

ON THE COST OF NOT KNOWING THE VARIANCE WHEN MAKING A FIXED-WIDTH CONFIDENCE INTERVAL FOR THE MEAN

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1. Summary. It is shown that the mean of a normal distribution with unknown variance σ^2 may be estimated to lie within an interval of given fixed width at a prescribed confidence level using a procedure which overcomes ignorance about σ^2 with no more than a finite number of observations. That is, the expected sample size exceeds the (fixed) sample size one would use if σ^2 were known by a finite amount, the difference depending on the confidence level α but *not* depending on the values of the mean μ , the variance σ^2 and the interval width $2d$. A number of unpublished results on the moments of the sample size are presented. Some do not depend on an assumption of normality.

2. Introduction. Let X, X_1, X_2, \dots be iid random variables with unknown mean μ and unknown variance $\sigma^2 < \infty$. Let $\bar{X}_n \equiv n^{-1} \sum_1^n X_i$. We desire to find a confidence interval for μ of width $2d$ ($d > 0$) for which the probability of coverage is at least as large as α ($0 < \alpha < 1$) for all values μ and σ^2 . N. Starr [6] and Chow-Robbins [4] have proposed using the interval $(\bar{X}_N - d, \bar{X}_N + d)$ where sample size N is to be sequentially determined. Let a be defined by $2\Phi(a) - 1 = \alpha$ where $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x e^{-u^2/2} du$. With X normally distributed, if σ^2 were known, one could use a fixed sample size $N \geq C \equiv a^2 \sigma^2 / d^2$. They reason that when σ^2 is unknown one might estimate σ^2 by some good estimator s_n^2 and use a sequential procedure of the basic form

$$(1) \quad N \equiv \text{smallest index } n \geq n_0 \geq 2 \text{ for which } n \geq a_n^2 s_n^2 / d^2,$$

where n_0 is an integer constant and where the a_n are chosen to be either identical to a or such that $0 < a_n \rightarrow a$. With $s_n^2 \equiv (n-1)^{-1} \sum_1^n (X_i - \bar{X}_n)^2$, Chow and Robbins showed that no matter what continuous distribution X might have,

$$(2) \quad \lim_{d \rightarrow 0} P\{|\bar{X}_N - \mu| < d\} = \alpha \quad (\text{"asymptotic consistency"})$$

and

$$(3) \quad \lim_{d \rightarrow 0} EN/C = 1 \quad (\text{"asymptotic efficiency"}).$$

H. Chernoff and the author (unpublished) have shown a stronger efficiency result which holds when $a_n \equiv a$, namely,

$$(4) \quad EN \leq C + n_0 + 1 \quad (\text{independent of } d, a, \text{ and the distribution of } X).$$

One is tempted to claim that the "cost of ignorance" in not knowing σ^2 is at

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most $n_0 + 1$ observations. However, the objective is not to achieve asymptotic consistency but rather to achieve

$$(5) \quad P\{|\bar{X}_N - \mu| < d\} \geq \alpha \quad \text{for all values of } \mu \text{ and } \sigma^2.$$

One must be able to satisfy (5) before one can properly assess the true cost of ignorance. Starr conducted a numerical study for normally distributed X using a particular sequence a_n (of the form $a + O(n^{-1})$). It appears he nearly achieves objective (5). One can show for $a_n = a + O(n^{-1})$ that

$$(6) \quad EN \leq C + O(1) \quad \text{as } d \rightarrow 0.$$

Thus it seems likely and we shall verify for normally distributed X that there exist stopping rules N for which (5) holds and for which the cost of ignorance, $EN - C$, is uniformly bounded for all μ , σ^2 and $d > 0$. Specifically, for some integer k , we can achieve these objectives by taking k more observations after rule (1) says to stop. We shall need $n_0 \geq 3$ (not 2) and we shall be satisfied with $a_n \equiv a$. Stopping rules of this type are suggested in Starr's paper but they were not analyzed mathematically.

