On the Coupling of Systems of Hyperbolic Conservation Laws with Ordinary Differential Equations

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Abstract

Motivated by applications to the piston problem, to a manhole model, to blood flow and to supply chain dynamics, this paper deals with a system of conservation laws coupled with a system of ordinary differential equations. The former is defined on a domain with boundary and the coupling is provided by the boundary condition. For each of the examples considered, numerical integrations are provided.

Keywords: Mixed P.D.E.–O.D.E. Problems, Conservation Laws, Ordinary Differential Equations

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1 Introduction

This paper deals with a mixed problem consisting of a 1D system of hyperbolic conservation laws coupled with a system of ordinary differential equations. The former is defined on the positive half-line $\mathbb{R}^+$ and the coupling is provided by the boundary condition, i.e.

$$\left\{\begin{array}{lrcl} \partial_t u + \partial_x f(u) = 0 & (t,x) & \in & \mathbb{R}^+ \times \mathbb{R}^+ \\ b\left(u(t,0)\right) = B\left(t, w(t)\right) & t & \in & \mathbb{R}^+ \\ \dot{w} = F\left(t, u(t,0), w\right) & t & \in & \mathbb{R}^+. \end{array}\right. \tag{1.1}$$

The boundary condition on the second line above is an algebraic relation acting as interface between the two evolutionary differential equations. Recall that systems of conservation laws admit solutions developing jump discontinuities and the sense in which the boundary condition is attained can be critical. Here, we restrict our attention to the *non characteristic case*, which allows us to require that the solution to (1.1) attains the value imposed at the boundary in the strong sense of the trace, for a.e. time, see 2. in Definition (2.1). We refer to [2, 3, 15] for further information on the analytical treatment of boundary value problems for systems of conservation laws.

In recent years, there has been an increasing interest in systems composed by ordinary differential equations and partial differential equations interacting together. The most famous example probably consists in the interaction of a fluid (liquid or gas) with a rigid body or with an elastic structure, like a membrane, see [25, 26]. The evolution of the rigid body is described by a system of ordinary differential equations, while the evolution of the fluid is subject to partial differential equations like Navier-Stokes or Euler equations. Up to now, the focus was mainly on the study of regular solutions and, hence, the phenomenon of the creation of discontinuities in finite time,

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typical of nonlinear conservation laws, was not considered. Among the first papers in the present context dealing with shock waves we recall [17], in the case of particles in a fluid.

The analytical results below ensure the existence of solutions to the mixed system (1.1). The technique used essentially relies on the classical theorems on the Cauchy problem for an ordinary differential equation, see [14], and separately on more recent results on the initial – boundary value problem for 1D systems of conservation laws [8]. We stress that in the analytical results, as well as in the applications below, the relation on the second line of (1.1) may not be inverted, so that system (1.1) may not be decoupled, see Remark 2.1.

As a first driving example for (1.1) we select the classical piston problem. Here, u is the pair of the fluid specific volume and speed; w is the position of piston; f states the conservation of mass and of linear momentum; F relates the fluid pressure, the outer pressure and the acceleration of the piston; the pair b, B ensures that the piston remains adjacent to the fluid, see Section 3.1.

Then, in Section 3.2, we pass to the model recently presented in [5]. Aiming at the description of a sewage system, it deals with the dynamics of water in a network of horizontal pipes receiving water from a vertical manhole. Now, u is the vector of pairs (A_i, Q_i), where A_i is the wet area of the i–th pipe and Q_i is the flow therein, with i ranging over all the tubes; w is the height of water in the vertical pipe. The interface between the manhole and the horizontal pipes is described by b, B, which are here physically justified, see [5].

In Section 3.3, we consider a supply chain model recently considered in the literature, see [4]. Now, u is the vector of the densities of goods in the processors and w are the loads of the processing queues.

Finally, in Section 3.4 a further example is provided by a model for blood flow, often considered in the present literature, see for instance [7, 13, 23]. In this case, u is the pair artery section, blood flow in the artery; w is the pair consisting of the blood pressure and flow in a vessel.

A further example of a mixed ordinary differential equation – conservation law is found in [19], where the motivating application is a traffic flow problem. There, no boundary is explicitely present and the two equations interact in the interior of the spatial domain.

The technical details of the proofs are collected in Section 4, while the various numerical integrations are presented in the paragraph of the model they are referring to.

2 Analytical Results

Throughout, we denote $\mathbb{R}^+ = [0, +\infty[$ and $\mathring{\mathbb{R}}^+ =]0, +\infty[$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

On system (1.1) we require the following conditions, where we refer to [6, 11] for the standard vocabulary about conservation laws.

(f) $f \in \mathbf{C^4}(\Omega; \mathbb{R}^n)$ is smooth and such that $Df(u)$ is strictly hyperbolic for all $u \in \Omega$, each characteristic field is either genuinely nonlinear or linearly degenerate.

Without loss of generality, we may assume that $0 \in \Omega$ and for all u in Ω, $Df(u)$ admits n real distinct eigenvalues $\lambda_1(u), \ldots, \lambda_n(u)$, ordered so that $\lambda_{i-1}(u) < \lambda_i(u)$ for all $u \in \Omega$ and $i = 2, \ldots, n$, with right eigenvectors $r_1(u), \ldots, r_n(u)$. We also introduce the following conditions.

(NC) There exist a $c > 0$ and $\ell \in \{1, 2, \ldots, n-1\}$ such that for all $u \in \Omega$, $\lambda_\ell(u) < -c$ and $\lambda_{\ell+1}(u) > c$.

The above Non Characteristic condition on f is coordinated with the following assumption on b, which describes how the boundary data is assigned.

(b) $b \in \mathbf{C^1}(\Omega; \mathbb{R}^{n-\ell})$ is such that $\det \begin{bmatrix} D_u b(0)\, r_{\ell+1}(0) & D_u b(0)\, r_{\ell+2}(0) & \cdots & D_u b(0)\, r_n(0) \end{bmatrix} \neq 0$.

Remark 2.1. If the boundary condition is invertible, in the sense that it is equivalent to $u(t, 0) = b^{-1}\Big(B\left(t, w(t)\right)\Big)$ for a suitable B, then system (1.1) can be decoupled, solving first the o.d.e. $\dot w = F\Big(t, b^{-1}\left(B(t, w)\right), w\Big)$ and then, separately, the balance law with boundary (2.1). In the applications considered below, b is *not* invertible in this sense. Moreover, condition **(b)** above essentially *imposes* b to be *not* invertible. The case of an invertible b would formally correspond to $\ell = 0$ in **(b)**.

The two existence results below, theorems 2.7 and 2.8, rely on the following two different assumptions on B, see Remark 3.2.

(B1) $B \in \mathbf{C^1}(\mathbb{R}^+ \times \mathbb{R}^m; \mathbb{R}^{n-\ell})$ is independent from the time variable, so that $B = B(w)$.

(B2) $B \in \mathbf{C^1}(\mathbb{R}^+ \times \mathbb{R}^m; \mathbb{R}^{n-\ell})$ is locally Lipschitz, i.e. for every compact subset $\tilde{K}$ of $\mathbb{R}^m$, there exists a constant $\tilde{C}_{\tilde{K}}$ such that, for every $t > 0$ and $w \in \tilde{K}$:

$$\left\| \frac{\partial}{\partial t} B(t, w) \right\|_{\mathbb{R}^{n-\ell}} + \left\| \frac{\partial}{\partial w} B(t, w) \right\|_{\mathbb{R}^{n-\ell}} \le \tilde{C}_{\tilde{K}} .$$

Finally, we impose to the ordinary differential equation in (1.1) to fit into the framework of Caratheodory equations, introducing the following conditions.

(F) The map $F \colon \mathbb{R}^+ \times \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is such that

1. For all $t > 0$ and $u \in \Omega$, the function $\begin{array}{ccc} \mathbb{R}^m & \longrightarrow & \mathbb{R}^m \\ w & \longmapsto & F(t, u, w) \end{array}$ is continuous.
2. For all $t > 0$ and $w \in \mathbb{R}^m$, the function $\begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R}^m \\ u & \longmapsto & F(t, u, w) \end{array}$ is continuous.
3. For all $u \in \Omega$ and $w \in \mathbb{R}^m$, the function $\begin{array}{ccc} \mathbb{R}^+ & \longrightarrow & \mathbb{R}^m \\ t & \longmapsto & F(t, u, w) \end{array}$ is Lebesgue measurable.
4. For all compact subset K of Ω, there exists a constant $C_K > 0$ such that

$$\left\| F(t, u, w_1) - F(t, u, w_2) \right\|_{\mathbb{R}^m} \le C_K \left\| w_1 - w_2 \right\|_{\mathbb{R}^m} .$$

(F1) There exists a function $C \in \mathbf{L^1_{loc}}(\mathbb{R}^+; \mathbb{R}^+)$ such that for all $t > 0$, $u \in \Omega$ and $w \in \mathbb{R}^m$

$$\left\| F(t, u, w) \right\|_{\mathbb{R}^m} \le C(t) \left\| w \right\|_{\mathbb{R}^m} .$$

(F2) There exists a function $C \in \mathbf{L^1_{loc}}(\mathbb{R}^+; \mathbb{R}^+)$ such that for all $t > 0$ and $u \in \Omega$ and $w \in \mathbb{R}^m$

$$\left\| F(t, u, w) \right\|_{\mathbb{R}^m} \le C(t) \left(1 + \left\| w \right\|_{\mathbb{R}^m} \right) .$$

Above, we used the notation $C(t)$ and C_K to denote quantities whose precise value is not relevant in the sequel. The distinction between **(F1)** and **(F2)** leads to the two theorems 2.7 and 2.8. A possible physical interpretation of the necessity of this distinction is provided by Remark 3.2.

First, we consider the conservation law with boundary

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ b\left(u(t, 0)\right) = B_*(t) & t \in \mathbb{R}^+ \\ u(0, x) = u_o(x) & x \in \mathbb{R}^+. \end{cases} \tag{2.1}$$

The above problem will be related to (1.1) setting $B_*(t) = B\left(t, w(t)\right)$. Following [8, Definition 3.1], we slightly modify the definition given in [15] of solution to (2.1) in the non characteristic case, see also [3]. Indeed, here we require the boundary condition to be satisfied by the solution only *almost everywhere*.

