On the Covariance Completion Problem under a Circulant Structure

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Abstract— **Covariance matrices with a circulant structure arise in the context of discrete-time periodic processes and their significance stems also partly from the fact that they can be diagonalized via a Fourier transformation. This note deals with the problem of completion of partially specified circulant covariance matrices. The particular completion that has maximal determinant (i.e., the so-called maximum entropy completion) was considered in Carli** *etal.* **[2] where it was shown that if** *a single band* **is unspecified and to be completed, the algebraic restriction that enforces the circulant structure is automatically satisfied and that the inverse of the maximizer has a band of zero values that corresponds to the unspecified band in the data—i.e., it has the Dempster property. The purpose of the present note is to develop an independent proof of this result which in fact extends naturally to any number of missing bands as well as arbitrary missing elements. More specifically, we show that this general fact is a direct consequence of the invariance of the determinant under the group of transformations that leave circulant matrices invariant. A description of the complete set of all positive extensions of partially specified circulant matrices is also given and certain connections between such sets and the factorization of certain polynomials in many variables, facilitated by the circulant structure, is highlighted.**

I. Introduction

The present work has been motivated by a recent study by Carli, Ferrante, Pavon and Picci [2] on discrete-time periodic processes. In particular, they studied reciprocal random processes on the cyclic group with n elements [11], [12]. Applications in the digital age abound, not least because of the computational simplicity in working with such processes and their representations ([10]).

The subject in [2] was the completion of partially specified covariances for such a process x_ℓ on the discrete finite "time-set" $\{0, 1, \ldots, n-1\}$. The assumption of (second-order) stationarity $R_\ell := E\{x_0 x_\ell^*\} = E\{x_k x^*_{(k+\ell)_{\mathrm{mod}\, n}}\}$ for all $k, \ell \in \mathbb{Z}/(n\mathbb{Z})$, together with the periodicity $R_\ell = R_{\ell\, \mathrm{mod}\, n}$ dictates that the covariance matrix

$$R := \begin{bmatrix} R_0 & R_1 & R_2 & \ldots & R_2^* & R_1^* \\ R_1^* & R_0 & R_1 & & R_3^* & R_2^* \\ R_2^* & R_1^* & R_0 & & R_4^* & R_3^* \\ \vdots & & & \ddots & & \vdots \\ R_2 & R_3 & R_4 & & R_0 & R_1 \\ R_1 & R_2 & R_3 & \ldots & R_1^* & R_0 \end{bmatrix}, \tag{1}$$

has the block-circulant structure [3], [8]. In practice, lack of data may result in missing values in estimating statistics, i.e., some of the R_ℓ's above.

To this end, for the completion of general statistics it is common practice (e.g., see [4], [9], [5]) to select values that maximize the determinant of a covariance matrix. The reason can be traced to the fact that, for Gaussian statistics, this gives the maximum likelihood distribution. Specializing to the case of circulant covariances, Carli *etal.* [2] showed that when there is *a single band* of unspecified values, the constraint that enforces the circulant structure when maximizing the determinant is automatically satisfied, and thereby, the maximizer shares the property of maximizers for more general problems in having a banded inverse with zero values corresponding to the location of unspecified elements, cf. [4].

The purpose of the present note is to develop a simple independent argument that explains this result and, at the same time, shows that the algebraic constraint for the completion to be circulant is automatically satisfied in *all* cases, i.e., for any number of missing bands as well as for any number of arbitrary missing elements in a block-circulant structure. More specifically, the proof of the key result relies on the observation that circulant and block-circulant matrices are stable points of a certain group. The group action preserves the value of the determinant. Hence, the maximizer of the determinant, which is unique and has the Dempster property [4], will generate an orbit under the group-action that preserves the specified elements in their original locations (since these are compatible with the circulant structure). The values at the unspecified locations will be varying and thus generating more than one maximizer unless they are also compatible with the circulant structure. Since the maximizer is unique, it follows that the maximizer is circulant. We note that the importance of invariance in establishing an alternative proof was already anticipated in a remark in [2].