A useful random variable related to N (defined by (1)) is the variable

$$(7) \quad \begin{aligned} M &\equiv \text{last index } m \geq n_0 \text{ for which } m < a_m^2 s_m^2 / d^2 \text{ if such an } m \text{ exists,} \\ &\equiv n_0 - 1 \text{ if } m \geq a_m^2 s_m^2 / d^2 \text{ for all } m \geq n_0, \\ &\equiv \infty \text{ if } m < a_m^2 s_m^2 / d^2 \text{ infinitely often.} \end{aligned}$$

Such a random variable is not a stopping variable but rather a *reverse stopping variable*, one that depends on the future and not on the past. If M and N are usually close we can hope to learn something about N by studying M .

In Section 3, we define and relate reverse stopping variables to (reverse) martingales. In Section 4, we derive some preliminary results involving moments of M and N , and in Section 5, we prove the true cost of ignorance concerning σ^2 is a finite number of observation.

It may be recalled that C. Stein [7] showed that (5) could be accomplished for normal X using a two-stage procedure but for his procedure $EN - C$ is not bounded.

3. Reverse stopping variables and some martingale lemmas. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and $\{\mathfrak{F}_j : \mathfrak{F}_j \subset \mathfrak{F}, j \in J\}$ be a non-increasing sequence of σ -fields where J is a continuous sequence of integers including possibly $\pm \infty$. We recall that a family $Z = \{Z_j, \mathfrak{F}_j, j \in J\}$ is called a *reverse martingale* if for all $j \in J$ (i) Z_j is an \mathfrak{F}_j -measurable random variable, (ii) $E|Z_j| < \infty$, and (iii) $\int_A Z_j = \int_A Z_k$ for all $k \in J, k \geq j, A \in \mathfrak{F}_k$. The appropriateness of the term "reverse" comes from the observation that if Z is a reverse martingale, then by "reversing" the usual ordering of the indexing set J , we obtain a martingale. Extending this terminology, we say that a random variable M with values a.s. in J is a *reverse stopping variable* if $\{M = j\} \in \mathfrak{F}_j$ for all $j \in J$. We shall use the following trivial generalization of a result of Doob ([5], p. 300):

LEMMA 1. Let Z be a reverse martingale and M a reverse stopping variable. If J has a first element j_0 (possibly $-\infty$), then

$$(8) \quad E|Z_M| \leq E|Z_{j_0}| < \infty \quad \text{and} \quad EZ_M = EZ_{j_0}.$$

A well known reverse martingale is $n^{-1}S_n$ where $S_n \equiv \sum_1^n X_i$ is a sum of n iid random variables and $E|X_1| < \infty$. More generally, sequences of U -statistics form reverse martingales.

LEMMA 2. (Berk [2]). Let X_1, X_2, \dots be iid and $U_{j_0}, U_{j_0+1}, \dots$ a sequence of U -statistics for some $j_0 \geq 1$. If $E|U_{j_0}| < \infty$ and \mathfrak{F}_j is the Borel field $\mathfrak{B}(U_j, U_{j+1}, \dots)$ for $j \geq j_0$, then $\{U_j, \mathfrak{F}_j, j \geq j_0\}$ is a reverse martingale.

PROOF. For $j \geq j_0$, let Y_j be the order statistic for the first j X 's and let \mathfrak{G}_j be the Borel field $\mathfrak{B}(Y_j, X_{j+1}, X_{j+2}, \dots)$. Since $U_j = E^{\mathfrak{G}_j}U_{j_0}$ and $\{\mathfrak{G}_j\}$ is a non-increasing sequence of σ -fields, it follows from [5], pg. 293, that $\{U_j, \mathfrak{F}_j, j \geq j_0\}$ is a reverse martingale.

LEMMA 3. Let Y_1, Y_2, \dots be a sequence of independent random variables with a common (two parameter) gamma distribution (having a density of the form $c(\theta, \beta)x^{\beta-1}e^{-x/\theta}$; $\beta, \theta > 0$). For given $\lambda > 0$, let $S_n \equiv \sum_1^n Y_i, Z_n \equiv (S_n)^\lambda/E(S_n)^\lambda$, and \mathfrak{F}_n be the Borel field $\mathfrak{B}(Z_n, Z_{n+1}, \dots)$ for $n = 1, 2, \dots$. Then $\{Z_n, \mathfrak{F}_n; n \geq 1\}$ is a reverse martingale.