Definition 2.2. *Let $T > 0$. A map $u = u(t, x)$ is a solution to (2.1) if*

1. $u \in \mathbf{C^0}\left([0, T]; \mathbf{L^1}(\mathbb{R}; \mathbb{R}^n)\right)$ *with* $u(t, x) \in \Omega$ *for a.e.* $t \in \mathbb{R}^+$, $x \in \mathbb{R}^+$ *and* $u(t, x) = 0$ *otherwise;*
2. $u(0, x) = u_o(x)$ *for a.e.* $x > 0$ *and* $\lim_{x \to 0+} b\left(u(t, x)\right) = B_*(t)$ *a.e.* $t \ge 0$*;*

3. for $x > 0$, u is a weak entropy solution to $\partial_t u + \partial_x f(u) = 0$.

We refer to [6, Chapter 4] for the entropy admissibility criterion in balance laws. Below, we construct solutions u such that $u(t) \in \mathbf{BV}(\mathbb{R}^+;\Omega)$, which ensures the existence of the trace at 2.

Theorem 2.3. *[8, Theorem 2.2] Let system (2.1) satisfy **(f)**, **(b)** and **(NC)**. Then, there exist positive δ, Δ and L such that for all $B_* \in \mathbf{BV}(\mathbb{R}^+;\mathbb{R}^{n-\ell})$ satisfying*

$$\left\|b(0) - B_*(0)\right\|_{\mathbb{R}^{n-\ell}} + \mathrm{TV}(B_*) < \delta\,, \tag{2.2}$$

there exists a family of closed domains $\mathcal{D}_t$ with

$$\left\{u \in \mathbf{L^1}(\mathbb{R}^+;\Omega) : \mathrm{TV}(u) < \delta\right\} \subseteq \mathcal{D}_t \subseteq \left\{u \in \mathbf{L^1}(\mathbb{R}^+;\Omega) : \mathrm{TV}(u) < \Delta\right\}$$

defined for all $t \geq 0$, and a process

$$P_{B_*}(t,t_o)\colon \mathcal{D}_{t_o} \to \mathcal{D}_{t_o+t}\,, \qquad \text{for all } t_o, t \geq 0,$$

such that

1) *for all $t_o \geq 0$ and $u \in \mathcal{D}_{t_o}$, $P_{B_*}(0,t_o)\,u = u$ while for all $t,s,t_o \geq 0$ and $u \in \mathcal{D}_{t_o}$, it holds that $P_{B_*}(t+s,t_o)\,u = P_{B_*}(t,t_o+s) \circ P_{B_*}(s,t_o)u$;*
2) *for any $u \in \mathcal{D}_{t_o}$, $v \in \bar{\mathcal{D}}_{t'_o}$ and for all $t'_o \geq t_o \geq 0$ and $t \geq t' \geq 0$, we have the following Lipschitz estimate:*

$$\left\|P_{B_*}(t,t_o)u - P_{\bar{B}_*}(t',t'_o)v\right\|_{\mathbf{L^1}} \leq L\left[\|u-v\|_{\mathbf{L^1}} + \left|t-t'\right| + \left|t_o - t'_o\right| + \int_{t_o}^{t_o+t} \left\|B_*(\tau) - \bar{B}_*(\tau)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}\tau\right];$$

3) *for all $u_o \in \mathcal{D}_0$, the map $u(t,x) = \left(P_{B_*}(t,0)u_o\right)(x)$ defined for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$, solves (2.1) in the sense of Definition 2.2;*

The proof amounts to an application of [8, Theorem 2.2], see Section 4.

Now we pass to the ordinary differential equation

$$\left\{\begin{array}{l} \dot{w} = F_*(t,w) \qquad t \in \mathbb{R}^+ \\ w(0) = w_o\,. \end{array}\right. \tag{2.3}$$

which is understood in the Caratheodory sense, see [14, § 1]. The link between (2.3) and (1.1) will consist below in the relation $F_*(t,w) = F\left(t, u(t,0+), w\right)$.

Definition 2.4. *Let (2.3) be a Caratheodory equation in the sense of [14, § 1]. A function $w \in \mathbf{W^{1,1}}(\mathbb{R}^+;\mathbb{R}^m)$ is a solution to (2.3) if, for a.e. $t \in \mathbb{R}^+$,*

$$w(t) = w_o + \int_0^t F_*\left(\tau, w(\tau)\right)\,\mathrm{d}\tau\,.$$

The following standard result ensures the well posedness of (2.3), see similar results in [14, Chapter 1], for instance.

Proposition 2.5. *Let $F_* = F_*(t,w)$ be a map measurable in t and such that there exist $A, B \in \mathbf{L^1_{loc}}(\mathbb{R}^+;\mathbb{R}^+)$ such that*

$$\left\|F_*(t)\right\|_{\mathbb{R}^m} \leq A(t) + B(t)\left\|w(t)\right\|_{\mathbb{R}^m} \qquad \text{for all } t \geq 0 \text{ and } w \in \mathbb{R}^m \tag{2.4}$$

and for any compact set $K \subset \mathbb{R}^m$ there exists a constant $C_K > 0$ satisfying

$$\left\|F_*(t,w_1) - F_*(t,w_2)\right\|_{\mathbb{R}^m} \leq C_K \left\|w_1 - w_2\right\|_{\mathbb{R}^m} \qquad \text{for all } t \geq 0 \text{ and } w \in K\,. \tag{2.5}$$

Then, problem (2.3) admits a unique solution w in the sense of Definition 2.4. Given a sequence of vector fields F^h_* all satisfying (2.4), (2.5) and converging a.e. to F_*, call w_h the corresponding solutions to (2.3). Then, we have the convergence $\lim_{h\to+\infty} w_h = w$ uniformly on any compact time interval.

The proof is elementary and is sketched in Section 4.

Now we pass to the full problem (1.1), providing a rigorous definition of solution to (1.1).

Definition 2.6. *A pair (u,w) with u in $\mathbf{C^0}\left(\mathbb{R}^+;\mathbf{L^1}(\mathbb{R}^+;\Omega)\right)$ such that $u(t)\in\mathbf{BV}(\mathbb{R}^+;\mathbb{R})$ for a.e. $t\in\mathbb{R}^+$ and w in $\mathbf{W^{1,1}}(\mathbb{R}^+;\mathbb{R}^m)$ is a solution to (1.1) with initial datum*

$$u(0,x)=u_o(x) \quad \text{for } x\geq 0 \qquad \text{and} \qquad w(0)=w_o(t) \quad \text{for } t\geq 0$$

if u solves (2.1) with $B_*(t)=B\left(t,w(t)\right)$ in the sense of Definition 2.2 and w solves (2.3) with $F_*(t)=F\left(t,u(t,0+),w\right)$ in the sense of Definition 2.4.

We are now ready to state the main results of this paper. First we provide an existence result that holds on any time interval $[0,T]$.

Theorem 2.7. *Let **(f)**, **(NC)**, **(b)**, **(B1)**, **(F)** and **(F1)** hold. Assume that $0\in\mathring{\Omega}$ and $b(0)=B(0)$. Then, for all $T>0$, there exists $\delta_T>0$ such that, if the initial data $u_o\in\mathbf{L^1}(\mathbb{R}^+;\Omega)$ and $w_o\in\mathbb{R}^m$ satisfy*

$$\mathrm{TV}(u_o)+\|w_o\|_{\mathbb{R}^m}<\delta_T \tag{2.6}$$

then problem (1.1) admits a solution in the sense of Definition 2.6.

The above result can not be applied as soon as $F(t,0,0)$ does not vanish, for in this case the total variation of $B\left(w(\cdot)\right)$ on the time interval $[0,T]$ may exceed the bound (2.2) preventing the solution to (1.1) to be defined on all $[0,T]$, see Remark 3.2. Therefore, the assumptions in the next theorem are weakened substituting **(F1)** with **(F2)**. An analogous observation holds for the distinction between **(B1)** and **(B2)**. Note however that in all the examples below, B does not depend explicitly on time.

Theorem 2.8. *Let **(f)**, **(NC)**, **(b)**, **(B2)**, **(F)** and **(F2)** hold. Assume that $0\in\mathring{\Omega}$ and $b(0)=B(0,0)$. Then, there exist $\delta>0$ and $T_\delta>0$ with the following property: for every initial data u_o and w_o satisfying*

$$\mathrm{TV}(u_o)+\|w_o\|_{\mathbb{R}^m}<\delta \tag{2.7}$$

problem (1.1) admits a solution in the sense of Definition 2.6 on $[0,T_\delta]$.

The proofs are deferred to Section 4.

3 Applications

As it is usual in applications, in the paragraphs below the role played by the state 0 in Ω will be played by a fixed reference state, say $\bar{u}$, in a neighborhood of which the various assumptions of Theorem 2.8 hold.

The numerical results are obtained with a local Lax-Friedrichs MUSCL scheme in case of the conservation law and with an explicit Euler method for the ordinary differential equation. The coupling is done after each time step. At the right boundary zero Neumann conditions are imposed. The CFL number is 0.45 in all the examples given.

3.1 The Piston Problem

Consider a rectilinear tube filled with fluid to the right of a piston, see Figure 1. In the isentropic, or isothermal, case this system can be described using Lagrangian coordinates, by the p-system coupled with an ordinary differential equation governing the piston. More precisely

$$\begin{cases} \partial_t\tau-\partial_x v=0\\ \partial_t v+\partial_x p(\tau)=0\\ V=v(t,0+)\\ \dot{V}=\alpha\cdot\Big(P(t)-p\left(\tau(t,0+)\right)\Big) \end{cases} \tag{3.1}$$

where t is time, x the Lagrangian mass coordinate, τ the specific volume, v the Lagrangian speed of the flow, p the pressure in the fluid, V the speed of the piston, $P(t)$ the (given) pressure to the left of the piston and α is the ratio between the section of the tube and the mass of the piston, see Figure 1. This problem has been widely considered in the literature, but mainly with pistons

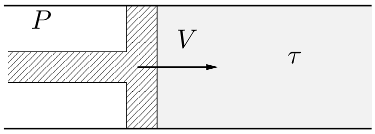

Figure 1: The notation used in the piston problem (3.1).

having *assigned* movements, see for instance [21, 22] or [18, § 99]. Here, on the contrary, the acceleration of the piston is due to the difference between the pressure of the fluid on its right and that of the outer environment on its left.

Below we prove that Theorem 2.8 can be applied to (3.1).

