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ON THE COUPON COLLECTOR'S WAITING TIME

BY BENGT ROSÉN

Royal Institute of Technology, Stockholm

1. Introduction, summary and notation. We shall introduce a set of random variables and give interpretations of them in terms of coupon collection.

A person collects coupons with different colors. Let there be in all N different colors, which we label $1, 2, \cdots, N$. The different colors may occur with different frequencies. The colors of successive coupons are independent. Let J_v be the color of the vth coupon. Our formal assumptions are:

$J_1, J_2, \cdots$ are independent random variables, all with the following distribution

$$P(J = s) = p_s, \qquad s = 1, 2, \cdots, N \tag{1.1}$$

where

$$p_s \geqq 0, \qquad p_1 + p_2 + \cdots + p_N = 1. \tag{1.2}$$

Thus, p_s is the probability that a coupon has color s. Let

$$M_n = \# \text{ different elements among } (J_1, J_2, \cdots, J_n), \qquad n = 1, 2, \cdots. \tag{1.3}$$

Thus, M_n is the number of different colors in the collection after n coupons. Let

$$T_n = \min\{v: M_v = n\}, \qquad n = 1, 2, \cdots, N. \tag{1.4}$$

T_n is the number of coupons needed in order to get a collection with n different colors in. Define

$$\begin{aligned} D_v &= 1 \quad \text{if} \quad J_v \notin (J_1, J_2, \cdots, J_{v-1}), \qquad v = 1, 2, \cdots \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{1.5}$$

Thus, D_v tells if the vth coupon adds a new color to the collection or not.

We shall assume that the coupons also carry a *bonus value*, which is a real number. All coupons with the same color have the same bonus value, while the bonus value may differ from color to color. Let a_s be the bonus value of coupons with color s, $s = 1, 2, \cdots, N$. *The bonus sum of a collection* of coupons is obtained by adding the bonus values of the different colors which are represented in the collection. Thus, duplicates do not count. Formally we define the bonus sum as follows.

$$Q_n = a_{J_1}D_1 + a_{J_2}D_2 + \cdots + a_{J_n}D_n, \qquad n = 1, 2, \cdots. \tag{1.6}$$

The random variable Q_n will be referred to as *the bonus sum after n coupons for a collector in the situation* $\Omega = ((p_1, a_1), (p_2, a_2), \cdots, (p_N, a_N))$.

We define for $B > 0$

$$W(B) = \min\{n: Q_n \geqq B\}. \tag{1.7}$$

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$W(B)$ will be referred to as *the waiting time to obtain bonus sum B for a coupon collector in the situation Ω.*

The following lemma, which is obvious, states that we have introduced a slight abundance of terminology and notation.

LEMMA 1.1. *The random variables M_n and T_n in* (1.3) *and* (1.4) *are respectively the bonus sum after n coupons and the waiting time to obtain bonus sum n for a coupon collector in the situation* $((p_1, 1), (p_2, 1), \cdots, (p_n, 1))$.

Our main concern will be to study the random variable $W(B)$ and its particular case T_n. We confine ourselves to the case when all bonus values, a_s, are positive. The main result is that $W(B)$, under general conditions, is asymptotically (as n and N increase simultaneously) normally distributed. We give a brief sketch of the idea of proof, which is well known. When all a's are positive, the distributions of the random variables $W(B)$ and Q_n are related according to the formula

$$P(W(B) > x) = P(Q_{[x]} < B), \qquad x, B > 0. \tag{1.8}$$

With the aid of formula (1.8) one can "invert" results concerning either of the random variables Q or W to yield results concerning the other variable. In [5] we showed that Q_n, under general conditions, is asymptotically normally distributed. The asymptotic normality of W will be derived by inversion of the results in [5].

The asymptotic behavior of the collector's waiting time has, to the best of our knowledge, earlier only been considered in the classical case, i.e. $p_s = 1/N$ and $a_s = 1$, $s = 1, 2, \cdots, N$. In [4] Section 3, Rényi derives results about M by first deriving results about T and then "inverting." His basic tool is the representation

$$T_n = U_1 + U_2 + \cdots + U_n \tag{1.9}$$

where U_ν is the waiting time from bonus sum $\nu - 1$ to bonus sum ν. In the classical case U_1, $U_2, \cdots$ are independent random variables and $P(U_\nu = k) = ((\nu-1)/N)^{k-1}(N-\nu+1)/N$, $k = 1, 2, \cdots$. Thus, results concerning the asymptotic behavior of sums of independent random variables can be applied. A complete investigation along these lines is given by Baum and Billingsley in [1]. A generalized version of the problem is considered by Ivchenko and Medvedev in [2]. In their problem, as in our problem here, a representation of the type (1.9) no longer holds. They proceed along the path we shall follow here, i.e. to obtain results about the waiting time by "inverting" results concerning the bonus sum.

The following notation will be used throughout the paper. E and σ^2 stand for expectation and variance. ^{c} denotes centering at expectation, i.e. $X^c = X - EX$. $X =_{\mathscr{L}} Y$ means that the random variables X and Y have the same distribution. $\Rightarrow$ denotes convergence in law. The normal distribution with mean μ and variance σ^2 is denoted by $N(\mu, \sigma^2)$. The integral part of a real number is denoted by [ ].

2. Formulation of the results concerning the asymptotic behavior of $W(B)$ and T_n. We shall consider the asymptotic behavior of the random variables W and T, as n and N increase simultaneously. To effect this limit procedure we consider a sequence

$\Omega_k = ((p_{ks}, a_{ks}),\ s = 1, 2, \cdots, N_k),\ k = 1, 2, \cdots$ of collector situations. We assume that

$$a_{ks} > 0, \qquad s = 1, 2, \cdots, N_k,\ k = 1, 2, \cdots \tag{2.1}$$

and we put

$$A_k = a_{k1} + a_{k2} + \cdots + a_{kN_k}, \qquad k = 1, 2, \cdots. \tag{2.2}$$

The random variables $Q_n^{(k)}$, $W_k(B)$ and $T_n^{(k)}$ are defined relative to Ω_k, according to (1.6), (1.7) and (1.4).

We define some functions related to the collector situation Ω_k, $k = 1, 2, \cdots$

$$d_k^2(x) = \sum_{s=1}^{N_k} a_{ks}^2 e^{-p_{ks}x}(1 - e^{-p_{ks}x}) - x(\sum_{s=1}^{N_k} a_{ks} p_{ks} e^{-p_{ks}x})^2, \qquad x \geqq 0. \tag{2.3}$$

The function $w_k(x)$ is defined implicitly by the relation

$$A_k - x = \sum_{s=1}^{N_k} a_{ks} e^{-p_{ks} w_k(x)}, \qquad 0 \leqq x < A_k. \tag{2.4}$$

Furthermore, we define

$$q_k^2(x) = d_k^2(w_k(x)) / (\sum_{s=1}^{N_k} a_{ks} p_{ks} e^{-p_{ks} w_k(x)})^2, \qquad 0 \leqq x < A_k. \tag{2.5}$$

Next we introduce some conditions on the sequence $\{\Omega_k\}_{k=1}^{\infty}$.

$$N_k \to \infty \quad \text{as} \quad k \to \infty \tag{2.6}$$

$$\limsup_{k\to\infty} \max_s p_{ks} / \min_s p_{ks} < \infty \tag{2.7}$$

$$\limsup_{k\to\infty} \max_s a_{ks} / \min_s a_{ks} < \infty. \tag{2.8}$$

As stated in the introduction, we shall derive results about W from earlier results about Q. The following result is included in Theorem 1 in [5].