The proof is routine if one first derives the conditional distribution of S_n given S_{n+1} .

4. Some preliminary results involving M and N . Let M and N be defined by (7) and (1), respectively, with $a_n \equiv a$. We proceed with the notation of Section 2.

THEOREM 1. The following results do not depend on X being normally distributed.

$$(9) \quad EM \leq C + (n_0 - 1)P\{M = n_0 - 1\} \leq C + n_0 - 1;$$

$$(10) \quad EN \leq C + 1 + (n_0 - 1)P\{M = n_0 - 1\} \leq C + n_0;$$

$$(11) \quad EM \geq C - 2 - 2n_0^{-1}.$$

PROOF. The fact $N \leq M + 1$ and (9) imply (10). Now, using indicator functions,

$$(12) \quad M \leq a^2 s_M^2 / d^2 + (n_0 - 1)I_{[M=n_0-1]}.$$

(Defining $s_1^2 \equiv s_2^2, s_M^2$ is well defined for $M = 1$. This can occur when $n_0 = 2$. The event $[M = \infty]$ is null.)

Using Lemma 2 and then Lemma 1, we conclude first that s_1^2, s_2^2, \dots is a reverse martingale and then $Es_M^2 = \sigma^2$.

Since $C \equiv a^2 \sigma^2 / d^2$, (9) follows. A reverse martingale argument has been used by Starr and Woodroffe to prove the extremes of (10) in a similar way. Now define $M' = \max(M, n_0 + 1)$, a reverse stopping variable. If $M = n_0 - 1$ or n_0 , then $M' = n_0 + 1$ and $M + 2 \geq a^2 s_{M'}^2 / d^2$. If $M \geq n_0 + 1$, then $M' = M$ and $M + 1 \geq a^2 s_{M+1}^2 / d^2 = a^2 M^{-1} \sum_1^{M+1} (X_i - \bar{X}_{M+1})^2 / d^2 \geq a^2 M^{-1} (M - 1) s_{M'}^2 / d^2$.

In general,

$$(13) \quad M \geq a^2 s^2_{M'} / d^2 - 2 - 2n_0^{-1}$$

from which (11) follows.

We remark that $P\{M = n_0 - 1\} = o(1)$ as $d \searrow 0$. When X is normally distributed $P\{M = n_0 - 1\} = o(d^k)$ for any k as $d \searrow 0$. These lead to strong asymptotic upper bounds for EM and EN .

THEOREM 2. *The following results apply to normally distributed X .*

$$(14) \quad EM^\lambda \leq C^\lambda + O(C^{\lambda-1}) \quad \text{as } C \rightarrow \infty \quad \text{for } \lambda = 1, 2, \dots;$$

$$(15) \quad EN^\lambda \leq C^\lambda + O(C^{\lambda-1}) \quad \text{as } C \rightarrow \infty \quad \text{for } \lambda = 1, 2, \dots;$$

$$(16) \quad E(M - N) = O(1) \quad \text{for any } n_0 \geq 3;$$

$$(17) \quad EM^\lambda \leq C^\lambda + O(C^{\lambda-1}) \quad \text{as } C \rightarrow \infty \quad \text{for } \lambda = 1, 2, \dots;$$

$$(18) \quad EN^\lambda \leq C^\lambda + O(C^{\lambda-1}) \quad \text{as } C \rightarrow \infty \quad \text{for any } n_0 \geq 3, \text{ for } \lambda = 1, 2, \dots.$$