The present note goes on to highlight certain connections between the structure of all positive extensions for partially specified circulant matrices and factorizations of certain polynomials in many variables. More specifically, since circulant $m\times m$-block matrices can be diagonalized by a Fourier transformation, they can be thought of as matrix-valued functions on the cyclic group $\mathbb{Z}/(n\mathbb{Z})$. Therefore, positivity of a partially specified such matrix gives rise to n, m-order curves, that delineate the admissible completion set. These curves represent factors of the determinant viewed as a polynomial in the unspecified coefficients—in case $m = 1$ they are lines and the solution set a polytope. Thus, a Fourier transformation allows factorization of polynomials which, without the knowledge that they can be written as the determinant of a circulant matrix with variable entries, would be challenging or impossible using rational techniques and elimination theory (e.g., Galois theory, Gröbner bases, symbolic manipulations).

The outline of the material is as follows. In Section II we introduce certain facts about circulant matrices. In Section III we discuss the Dempster property (Theorem 4), namely the property of the inverse of the maximum entropy completion to have zero entries in places where the covariance matrix is unspecified. In the same section we present our main result (Theorem 5) for maximum-determinant completions of matrices with a circulant structure. Finally Section IV contains two examples which provide insight into the structure of the completion-set, and highlight a connection with the factorization of certain polynomials in many variables.

II. Technical preliminaries & notation

We work in the space $\mathbb{C}^{m\times m}$ of $(m\times m)$ complex-valued matrices. As usual, for any $a\in\mathbb{C}^{m\times m}$, the complex-conjugate-transpose (adjoint) is denoted by a^*. At times the size of matrices is $(nm\times nm)$, in which case these are typically of the form $b\otimes a$ with $b\in\mathbb{C}^{n\times n}$, $a\in\mathbb{C}^{m\times m}$ and $\otimes$ denoting the Kronecker product, i.e., these are "block-matrices."

We define the circulant (up) $(n\times n)$-shift

$$S := \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \\ 1 & 0 & \ldots & 0 \end{bmatrix},$$

and the notation I_n for the $(n\times n)$-identity matrix. Clearly

Supported by the National Science Foundation and by the Air Force Office of Scientific Research.

$S^n = I_n$, and as is well-known (and easy to check), S^k has the (eigenvalue-eigenvector) decomposition

$$S^k U = U W^k,$$

where U is the Fourier-matrix with elements $U_{p,q} = w^{pq}$ for $j := \sqrt{-1}$, $w = e^{-j\frac{2\pi}{n}}$ and

$$W \quad = \quad \mathrm{diag}\{1, w, w^2, \ldots, w^{n-1}\}.$$

Note that diag$\{\cdot\}$ denotes a diagonal (or, possibly, block-diagonal) matrix with its entries listed within the brackets. We now consider the space of $(n \times n)$-circulant matrices

$$\mathcal{C}_n := \left\{ a(S) := \sum_{k=0}^{n-1} S^k a_k \mid a_k \in \mathbb{C} \right\},$$

and the space of $(n \times n)$-circulant m-block matrices

$$\mathcal{C}_{n;m} := \left\{ a(S) := \sum_{k=0}^{n-1} S^k \otimes a_k \mid a_k \in \mathbb{C}^{m\times m} \right\}.$$

It can be seen that a (block-)circulant matrix is completely defined by its first (block-)row, and also that a circulant matrix $a(S)$ is Hermitian if $a_0 = a_0^*$ as well as

$$a_k = a_{n-k}^* \text{ for } k = 1, \ldots \left\lfloor \frac{n}{2} \right\rfloor.$$

Thus, when n is even, $a_{\lfloor \frac{n}{2} \rfloor}$ needs to be Hermitian as well for $a(S)$ to be Hermitian. The cone of Hermitian non-negative elements in these two spaces will be denoted by $\mathcal{C}_n^+$ and $\mathcal{C}_{n;m}^+$, respectively. E.g.,

$$\mathcal{C}_n^+ := \{a(S) \geq 0 \mid a(S) = a(S)^* \in \mathcal{C}_n\}$$

where ≥ 0 denotes non-negative definiteness while 0 denotes the zero matrix of suitable size. Next, we provide certain basic facts about circulant/block-circulant matrices.