THEOREM 1. (a)

$$EQ_n^{(k)} = \sum_{s=1}^{N_k} (1 - (1 - p_{ks})^n) a_{ks}, \qquad n, k = 1, 2, \cdots \tag{2.9}$$

(b) *Let* $\{\Omega_k\}_{k=1}^{\infty}$ *satisfy* (2.6), (2.7) *and* (2.8), *and let* $\{n_k\}_{k=1}^{\infty}$ *satisfy*

$$0 < \liminf_{k\to\infty} n_k / N_k \leqq \limsup_{k\to\infty} n_k / N_k < \infty. \tag{2.10}$$

Then, for d_k *according to* (2.3), *we have*

$$\mathscr{L}((Q_{n_k}^{(k)} - EQ_{n_k}^{(k)}) / d_k(n_k)) \Rightarrow N(0, 1) \qquad \textit{as} \quad k \to \infty. \tag{2.11}$$

Next we formulate the results concerning W.

THEOREM 2. *Let* $\{\Omega_k\}_{k=1}^{\infty}$ *satisfy* (2.6), (2.7) *and* (2.8), *and let* $\{B_k\}_{k=1}^{\infty}$ *satisfy*

$$0 < \liminf_{k\to\infty} B_k / A_k \leqq \limsup_{k\to\infty} B_k / A_k < 1. \tag{2.12}$$

Then, for w_k *and* q_k *defined according to* (2.4) *and* (2.5), *we have*

$$\mathscr{L}((W_k(B_k) - w_k(B_k)) / q_k(B_k)) \Rightarrow N(0, 1) \qquad \textit{as} \quad k \to \infty. \tag{2.13}$$

THEOREM 3. *Let w_k and q_k be the functions which are defined in* (2.4) *and* (2.5). *We write*

$$EW_k(B) = w_k(B) + q_k(B) \cdot R_k^{(1)}(B, \Omega_k), \quad B > 0, k = 1, 2, \cdots \tag{2.14}$$

$$\sigma^2(W_k(B)) = q_k^2(B)(1 + R_k^{(2)}(B, \Omega_k)), \quad B > 0, k = 1, 2, \cdots. \tag{2.15}$$

If $\{\Omega_k\}_{k=1}^{\infty}$ *satisfies* (2.6), (2.7) *and* (2.8), *we have for every* $0 < \tau_1 \leqq \tau_2 < 1$,

$$\text{(a) } \lim_{k\to\infty} \sup_{\tau_1 \leqq B/A_k \leqq \tau_2} |R_k^{(1)}(B, \Omega_k)| = 0 \tag{2.16}$$

$$\text{(b) } \lim_{k\to\infty} \sup_{\tau_1 \leqq B/A_k \leqq \tau_2} |R_k^{(2)}(B, \Omega_k)| = 0. \tag{2.17}$$

As stated in Lemma 1.1, the random variable T_n is only a special case of the random variable W. Thus, Theorem 2 and Theorem 3 contain information about T_n. However, we find it worthwhile to write down the results for T_n explicitly. First we introduce notations for the special cases of the functions d_k, w_k and q_k, which are obtained when all a's equal 1. We define $u_k(x)$, $t_k(x)$ and $v_k(x)$ by the following relations

$$u_k^2(x) = \textstyle\sum_{s=1}^{N_k} e^{-p_{ks}x}(1 - e^{-p_{ks}x}) - x(\sum_{s=1}^{N_k} p_{ks} e^{-p_{ks}x})^2, \qquad x \geqq 0 \tag{2.18}$$

$$N_k - x = \textstyle\sum_{s=1}^{N_k} e^{-p_{ks}t_k(x)}, \qquad 0 \leqq x < N_k \tag{2.19}$$

$$v_k^2(x) = u_k^2(t_k(x))/(\textstyle\sum_{s=1}^{N_k} p_{ks} e^{-p_{ks}t_k(x)})^2, \qquad 0 \leqq x < N_k. \tag{2.20}$$

Furthermore, let $\mathbf{p}_k = (p_{k1}, p_{k2}, \cdots, p_{kN_k})$, $k = 1, 2, \cdots$.

THEOREM 4. *Let* $\{\mathbf{p}_k\}_{k=1}^{\infty}$ *satisfy* (2.6) *and* (2.7), *and let* $\{n_k\}_{k=1}^{\infty}$ *satisfy*

$$0 < \liminf n_k/N_k \leqq \limsup_{k\to\infty} n_k/N_k < 1. \tag{2.21}$$

Then, for t_k and v_k according to (2.19) *and* (2.20), *we have*

$$\mathscr{L}((T_{n_k}^{(k)} - t_k(n_k))/v_k(n_k)) \Rightarrow N(0, 1) \qquad \text{as} \quad k \to \infty. \tag{2.22}$$

THEOREM 5. *Let t_k and v_k be defined by* (2.19) *and* (2.20). *We write for* $n = 1, 2, \cdots, N_k$, $k = 1, 2, \cdots$

$$ET_n^{(k)} = t_k(n) + n^{\frac{1}{2}} R_k^{(1)}(n, \mathbf{p}_k) \tag{2.23}$$

$$\sigma^2(T_n^{(k)}) = v_k^2(n)(1 + R_k^{(2)}(n, \mathbf{p}_k)). \tag{2.24}$$

If $\{\mathbf{p}_k\}_{k=1}^{\infty}$ *satisfies* (2.6) *and* (2.7), *we have for every* $0 < \tau_1 \leqq \tau_2 < 1$

$$\text{(a) } \lim_{k\to\infty} \max_{\tau_1 N_k \leqq n \leqq \tau_2 N_k} |R_k^{(1)}(n, \mathbf{p}_k)| = 0 \tag{2.25}$$

$$\text{(b) } \lim_{k\to\infty} \max_{\tau_1 N_k \leqq n \leqq \tau_2 N_k} |R_k^{(2)}(n, \mathbf{p}_k)| = 0. \tag{2.26}$$

REMARK. Theorem 4 follows immediately from Theorem 2, while Theorem 5 follows from Theorem 3 and an easy estimate of $v_k^2(x)$, which is derived in Lemma 3.8.

The ideas of proof in Theorem 2 and Theorem 3 are simple and well known. However, we will run into technical difficulties. To motivate the following, somewhat lengthy, estimates we start out on the proofs and see where we get stuck.

START OF THE PROOF OF THEOREM 2. According to (1.8) we have for $-\infty < x < \infty$

$$P\left(\frac{W_k(B_k)-w_k(B_k)}{q_k(B_k)} \leqq x\right) = P(W_k(B_k) \leqq w_k(B_k)+xq_k(B_k)) \tag{2.27}$$

$$= P\left(\frac{Q^c_{[w_k(B_k)+xq_k(B_k)]}}{d_k([w_k(B_k)+xq_k(B_k)])} \geqq x_k(x, B_k)\right).$$

where

$$x_k(x, B_k) = \frac{B_k - EQ_{[w_k(B_k)+xq_k(B_k)]}}{d_k([w_k(B_k)+xq_k(B_k)])}. \tag{2.28}$$

(2.13) will follow from (2.27) and (2.11) if we prove that

$$x_k(x, B_k) \to -x \qquad \text{as} \quad k \to \infty. \tag{2.29}$$

The technical part of the proof will be to show that (2.29) actually holds.