Proposition 3.1. *Fix any $(\bar\tau, \bar v) \in \mathring{\mathbb{R}}^+ \times \mathbb{R}$. Let $n = 2$, $\ell = 1$, $m = 1$ and, with reference to (1.1), define*

$$\begin{array}{rclrcl} u & = & \begin{bmatrix} \tau - \bar\tau \\ v - \bar v \end{bmatrix} & f(u) & = & \begin{bmatrix} -v \\ p(\tau) \end{bmatrix} \\ w & = & V - \bar v & F(t,u,w) & = & \alpha\left(P(t) - p(\tau)\right) \\ b(u) & = & v & B(t,w) & = & V\,. \end{array} \tag{3.2}$$

Then, problem (3.1) is of the type (1.1). Moreover, assume that

$$\begin{array}{rcl} p & \in & \mathbf{C^4}(\mathring{\mathbb{R}}^+; \mathring{\mathbb{R}}^+) \text{ with } p'(\tau) < 0 \text{ and } p''(\tau) > 0 \text{ for all } \tau \in \mathring{\mathbb{R}}^+ \\ P & \in & \mathbf{L^1_{loc}}(\mathbb{R}^+; \mathring{\mathbb{R}}^+) \end{array}$$

then, there exists a positive δ and a positive T such that for any initial datum $(\tau_o, v_o) \in (\bar\tau, \bar v) + \mathbf{L^1}(\mathbb{R}^+; \mathbb{R}^2)$ and $V_o \in \mathbb{R}$ with $\mathrm{TV}(\tau_o, v_o) + \left|V_o - \bar V\right| < \delta$ problem (3.1) admits a solution on $[0, T]$ in the sense of Definition 2.6.

Above, the requirement on p is a standard assumption on the pressure law for an ideal gas. For instance, the standard γ–law $p(\tau) = k/\tau^\gamma$ meets this requirement, for $k > 0$ and $\gamma \geq 1$.

Remark 3.2. This example also shows the physical meaning of the difference between conditions **(F1)** and **(F2)**. Indeed, **(F1)** holds only at time $t = 0$ in the case where P is constant and equal to $p\left(\tau_o(0+)\right)$. Otherwise, consider, for instance, the case $p(\tau) = k/\tau^\gamma$ with $\gamma > 1$, τ_o constant, $v_o = 0$ and $P(t) = p(\tau_o) - \varepsilon$. Then, the piston accelerates to the left eventually creating vacuum, preventing the existence of solution to (3.1) on an *a priori* fixed interval $[0, T]$.

As an example of solution to (3.1), we consider a case in which the piston's movement is determined by the fluid. Indeed, in (3.1) we choose the ratio between the pipe's section and mass to be $\alpha = 1$, the γ–law $p(\tau) = \tau^{-1.4}$ and $P(t) = 0.6^{1.4}$ as outer pressure. As initial datum we choose a shock approaching a piston at rest, that is

$$\tau_o = \begin{cases} 1 & \text{if } x \in [0, 0.5[\\ 1.429 & \text{if } x \in [0.5, +\infty[\end{cases} \qquad v_o = \begin{cases} 0 & \text{if } x \in [0, 0.5[\\ -0.441 & \text{if } x \in [0.5, +\infty[\end{cases} \qquad V_o = 0\,. \tag{3.3}$$

The results shown in Figure 2 refer to the time interval $[0, 1.2]$. At the beginning the outer pressure pushes the piston to the right. When the shock hits the piston, its acceleration suddenly changes sign. After that, the piston slows down, until it finally moves backwards.

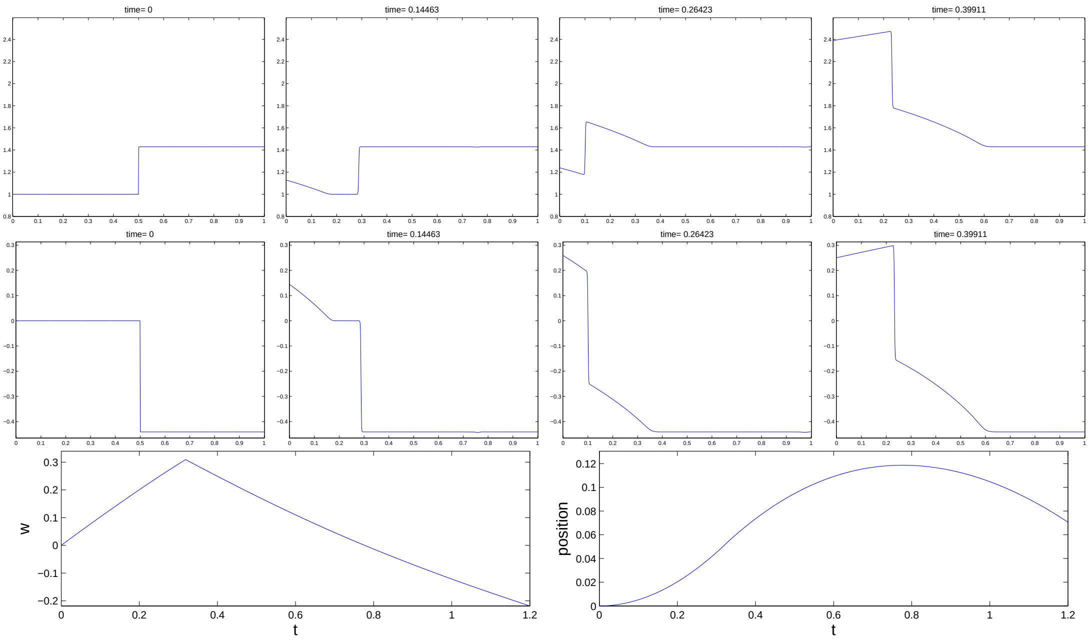

Figure 2: On the first line, the τ component and, on the second line, the v component of the solution to (3.1)–(3.3) plotted as a function of the space variable x. On the third line, the piston's speed, left, and position, right, as a function of time. In the first two lines, on the left, the initial datum with a shock approaching the piston; on the second column: due to the outer pressure on the piston, a rarefaction arises and the pisyon's speed changes sign; third column: the rarefaction and the shock are interacting; fourth column: the interaction finished and the piston is decelerating. Last line: first, the piston accelerates to the right. Then, when it is hit by the shock, its acceleration suffers a jump discontinuity and it slows down, eventually starting moving leftward.

3.2 Flow in a Sewer System with a Manhole

Consider a single junction in a sewer network. At $x = 0$, a junction joins k horizontal pipes to one vertical manhole. All pipes start at the junction, so that each of them is referred to an abscissa $x \in \mathbb{R}^+$. The flow in the i–th tube, for $i = 1, \ldots, k$, can be described by the Saint Venant equations [12], see also [10] or [18, formula (108.1)],

$$\begin{cases} \partial_t A_i + \partial_x Q_i = 0 \\ \partial_t Q_i + \partial_x \left(\frac{Q_i^2}{A_i} + p_i(A_i)\right) = 0 \end{cases} \tag{3.4}$$

ensuring the conservation of mass and momentum. A_i is the wet cross sectional area, Q_i the flow in the x direction and $p_i \in \mathbf{C^4}(\mathbb{R}^+; \mathbb{R}^+)$ is a function representing the hydrostatic pressure, with $p_i' > 0$ and $p_i'' > 0$. In the following we assume the flow in the pipes to be subsonic, i.e. $\left|Q_i/A_i\right| < \sqrt{p_i'(A_i)}$, and far away from the dry state, so that $A_i > 0$. Following [20], as

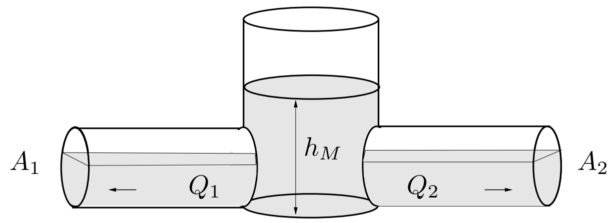

Figure 3: A vertical manhole with two horizontal tubes that exit from it, with the notation used in (3.5). The Preissmann slots are not shown.

boundary condition, we require the equality of all the hydraulic heads at the junction, that is

$$\hat{h}(t) = \frac{1}{2g} \frac{Q_i(t, 0+)^2}{A_i(t, 0+)^2} + h_i\left(A_i(t, 0+)\right) \qquad \text{for all } i = 1, \ldots, k \text{ and all } t \geq 0$$

where $h_i(A_i)$ is the height of water in the i-th tube and g is the gravitational acceleration. The conservation of mass at the junction is expressed by

$$Q_M\left(\hat{h}(t), h_M(t)\right) + \sum_{i=1}^{k} Q_i(t, 0+) = 0$$

where $Q_M(\hat{h}, h_M)$ is the flow into the manhole. The level h_M, which is the height of the water level inside the storage, changes according to

$$\dot{h}_M(t) = \frac{Q_M\left(\hat{h}(t), h_M(t)\right) + Q_{\text{ext}}(t)}{A_M}.$$

Here, A_M is the horizontal cross section of the manhole and $Q_{\text{ext}}(t)$ is a given external inflow into the manhole. Finally, energy conservation gives

$$Q_M\left(\hat{h}(t), h_M(t)\right) = \text{sign}\left(\hat{h}(t) - h_M(t)\right) A_M \sqrt{2g\left|\hat{h}(t) - h_M(t)\right|}.$$

We are thus lead to study the system [5]

$$\begin{cases} \partial_t A_i + \partial_x Q_i = 0 & i = 1, \ldots, k \\ \partial_t Q_i + \partial_x \left(\dfrac{Q_i^2}{A_i} + p_i(A_i) \right) = 0 & i = 1, \ldots, k \\ \hat{h}(t) = \dfrac{1}{2g} \dfrac{Q_i(t, 0+)^2}{A_i(t, 0+)^2} + h_i \left(A_i(t, 0+) \right) & i = 1, \ldots, k \\ h_M(t) = -\dfrac{1}{2g\, A_M^2} \left(\displaystyle\sum_{i=1}^k Q_i(t, 0+) \right) \left| \displaystyle\sum_{i=1}^k Q_i(t, 0+) \right| + \hat{h}(t) \\ \dot{h}_M(t) = \dfrac{1}{A_M} \left(Q_{\text{ext}}(t) - \displaystyle\sum_{i=1}^k Q_i(t, 0+) \right) \end{cases} \tag{3.5}$$

which falls within the class (1.1). The next result shows that Theorem 2.8 applies also to (3.5).