PROOF. The fact $N \leq M + 1$ and (14) imply (15). Jensen's inequality and (11) imply (17). Jensen's inequality, (11) and (16) imply (18). If X is normally distributed, we can write $\sum_1^n (X_i - \bar{X}_n)^2 = \sigma^2 \sum_2^n u_i$ where u_2, u_3, \dots are iid chi-square random variables with one degree of freedom. According to Lemma 3,

$$Z_n^{(\lambda)} = (\sum_2^n u_i)^\lambda / E(\sum_2^n u_i)^\lambda = \Gamma((n-1)/2) (\sum_2^n u_i)^\lambda / (2^\lambda \Gamma((n-1)/2 + \lambda))$$

is a reverse martingale with $EZ_n^{(\lambda)} = 1$ for $n = 2, 3, \dots$, and fixed $\lambda > 0$. It easily follows (by definition), for (positive) integer valued λ , that

$$(19) \quad (s_n^2)^\lambda = \sigma^{2\lambda} Z_n^{(\lambda)} (1 + O(1/n)).$$

(12) and (19) combine to give $M^\lambda \leq C^\lambda Z_M^{(\lambda)} + O(M^{\lambda-1})$. Using Lemma 1 and trivial induction we derive (14). Finally (16) follows directly from

LEMMA 4. *For $0 < \theta < 1$,*

$$(20) \quad P\{N \leq \theta C\} = O_e(C^{-(n_0-1)/2}) \quad \text{as } C \rightarrow \infty,$$

where O_e denotes exact order;

$$(21) \quad E(M - N | N = n) \leq C + 1 \quad \text{for all } n \geq n_0;$$

and for $n > \theta C, \theta > \frac{1}{2}$,

$$(22) \quad E(M - N | N = n) \leq K(\theta), \quad \text{' a constant (independent of } C \text{)}.$$

PROOF. Let u_1, u_2, \dots be a sequence of iid random variables distributed as χ_1^2 (chi-square with one degree of freedom) which we will use in various contexts below.

PROOF of (20). Let $0 < \theta < 1$.

$$P\{N \leq \theta C\} \leq P_1 + P_2 + P_3,$$

where

$$\begin{aligned} P_1 &\equiv P\{n_0 \leq N \leq 2n_0\}, \\ P_2 &\equiv P\{2n_0 < N \leq C^{\frac{1}{2}}\}, \\ P_3 &\equiv P\{C^{\frac{1}{2}} < N \leq \theta C\}. \end{aligned}$$

Now for $n \geq n_0$,

$$\begin{aligned} P\{N = n\} &\leq P\{n \geq a^2 s_n^2 / d^2\} = P\{n(n-1) \\ &\geq C\chi_{n-1}^2\} = O_e(C^{-(n-1)/2}) \quad \text{as } C \rightarrow \infty. \end{aligned}$$

(The last equality may be easily verified.) Thus for large C ,

$$P\{N \leq \theta C\} > P\{N = n_0\} = O_e(C^{-(n_0-1)/2}),$$

and

$$P_1 = O_e(C^{-(n_0-1)/2}).$$

Using a fairly well-known result concerning the probability of a random walk crossing a linear boundary (e.g., Section 2.1 Bartlett [1]), we find for $\gamma < 0 < \beta < 1$, that

$$(23) \quad P\{\sum_1^m u_i \leq \beta m + \gamma \text{ for some (positive) } m\} \leq e^{-\gamma h},$$

where $e^{-2\beta h} = 1 - 2h$ defines negative valued h .

Now for $0 < \beta < 1$,

$$(24) \quad (a) \quad h < \beta - 1; \quad (b) \quad h < (2\beta)^{-1} \log \beta.$$

The first inequality is immediate upon an expansion of $\log(1 - 2h)$ about $h = 0$. By substituting $(2\beta)^{-1} \log \beta$ for h in $e^{-2\beta h} = 1 - 2h$ one can easily verify (24b). Hence (for large C)

$$\begin{aligned} P_2 &= P\{2n_0 < N \leq C^{\frac{1}{2}}\} \leq P\{\sum_1^m u_i \leq C^{-1}m(m+1) \text{ for some } m, 2n_0 \leq m \\ &< C^{\frac{1}{2}}\} \leq P\{\sum_1^m u_i \leq C^{-1}(2n_0 + C^{\frac{1}{2}} + 1)m - 2C^{-\frac{1}{2}}n_0 \text{ for some } m \geq 1\}. \end{aligned}$$

By (23) and (24b),

$$P_2 \leq O(C^{-n_0/2}) \leq O_e(C^{-(n_0-1)/2}).$$

P_3 is shown to be of a sufficiently small order in the same manner if we use (24a) instead of (24b).