START OF THE PROOF OF THEOREM 3. We introduce a condition: For some $r > 2$ we have for every $0 < \tau_1 \leqq \tau_2 < 1$

$$\limsup_{k\to\infty} \sup_{\tau_1 \leqq B/A_k \leqq \tau_2} E\left|\frac{W_k(B)-w_k(B)}{q_k(B)}\right|^r < \infty. \tag{2.30}$$

We shall prove (2.16) under the assumption that Theorem 2 and (2.30) are true. We give an indirect proof, and we assume that (2.16) does not hold. Then, there exists a sequence $\{B_k\}_{k=1}^{\infty}$ and τ_1, τ_2 such that $0 < \tau_1 \leqq B_k/A_k \leqq \tau_2 < 1$ and

$$\limsup_{k\to\infty} |R_k^{(1)}(B_k, \Omega_k)| > 0. \tag{2.31}$$

According to (2.30), $\{(W_k(B_k)-w_k(B_k))/q_k(B_k)\}_{k=1}^{\infty}$ is uniformly integrable. By combining this with (2.13) and the fact that $N(0, 1)$ has mean 0, we get

$$E\left(\frac{W_k(B_k)-w_k(B_k)}{q_k(B_k)}\right) = \frac{EW_k(B_k)-w_k(B_k)}{q_k(B_k)} \to 0 \qquad \text{as} \quad k \to \infty. \tag{2.32}$$

Now, as is easily realized, (2.32) and (2.31) contradict each other, and (2.16) is thus proved.

In a very similar manner we can prove (2.17) under the assumption that Theorem 2 and (2.30) hold. Assume that (2.17) is false, and select a sequence $\{B_k\}_{k=1}^{\infty}$, $0 < \tau_1 \leqq B_k/A_k \leqq \tau_2 < 1$, such that

$$\limsup_{k\to\infty} |R_k^{(2)}(B_k, \Omega_k)| > 0. \tag{2.33}$$

As (2.30) implies uniform integrability of $\{(W_k(B_k)-w_k(B_k))^2/q_k^2(B_k)\}_{k=1}^{\infty}$, we get from (2.13)

$$E\left(\frac{W_k(B_k)-w_k(B_k)}{q_k(B_k)}\right)^2 \to 1 \qquad \text{as} \quad k \to \infty. \tag{2.34}$$

Furthermore, in the notation of (2.14),

$$E\left(\frac{W_k(B_k)-w_k(B_k)}{q_k(B_k)}\right)^2 = q_k^{-2}(B_k)\cdot\sigma^2(W_k(B_k))+R_k^{(1)}(B_k,\Omega_k)^2. \tag{2.35}$$

Now (2.34), (2.35) and (2.16) yield that $R_k^{(2)}(B_k,\Omega_k)\to 0$ as $k\to\infty$. This contradicts (2.33) and concludes the proof.

The hard part of the proof will be to show that condition (2.30) actually holds.

3. Some auxiliary results. Here we shall collect, for future use, some results concerning the functions w, d and q, which were introduced in the previous section. First we list some notation and assumptions, which will be used throughout this section.

$p_1, p_2, \cdots, p_N$ is a set of probabilities, i.e. $p_s \geqq 0$, $s = 1, 2, \cdots, N$ and $p_1+p_2+\cdots+p_N = 1$. $a_1, a_2, \cdots, a_N$ are positive real numbers. To obtain simpler expressions in the sequel, we introduce the following notation.

$$A = a_1+a_2+\cdots+a_N \quad\text{and}\quad \bar{a} = A/N \tag{3.1}$$

$$m = \min_s a_s \quad\text{and}\quad M = \max_s a_s \tag{3.2}$$

$$\rho_1 = \min_s Np_s \quad\text{and}\quad \rho_2 = \max_s Np_s. \tag{3.3}$$

We define

$$\varphi^*(n) = \textstyle\sum_{s=1}^N (1-(1-p_s)^n)a_s, \qquad n = 1, 2, \cdots, \tag{3.4}$$

and

$$\varphi(x) = \textstyle\sum_{s=1}^N (1-e^{-p_s x})a_s, \qquad x \geqq 0. \tag{3.5}$$

Our interest in the function φ^* is explained by the formula (2.9), and φ will be used as an approximation of φ^*.

LEMMA 3.1. *For $n = 1, 2, \cdots$ we have*

$$0 \leqq \varphi^*(n)-\varphi(n) \leqq M. \tag{3.6}$$

PROOF. By using the elementary inequalities $0 \leqq e^{-nx}-(1-x)^n \leqq nx^2 e^{-nx}$, $0 \leqq x \leqq 1$, and $x\,e^{-x} \leqq 1/e \leqq 1$ we get, as $a_s > 0$, $s = 1, 2, \cdots, N$

$$0 \leqq \varphi^*(n)-\varphi(n) = \textstyle\sum_{s=1}^N (e^{-np_s}-(1-p_s)^n)a_s$$
$$\leqq \textstyle\sum_{s=1}^N p_s(np_s)\,e^{-np_s}a_s \leqq M\,e^{-1}\sum_{s=1}^N p_s \leqq M.$$

Thus, the lemma is proved. The proofs of the inequalities in the next lemma are straightforward, and we omit them.

LEMMA 3.2. *For $x \geqq 0$ we have*

$$\text{(a)}\ m\,e^{-\rho_2 x/N} \leqq \varphi'(x) \leqq M\,e^{-\rho_1 x/N} \tag{3.7}$$

$$\text{(b)}\ 0 \leqq -\varphi''(x) \leqq \rho_2/(N)M\,e^{-\rho_1 x/N}. \tag{3.8}$$

LEMMA 3.3. *For $x, y > 0$ we have*

$$|\varphi(x)-\varphi(y)| \leqq |x-y| \cdot M. \tag{3.9}$$

PROOF. According to the mean value theorem and (3.7) we have, $0 \leqq \theta \leqq 1$,

$$|\varphi(x)-\varphi(y)| = |x-y|\,\varphi'(x+\theta(y-x)) \leqq |x-y| \cdot M$$

and the lemma is proved.

Let $w(y)$, $0 \leqq y < A$, be the inverse of $\varphi(x)$, i.e. $w(y)$ is defined implicitly by the following relation (cf. (2.4)).

$$y = A-\textstyle\sum_{s=1}^{N} a_s e^{-p_s w(y)}, \qquad 0 \leqq y < A. \tag{3.10}$$

LEMMA 3.4. *For $0 \leqq y < A$ we have*

$$\text{(a)}\ w(y) \leqq \frac{N}{\rho_1}\log\left(1-\frac{y}{A}\right)^{-1} \leqq \frac{y}{\bar{a}}\frac{1}{\rho_1}\left(1-\frac{y}{A}\right)^{-1} \tag{3.11}$$

$$\text{(b)}\ w(y) \geqq \frac{N}{\rho_2}\log\left(1-\frac{y}{A}\right)^{-1} \geqq \frac{y}{\bar{a}}\cdot\frac{1}{\rho_2}. \tag{3.12}$$

PROOF. We have

$$\varphi(x) = \textstyle\sum_{s=1}^{N}(1-e^{-p_s x})a_s \geqq (1-e^{-\rho_1 x/N})A. \tag{3.13}$$

From (3.13) we conclude that $w(y)$ is at most as big as the solution of the equation $y = (1-\exp(-\rho_1 x/N))A$. The solution is $x = N\rho_1^{-1}\log(1-y/A)^{-1}$. By combining this with the inequality $-\log(1-z) \leqq z/(1-z), 0 \leqq z < 1$, (3.11) follows.

Quite analogously we get $\varphi(x) \leqq (1-\exp(-\rho_2 x/N))A$, which yields that $w(y)$ is at least as big as the solution of $y = (1-\exp(-\rho_2 x/N))A$, which is $x = N\rho_2^{-1}\log(1-y/N)^{-1}$. Now, apply the inequality $-\log(1-z) \geqq z$, $0 \leqq z < 1$, and (3.12) is proved.