Proposition 3.3. *Let $k \in \mathbb{N} \setminus \{0\}$ and fix $n = 2k$, $\ell = k$, $m = 1$. Choose, the strictly positive areas $\bar{A}_1, \ldots, \bar{A}_k$, the flows $\bar{Q}_1, \ldots, \bar{Q}_k$ and the strictly positive height $\bar{h}_M$. Define*

$$u = \begin{bmatrix} A_1 - \bar{A}_1 \\ Q_1 - \bar{Q}_1 \\ \vdots \\ A_k - \bar{A}_k \\ Q_k - \bar{Q}_k \end{bmatrix} \qquad f(u) = \begin{bmatrix} Q_1 \\ \frac{Q_1^2}{A_1} + p_1(A_1) \\ \vdots \\ Q_k \\ \frac{Q_k^2}{A_k} + p_k(A_k) \end{bmatrix}$$

$$w = h_M - \bar{h}_M \qquad F(t, u, w) = \operatorname{sign}(\hat{h} - h_M) \sqrt{2g \left| \hat{h} - h_M \right|} + \tfrac{1}{A_M} Q_{\text{ext}}(t)$$

$$B(t, w) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ h_M \end{bmatrix} \qquad b(u) = \begin{bmatrix} \frac{1}{2g} \frac{Q_2^2}{A_2^2} + h_2(A_2) - \hat{h} \\ \vdots \\ \frac{1}{2g} \frac{Q_k^2}{A_k^2} + h_k(A_k) - \hat{h} \\ -\frac{1}{2g\, A_M^2} \left(\sum_{i=1}^k Q_i \right) \left| \sum_{i=1}^k Q_i \right| + \hat{h} \end{bmatrix}$$

where $\hat{h} = \frac{1}{2g} \frac{Q_1^2}{A_1^2} + h_1(A_1)$. Then, problem (3.5) is of type (1.1). Moreover, assume that

$$\frac{1}{2g} \frac{\bar{Q}_1^2}{\bar{A}_1^2} + h_1(\bar{A}_1) = \cdots = \frac{1}{2g} \frac{\bar{Q}_k^2}{\bar{A}_k^2} + h_k(\bar{A}_k) = \bar{h}_M + \frac{1}{2g\, \bar{A}_M^2} \left(\sum_{i=1}^k \bar{Q}_i \right) \left| \sum_{i=1}^k \bar{Q}_i \right|$$

$$\left| \bar{Q}_i \right| < \bar{A}_i \sqrt{p_i(\bar{A}_i)} \quad \textit{for } i = 1, \ldots, k \tag{3.6}$$

$$\left| \sum_{i=1}^k \bar{Q}_i \right| \sum_{i=1}^k \frac{\bar{A}_i}{\sqrt{p_i'(\bar{A}_i)}} \neq 1 \tag{3.7}$$

$$p_i \in \mathbf{C^4}(\mathring{\mathbb{R}}^+; \mathring{\mathbb{R}}^+) \ \textit{with } p_i' > 0 \textit{ and } p_i'' > 0 \textit{ for } i = 1, \ldots, k,$$

$$h_i \in \mathbf{C^1}(\mathbb{R}^+; \mathbb{R}^+) \tag{3.8}$$

$$Q_{\text{ext}} \in \mathbf{L^1_{loc}}(\mathbb{R}^+; \mathbb{R}^+) \tag{3.9}$$

then, there exist positive δ and T such that for any initial datum

$$\left((A_1^o, Q_1^o), \ldots, (A_k^o, Q_k^o) \right) \in \left((\bar{A}_1, \bar{Q}_1), \ldots, (\bar{A}_k, \bar{Q}_k) \right) + \mathbf{L^1}(\mathbb{R}^+; \mathbb{R}^n) \quad \textit{and} \quad h_M^o \in \mathbb{R}$$

with

$$\mathrm{TV}\left((A_1^o, Q_1^o), \ldots, (A_k^o, Q_k^o)\right) + \left|h_M^o - \bar{h}_M\right| < \delta \tag{3.10}$$

problem (3.1) admits a solution on $[0, T]$ *in the sense of Definition 2.6.*

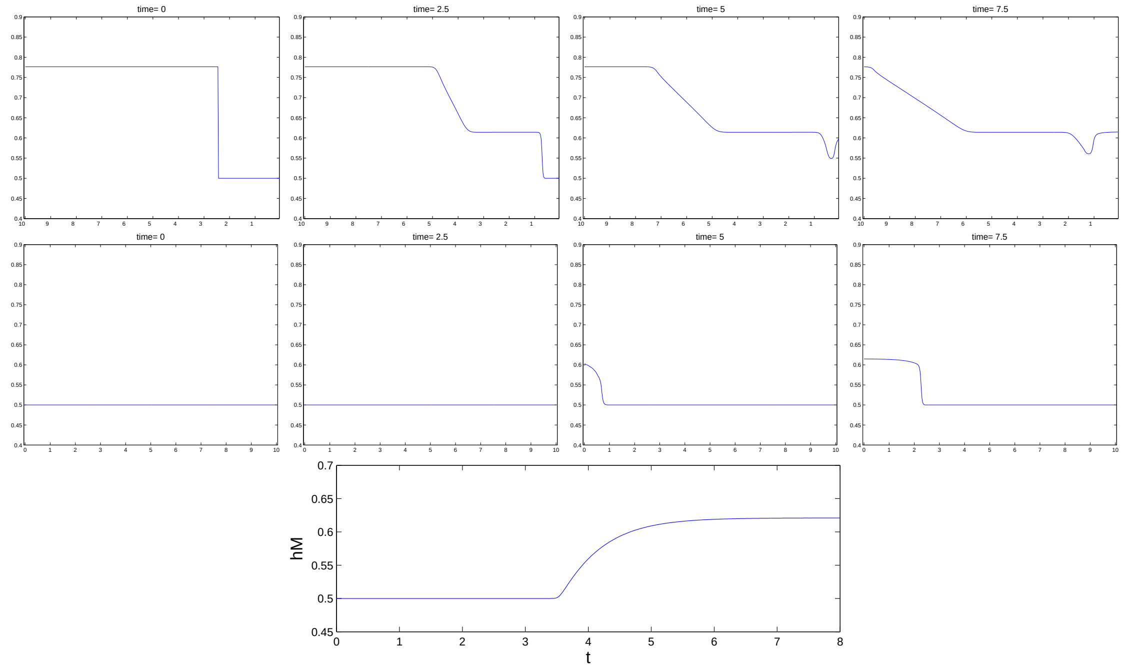

Figure 4: Integration of (3.5)–(3.11) with $k = 2$, as in Figure 3. On the first line, the water level h_1 in the left pipe is plotted vs. x at times 0, 2.5, 5 and 7.5. On the second line, the water level h_2 in the right pipe is plotted vs. x at the same times. On the third line, the water level h_M in the manhole is plotted vs. t. A shock from the left pipe hits the junction at about time $t = 3.5$. This interaction results in water entering the manhole and in a compression wave propagating in the right pipe, eventually developing into a shock. To ease the comparison, we plotted the water level h_i and not the wet surface A_i.

The proof is deferred to Section 4.

As an example, we consider two horizontal pipes as in Figure 3 and plot the result in Figure 4. We set $g = 9.81$, $Q_{\mathrm{ext}} = 0$, the manhole area $A_M = 1$, the pipe radius to $r = 0.5$, the width of the Preissmann slot to $d = 0.1$ and

$$\begin{aligned} h_i(A) &= \begin{cases} \sqrt{\frac{2}{\pi} A} & A \in [0, \frac{\pi}{2} r^2] \\ 2r - \sqrt{2r^2 - \frac{2}{\pi} A} & A \in \left]\frac{\pi}{2} r^2, \pi r^2 - \frac{d^2}{2\pi}\right] \\ \frac{1}{d} A - \frac{d}{2\pi} + 2r - \frac{\pi}{d} r^2 & A \in \left]\pi r^2 - \frac{d^2}{2\pi}, +\infty\right[\end{cases} \\ p_i(A) &= g \int_0^A \big(h(A) - h(a)\big)\, \mathrm{d}a \end{aligned}$$

for $i = 1, 2$. At time $t = 0$, we impose the initial datum

$$\begin{aligned} A_1 &= \begin{cases} \frac{1}{2} \pi r^2 & x \in [0, 2.5] \\ 0.9 \pi r^2 & x \in]2.5, +\infty[\end{cases} & Q_1 &= 0 \\ A_2 &= \frac{1}{2} \pi r^2 & Q_2 &= 0\,. \end{aligned} \tag{3.11}$$

3.3 Supply Chains

We now consider a single node of a network of supply chains, connecting n suppliers. Assume ℓ incoming suppliers and $n-\ell$ outgoing ones. Each outgoing supplier is composed out of a processor and a queue in front of it. For the incoming ones we consider only a processor, as in Figure 5. The work done by a processor is modeled by

$$\partial_t \rho_i(t,x) + \partial_x \big(v_i \rho_i(t,x)\big) = 0 \qquad x \in \mathbb{R}^+,\ t \in \mathbb{R}^+,\ i = 1,\ldots,n$$

where ρ_i is the density of goods in the i-th processor and v_i is the (constant) processing velocity. To guarantee the correct orientation of flow, we have $v_j < 0$ for $j = 1,\ldots,\ell$ and $v_k > 0$ for $k = \ell+1,\ldots,n$ respectively. For the load q_k of goods stored in the k-th queue, we impose the conservation of mass:

$$\dot q_k(t) = f_k^{in}\big(t, u(t,0)\big) - f_k^{out}(q_k) \qquad k = \ell+1,..,n$$

where f_k^{in} is the inflow and f_k^{out} is the outflow from the queue to the k-th processor. The distribution matrix $A(t) = \big(a_{jk}(t)\big)$, for $j = 1,\ldots,\ell$ and $k = \ell+1,\ldots,n$, assigns the percentage a_{jk} of the goods exiting processor j and lining up into the k-th queue. Thus, the queue is filled by

$$f_k^{in}\big(t, \rho(t,0)\big) = \sum_{j=1}^{\ell} a_{jk}(t)\, v_j\, \rho_j(t,0)\,. \qquad k = \ell+1,\ldots,n\,.$$

For the outflow we use the relaxed formulation, as presented in [4],

$$f_k^{out}(q_k) = \min\left(\frac{q_k(t)}{\varepsilon}, \mu_k\right) \qquad k = \ell+1,\ldots,n$$

with the relaxation parameter $\varepsilon > 0$ small and μ_k the maximal capacity of the k-th processor. Existence and uniqueness for the non relaxed case were shown in [16], by directly using wave front tracking. Summarizing the above equations, we are left with the following problem:

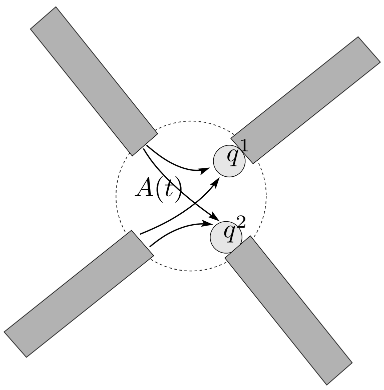

Figure 5: One node in a network of supply chains

$$\begin{cases} \partial_t \rho_i(t,x) + \partial_x \big(v_i \rho_i(t,x)\big) = 0 & i = 1,\ldots,n \\ v_k\, \rho_k(t,0+) = \min\left(\frac{q_k(t)}{\varepsilon}, \mu_k\right) & k = \ell+1,\ldots,n \\ \dot q_k(t) = \sum_{j=1}^{\ell} a_{jk}(t)\, v_k\, \rho_k(t,0) - \min\left(\frac{q_k(t)}{\varepsilon}, \mu_k\right) & k = \ell+1,\ldots,n\,. \end{cases} \tag{3.12}$$

Proposition 3.4. *Let $n \in \mathbb{N}$, with $n \geq 2$, $\ell \in \{1, \ldots, n-1\}$, $m = n - \ell$ and define*

$$
\begin{array}{rclcrcl}
u & = & \begin{bmatrix} \rho_1 - \bar\rho_1 \\ \vdots \\ \rho_n - \bar\rho_n \end{bmatrix} & & f(u) & = & \begin{bmatrix} v_1\rho_1 \\ \vdots \\ v_n\rho_n \end{bmatrix} \\
w & = & \begin{bmatrix} q_{\ell+1} - \bar q_{\ell+1} \\ \vdots \\ q_n - \bar q_n \end{bmatrix} & & F(t,u,w) & = & \begin{bmatrix} \sum_{j=1}^{\ell} a_{j\ell+1}(t)\, v_j\, \rho_j - \min\left\{\frac{1}{\varepsilon} q_{\ell+1}, \mu_{\ell+1}\right\} \\ \vdots \\ \sum_{j=1}^{\ell} a_{jn}(t)\, v_j \rho_j - \min\left\{\frac{1}{\varepsilon} q_n, \mu_n\right\} \end{bmatrix} \\
b(u) & = & \begin{bmatrix} v_{\ell+1}\rho_{\ell+1} \\ \vdots \\ v_n\rho_n \end{bmatrix} & & B(t,w) & = & \begin{bmatrix} \min\left\{\frac{1}{\varepsilon} q_{\ell+1}, \mu_{\ell+1}\right\} \\ \vdots \\ \min\left\{\frac{1}{\varepsilon} q_n, \mu_n\right\} \end{bmatrix}.
\end{array}
$$

Then, problem (3.12) is of type (1.1). Moreover, assume that

$$
\begin{array}{rcl}
v_1, \ldots, v_\ell < 0 & \text{and} & v_{\ell+1}, \ldots, v_n > 0 \\
\displaystyle\sum_{k=\ell+1}^{n} a_{jk}(t) & = & 1 \qquad \text{for all } j = 1, \ldots, \ell \\
a_{ij} & \in & \mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}), \qquad i = l+1, .., n \;\; j = 1, .., n
\end{array}
\tag{3.13}
$$

then in a neighborhood of any initial state $u_o \equiv (\rho_1^o, \ldots, \rho_n^o)$ and $w_o \equiv (q_{\ell+1}^o, \ldots, q_n^o)$ with $\rho_i^o > 0$ for $i = 1, \ldots, n$ and $q_k^o > 0$ for $k = \ell+1, \ldots, n$, Theorem 2.8 can be applied to (3.12).

The example for the numerical computation displayed in Figure 6 has two processors,$n = 2$ and a single queue,$m = 1, \ell = 1$. One processor is incoming, e.g. $v_1 = -2$, the second one has $v_2 = 1$. The remaining parameters have the following values $\mu_2 = 1$, $a_{12} = 1$ and $\varepsilon = 0.01$.

At $t = 0$, the outgoing supplier is empty, $\rho_2^o = 0$ and $q_2^o = 0$, but the first processor contains an incoming shock, $\rho_1^o = 0$ on $(0, 1/2)$ and $\rho_1^o = 1$ elsewhere. The shock approaches the node. Clearly

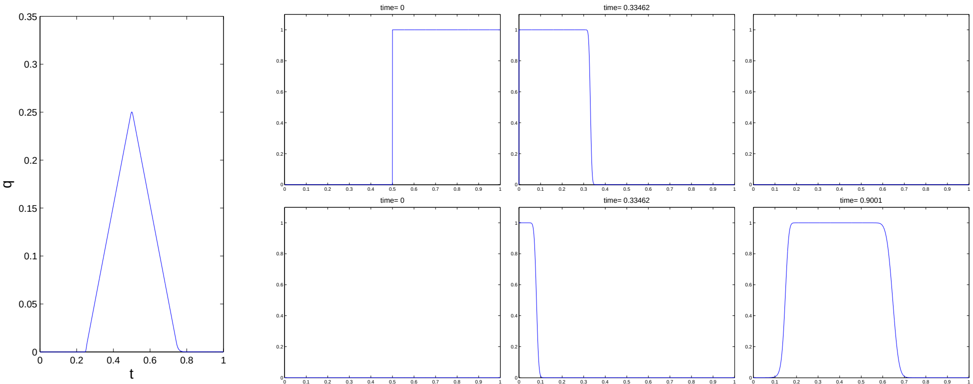

Figure 6: Solution to (3.12). The queue length q_2 is on the left, plotted against t. On the right: ρ_1 on top, ρ_2 below, at three different times plotted as function of the space variable x. Note that the initial load of processor 1 fills the queue and then passes to processor 2.

the second processor can not carry all the incoming load, thus some goods are stored in the queue. While the second processor is running at maximum capacity, the first one is emptying. When the inflow breaks, the queue begins to clear. Finally all goods are located in the second processor.

Clearly the numerical diffusion is remarkable, but the qualitative behavior is captured.

3.4 Blood Flow

Following [13, formulæ (2.3), (2.12), (2.14)], [1, Section 2] and [7], we consider the following model for blood flowing through an artery and a vessel, see also [23] and the references therein. The former is described through a 1D conservation law, while an ordinary differential equation is used for the latter.

$$\begin{cases} \partial_t a + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{a} + \pi(a)\right) = 0 \\ a(t,0+) = a_S \left(1 + \frac{\sqrt{a_s}}{\beta} P\right)^2 \\ \dot P = -\frac{1}{C} Q - \frac{1}{C} q(t,0) \\ \dot Q = \frac{1}{L} P - \frac{R}{L} Q - \frac{1}{L}\Pi(t) \end{cases} \tag{3.14}$$

where the minus sign in the third line above is due to our choice of the orientation of the x axis. Here, we used the following notation:

Independent Variables:	t	time	x	length along the artery
Known Functions:	p	artery blood pressure	π	$= (1/\rho)\int_{a_S}^a \alpha\, p'(\alpha)\, d\alpha$
	Π	vessel external blood pressure		
Known Constants:	ρ	blood density	a_S	artery size at rest
	β	artery elasticity	R	vessel resistance
	C	vessel capacitance	L	vessel inductance
Unknown Functions:	a	area of artery section	q	artery blood flow
	P	vessel blood pressure	Q	vessel blood flow.

A typical choice for the artery blood pressure is

$$p(a) = \frac{\beta}{\sqrt{a_S}} \left(\sqrt{\frac{a}{a_S}} - 1\right),$$

see [1, Paragraph 2.3]. As it is standard in this context, we assume throughout that all constants are real positive numbers. The above model fits in the framework constructed in Section 2.

Proposition 3.5. *Let $n = 2$, $\ell = 1$, $m = 2$. Choose $(\bar a, \bar q) \in \mathring{\mathbb{R}}^+ \times \mathbb{R}$ and $(\bar P, \bar Q) \in \mathring{\mathbb{R}}^+ \times \mathbb{R}$. Define*

$$\begin{aligned} u &= \begin{bmatrix} a - \bar a \\ q - \bar q \end{bmatrix} & f(u) &= \begin{bmatrix} q \\ \frac{q^2}{a} + \pi(a) \end{bmatrix} \\ w &= \begin{bmatrix} P - \bar P \\ Q - \bar Q \end{bmatrix} & F(t,u,w) &= \begin{bmatrix} -\frac{1}{C} Q + \frac{1}{C} q \\ \frac{1}{L} P - \frac{R}{L} Q + \frac{1}{L}\Pi(t) \end{bmatrix} \\ b(u) &= a & B(t,w) &= a_S \left(1 + \frac{\sqrt{a_s}}{\beta} P\right)^2 . \end{aligned}$$

Then, problem (3.14) is of type (1.1). Moreover, assume that

$$|\bar q| < \bar a \sqrt{\bar a\, p'(\bar a)} \tag{3.15}$$

$$\bar a = a_S \left(1 + \frac{\sqrt{a_s}}{\beta} \bar P\right)^2$$

$$p \in \mathbf{C^4}(\mathring{\mathbb{R}}^+; \mathbb{R}) \text{ with } p'(a) > 0 \text{ and } p'(a) + a p''(a) > 0 \text{ for all } a > 0 \tag{3.16}$$

$$\Pi \in \mathbf{L^1_{loc}}(\mathring{\mathbb{R}}^+; \mathring{\mathbb{R}}^+), \tag{3.17}$$

then, there exist positive δ and T such that for any initial datum $(a_o, q_o) \in (\bar a, \bar q) + \mathbf{L^1}(\mathbb{R}^+; \mathbb{R}^2)$, $(P_o, Q_o) \in \mathbb{R}^+ \times \mathbb{R}$ with

$$\mathrm{TV}(a_o, q_o) + \left\|(P_o, Q_o) - (\bar P, \bar Q)\right\|_{\mathbb{R}^2} < \delta \tag{3.18}$$

problem (3.14) admits a solution on $[0, T]$ in the sense of Definition 2.6.

Several numerical results about of (3.14) can be found in the current literature. We refer for instance to [23] and to the references therein.

4 Technical Details

For later use, we state here without proof the Gronwall type lemma used in the sequel.