PROOF OF (21). If $N = n$, the point $(n, \sum_2^n u_i)$ is below the parabola $C^{-1}x(x-1)$ and either $M - n = -1$ or $M - n = \text{last time } k \geq 1 \text{ such that } (k, \sum_{n+1}^{n+k} u_i)$ is above the parabola $C^{-1}(x+n)(x+n-1) - \sum_2^n u_i$. In either case, $M - n \leq M^*$ where $M^* \equiv \text{last } k \geq 1 \text{ such that } (k, \sum_{n+1}^{n+k} u_i)$ is above the parabola $C^{-1}x(x-1)$. But $EM^* \leq C + 1$ (cf. (9)) and (21) follows.

PROOF OF (22). Let $n > \theta C$, $\theta > \frac{1}{2}$.

$$\begin{aligned}
 P\{M = n + k \mid N = n\} &\leq P\{n + k < a^2 s_{n+k}^2 / d^2 \mid N = n\} \\
 &\leq P\{(n + k)(n + k - 1) < C \sum_2^{n+k} u_i \mid n(n - 1) = C \sum_2^n u_i\} \\
 &\leq P\{C \sum_1^{n+k} u_i > (n + k)(n + k - 1) - n(n - 1)\} \\
 &\leq P\{C \sum_1^k u_i > 2nk\} \\
 &\leq P\{\sum_1^k u_i > 2\theta k\}.
 \end{aligned}$$

Since $2\theta > EU_1 = 1$, we know from (for instance) H. Chernoff [3] that there exists a constant $b = b(\theta) > 0$ for which the latter probability is bounded above by e^{-bk} . Then

$$(25) \quad E(M - N \mid N = n) \leq \sum_{k=1}^{\infty} k P\{M = n + k \mid N = n\} \leq \sum_{k=1}^{\infty} k e^{-bk} < b^{-2}$$

and hence (22) holds.

5. The cost of ignorance is a finite number of observations. Here we assume that X, X_1, X_2, \dots are normal iid random variables with mean μ and variance σ^2 and as before $d > 0$. Let $r \equiv d/\sigma$, so $C = a^2/r^2$.

MAIN THEOREM. *If the value of any stopping variable N is determined by s_2^2, s_3^2, \dots , then*

$$(26) \quad P\{|\bar{X}_N - \mu| < d\} = 2E\Phi(rN^{\frac{1}{2}}) - 1 \quad \text{for all } \mu, \sigma^2.$$

For N defined by (1) with $a_n \equiv a$ and $n_0 \geq 3$, we have for some finite integer $k \geq 0$,

$$(27) \quad E\Phi(r(N + k)^{\frac{1}{2}}) \geq \Phi(a) = (1 + \alpha)/2 \quad \text{for all } \mu, \sigma^2, \text{ and } d.$$

Then

$$(28) \quad P\{|\bar{X}_{N+k} - \mu| < d\} \geq \alpha \quad \text{for all } \mu, \sigma^2 \text{ and } d,$$

and

$$(29) \quad E(N + k) \leq C + n_0 + k \quad \text{for all } \mu, \sigma^2 \text{ and } d.$$

PROOF. The random variables \bar{X}_n and $Y_n = (s_2^2, \dots, s_n^2)$ are independent for $n = 2, 3, \dots$ when X is normal. (See for instance N. Starr [6].) Thus, if the events $[N = n] \in \mathcal{B}(s_2^2, \dots, s_n^2)$, then

$$\begin{aligned}
 P\{|\bar{X}_N - \mu| < d\} &= \sum_{n=n_0}^{\infty} P\{|\bar{X}_n - \mu| < d \text{ and } N = n\} \\
 &= \sum_{n=n_0}^{\infty} P\{|\bar{X}_n - \mu| < d\} P\{N = n\} = 2E\Phi(rN^{\frac{1}{2}}) - 1.
 \end{aligned}$$