LEMMA 3.5. *For $0 \leqq y < A$ and $u \geqq 0$ we have*

$$\text{(a)}\ \varphi(w(y)+u)-y \geqq u \cdot m \cdot e^{-\rho_2 u/N}\left(1-\frac{y}{A}\right)^{\rho_2/\rho_1} \tag{3.14}$$

$$\text{(b)}\ y-\varphi(w(y)-u) \geqq u \cdot m\left(1-\frac{y}{A}\right)^{\rho_2/\rho_1} \qquad 0 \leqq u \leqq w(y). \tag{3.15}$$

PROOF. Remember that $\varphi(w(y)) = y$. As $\varphi'(x)$ decreases when x increases, we have, $0 \leqq \theta \leqq 1$,

$$\varphi(w(y)+u) = \varphi(w(y))+u\varphi'(w(y)+\theta u) \geqq y+u\varphi'(w(y)+u). \tag{3.16}$$

By combining the estimates (3.7) and (3.11) we get

$$\varphi'(w(y)+u) \geqq m\exp\left\{-\frac{\rho_2}{N}\left(\frac{N}{\rho_1}\log\left(1-\frac{y}{A}\right)^{-1}+u\right)\right\} \tag{3.17}$$

$$= m\,e^{-\rho_2 u/N}\left(1-\frac{y}{A}\right)^{\rho_2/\rho_1}.$$

Now (3.14) follows from (3.16) and (3.17). Similarly we get

$$y-\varphi(w(y)-u) = y-\varphi(w(y))+u\varphi'(w(y)-\theta u) \geqq u\varphi'(w(y))$$

$$\geqq u \cdot m \cdot \exp\left\{-\frac{\rho_2}{N}\cdot\frac{N}{\rho_1}\log\left(1-\frac{y}{A}\right)^{-1}\right\} = u\cdot m\left(1-\frac{y}{A}\right)^{\rho_2/\rho_1}.$$

Thus, the lemma is proved.

Next we shall derive some estimates concerning the function d below (cf. (2.3)).

(3.18) $$d^2(x) = \sum_{s=1}^N a_s^2 e^{-p_s x}(1-e^{-p_s x}) - x\left(\sum_{s=1}^N a_s p_s e^{-p_s x}\right)^2, \qquad x \geqq 0.$$

LEMMA 3.6. *For $x \geqq 0$ we have*

(3.19) $$\text{(a)}\ d^2(x) \leqq x \cdot M^2$$

(3.20) $$\text{(b)}\ d^2(x) \geqq x \cdot m^2 \psi(\rho_2 x/N)\xi(\rho_1 x/N)$$

where

(3.21) $$\psi(z) = e^{-z}(1-e^{-z})/z, \qquad 0 \leqq z < \infty$$

and

(3.22) $$\xi(z) = 1 - z e^{-z}/(1-e^{-z}), \qquad 0 \leqq z < \infty.$$

PROOF. By using the inequality, $1-e^{-z} \leqq z$, $z \geqq 0$, we get

$$d^2(x) \leqq \sum_{s=1}^N a_s^2 e^{-p_s x}(1-e^{-p_s x}) \leqq \sum_{s=1}^N a_s^2 p_s x \leqq xM^2 \sum_{s=1}^N p_s,$$

and (3.19) is proved. According to Schwarz's inequality we have

(3.23) $$\left(\sum_{s=1}^N a_s p_s e^{-p_s x}\right)^2 \leqq \sum_{s=1}^N a_s^2 e^{-p_s x}(1-e^{-p_s x}) \cdot \sum_{s=1}^N p_s^2 \frac{e^{-p_s x}}{1-e^{-p_s x}}.$$

From (3.18) and (3.23) we get

(3.24) $$d^2(x) \geqq \left(\sum_{s=1}^N a_s^2 e^{-p_s x}(1-e^{-p_s x})\right)\left(1-\sum_{s=1}^N p_s^2 x \frac{e^{-p_s x}}{1-e^{-p_s x}}\right).$$

It is easily verified that the function $\psi(z)$ in (3.21) decreases from 1 to 0 as z increases from 0 to ∞. Thus, we get the following estimate for the first factor in (3.24)

(3.25) $$\sum_{s=1}^N a_s^2 e^{-p_s x}(1-e^{-p_s x}) \geqq m^2 \cdot x \cdot \psi(\rho_2 x/N) \sum_{s=1}^N p_s.$$

By using the easily verified fact that the function $\xi(z)$ increases from 0 when z increases from 0 we get the following bound on the second factor in (3.24)

(3.26) $$1-\sum_{s=1}^N p_s^2 x \frac{e^{-p_s x}}{1-e^{-p_s x}} = \sum_{s=1}^N p_s\left(1-\frac{p_s x \cdot e^{-p_s x}}{1-e^{-p_s x}}\right) \geqq \xi(\rho_1 x/N).$$

Now (3.24)–(3.26) yield (3.20) and the lemma is proved.

LEMMA 3.7. *For $x, y > 0$ we have*

$$|d(x)-d(y)| \leqq \frac{|x-y|}{x^{\frac{1}{2}}+y^{\frac{1}{2}}} \cdot \frac{M^2}{m} \cdot 3 \cdot \psi\left(\rho_2 \frac{\max(x, y)}{N}\right)^{-\frac{1}{2}} \xi\left(\rho_1 \frac{\min(x, y)}{N}\right)^{-\frac{1}{2}}, \tag{3.27}$$

where ψ and ξ are the functions in (3.21) *and* (3.22).

PROOF. We first give an upper bound for $|d^2(x)-d^2(y)|$. From (3.18) we get

$$\begin{aligned} |d^2(x)-d^2(y)| &\leqq \textstyle\sum_{s=1}^N a_s^2|(e^{-p_s x}-e^{-2p_s x})-(e^{-p_s y}-e^{-2p_s y})| \\ &\quad + x|(\textstyle\sum_{s=1}^N a_s p_s e^{-p_s x})^2-(\sum_{s=1}^N a_s p_s e^{-p_s y})^2| \\ &\quad + |x-y|(\textstyle\sum_{s=1}^N a_s p_s e^{-p_s y})^2 \\ &= Q_1+Q_2+Q_3. \end{aligned} \tag{3.28}$$

From the mean value theorem, and the inequality $|\exp(-x)-2\exp(-2x)| \leqq 1$, $x \geqq 0$, we get, $0 \leqq \theta \leqq 1$

$$Q_1 \leqq M^2 \textstyle\sum_{s=1}^N |x-y|\, p_s|e^{-p_s(x+\theta(y-x))}-2e^{-2p_s(x+\theta(y-x))}| \leqq M^2|x-y|. \tag{3.29}$$

Assume, without loss of generality, that $x < y$. Then,

$$\begin{aligned} Q_2 &\leqq x|\textstyle\sum_{s=1}^N a_s p_s(e^{-p_s x}-e^{-p_s y})| \cdot \sum_{s=1}^N a_s p_s(e^{-p_s x}+e^{-p_s y}) \\ &\leqq M|x-y|\textstyle\sum_{s=1}^N p_s(p_s x)e^{-p_s x} \cdot M \cdot 2\sum_{s=1}^N p_s \\ &\leqq M^2|x-y|\,e^{-1}2 \leqq M^2|x-y|. \end{aligned} \tag{3.30}$$

Furthermore,

$$Q_3 \leqq M^2|x-y|. \tag{3.31}$$

Now, (3.28)–(3.31) yield that

$$|d^2(x)-d^2(y)| \leqq |x-y| \cdot 3M^2. \tag{3.32}$$

We have

$$|d(x)-d(y)| = |d^2(x)-d^2(y)|/(d(x)+d(y)). \tag{3.33}$$

From (3.20) and the monotonicity of ψ and ξ we get

$$\begin{aligned} d(x)+d(y) &\geqq m([x\psi(\rho_2 x/N)\xi(\rho_1 x/N)]^{\frac{1}{2}}+[y\psi(\rho_2 y/N)\xi(\rho_1 y/N)]^{\frac{1}{2}}) \\ &\geqq m(x^{\frac{1}{2}}+y^{\frac{1}{2}})\psi\left(\rho_2 \frac{\max(x, y)}{N}\right)^{\frac{1}{2}} \xi\left(\rho_1 \frac{\min(x, y)}{N}\right)^{\frac{1}{2}}. \end{aligned} \tag{3.34}$$

Now, (3.32), (3.33) and (3.34) yield (3.27) and the lemma is proved.