Proposition 1: A given $M \in \mathbb{C}^{nm\times nm}$ is in $\mathcal{C}_{n;m}$ if and only if

$$(S \otimes I_m) M (S^* \otimes I_m) = M. \tag{2}$$

Sketch of the proof: It suffices to expand (2) and note that this constrains the $(m \times m)$-block entries of M to have the circulant symmetry. □

Remark 1: Elements of the form $(S \otimes I_m)$ generate a group which acts on arbitrary matrices via

$$\text{shift}: \; M \mapsto (S \otimes I_m) M (S^* \otimes I_m).$$

This is clearly a finite group, and the above proposition simply states that an orbit consists of a single matrix M if and only if this matrix already belongs to $\mathcal{C}_{n;m}$.

Proposition 2: The matrix $a(S) \in \mathcal{C}_{n;m}$ is invertible if and only if the determinant of the polynomial-matrix

$$a(x) = \sum_{k=0}^{n-1} x^k a_k$$

does not vanish at the nth roots of unity $\{w^k \mid k = 0, 1, \ldots, n-1\}$. In case $a(S) \in \mathcal{C}_{n;m}$ is invertible, its inverse matrix is also in $\mathcal{C}_{n;m}$.

Proof: For the invertibility condition it suffices to note that

$$\begin{aligned}(U^* \otimes I_m)(\sum_{k=0}^{n-1} S^k \otimes a_k)(U \otimes I_m) &= \sum_{k=0}^{n-1} W^k \otimes a_k \\ &= \sum_{k=0}^{n-1} \mathrm{diag}\{w^0 a_k, w^k a_k, w^{2k} a_k, \ldots, w^{(n-1)k} a_k\} \\ &= \mathrm{diag}\{a(w^0), a(w^1), a(w^2), \ldots, a(w^{n-1})\}. \end{aligned} \tag{3}$$

In case the inverse exists, then it is

$$\begin{aligned} a(S)^{-1} = (U \otimes I_m)\big(&\mathrm{diag}\{a(w^0)^{-1}, a(w^1)^{-1}, a(w^2)^{-1}, \\ &\ldots, a(w^{n-1})^{-1}\}\big)(U^* \otimes I_m).\end{aligned}$$

Define the notation

$$E_k = \mathrm{diag}\{0, \ldots, 0, 1, 0, \ldots, 0\}$$

with a 1 at the kth diagonal entry. Since $(S \otimes I_m)(U \otimes I_m) = (U \otimes I_m)(W \otimes I_m)$ while $W^{-1} = W^*$, we have that

$$\begin{aligned}(S \otimes I_m)\left(a(S)^{-1}\right)(S^* \otimes I_m) &= (U \otimes I_m)(W \otimes I_m) \\ &\times \left(\sum_{k=0}^{n-1} E_k \otimes a(w^k)^{-1}\right)(W^{-1} \otimes I_m)(U^* \otimes I_m) \\ &= (U \otimes I_m)\left(\sum_{k=0}^{n-1} E_k \otimes a(w^k)^{-1}\right)(U^* \otimes I_m) \\ &= a(S)^{-1}.\end{aligned}$$

Therefore, the inverse is also circulant/block-circulant by Proposition 1. □

Corollary 3: A Hermitian matrix $a(S) \in \mathcal{C}_{n;m}$ is non-negative definite if and only if the following $m\times m$ matrices are non-negative,

$$a(e^{-j2\pi\ell/n}) = \sum_{k=0}^{n-1} e^{-j2\pi\ell k/n} a_k \geq 0, \text{ for } \ell = 0, \ldots, n-1.$$

Proof: Follows readily from equation (3). □

Of course, in the above, if $m = 1$ then $a(x)$ is a polynomial and $\det(a(S)) = \prod_{\ell=1}^{n} a(e^{-j2\pi\ell/n})$. However, in general,

$$\det(a(S)) = \prod_{\ell=1}^{n} \det(a(e^{-j2\pi\ell/n}))$$

where $a(e^{-j2\pi\ell/n})$, $\ell = 0, 1, \ldots, n-1$, are $m \times m$ matrices.