Lemma 4.1. *Let* $\delta \in \mathbf{C^0}\left([0,T];\mathbb{R}^+\right)$*,* $\alpha \in \mathbf{L^\infty_{loc}}\left([0,T];\mathbb{R}^+\right)$ *and* $\beta \in \mathbf{L^1_{loc}}\left([0,T];\mathbb{R}^+\right)$*. If*

$$\delta(t) \le \alpha(t) + \int_0^t \beta(\tau)\,\delta(\tau)\,\mathrm{d}\tau$$

then

$$\delta(t) \le \alpha(t) + \int_0^t \alpha(\tau)\,\beta(\tau)\,e^{\int_\tau^t \beta(s)\,\mathrm{d}s}\,\mathrm{d}\tau\ .$$

Proof of Theorem 2.3. We simply observe that [8, Theorem 2.2] is applicable. Indeed, conditions **(f)** and **(b)** therein are the same as the ones here in Section 2. Condition **($\boldsymbol{\gamma}$)** is here trivially satisfied, setting $\gamma(t) = 0$ for all t and thanks to **(NC)**. Finally, we note that the upper bound on the total variation of the functions in $\mathcal{D}_t$ follows from the definition of $\mathcal{D}_t$ in the proof of [8, Proposition 4.6] and by [8, formula (4.9)]. □

Proof of Proposition 2.5. The existence and uniqueness of a global solution to (2.3) follow from [14, § 1]. To prove the continuous dependence from the vector field, find first the *a priori* estimate by means of (2.4)

$$\left\|w(t)\right\|_{\mathbb{R}^m} \le \|w_o\|_{\mathbb{R}^m} + \int_0^t \left\|F\left(\tau, w(\tau)\right)\right\|_{\mathbb{R}^m} \mathrm{d}\tau \le \|w_o\|_{\mathbb{R}^m} + \int_0^t \left(A(\tau) + B(\tau)\left\|w(\tau)\right\|_{\mathbb{R}^m}\right)\mathrm{d}\tau$$

so that by Lemma 4.1 with $\alpha(t) = \|w_o\|_{\mathbb{R}^m} + \int_0^t A(\tau)\,\mathrm{d}\tau$ and $\beta(t) = B(t)$,

$$\left\|w(t)\right\|_{\mathbb{R}^m} \le \|w_o\|_{\mathbb{R}^m} + \int_0^t \left(A(\tau) + \left(\|w_o\|_{\mathbb{R}^m} + \int_0^\tau A(s)\,\mathrm{d}s\right) B(\tau) e^{\int_\tau^t B(s)\mathrm{d}s}\right)\mathrm{d}\tau\ .$$

Define

$$R_t = \|w_o\|_{\mathbb{R}^m} + \int_0^t \left(A(\tau) + \left(\|w_o\|_{\mathbb{R}^m} + \int_0^\tau A(s)\,\mathrm{d}s\right) B(\tau) e^{\int_\tau^t B(s)\mathrm{d}s}\right)\mathrm{d}\tau\ .$$

Now, following usual procedures based on Gronwall Lemma

$$\begin{aligned}
\left\|w_h(t) - w(t)\right\|_{\mathbb{R}^m} &\le \int_0^t \left\|F_*^h\left(\tau, w_h(\tau)\right) - F_*\left(\tau, w(\tau)\right)\right\|_{\mathbb{R}^m}\mathrm{d}\tau \\
&\le \int_0^t \left\|F_*\left(\tau, w_h(\tau)\right) - F_*\left(\tau, w_h(\tau)\right)\right\|_{\mathbb{R}^m}\mathrm{d}\tau \\
&\quad + \int_0^t \left\|F_*^h\left(\tau, w_h(\tau)\right) - F_*\left(\tau, w(\tau)\right)\right\|_{\mathbb{R}^m}\mathrm{d}\tau\ .
\end{aligned}$$

Let $K = \left\{w \colon \|w\|_{\mathbb{R}^m} \le R_t\right\}$ and call C_{R_t} the corresponding constant in 4. of **(F)**. Call $A_h(t)$ the latter summand above, apply 4. in **(F)** and Lemma 4.1 with $\alpha = A_h$ and $\beta = C_{R_t}$ to obtain

$$\begin{aligned}
\sup_{t\in[0,T]} \left\|w_h(t) - w(t)\right\|_{\mathbb{R}^m} &\le \sup_{t\in[0,T]} \left(A_h(t) + C_{R_t}\int_0^t A_h(\tau)\,e^{C_{R_t}(t-\tau)}\,\mathrm{d}\tau\right) \\
&\le A_h(T) + C_{R_T}\int_0^T A_h(\tau)\,e^{C_{R_t}(T-\tau)}\,\mathrm{d}\tau\ .
\end{aligned}$$

At the limit $h \to 0$, by Lebesgue Dominated Convergence Theorem we have that $A_h(t) \to 0$ on any compact time interval and the proof is completed. □

Proof of Theorem 2.7. The proof is obtained by an iterative method through several steps.

1. Definition of $\tilde{u}_k$ and w_k. By assumptions **(B1)** and (2.2), there exists a positive $\tilde{\delta}$ such that $\left\|B(w) - b(0)\right\|_{\mathbb{R}^{n-\ell}} < \delta/2$ for every w such that $\|w\|_{\mathbb{R}^m} < \tilde{\delta}$, where δ is the constant defined in Theorem 2.3. Using Δ as in Theorem 2.3 and C as in **(F1)**, define

$$
\begin{aligned}
K_1 &= \left\{ w \in \mathbb{R}^m \colon \|w\|_{\mathbb{R}^m} \leq 1 + \|C\|_{\mathbf{L^1}([0,T])} \, e^{\|C\|_{\mathbf{L^1}([0,T])}} \right\} \\
\delta_T &= \min \left\{ \delta, \tilde{\delta}, \frac{\delta}{2 \left[1 + \|C\|_{\mathbf{L^1}([0,T])} \, e^{\|C\|_{\mathbf{L^1}([0,T])}} \right] \tilde{C}_{K_1} \|C\|_{\mathbf{L^1}([0,T])}} \right\} .
\end{aligned}
\tag{4.1}
$$

Fix $u_o \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)\left(\mathbb{R}^+;\Omega\right)$ and $w_o \in \mathbb{R}^m$ satisfying (2.6). Define $\tilde{u}_0(t,x) = u_o$ and $w_0(t) = w_o$ for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$. We easily get $\mathrm{TV}\left(B(w_o)\right) < \delta/2$ and, by (4.1) and (2.6), $\tilde{u}_0(0,\cdot) \in \mathcal{D}_0$. Hence, by Theorem 2.3, there exists a solution $\tilde{u}_1(t,x) = \left(P_{B_*}(t,0)\tilde{u}_0(0,\cdot)\right)(x)$ defined on all $[0,T]$ to

$$
\left\{
\begin{array}{l}
\partial_t u + \partial_x f(u) = 0 \\
b\left(u(t,0)\right) = B\left(w_0(t)\right) \\
u(0,x) = u_o(x)
\end{array}
\right.
$$

with $\left\|u_1(t,x)\right\|_{\mathbb{R}^n} \leq \Delta$ for a.e. $t \in [0,T]$ and $x > 0$. Hypotheses **(F)** and **(F1)** imply that there exists a unique solution w_1 to the Cauchy problem

$$
\left\{
\begin{array}{l}
\dot{w} = F\left(t, \tilde{u}_1(t,0), w\right) \\
w(0) = w_o \, .
\end{array}
\right.
$$

By **(F1)**, we get that, for every $t \in [0,T]$,

$$
\left\|w_1(t)\right\|_{\mathbb{R}^m} \leq \|w_o\|_{\mathbb{R}^m} + \int_0^t \left\|F\left(s, \tilde{u}_0(s,0), w_1(s)\right)\right\|_{\mathbb{R}^m} \mathrm{d}s \leq \|w_o\|_{\mathbb{R}^m} + \int_0^t C(s) \left\|w_1(s)\right\|_{\mathbb{R}^m} \mathrm{d}s
$$

and, by Lemma 4.1,

$$
\left\|w_1(t)\right\|_{\mathbb{R}^m} \leq \|w_o\|_{\mathbb{R}^m} \left[1 + \int_0^t C(s) e^{\int_s^t C(r)\mathrm{d}r} \, \mathrm{d}s\right] \leq \left[1 + \|C\|_{\mathbf{L^1}([0,T])} e^{\|C\|_{\mathbf{L^1}([0,T])}}\right] \|w_o\|_{\mathbb{R}^m} . \tag{4.2}
$$

Introduce recursively, for $k \geq 2$,

$$
\begin{array}{lll}
\tilde{u}_k & \text{as the solution of} & \left\{
\begin{array}{l}
\partial_t u + \partial_x f(u) = 0 \\
b\left(u(t,0)\right) = B\left(w_{k-1}(t)\right) \\
u(0,x) = u_o(x)
\end{array}
\right. \\
w_k & \text{as the solution of} & \left\{
\begin{array}{l}
\dot{w} = F(t, \tilde{u}_k(t,0+), w) \\
w(0) = w_o \, .
\end{array}
\right.
\end{array}
$$

Note that the same estimate as (4.2) holds for all $\left\|w_k(t)\right\|_{\mathbb{R}^m}$ ($k \geq 2$), provided $\tilde{u}_k$ exists and $\left\|\tilde{u}_k(t,x)\right\|_{\mathbb{R}^n} \leq \Delta$. Moreover, by **(B1)** and **(F1)** and since the function $t \mapsto B\left(w_{k-1}(t)\right)$ is absolutely continuous, then

$$
\begin{aligned}
\mathrm{TV}\left(B\left(w_{k-1}(\cdot)\right)\right) &= \int_0^T \left\|\frac{d}{dt} B\left(w_{k-1}(t)\right)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}t \;=\; \int_0^T \left\|\frac{\partial}{\partial w} B\left(w_{k-1}(t)\right) w'_{k-1}(t)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}t \\
&\leq \tilde{C}_{K_1} \int_0^T \left\|w'_{k-1}(t)\right\|_{\mathbb{R}^m} \mathrm{d}t \;=\; \tilde{C}_{K_1} \int_0^T \left\|F(t, \tilde{u}_{k-1}(t,0), w_{k-1}(t))\right\|_{\mathbb{R}^m} \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\leq \tilde{C}_{K_1} \int_0^T C(t) \left\| w_{k-1}(t) \right\|_{\mathbb{R}^m} \mathrm{d}t \\
&\leq \tilde{C}_{K_1} \left[1 + \|C\|_{\mathbf{L^1}([0,T])} \, e^{\|C\|_{\mathbf{L^1}([0,T])}} \right] \|w_o\|_{\mathbb{R}^m} \|C\|_{\mathbf{L^1}([0,T])} \; < \; \frac{\delta}{2}
\end{aligned}
$$

and, by Theorem 2.3, also $\tilde{u}_k$ is well defined.