Finally we shall consider the following function (cf. (2.5) and (3.5))

$$q^2(x) = \frac{d^2(w(x))}{\varphi'(w(x))^2} = d^2(w(x))/(\textstyle\sum_{s=1}^N a_s p_s e^{-p_s w(x)})^2, \qquad 0 \leqq x < A. \tag{3.35}$$

LEMMA 3.8. *For $0 \leqq x < A$ we have*

(3.36) $$\text{(a)}\quad q^2(x) \leqq \frac{x}{a}\left(\frac{M}{m}\right)^2 \frac{1}{\rho_1}\left(1-\frac{x}{A}\right)^{-(1+2\rho_2/\rho_1)}$$

(3.37) $$\text{(b)}\quad q^2(x) \geqq \frac{x}{a}\left(\frac{m}{M}\right)^2 \frac{1}{\rho_2}\left(1-\frac{x}{A}\right)^{-2\rho_1/\rho_2} \psi\left(\frac{\rho_2}{\rho_1}\left(1-\frac{x}{A}\right)^{-1}\right)\xi\left(\frac{\rho_1}{\rho_2}\cdot\frac{x}{A}\right),$$

where ψ and ξ are defined according to (3.21) *and* (3.22).

PROOF. By combining the estimates in (3.19), (3.7) and (3.11) we get

$$q^2(x) \leqq w(x)\left(\frac{M}{m}\right)^2 e^{2\rho_2 N^{-1}w(x)} \leqq \frac{x}{a}\frac{1}{\rho_1}\left(1-\frac{x}{A}\right)^{-1}\cdot\left(\frac{M}{m}\right)^2 e^{2(\rho_2/\rho_1)\log(1-x/A)^{-1}}$$
$$= \frac{x}{a}\left(\frac{M}{m}\right)^2 \frac{1}{\rho_1}\left(1-\frac{x}{A}\right)^{-(1+2\rho_2/\rho_1)}$$

and (3.36) is proved. Quite analogously we get from (3.20), (3.7), (3.11) and (3.12), remembering that ψ is decreasing and ξ is increasing.

$$q^2(x) \geqq w(x)\left(\frac{m}{M}\right)^2 \psi\left(\rho_2\frac{w(x)}{N}\right)\xi\left(\rho_1\frac{w(x)}{N}\right)e^{2\rho_1(w(x)/N)}$$
$$\geqq \frac{x}{a}\cdot\frac{1}{\rho_2}\left(\frac{m}{M}\right)^2 \psi\left(\rho_2\frac{x}{A}\cdot\frac{1}{\rho_1}\left(1-\frac{x}{A}\right)^{-1}\right)\xi\left(\rho_1\frac{x}{A\rho_2}\right)e^{2\rho_1/\rho_2\log(1-x/A)^{-1}}$$
$$\geqq \frac{x}{a}\left(\frac{m}{M}\right)^2 \frac{1}{\rho_2}\left(1-\frac{x}{A}\right)^{-2\rho_2/\rho_1} \psi\left(\frac{\rho_2}{\rho_1}\left(1-\frac{x}{A}\right)^{-1}\right)\xi\left(\frac{\rho_1}{\rho_2}\cdot\frac{x}{A}\right),$$

and (3.37), and thus the lemma is proved.

4. Completion of the proof of Theorem 2. We are now prepared to finish the proof of Theorem 2, i.e. to prove (2.29). Let d_k, w_k and q_k be the functions which are defined in (2.3), (2.4) and (2.5). Furthermore, let $\varphi_k{}^*$ and φ_k be defined, relative Ω_k, according to (3.4) and (3.5). According to (2.9) we have

(4.1) $$B_k - EQ_{[w_k(B_k)+xq_k(B_k)]} = B_k - \varphi_k{}^*([w_k(B_k)+xq_k(B_k)]).$$

Furthermore,

(4.2) $$\begin{aligned}&\varphi_k{}^*([w_k(B_k)+xq_k(B_k)])\\ &= \varphi_k(w_k(B_k)+xq_k(B_k)) + \{\varphi_k{}^*([w_k(B_k)+xq_k(B_k)]) - \varphi_k([w_k(B_k)+xq_k(B_k)])\}\\ &\quad + \{\varphi_k([w_k(B_k)+xq_k(B_k)]) - \varphi_k(w_k(B_k)+xq_k(B_k))\}.\end{aligned}$$

By applying the estimates in Lemma 3.1 and Lemma 3.3 to the last two terms in (4.2), we get

(4.3) $$|\varphi_k{}^*([w_k(B_k)+xq_k(B_k)]) - \varphi_k(w_k(B_k)+xq_k(B_k))| \leqq 2\max_s a_{ks}.$$

We have the following Taylor expansion, $0 \leqq \theta \leqq 1$, (cf. (2.5) and (3.35))

$$\begin{aligned} &\varphi_k(w_k(B_k)+xq_k(B_k)) \\ (4.4) \qquad &= \varphi_k(w_k(B_k))+xq_k(B_k)\varphi_k'(w_k(B_k)) \\ &\qquad +\tfrac{1}{2}x^2q_k^2(B_k)\varphi_k''(w_k(B_k)+\theta xq_k(B_k)) \\ &= B_k+xd_k(w_k(B_k))+\tfrac{1}{2}x^2q_k^2(B_k)\varphi_k''(w_k(B_k)+\theta xq_k(B_k)). \end{aligned}$$

According to Lemma 3.2(b) we have, $\rho_2^{(k)}$ being defined in accord with (3.3)

$$(4.5) \qquad |\varphi_k''(w_k(B_k)+\theta xq_k(B_k))| \leqq \frac{\rho_2^{(k)}}{N_k} \max_s a_{ks}.$$

From (4.1)–(4.5) we get

$$(4.6) \qquad \left|\frac{B_k-EQ_{[w_k(B_k)+xq_k(B_k)]}}{d_k(w_k(B_k))}+x\right| \leqq \frac{\max_s a_{ks}}{d_k(w_k(B_k))}\left(2+\tfrac{1}{2}x^2q_k^2(B_k)\frac{\rho_2^{(k)}}{N_k}\right).$$

By using the estimates in Lemmas 3.6(b), 3.4(b), and 3.8(a) it is quite straightforward to verify that, under the assumptions in Theorem 2, the right-hand side in (4.6) tends to 0 as k tends to infinity. Thus, we obtain

$$(4.7) \qquad (B_k-EQ_{[w_k(B_k)+xq_k(B_k)]})/d_k(w_k(B_k)) \to -x \qquad \text{as} \quad k\to\infty.$$

Now (2.29) follows from (4.7) if we show that

$$(4.8) \qquad \lim_{k\to\infty} d_k([w_k(B_k)+xq_k(B_k)])/d_k(w(B_k)) = 1.$$

Again the verification is quite straightforward by using the estimates which were derived in the previous section, in particular the estimate in Lemma 3.7. This concludes the proof of Theorem 2.