Remark 2: It is conceptually advantageous to think of conjugation, conj : $M \mapsto M^*$, as an element of the binary group that leaves Hermitian matrices unchanged. Thus, Hermitian circulant matrices can be seen as those matrices that stay invariant under the action of the group generated by $\{\text{conj, shift}\}$. Since, $(S \otimes I_m) M^* (S \otimes I_m)^* = ((S \otimes I_m) M (S \otimes I_m)^*)^*$, this is a commutative finitely generated group that, for ease of reference, we denote by $\mathcal{G} = \langle\{\text{conj, shift}\}\rangle$.

III. The covariance completion problem

Denote by $\mathcal{H}_n$ the set of $n \times n$ Hermitian matrices, by $\mathcal{H}_n^+ \subset \mathcal{H}_n$ the cone of positive definite $n \times n$ matrices, and consider a partially specified matrix $M \in \mathcal{H}_n$. As long as a positive definite completion for M exists, and as long as the set of such positive completions is bounded, a completion with maximal determinant is uniquely defined because the determinant is a strictly log-concave function of its argument. In such a case we denote

$$M_{\mathrm{me}} := \mathrm{argmax}\{\det(M) \mid M \in \mathcal{H}_n^+ \text{ satisfies (5)}\}, \tag{4}$$

where for a specified symmetric selection $\mathcal{S}$ of pairs of indices (i.e., if $(k, \ell) \in \mathcal{S}$, then $(\ell, k) \in \mathcal{S}$) it is required that the corresponding entries of M have the specified value $m_{k,\ell}$; more explicitely,

$$e_k M e_\ell^* = m_{k,\ell}, \quad \text{for } (k, \ell) \in \mathcal{S} \tag{5}$$

where

$$e_k := [\overbrace{0,\ \dots\ 0,\ 1}^{k},\ 0,\ \dots\ 0]$$

is the row vector with a 1 at the kth entry. Naturally, the data

$$\mathcal{M} := \{m_{k,\ell} \mid (k,\ell) \in \mathcal{S}\}$$

must be consistent with the hypothesis that M is Hermitian, i.e., $m_{k,\ell} = m^*_{\ell,k}$ for all entries in $\mathcal{M}$. For notational convenience, we also define the subspace

$$\mathcal{L}_n := \{M \in \mathcal{H}_n \mid (5) \text{ holds}\}$$

that contains matrices with the specified elements.

Theorem 4: Consider an index set $\mathcal{S}$ and a corresponding data set $\mathcal{M}$ consistent with the hypothesis that $M \in \mathcal{H}_n$ and assume that the set $\mathcal{H}^+_n \cap \mathcal{L}_n \neq \emptyset$ and bounded. The following holds:

$$e_k M_{\rm me}^{-1} e^*_\ell = 0 \text{ for } (k,\ell) \notin \mathcal{S}.$$

Proof: Clearly $M_{\rm me}$ exists, it is uniquely defined, and it is also a maximal point of $\log\det(M)$ on $\mathcal{L}_n$. Hence, it is a stationary point of the Lagrangian

$$\mathbb{L}(M, \lambda_{k,\ell}) := \log\det(M) + \sum_{(k,\ell)\in\mathcal{S}} \lambda_{k,\ell}(m_{k,\ell} - e_k M e^*_\ell). \quad (6)$$

When we set the derivative of $\mathbb{L}$ with respect to the entries of M equal to zero, we readily obtain that

$$M^{-1} = \sum_{(k,\ell)\in\mathcal{S}} \lambda_{k,\ell} e^*_k e_\ell \quad (7)$$

which completes the proof. ■

The above establishes a key result in [4], see also [9], [6] for other manifestations and applications of this property.