2. The w_k satisfy Ascoli–Arzelà Theorem. In this part we prove that the sequence $w_k \in \mathbf{C^0}\left([0,T]; \mathbb{R}^m\right)$ satisfies the hypotheses of Ascoli-Arzelà Theorem, see [24, Theorem A5]. In the previous step, we proved that, for every $k \in \mathbb{N}$,

$$
\left\| w_k(t) \right\|_{\mathbb{R}^m} \leq \left[1 + \|C\|_{\mathbf{L^1}([0,T])} \, e^{\|C\|_{\mathbf{L^1}([0,T])}} \right] \|w_o\|_{\mathbb{R}^m} \,, \tag{4.3}
$$

see (4.2), which implies that the sequence w_k is bounded. Moreover we easily get that, for every $k \in \mathbb{N}$ and $0 \leq s < t \leq T$,

$$
\begin{aligned}
\left\| w_k(t) - w_k(s) \right\|_{\mathbb{R}^m} &= \left\| \int_0^t F(r, \tilde{u}_k(r,0), w_k(r)) \, \mathrm{d}r - \int_0^s F(r, \tilde{u}_k(r,0), w_k(r)) \, \mathrm{d}r \right\|_{\mathbb{R}^m} \\
&\leq \int_s^t \left\| F\left(r, \tilde{u}_k(r,0), w_k(r)\right) \right\|_{\mathbb{R}^m} \mathrm{d}r
\end{aligned}
$$

and by **(F1)**,

$$
\left\| w_k(t) - w_k(s) \right\|_{\mathbb{R}^m} \leq \int_s^t C(r) \left\| w_k(r) \right\|_{\mathbb{R}^m} \mathrm{d}r \,.
$$

By (4.3), we deduce that, for every $k \in \mathbb{N}$ and $0 \leq s < t \leq T$,

$$
\left\| w_k(t) - w_k(s) \right\|_{\mathbb{R}^m} \leq \|w_o\|_{\mathbb{R}^m} \left[1 + \|C\|_{\mathbf{L^1}([0,T])} \, e^{\|C\|_{\mathbf{L^1}([0,T])}} \right] \int_s^t C(r) \, \mathrm{d}r
$$

and we conclude that the sequence w_k is equicontinuous. Thus Ascoli-Arzelà Theorem implies that there exists a subsequence w_{k_h} and a function $w_* \in \mathbf{C^0}\left([0,T]; \mathbb{R}^m\right)$ such that w_{k_h} converges to w_* in $\mathbf{C^0}\left([0,T]; \mathbb{R}^m\right)$.

3. Definition of u With a slight abuse of notation, call w_k the convergent subsequence constructed in the previous step. Define for $k \geq 1$

$$
u_k \quad \text{as the solution of} \quad \left\{ \begin{array}{l} \partial_t u + \partial_x f(u) = 0 \\ b\left(u(t,0)\right) = B_*(t) \\ u(0,x) = u_o(x) \end{array} \right. \quad \text{with} \quad B_*(t) = B\left(t, w_k(t)\right) \,,
$$

which is uniquely defined on all $[0,T]$ by Theorem 2.3. Observe that the sequence u_k converges in $\mathbf{C^0}\left([0,T]; \mathbf{L^1}(\mathbb{R}^+; \mathbb{R}^n)\right)$. Indeed, by 2 in Theorem 2.3, (4.3) and **(B1)**, setting $R = \left[1 + \|C\|_{\mathbf{L^1}([0,T])} e^{\|C\|_{\mathbf{L^1}([0,T])}}\right] \|w_o\|_{\mathbb{R}^m}$,

$$
\begin{aligned}
\left\| u_k(t) - u_h(t) \right\|_{\mathbf{L^1}} &\leq L \int_0^t \left\| B\left(\tau, w_k(\tau)\right) - B\left(\tau, w_h(\tau)\right) \right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}\tau \\
&\leq L \sup_{\|w\| \leq R} \left\| \frac{\partial B}{\partial w} \right\|_{\mathbb{R}^{m \times (n-\ell)}} \int_o^t \left\| w_k(\tau) - w_h(\tau) \right\|_{\mathbb{R}^m} \mathrm{d}\tau \\
&\leq L \, T \, C_{\{w \colon \|w\|_{\mathbb{R}^m} \leq R\}} \left\| w_k - w_h \right\|_{\mathbf{C^0}}
\end{aligned}
$$

which shows that the u_k form a Cauchy sequence in $\mathbf{C^0}\left([0,T]; \mathbf{L^1}(\mathbb{R}^+; \mathbb{R})\right)$. Let $u = \lim_{k \to +\infty} u_k$.

4. Definition of w**.** Let $\bar w$ be the solution to (2.3) with $F_*(\tau,w) = F\left(\tau, u(\tau,0+), w\right)$. We now prove that $w_* = \bar w$. Indeed, let $F_*^k(t,w) = F\left(t, u_k(t,0+), w\right)$ and apply the last part of Proposition 2.5. This is possible, since $u_k(t,0+) \to u(t,0+)$ for a.e. $t \in [0,T]$, which is shown as in the proof of [2, Theorem 1.2], thanks to **(NC)**. In the sequel we denote $w = \bar w = w_*$.

5. The Pair (u,w) **Solves (1.1).** Thanks to what was proved in the previous step, it is now sufficient to verify that u satisfies (2.1) with $B_*(t) = B\left(w(t)\right)$. Indeed, with the notation of Theorem 2.3,

$$\begin{aligned}
\left\| P_{B_*}(t,0)u_o - u(t)\right\|_{\mathbf{L^1}} &= \lim_{k\to+\infty} \left\| P_{B_*}(t,0)u_o - u_k(t)\right\|_{\mathbf{L^1}} \\
&\le L \lim_{k\to+\infty} \int_0^t \left\| B\left(w(\tau)\right) - B\left(w_k(\tau)\right)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}\tau \\
&= 0
\end{aligned}$$

where we used **(B1)** and the uniform convergence of w_k to w. □

Remark that in the previous proof we are not able to verify that $\lim_k \tilde u_k$ and w do solve (1.1). Indeed, the extraction of a subsequence of the w_k destroys the link between $\tilde u_{k-1}$ and w_k.

We also note that the roles of u and w in the previous proof are symmetric. An entirely analogous proof can be obtained beginning with the definition of the sequences u_k and $\tilde w_k$, using Helly Compactness Theorem on the u_k, define a new sequence w_k and pass to the limit.

Proof of Theorem 2.8. The proof is similar to the one of Theorem 2.7. Therefore, below we present only the relevant modifications.

Let δ_1 be equal to the δ exhibited in Theorem 2.3. By **(B2)**, there exists $\tilde\delta > 0$ such that $\left\| B(0,w) - b(0)\right\|_{\mathbb{R}^m} < \delta_1/2$ for every $\|w\|_{\mathbb{R}^m} < \tilde\delta$. Define

$$\begin{aligned}
\delta &= \min\left\{\tilde\delta, \delta_1\right\} \\
H &= \left[1 + \|C\|_{\mathbf{L^1}([0,t])} e^{\|C\|_{\mathbf{L^1}([0,t])}}\right] \left[\tilde\delta + \|C\|_{\mathbf{L^1}([0,t])}\right] \\
K_1 &= \left\{w \in \mathbb{R}^m : \|w\|_{\mathbb{R}^m} \le H\right\}
\end{aligned}$$

where Δ is defined in Theorem 2.3 and C in **(F2)**. Let $\tilde C_{K_1}$ be as in **(B2)**. Choose $T_\delta \in \left]0,1\right[$ such that

$$T_\delta < \frac{\delta_1}{4\tilde C_{K_1}} \qquad \text{and} \qquad \|C\|_{\mathbf{L^1}(0,T_\delta)} < \frac{\delta_1}{4\left(1+H\right)\tilde C_{K_1}}\,. \tag{4.4}$$

Fix $u_o \in \left(\mathbf{L^1} \cap \mathbf{BV}\right)\left(\mathbb{R}^+;\Omega\right)$ and $w_o \in \mathbb{R}^m$ such that (2.7) holds. Define $\tilde u_0(t,x) = u_o$ and $w_0(t) = w_o$ for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$.

By (2.7), we easily get $w_o \in K_1$, $u_o \in \mathcal{D}_0$ and, since the function $t \mapsto B(t,w_o)$ is absolutely continuous, then

$$\begin{aligned}
\mathrm{TV}(B(\cdot,\, w_o(\cdot))_{|[0,T_\delta]}) + \left\| B(0,w_o) - b(0)\right\|_{\mathbb{R}^m} &< \int_0^{T_\delta} \left\| \frac{\partial}{\partial s} B(s,\, w_o)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}s + \frac{\delta_1}{2} \\
&\le \tilde C_{K_1} T_\delta + \frac{\delta_1}{2} \\
&< \delta_1
\end{aligned}$$

by (4.4). Use now Theorem 2.3, for every $t \in [0,T_\delta]$, there exists $\tilde u_1(t,x) = \left(P_{B_*}(t,0)\tilde u_0(0,\cdot)\right)(x)$ solving

$$\begin{cases}
\partial_t u + \partial_x f(u) = 0 \\
b\left(u(t,0)\right) = B\left(t, w_0(t)\right) \\
u(0,x) = u_o(x)
\end{cases}$$

Note that $\left\|\tilde{u}_1(t,x)\right\|_{\mathbb{R}^n} \leq \Delta$ for a.e. $t > 0$ and $x > 0$. Hypotheses **(F)** and **(F2)** imply that there exists a unique solution w_1 on $[0, T_\delta]$ to the Cauchy problem

$$\left\{\begin{array}{l} \dot{w} = F(t, \tilde{u}_0(t,0), w) \\ w(0) = w_o \,. \end{array}\right.$$