5. On the absolute central moments of Q_n. The remaining part of the proof of Theorem 3, i.e. (2.30), concerns the absolute moments of $W(B)-w(B)$. We shall obtain information about these moments by first deriving results about the absolute central moments of Q_n and then "inverting" these results by (1.8). Our aim in this section is to prove the following theorem.

THEOREM 6. *Let Q_n be the coupon collector's bonus sum after n coupons in the situation* $((p_1, a_1), (p_2, a_2), \cdots, (p_N, a_N))$, *(cf.* (1.6)). *Then, we have for every* $r > 0$ *and* $n = 1, 2, \cdots$

$$(5.1) \qquad E|Q_n^c|^r \leqq C_r n^{r/2}(\max_s |a_s|)^r,$$

where C_r is a number which only depends on r.

COROLLARY. *For $\varphi(n)$ according to* (3.5) *we have for* $r > 0$ *and* $n = 1, 2, \cdots$

$$(5.2) \qquad E|Q_n-\varphi(n)|^r \leqq C_r n^{r/2}(\max_s a_s)^r.$$

Derivation of the corollary from the theorem. Let $\varphi^*(n)$ be defined according to (3.4). In virtue of (2.9) we have

$$E|Q_n-\varphi(n)|^r \leqq 2^{r-1}(E|Q_n^c|^r+|\varphi(n)-\varphi^*(n)|^r). \tag{5.3}$$

Now (5.2) follows from (5.3), (5.1) and (3.6).

Before we can prove the theorem, we need some auxiliary results. We shall use a representation of the random variable Q, which was introduced by S. Karlin in [3].

Let $X_1(t), X_2(t), \cdots, X_N(t)$, $t \geqq 0$, be independent Poisson processes with right-continuous trajectories, all starting at the origin at $t = 0$. Let $X_s(t)$ have intensity parameter p_s, $s = 1, 2, \cdots, N$. Let

$$X(t) = X_1(t)+X_2(t)+\cdots+X_N(t), \qquad t \geqq 0. \tag{5.4}$$

Then, $X(t)$ is a Poisson process with intensity parameter $p_1+p_2+\cdots+p_N = 1$, and $X(0) = 0$. We define

$$H_n = \inf\{t : X(t) = n\}. \tag{5.5}$$

Furthermore, let

$$\chi(x) = 1 \quad \text{for} \quad x > 0; \qquad = 0 \quad \text{for} \quad x \leqq 0. \tag{5.6}$$

The following representation of Q_n is easily realized.

LEMMA 5.1. *For $n = 1, 2, \cdots$ we have*

$$Q_n =_{\mathscr{L}} \textstyle\sum_{s=1}^N \chi(X_s(H_n))\cdot a_s. \tag{5.7}$$

We re-write (5.7) as follows

$$Q_n =_{\mathscr{L}} \textstyle\sum_{s=1}^N \chi(X_s(n))a_s+R_n \tag{5.8}$$

where

$$R_n = \textstyle\sum_{s=1}^N \{\chi(X_s(H_n))-\chi(X_s(n))\}\, a_s. \tag{5.9}$$

LEMMA 5.2. *For $n, p = 1, 2, \cdots$ we have*

$$E|R_n|^p \leqq C_p n^{p/2}(\max_s |a_s|)^p \tag{5.10}$$

where C_p is a number which only depends on p.

Again, we first need some auxiliary results.

LEMMA 5.3. *For $n = 1, 2, \cdots$, we have*

$$|R_n| \leqq |X(H_n)-X(n)|\cdot \max_s |a_s|. \tag{5.11}$$

PROOF. As $X_s(t)$ is non-decreasing when t increases, and increases with jumps of size 1, we have on the event $\{H_n \geqq n\}$, for $s = 1, 2, \cdots, N$,

$$0 \leqq \chi(X_s(H_n))-\chi(X_s(n)) \leqq X_s(H_n)-X_s(n). \tag{5.12}$$

From (5.9), (5.12) and (5.4) we get, on $\{H_n \geqq n\}$

$$|R_n| \leqq \max_s |a_s| \cdot \sum_{s=1}^N |\chi(X_s(H_n)) - \chi(X_s(n))|$$
$$\leqq \max_s |a_s| \cdot \sum_{s=1}^N (X_s(H_n) - X_s(n)) = \max_s |a_s|(X(H_n) - X(n)).$$

Thus, (5.11) holds on the event $\{H_n \geqq n\}$. By similar arguments it is easily shown that (5.11) is true also on the event $\{H_n < n\}$. Thus, Lemma 5.3 is proved.

The results in the following three lemmas are well known.

LEMMA 5.4. *Let $X(t)$, $t \geqq 0$, be a Poisson process with right-continuous trajectories, having intensity parameter* 1 *and* $X(0) = 0$, *and let H_n be defined by* (5.5). *Then,*

(a) *H_n has a $\Gamma(n)$-distribution, i.e.*

$$P(H_n \leqq t) = \int_0^t \frac{e^{-x}x^{n-1}}{(n-1)!}\,dx, \qquad t > 0,\ n = 1, 2, \cdots. \tag{5.13}$$

(b) *The conditional distribution of $|X(H_n) - X(n)|$, given that $H_n = t$, is*

(i) *for $t < n$: a Poisson distribution with parameter $(n-t)$;*
(ii) *for $t > n$: the distribution of $1 + Y$, where Y has a binomial distribution with parameters $(n-1, 1-n/t)$.*

LEMMA 5.5. *Let Y have a Poisson distribution with parameter λ. Then, for* $p = 1, 2, \cdots$

$$EY^p \leqq C_p(\lambda^p + \lambda) \leqq C_p'(\lambda^p + 1) \tag{5.14}$$

where C_p and C_p' are numbers which only depend on p.

LEMMA 5.6. *Let Y have a binomial distribution with parameters (n, π). Then, for* $0 \leqq \pi \leqq 1$, $n, p = 1, 2, \cdots$ *we have*

$$E(1 + Y)^p \leqq C_p(1 + (n\pi)^p) \tag{5.15}$$

where C_p is a number, which only depends on p.

PROOF OF LEMMA 5.2. C_p and C_p' denote numbers which only depend on p. From Lemmas 5.4, 5.5 and 5.6 we get

$$E|X(n) - X(H_n)|^p = \int_0^\infty E(|X(n) - X(H_n)|^p | H_n = t)\frac{e^{-t}t^{n-1}}{(n-1)!}\,dt$$

$$\leqq C_p \int_0^n (|n-t|^p + 1)\frac{e^{-t}t^{n-1}}{(n-1)!} + C_p \int_n^\infty \left(1 + \left((n-1)\left(1 - \frac{n}{t}\right)\right)^p\right)\frac{e^{-t}t^{n-1}}{(n-1)!}\,dt \tag{5.16}$$

$$\leqq C_p \int_0^\infty (1 + |n-t|^p)\frac{e^{-t}t^{n-1}}{(n-1)!}\,dt \leqq C_p(1 + C_p' n^{p/2}).$$

Now (5.10) follows from (5.11) and (5.16), and Lemma 5.2 is proved.

LEMMA 5.7. *Let* $Y_1, Y_2, \cdots, Y_N$ *be independent Bernoulli random variables* $P(Y_s = 1) = 1 - P(Y_s = 0) = \pi_s$, $s = 1, 2, \cdots, N$. *Then, for* $p = 1, 2, \cdots$, *we have*

$$E|\sum_{s=1}^{N} Y_s^c a_s|^{2p} \leqq (\max_s |a_s|)^{2p} \cdot C_p \cdot \max((\sum_{s=1}^{N} \pi_s(1-\pi_s))^p, \sum_{s=1}^{N} \pi_s(1-\pi_s)), \tag{5.17}$$

where C_p *is a number which only depends on* p.

