psi_K}{du_K}(u_K),\end{aligned}$$

and since

$$\frac{(-\log \psi_m(u_m))^{m-1}}{(m-1)!}\frac{d\log\psi_m}{du_m}(u_m) = \frac{1}{\psi_m(u_m)},$$

we have that the integral in (3.7) is equal to

$$\int_0^1 \int_{\psi_{K-1}^{-1}(\psi_K(u_K))}^1 \cdots u_l \int_{\psi_m^{-1}(\psi_{m+1}(u_{m+1}))}^1 \frac{u_m}{\psi_m(u_m)}\,du_m \\ \times \left(\prod_{j=m+1}^{K-1} \frac{d\log\psi_j}{du_j}(u_j)\right)\frac{d\psi_K}{du_K}(u_K)\,du_{m+1}\cdots du_K.$$

Now, note that, since $u_m \geq \psi_m^{-1}(\psi_{m+1}(u_{m+1}))$,

$$\frac{u_m}{\psi_m(u_m)} = \sum_{j=0}^{m-1}\frac{(-\log\psi_m(u_m))^j}{j!} \leq \sum_{j=0}^{m-1}\frac{(-\log\psi_{m+1}(u_{m+1}))^j}{j!} \\ \leq \sum_{j=0}^{m}\frac{(-\log\psi_{m+1}(u_{m+1}))^j}{j!} = \frac{u_{m+1}}{\psi_{m+1}(u_{m+1})}.$$

Therefore,

$$\int_{\psi_m^{-1}(\psi_{m+1}(u_{m+1}))}^1 \frac{u_m}{\psi_m(u_m)}\,du_m \\ \leq \frac{u_{m+1}}{\psi_{m+1}(u_{m+1})}[1-\psi_m^{-1}(\psi_{m+1}(u_{m+1}))] \\ \leq \frac{u_{m+1}}{\psi_{m+1}(u_{m+1})}.$$

But replacing the integral in $\cdots$ by the $\frac{u_{m+1}}{\psi_{m+1}(u_{m+1})}$, we have $\rho_{m+1,l}$. □

Proof of Lemma 2.7. Let $(U_1, U_2, \ldots, U_K)$ be a random vector whose distribution function is C_K. The conditional distribution of U_m given $U_1, U_2, \ldots, U_{m-1}$ has distribution function

$$C_m(u_m|u_1,\ldots,u_{m-1}) = \mathbb{P}(U_m \leq u_m|U_1 = u_1,\ldots,U_{m-1} = u_{m-1}) \\ = \left(\frac{\partial^{m-1}C_m(u_1,\ldots,u_m)}{\partial u_1\cdots\partial u_{m-1}}\right) \\ \Big/\left(\frac{\partial^{m-1}C_{m-1}(u_1,\ldots,u_{m-1})}{\partial u_1\cdots\partial u_{m-1}}\right) \tag{3.8}$$

for every $m = 2, \ldots, k$.

We first deal with the numerator in (3.8) which by the formula in Proposition 2.3 can be written as

$$\partial^{m-1}\Bigg[-\psi_m(u_m)\sum_{j=1}^{m-1}\frac{-\log(\psi_j(u_j))^j}{j!} \times J_{m-j}(-\log\psi_{j+1}(u_{j+1}),\dots,-\log\psi_m(u_m))\Bigg] \Big/ \partial u_1\cdots\partial u_{m-1}.$$

If we remove the terms that do not depend on all the variables $u_1,\dots,u_{m-1}$, we obtain that the last partial derivative is equal to

$$\frac{\partial^{m-1}[-\psi_m(u_m)\prod_{j=1}^{m-1}(-\log(\psi(u_j)))]}{\partial u_1\cdots\partial u_{m-1}}. \tag{3.9}$$

Using that

$$\frac{d\log\psi_m}{du}=(-1)^{m-1}\left(\psi_m\frac{(\log\psi_m)^{m-1}}{(m-1)!}\right)^{-1},$$

we obtain that (3.9) is equal to

$$(-1)^m\psi_m(u_m)(-1)^{m-1}\prod_{j=1}^{m-1}(-1)^{j-1}\left(\psi_j(u_j)\frac{\log(\psi_j(u_j))^{j-1}}{(j-1)!}\right)^{-1}. \tag{3.10}$$

Now we consider the denominator in (3.8) which is equal to the density function of the $(m-1)$-extremal copula. Hence, it is equal to

$$\left(\prod_{j=1}^{m-2}(-1)^{j-1}\psi_j(u_j)\frac{(\log\psi_j(u_j))^{j-1})}{(j-1)!}\right)^{-1} \times\left(-\frac{(\log\psi_{m-1}(u_{m-1}))^{m-2}}{(m-2)!}\right)^{-1}. \tag{3.11}$$

Finally, replace the expressions in (3.10) and (3.11), respectively, in the numerator and denominator in (3.8) to obtain that

$$C_m(u_m|u_1,\dots,u_{m-1})=\frac{\psi_m(u_m)}{\psi_{m-1}(u_{m-1})}.$$ □

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References

Aas, K., Czado, C., Frigessi, A. and Bakken, H. (2009). Pair copula constructions of multiple dependence. *Insurance: Mathematics and Economics* **44**, 182–198. MR2517884

Averous, J., Genest, C. and Kochar, S. (2005). On dependence structure of order statistics. *Journal of Multivariate Analysis* **94**, 159–171. MR2161215

Billingsley, P. (1968). *Convergence of Probability Measures*. New York: Wiley. MR0233396

Chen, Y. (2007). A note on the relantionship between Spearman's ρ and Kendall's τ for extreme order statistics. *Journal of Statistical Planning and Inference* **137**, 2165–2171. MR2325424

Cherubini, U., Luciano, E. and Vecchiato, W. (2004). *Copula Methods in Finance*. *Wiley Finance Series*. Chichester: Wiley. MR2250804

Drouet-Mari, D. and Kotz, S. (2001). *Correlation and Dependence*. London: Imperial College Press.

Embrechts, P., Kluppelberg, C. and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*. *Applications of Mathematics* **33**. Berlin: Springer. MR1458613

Fredricks, G. and Nelsen, R. (2007). On the relation between Spearman's rho and Kendall's tau for pairs of continuous random variables. *Journal of Statistical Planning and Inference* **137**, 2143–2150. MR2325421

Joe, H. (1997). *Multivariate Models and Dependence Concepts*. London: Chapman & Hall. MR1462613

Leadbetter, M. R., Lindgreen, G. and Rootzen, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. New York: Springer. MR0691492

Liebscher, E. (2008). Construction of asymetric multivariate copulas. *Journal of Multivariate Analysis* **99**, 2234–2250. MR2463386

Mendes, B. and Sanfins, M. (2007). The limiting copula of the two largest order statistics of iid samples. *Brazilian Jounal of Probability and Statistics* **21**, 85–101. MR2396715

Morillas, P. M. (2005). A method to obtain new copulas from a given one. *Metrika* **61**, 169–184. MR2159414

Nelsen, R. (2006). *An Introduction to Copulas*, 2nd ed. *Springer Series in Statistics*. Springer. MR2197664

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