When additional restrictions are placed on M then, in general, this property of $M_{\rm me}$ no longer holds. For instance, if M is required to have a Toeplitz structure, then elements not in $\mathcal{S}$ that are on the same diagonal are constrained to have the same value. In this case, an additional set of Lagrange multipliers $(k,\ell) \notin \mathcal{S}$ is needed to enforce the Toeplitz structure via terms of the form $\lambda_{k,\ell}(e^*_k M e_\ell - e^*_{k+1} M e_{\ell+1})$. As a consequence, the statement of Theorem 4 fails in such cases.

Because of the above, it is interesting and surprising at first, that the statement of the proposition is valid when M is required to have a circulant structure. This is the main result of Carli etal. [2] that was shown by a direct algebraic verification when a single band is unspecified. Proposition 5 below gives an independent proof which at the same time shows this to be a general fact for arbitrary sets of interpolation conditions on the circulant structure.

For the remaining of this section we consider a partially specified block-matrix $M \in \mathcal{C}_{n;m}$, and assume that the linear constraints are consistent with the block-circulant Hermitian structure and that a positive completion exists. We show that the property of Theorem 4 holds true in general. To this end, we define $\mathcal{S}$ to be a set of index-pairs (k,ℓ) *consistent with the $\mathcal{C}_{n;m}$-circulant Hermitian structure* if

$$(k,\ell) \in \mathcal{S} \Rightarrow (\ell,k) \in \mathcal{S} \quad (8a)$$
$$(k,\ell) \in \mathcal{S} \Rightarrow ((\ell+m)_{\mathrm{mod}\,nm}, (k+m)_{\mathrm{mod}\,nm}) \in \mathcal{S}. \quad (8b)$$

Throughout, $k_{\mathrm{mod}\,nm}$ represents the remainder when k is divided by nm. Similarly, we define a data-set

$$\mathcal{M} := \{m_{k,\ell} \mid (k,\ell) \in \mathcal{S}\}$$

of $m_{k,\ell}$-values to be *consistent with the $\mathcal{C}_{n;m}$-circulant Hermitian structure* if the corresponding set of indices $\mathcal{S}$ is consistent and the values of its entries satisfy (8) and the $m_{k,\ell}$ values in $\mathcal{M}$ satisfy

$$m_{k,\ell} = m^*_{\ell,k} \quad (9a)$$
$$m_{k,\ell} = m^*_{(\ell+m)_{\mathrm{mod}\,nm},(k+m)_{\mathrm{mod}\,nm}} \quad (9b)$$

for all pairs of indices. Thus, strictly speaking, the entries of $\mathcal{M}$ are triples $(m_{k,\ell}, k, \ell)$ but we refrain from such overburdened notation as superfluous.

Theorem 5: Let $\mathcal{S}, \mathcal{M}$ be sets of indices and corresponding values consistent with the $\mathcal{C}_{n;m}$-circulant structure and assume that there exists a positive completion (not necessarily circulant), and that this set is bounded. Then

i) there is a positive circulant completion,
ii) the completion $M_{\rm me}$ in (4) is in fact circulant,
iii) $e_k M_{\rm me}^{-1} e^*_\ell = 0$ for $(k,\ell) \notin \mathcal{S}$.

Clearly ii) implies i) as well as iii), by Theorem 4. Thus, the essence is to show that $M_{\rm me}$ is indeed circulant. One rather direct proof can be based on the significance of the Lagrange multipliers as representing the sensitivity of the functional to be maximized, in this case the determinant, on the corresponding constraints. Because the circulant structure dictates that all values linked via (8a) and (8b) impact the determinant in the same way (since $\det((S \otimes I_m) M^* (S^* \otimes I_m)) = \det(M)$ and hence the value of the determinant is not affected by action of any $\mathcal{G}$-element), the sensitivity to each value $m_{k,\ell}$ is the same, and therefore the corresponding values for the Lagrange multipliers $\lambda_{k,\ell}$ at the stationary point (see equation (6)) are equal. Thus, $M_{\rm me}^{-1}$ in (7) has a circulant structure and so does $M_{\rm me}$ by Proposition 2. An alternative and almost immediate proof of ii) is given below.