PROOF. This inequality is included in the Theorem in [6]. We have, for $k = 1, 2, \cdots, s = 1, 2, \cdots, N$, $E|Y_s^c|^{2k} = \pi_s(1-\pi_s)^{2k} + (1-\pi_s)\pi_s^{2k} \leqq \pi_s(1-\pi_s)$. Thus,

$$E|Y_s^c a_s|^{2k} \leqq a_s^{2k} \pi_s(1-\pi_s). \tag{5.18}$$

From (5.18) we conclude that Condition (1) in [6] is met for $\lambda_s(p) = |a_s|$ and $\rho_s(p) = \pi_s(1-\pi_s)$. Now (5.17) follows easily from (2) in [6].

PROOF OF THEOREM 6. From the fact that $(E|X|^r)^{1/r}$, $r \geqq 0$, is non-decreasing as r increases, it follows that it suffices to prove (5.1) for a sequence of r-values, which tend to infinity. From (5.8) we get, for $p = 1, 2, \cdots$, C_p denoting a number which only depends on p.

$$E|Q_n^c|^{2p} \leqq C_p E|\sum_{s=1}^{N} \chi(X_s(n))^c a_s|^{2p} + C_p E|R_n|^{2p}. \tag{5.19}$$

The random variables $\chi(X_1(n)), \chi(X_2(n)), \cdots, \chi(X_N(n))$ are independent Bernoulli random variables, such that

$$P(\chi(X_s(n)) = 0) = e^{-p_s n}, \qquad s = 1, 2, \cdots, N. \tag{5.20}$$

From Lemma 5.7, (5.20) and the inequality $1 - e^{-x} \leqq x$, $0 \leqq x$, we get

$$\begin{aligned} E|\sum_{s=1}^{N} \chi(X_s(n))^c a_s|^{2p} &\leqq (\max_s |a_s|)^{2p} \cdot C_p \cdot \max((\sum_{s=1}^{N} (1-e^{-p_s n}))^p, \sum_{s=1}^{N} (1-e^{-p_s n})) \\ &\leqq (\max_s |a_s|)^{2p} \cdot C_p \cdot \max((\sum_{s=1}^{N} np_s)^p, \sum_{s=1}^{N} np_s) \\ &= (\max_s |a_s|)^{2p} \cdot C_p \cdot n^p. \end{aligned}$$

Thus, Theorem 6 is proved for $r = 2, 4, 6, \cdots$, and thus in general.

6. On the absolute moments of $W(B) - w(B)$. First we introduce a notational convention, which will be used throughout this section. C_p denotes a number which only depends on p, while $C_p(u, t)$ and $C_p(u, s, t)$, $0 < p < \infty$, $1 \leqq u < \infty$, $1 \leqq s < \infty$, $0 \leqq t \leqq 1$, denote functions which for every p are bounded on every rectangle $1 \leqq u \leqq u_0 < \infty$, $0 \leqq t \leqq t_0 < 1$, respectively on every rectangle $1 \leqq u \leqq u_0 < \infty$, $1 \leqq s \leqq s_0 < \infty$, $0 \leqq t \leqq t_0 < 1$.

Furthermore, we continue to use the assumptions and notations, that were introduced in Section 3.

Our purpose in this section is to derive the following estimate.

THEOREM 7. *Let* $W(B)$ *be defined according to* (1.7). *Then, for*

$$\bar{a} \leqq B < A \tag{6.1}$$

we have for every $p > 0$,

$$E|W(B)-w(B)|^p \leqq \left(\frac{B}{\bar{a}}\right)^{p/2} C_p\left(\frac{\rho_2}{\rho_1}, \frac{M}{m}, \frac{B}{A}\right) \tag{6.2}$$

where $w(B)$, $\bar{a}$, A, m, M, ρ_1 *and* ρ_2 *are defined in* (3.10), (3.1), (3.2) *and* (3.3).

Again, we write down the particular result concerning T_n, which is included in this theorem, and which is an immediate consequence of it.

THEOREM 8. *Let* T_n *be defined according to* (1.4). *Then, for every* $p > 0$ *we have for* $n = 1, 2, \cdots$

$$E|T_n - t(n)|^p \leqq n^{p/2} C_p\left(\frac{\rho_2}{\rho_1}, \frac{n}{N}\right) \tag{6.3}$$

where $t(n)$ *is defined in accord with* (2.19).

PROOF OF THEOREM 7. Throughout the proof let

$$W^*(B) = (W(B)-w(B))/(B/\bar{a})^{\frac{1}{2}}. \tag{6.4}$$

In virtue of (1.8) and the Markov inequality we have for $u > 0$ and $r > 0$

$$\begin{aligned} P(W^*(B) \leqq -u) & \\ &= P(W(B) \leqq w(B) - u(B/\bar{a})^{\frac{1}{2}}) \\ &= P(Q_{[w(B)-u(B/\bar{a})^{1/2}]} - \varphi([w(B)-u(B/\bar{a})^{\frac{1}{2}}]) \geqq B - \varphi([w(B)-u(B/\bar{a})^{\frac{1}{2}})) \\ &\leqq \frac{E|Q_{[w(B)-u(B/\bar{a})^{1/2}]} - \varphi([w(B)-u(B/\bar{a})^{\frac{1}{2}}])|^r}{(B-\varphi([w(B)-u(B/\bar{a})^{\frac{1}{2}}]))^r}. \end{aligned} \tag{6.5}$$

According to the corollary of Theorem 6 and (3.11) we have

$$\text{Numerator in } (6.5) \leqq C_r w(B)^{r/2} M^r \leqq C_r \left(\frac{B}{\bar{a}}\right)^{r/2} \left(\frac{1}{\rho_1}\right)^{r/2} \left(1-\frac{B}{A}\right)^{-r/2} M^r. \tag{6.6}$$

By using the fact that $\varphi(x)$ increases with x, and (3.15) we obtain

$$\begin{aligned} (B-\varphi([w(B)-u(B/\bar{a})^{\frac{1}{2}}]))^r &\geqq (B-\varphi(w(B)-u(B/\bar{a})^{\frac{1}{2}}))^r \\ &\geqq (u(B/\bar{a})^{\frac{1}{2}} m(1-B/A)^{\rho_2/\rho_1})^r. \end{aligned} \tag{6.7}$$

From (6.5), (6.6) and (6.7) we get

$$P(W^*(B) \leqq -u) \leqq \left(\frac{1}{u}\right)^r C_r\left(\frac{\rho_2}{\rho_1}, \frac{M}{m}, \frac{B}{A}\right), \qquad r, u > 0. \tag{6.8}$$

Let $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$. As $W(B) \geqq 0$, $W^*(B)^-$ has finite absolute moments of all orders. We have

$$\begin{aligned} E|W^*(B)^-|^p &= p\textstyle\int_0^\infty u^{p-1} P(W^*(B) \leqq -u)\, du \\ &\leqq 1 + p\textstyle\int_1^\infty u^{p-1} P(W^*(B) \leqq -u)\, du. \end{aligned} \tag{6.9}$$

By inserting the estimate (6.8) with $r = p+1$ into (6.9) we get

$$E|W^*(B)^-|^p \leqq C_p\left(\frac{\rho_2}{\rho_1}, \frac{M}{m}, \frac{B}{A}\right), \qquad p > 0. \tag{6.10}$$

Next we shall derive an estimate for $E|W^*(B)^+|^p$ in an analogous way. However, things become a bit more intricate in this case.