For $g(x) \equiv \Phi(rx^{\frac{1}{2}})$, $g'(x) = r\varphi(rx^{\frac{1}{2}})/(2x^{\frac{1}{2}})$ and $g''(x) = -r(r^2x + 1)\varphi(rx^{\frac{1}{2}})/(4x^{\frac{3}{2}})$, where $\varphi(y) \equiv (2\pi)^{-\frac{1}{2}}e^{-y^2/2}$. Expand $g(x)$ in a Taylor series about $x = C$ with a second degree remainder term. We find for arbitrary θ , $0 < \theta < 1$, that

$$\begin{aligned}
 E\Phi(r(N + k)^{\frac{1}{2}}) &\geq E\Phi(r(N + k)^{\frac{1}{2}})I_{[N+k \geq \theta^2 C]} \\
 &\geq \Phi(a)P\{N + k \geq \theta^2 C\} + a\varphi(a)(2C)^{-1}E(N + k - C)I_{[N+k > \theta^2 C]} \\
 &\quad - a(a^2\theta^2 + 1)\varphi(a\theta)(8\theta^3 C^2)^{-1}E(N + k - C)^2 I_{[N+k > \theta^2 C]}
 \end{aligned}$$

$$\begin{aligned} &\geq \Phi(a) + a\varphi(a)(2C)^{-1}E(N + k - C) \\ &\quad - a(a^2\theta^2 + 1)\varphi(a\theta)(8\theta^3C^2)^{-1}E(N + k - C)^2 \\ &\quad + \{-\Phi(a) + a\varphi(a)(1 - \theta^2)/2 + a(a^2\theta^2 + 1)\varphi(a\theta)(1 - \theta^2)/(8\theta^3)\} \\ &\quad \cdot P\{N + k \leq \theta^2C\}. \end{aligned}$$

For small $\theta > 0$ the coefficient of $P\{N + k \leq \theta^2C\}$ is positive and for such θ , $E\Phi(r(N + k)^{\frac{1}{2}}) \geq \Phi(a) + a\varphi(a)(2C)^{-1}\{k + E(N - C)\} - a(a^2\theta^2 + 1)\varphi(a\theta)(8\theta^3C^2)^{-1}\{k^2 + 2kE(N - C) + E(N - C)^2\}$.

By (15) and (18),

$$\begin{aligned} E\Phi(r(N + k)^{\frac{1}{2}}) - \Phi(a) &\geq O(C^{-2})k^2 + \{a\varphi(a)(2C)^{-1} + O(C^{-2})\}k + O(C^{-1}) \\ &= \{a\varphi(a)(2C^{-1}) + O(C^{-2})\}\{O(C^{-1})k^2 + k + O(1)\}. \end{aligned}$$

Thus for some large k and for all large C (say $C \geq C_0$) (27) holds and clearly for some large k (27) holds for all $C < C_0$. Thus (27) holds for some integer $k \geq 0$. (28) follows from (26) and (27), and (29) from (10).

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REFERENCES

- [1] BARTLETT, M. S. (1955). *An Introduction to Stochastic Processes*. Cambridge Univ. Press.
- [2] BERK, R. H. (1966). Limiting behavior of posterior distributions when the model is incorrect. *Ann. Math. Statist.* **37** 51-58.
- [3] CHERNOFF, H. (1952). A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Ann. Math. Statist.* **23** 493-507.
- [4] CHOW, Y. S. and ROBBINS, H. (1965). On the asymptotic theory of fixed-width sequential confidence intervals for the mean. *Ann. Math. Statist.* **36** 457-462.
- [5] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.
- [6] STARR, N. (1966). The performance of a sequential procedure for the fixed-width interval estimation of the mean. *Ann. Math. Statist.* **37** 36-50.
- [7] STEIN, C. (1945). A two-sample test for a linear hypothesis whose power is independent of the variance. *Ann. Math. Statist.* **16** 243-258.