By **(F2)**, we get that, for every $t \in [0, T_\delta]$,

$$\begin{aligned}
\left\|w_1(t)\right\|_{\mathbb{R}^m} &\leq \left\|w_o\right\|_{\mathbb{R}^m} + \int_0^t \left\|F(s, \tilde{u}_0(s,0), w_1(s))\right\|_{\mathbb{R}^m} \mathrm{d}s \\
&\leq \left\|w_o\right\|_{\mathbb{R}^m} + \left\|C\right\|_{\mathbf{L^1}([0,t])} + \int_0^t C(s) \left\|w_1(s)\right\|_{\mathbb{R}^m} \mathrm{d}s
\end{aligned}$$

and so, by Lemma 4.1,

$$\begin{aligned}
\left\|w_1(t)\right\|_{\mathbb{R}^m} &\leq \left[\left\|w_o\right\|_{\mathbb{R}^m} + \left\|C\right\|_{\mathbf{L^1}([0,t])}\right] \left[1 + \int_0^t C(s) e^{\int_s^t C(r)\mathrm{d}r} \mathrm{d}s\right] \\
&\leq \left[1 + \left\|C\right\|_{\mathbf{L^1}([0,t])} e^{\left\|C\right\|_{\mathbf{L^1}([0,t])}}\right] \left[\left\|w_o\right\|_{\mathbb{R}^m} + \left\|C\right\|_{\mathbf{L^1}([0,t])}\right] \\
&\leq H. \qquad (4.5)
\end{aligned}$$

Introduce recursively for $k \geq 2$ on the time interval $[0, T_\delta]$

$$\begin{array}{lll}
\tilde{u}_k & \text{as the solution of} & \left\{\begin{array}{l} \partial_t u + \partial_x f(u) = 0 \\ b\left(u(t,0)\right) = B\left(t, w_{k-1}(t)\right) \\ u(0,x) = u_o(x) \end{array}\right. \\
w_k & \text{as the solution of} & \left\{\begin{array}{l} \dot{w} = F(t, \tilde{u}_{k-1}(t,0), w) \\ w(0) = w_o \,. \end{array}\right.
\end{array}$$

Note that the same estimate as (4.5) holds on $\left\|w_k(t)\right\|_{\mathbb{R}^m}$ for all $k \geq 2$, provided $\tilde{u}_k$ exists and $\left\|\tilde{u}_k(t,x)\right\|_{\mathbb{R}^n} \leq \Delta$. Moreover, by **(B2)**, **(F2)** and since the function $t \mapsto B\left(t, w_{k-1}(t)\right)$ is absolutely continuous, we have

$$\begin{aligned}
\mathrm{TV}\left(B\left(\cdot,\, w_{k-1}(\cdot)\right)_{|[0,T_\delta]}\right) &= \int_0^{T_\delta} \left\|\frac{d}{ds} B\left(s,\, w_{k-1}(s)\right)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}s \\
&= \int_0^{T_\delta} \left\|\frac{\partial}{\partial s} B\left(s,\, w_{k-1}(s)\right) + \frac{\partial}{\partial w} B\left(s,\, w_{k-1}(s)\right) \circ w'_{k-1}(s)\right\|_{\mathbb{R}^{n-\ell}} \mathrm{d}s \\
&\leq \tilde{C}_{K_1} \left[T_\delta + \int_0^{T_\delta} \left\|w'_{k-1}(s)\right\|_{\mathbb{R}^m} \mathrm{d}s\right] \\
&= \tilde{C}_{K_1} \left[T_\delta + \int_0^{T_\delta} \left\|F\left(s, \tilde{u}_{k-2}(s,0), w_{k-1}(s)\right)\right\|_{\mathbb{R}^m} \mathrm{d}s\right] \\
&= \tilde{C}_{K_1} \left[T_\delta + \int_0^{T_\delta} C(s) \left[1 + \left\|w_{k-1}(s)\right\|_{\mathbb{R}^m}\right] \mathrm{d}s\right] \\
&= \tilde{C}_{K_1} \left[T_\delta + (1+H) \left\|C\right\|_{\mathbf{L^1}(0,T_\delta)}\right]
\end{aligned}$$

Then, by (4.4), we deduce that

$$\mathrm{TV}\left(B\left(\cdot,\, w_{k-1}(\cdot)\right)_{|[0,T_\delta]}\right) + \left\|B(0, w_o) - b(0)\right\|_{\mathbb{R}^m} < \delta_1$$

and so, by Theorem 2.3, $\tilde{u}_k$ exists in the time interval $[0, T]$.

The proof is now completed following exactly the steps from **2.** to **5.** of the proof of Theorem 2.7. □

In this paragraph we verify that the examples, presented in Section 3, have the properties required to apply Theorem 2.8.

Lemma 4.2. *The eigenvalues and right eigenvectors of problem (3.1) are:*

$$\begin{array}{rcl}\lambda_1 &=& -\sqrt{-p'(\tau)}\\ \lambda_2 &=& \sqrt{-p'(\tau)}\end{array} \quad r_1=\begin{bmatrix}-1\\-\lambda_1(\tau)\end{bmatrix} \quad r_2=\begin{bmatrix}1\\ \lambda_2(\tau)\end{bmatrix} \quad \begin{array}{rcl}\nabla\lambda_1\cdot r_1 &=& p''(\tau)/\left(2\sqrt{-p'(\tau)}\right)\\ \nabla\lambda_2\cdot r_2 &=& p''(\tau)/\left(2\sqrt{-p'(\tau)}\right)\end{array}$$

so that the conservation law in (3.1) is strictly hyperbolic and both characteristic fields are genuinely nonlinear.

The proof is immediate and hence omitted.

Proof of Proposition 3.1. We check that the assumptions of Theorem 2.8 are fulfilled.

(f): Holds by the standard properties of the p-system, see also Lemma 4.2.

(NC): Holds by Lemma 4.2.

(b): The determinant in **(b)** equals $\lambda_2(\bar\tau)$, by Lemma 4.2 and (3.2).

(B2): Note that $B(w)=w+\bar v$, hence **(B2)** holds with $\tilde C_{\tilde K}=1$ for any compact $\tilde K$.

(F): F does not depend on w, hence **(F)** holds by (3.2) with $C_K=0$ for any compact K.

(F2): Holds by (3.2) with $C(t)=\alpha\left[P(t)+\sup_{|\tau-\bar\tau|<\delta_o}p(\tau)\right]$ for, say, $\delta_o\in\left]0,\bar\tau/2\right[$.

□

Lemma 4.3. *For $i=1,\ldots,n$, in the sewer problem (3.4) we have the following eigenvalues and right eigenvectors:*

$$\begin{array}{l}\lambda_{i,1}=\frac{Q_i}{A_i}-\sqrt{p'(A_i)}\\ \lambda_{i,2}=\frac{Q_i}{A_i}+\sqrt{p'(A_i)}\end{array} \quad r_{i,1}=\begin{bmatrix}-1\\-\lambda_{i,1}\end{bmatrix} \quad r_{i,2}=\begin{bmatrix}1\\ \lambda_{i,2}\end{bmatrix} \quad \begin{array}{l}\nabla\lambda_{i,1}\cdot r_{i,1}=\frac{-p''(A_i)}{2\sqrt{p'(A_i)}}-\frac{\sqrt{p'(A_i)}}{A_i}\\ \nabla\lambda_{i,2}\cdot r_{i,2}=\frac{p''(A_i)}{2\sqrt{p'(A_i)}}+\frac{\sqrt{p'(A_i)}}{A_i}\end{array}$$

so that the conservation law in (3.5) *is strictly hyperbolic and both characteristic fields are genuinely nonlinear.*

The computations for **(b)** are just technical, but lengthy, so we state the following lemma.

Proof of Proposition 3.3. We check that the assumptions of Theorem 2.8 are fulfilled.

(f): Along each pipe, the p-system is strictly hyperbolic, with both characteristic fields genuinely nonlinear, see Lemma 4.3. However, as it is standard in the framework of networked conservation laws, strict hyperbolicity may fail in (3.5). Indeed, in the simplest case $p_1=p_2=\cdots=p_k$, the eigenvalues computed in Lemma 4.3 in the different pipes may well coincide. Different rescalings of the x axis in the different pipes allow to recover strict hyperbolicity, see [9, Lemma 4.1].

(NC): Follows from Lemma 4.3 and (3.6).

(b): The regularity of b follows from (3.8). The determinant in **(b)** is proportional to

$$\left(-1+\frac{\left|\sum_{i=1}^k\bar Q_i\right|}{A_M^2}\sum_{i=1}^k\sqrt{\frac{\bar A_i}{g\,h'(\bar A_i)}}\right)\prod_{j=1}^k\left(\sqrt{\frac{h'(\bar A_j)}{g\bar A_j}}\,\lambda_{j,2}(\bar A_j)\right)$$

and it does not vanish by (3.7).

(B2): Here, $B(w) = [0 \cdots 0 w + \bar{h}_M]^T$, hence **(B2)** holds true with $\tilde{C}_{\tilde{K}} = 1$ for any compact K.

(F): The regularity conditions are fulfilled, by the definition of F and (3.9). The Lipschitz regularity is immediate, since $\hat{h} \geq 0$ by its definition and $\bar{h}_M > 0$ by assumption.

(F2): Sublinearity is ensured by (3.9) and by what was noted in the step above.

□

Proof of Proposition 3.4. We check that the assumptions of Theorem 2.8 are fulfilled.

(f): Clearly we have $\lambda_i = v_i$, $i = 1, .., n$, so that the different conservation laws are decoupled and linear.

(NC): Since $v_i \neq 0$, $i = 1, .., n$, this holds true for e.g. $c = \frac{1}{2} \min(|v_i|)$.

(b): We directly see $\det \left[D_u b(0)\, r_{1,2} \quad \cdots \quad D_u b(0)\, r_{n,2} \right] = \prod_{i=l+1}^{n} v_i > 0$.

(B2): This we obtain immediately with e.g. $\tilde{C}_{\tilde{K}}(t) = 1/\varepsilon$.

(F): The regularity condition is met by (3.13). The Lipschitz constant is $1/\varepsilon$.

(F2): Here, $C(t) = \max_{i,j} \left| a_{ij}(t) \right| \cdot \max_i |v_i| \cdot \sup_i \rho_i + \sup_i \mu_i$.

□

Proof of Proposition 3.5. We check that the assumptions of Theorem 2.8 are fulfilled.

(f): The conservation law is a p-system, with eigenvalues $\lambda_{1,2} = q/a \pm \sqrt{\pi'}$, which are real thanks to (3.16), see also Lemma 4.3.

(NC): Is immediate, since $\lambda_1 < 0 < \lambda_2$ by (3.15).

(b): The determinant takes the value $\pi'(\bar{a})/\bar{a}$, which does not vanish by (3.16).

(B2): Is immediate since B is a second order polynomial.

(F): F is linear in w.

(F2): Immediate, by (3.17).

□

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