We assume that $u \geqq 2$. In virtue of (6.1) we then have the following estimate

$$[w(B)+u(B/\bar{a})^{\frac{1}{2}}] \geqq w(B)+\tfrac{1}{2}u(B/\bar{a})^{\frac{1}{2}}. \tag{6.11}$$

By arguing as before, and by paying regard to (5.1), (6.11), (3.11) and (3.14) we get for $u \geqq 2$

$$\begin{aligned} P(W^*(B) > u) &\leqq \frac{E|Q_{[w(B)+u(B/\bar{a})^{1/2}]} - \varphi([w(B)+u(B/\bar{a})^{\frac{1}{2}}])|^r}{(\varphi([w(B)+u(B/\bar{a})^{\frac{1}{2}}])-B)^r} \\ &\leqq \frac{C_r(w(B)+u(B/\bar{a})^{\frac{1}{2}})^{r/2}M^r}{(\varphi(w(B)+\frac{1}{2}u(B/\bar{a})^{\frac{1}{2}})-B)^r} \\ &\leqq C_r' \frac{(w(B)^{r/2}+(u(B/\bar{a})^{\frac{1}{2}})^{r/2})\cdot M^r}{(u(B/\bar{a})^{\frac{1}{2}})^r m^r e^{-(r\rho_2 u/2N)(B/\bar{a})^{1/2}}(1-B/A)^{r\rho_2/\rho_1}}. \end{aligned} \tag{6.12}$$

(6.12) yields the following estimate

$$P(W^*(B) > u) \leqq \left(\frac{1}{u}\right)^{r/2} C_r\left(\frac{\rho_2}{\rho_1}, \frac{M}{m}, \frac{B}{A}\right), \qquad 2 \leqq u \leqq N(B/\bar{a})^{-\frac{1}{2}}. \tag{6.13}$$

However, (6.12) will give a poor estimate of $P(W^*(B) > u)$ when u is considerably larger than $N(B/\bar{a})^{-\frac{1}{2}}$. The following rather crude estimate will give us a better bound for large values of u

$$P(W(B) > m) \leqq N e^{-\rho_1(m/N)}, \qquad 0 < B \leqq A,\ m = 1, 2, \cdots. \tag{6.14}$$

To prove (6.14) we introduce the following events. $E(s, m)$: The color s does not occur among the m first coupons, $s = 1, 2, \cdots, N$, $m = 1, 2, \cdots$.

We have $\{W(A) > m\} = \bigcup_{s=1}^N E(s, m)$ which yields

$$\begin{aligned} P(W(A) > m) &= P(\textstyle\bigcup_{s=1}^N E(s, m)) \leqq \sum_{s=1}^N P(E(s, m)) \\ &= \textstyle\sum_{s=1}^N (1-p_s)^m \leqq \sum_{s=1}^N e^{-p_s m} \leqq N e^{-\rho_1(m/N)}. \end{aligned} \tag{6.15}$$

By combining (6.15) and $P(W(B) > m) \leqq P(W(A) > m)$, we obtain (6.14). From (6.14) and (6.11) we get for $u \geqq 2$

$$P(W^*(B) > u) \leqq N e^{-(\rho_1/N)(w(B)+u(B/\bar{a})^{1/2})} \leqq N e^{-\frac{1}{2}\rho_1(u/N)(B/\bar{a})^{1/2}} \qquad u \geqq 2. \tag{6.16}$$

From (6.16) we conclude that $W^*(B)^+$ has finite moments of all orders. Thus, we have the following formula, where α is a positive number to be specified later on

$$
\begin{aligned}
E|W^*(B)^+|^p &= p\int_0^\infty u^{p-1}P(W^*(B)>u)\,du \\
&\leqq C_p + p\Big(\int_2^{N(B/\bar a)^{-1/2}} + \int_{N(B/\bar a)^{-1/2}}^{\alpha N\log N(B/\bar a)^{-1/2}} \\
&\qquad + \int_{\alpha N\log N(B/\bar a)^{-1/2}}^\infty\Big)u^{p-1}P(W^*(B)>u)\,du \\
&= C_p + p(I_1+I_2+I_3).
\end{aligned} \tag{6.17}
$$

By using (6.16), (6.1) and the estimate $\int_x^\infty u^{p-1}e^{-\rho u}\,du \leqq C_p x^p e^{-\rho x}$, $\rho x \geqq 1$ we get, for $\frac{1}{2}\alpha\rho_1 \log N \geqq 1$

$$
\begin{aligned}
I_3 &\leqq N\int_{\alpha N\log N(B/\bar a)^{-1/2}} u^{p-1}e^{-\frac{1}{2}\rho_1(u/N)(B/\bar a)^{1/2}}\,du \leqq NC_p\left(\frac{\alpha N\log N}{(B/\bar a)^{\frac{1}{2}}}\right)^p e^{-\frac{1}{2}\alpha\rho_1\log N} \\
&\leqq C_p N^{p+1-\frac{1}{2}\rho_1\alpha}(\log N)^p\alpha^p.
\end{aligned} \tag{6.18}
$$

We now fix α to be $\alpha = 2(p+2)/\rho_1$. Then, (6.18) yields, that for this choice of α, we have

$$
I_3 \leqq C_p\rho_1^{-p}. \tag{6.19}
$$

From (6.13) and (6.1) we get for $p \geqq 1$

$$
\begin{aligned}
I_2 &= \int_{N(B/\bar a)^{-1/2}}^{\alpha N\log N(B/\bar a)^{-1/2}} u^{p-1}P(W^*(B)>u)\,du \\
&\leqq (\alpha N\log N)^{p-1}P(W^* > N(B/\bar a)^{-\frac{1}{2}}) \\
&\leqq (\alpha N\log N)^{p-1}((B/\bar a)^{\frac{1}{2}}/N)^{r/2}C_r\left(\frac{\rho_2}{\rho_1},\frac{M}{m},\frac{B}{A}\right) \\
&\leqq N^{p-1-r/4}(\log N)^{p-1}\alpha^{p-1}C_r\left(\frac{\rho_2}{\rho_1},\frac{M}{m},\frac{B}{A}\right).
\end{aligned} \tag{6.20}
$$

We now choose $r = 4p$. Then, (6.20) and our previous choice of α yield

$$
I_2 \leqq C_p\left(\frac{\rho_2}{\rho_1},\frac{M}{m},\frac{B}{A}\right). \tag{6.21}
$$

Thus, (6.21) is established for $p \geqq 1$. It is not difficult to modify the proof so as to obtain that (6.21) is true also for $0 < p < 1$.

From (6.13) and the choice $r = 2p+1$ we get

$$
I_1 \leqq \left(\int_2^{N(B/\bar a)^{-1/2}} u^{p-1-r/2}\,du\right)C_r\left(\frac{\rho_2}{\rho_1},\frac{M}{m},\frac{B}{A}\right) \leqq C_p\left(\frac{\rho_2}{\rho_1},\frac{M}{m},\frac{B}{A}\right). \tag{6.22}
$$

Now, (6.17), (6.19), (6.21) and (6.22) yield

$$
E|W^*(B)^+|^p \leqq C_p\left(\frac{\rho_2}{\rho_1},\frac{M}{m},\frac{B}{A}\right). \tag{6.23}
$$

From (6.10) and (6.23) we conclude that

$$E|W^*(B)|^p \leqq C_p\left(\frac{\rho_2}{\rho_1}, \frac{M}{m}, \frac{B}{A}\right). \qquad p > 0. \tag{6.24}$$

Now, (6.24) and (6.2) are equivalent. This concludes the proof of Theorem 7.

7. Completion of the proof of Theorem 3. In Section 2 we reduced the proof of Theorem 3 to the verification of (2.30). The truth of (2.30) under the conditions (2.7) and (2.8) follows easily from Theorem 7 and Lemma 3.8. This concludes the proof of Theorem 3.

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