Proof: Once again, observe that for any $M \in \mathcal{H}_{nm}$ it holds that $\det(\mathrm{shift}M) = \det(\mathrm{conj}M) = \det(M)$ as neither the circulant block-shift nor the conjugation of Hermitian matrices changes the value of the determinant. Furthermore, observe that if M satisfies

$$e_k M e^*_\ell = m_{k,\ell}, \text{ for } (k,\ell) \in \mathcal{S} \text{ and } m_{k,\ell} \in \mathcal{M}, \quad (10)$$

then the same is true for $\mathrm{shift}M$ as well as $\mathrm{conj}M$. This is due to the fact that the constraints are consistent with the block-circulant-Hermitian structure as well. Now since $\det(\cdot)$ is a strictly log-concave on $\mathcal{H}^+_{nm}$, it has a unique maximum subject to (10) (disregarding for the moment any restriction for the maximizer to belong to $\mathcal{C}_{n;m}$). But, this unique maximizing point $M_{\rm me}$ must be invariant under the group $\mathcal{G}$ generated by {conj and shift}, for otherwise, there would be multiple maxima. This proves directly that $M_{\rm me}$ is in $\mathcal{C}_{n;m}$. ■

Remark 3: The above argument applies to maximizers that may be restricted further by bounding individual elements, or in combination, to lie in a convex set in a way that is consistent with the circulant structure. More specifically and in a very general setting, if a maximizer exists over $\mathcal{H}^+_{nm}$ and if the constraints, of whatever nature, are consistent with the $\mathcal{C}_{n;m}$-structure, then the maximizer necessarily belongs to $\mathcal{C}_{n;m}$. Thus, the essence of this result is a rather general invariance principle that the maximizer of a concave functional when restricted to points that individually remain invariant under the action of a certain group, it is identical to the unconstrained one —assuming that the domain of the functional is convex and invariant under the group.

IV. Structure of solutions and factorization of polynomials in several variables

We now provide some insight on the structure of positive block-circulant completions of partially specified covariance matrices. The set is in general semi-algebraic, delineated by m-order curves. Thus, in case $m=1$ the solution set is a convex polyhedron whereas, in case $m=2$, it is the intersection of conic sections, etc. This can be seen from Corollary 3 where, in case $m=2$, the solution set is the intersection of the m-order surfaces specified via the conditions $a(e^{-j2\pi\ell/n})\geq 0$, for $\ell=0,1,\ldots,n-1$, where each matrix is *linear* in the unknown parameters. We illustrate the above with two examples.

Example 1: Consider $a(S)\in\mathcal{C}_{n;m}$, with $n=4$, $m=2$, where

$$a_0=\begin{bmatrix}2 & \frac{1}{2}\\ \frac{1}{2} & 2\end{bmatrix},\ a_1=a_3^\top=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\ \text{ while } a_2=\begin{bmatrix}x & y^*\\ y & z\end{bmatrix} \quad (11)$$

is left unspecified. The parameters in a_2 are indicated by x,y,z and, for ease of displaying the solution set, we further restrict our attention to the case where $y=y^*$ is real and, hence, a_2 and $a(S)$ are in fact symmetric. The maximum entropy solution can be computed using any general semi-definite programming solver (e.g., `yalmip, SDPT, SeDuMi`). In particular, in our work, we have used the interface in cvx [7] and we obtained $x=z=0.4853$, $y=0.4789$. The inverse of $a(S)$ is $\lambda(S)$ with

$$\lambda_0=\begin{bmatrix}1.1707 & -0.0163\\ -0.0163 & 1.1707\end{bmatrix},\ \lambda_1=\begin{bmatrix}-0.4469 & -0.4394\\ 0.3335 & -0.4469\end{bmatrix},$$
$$\lambda_2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\ \lambda_3=\lambda_1^\top,$$

where λ_2 is the 2×2 zero matrix, as claimed.

We now describe the complete solution set. We evaluate

$$\begin{aligned} a(w^0)&=\begin{bmatrix}4+x & \frac{3}{2}+y\\ \frac{3}{2}+y & 4+z\end{bmatrix},\\ a(w^1)&=\begin{bmatrix}2-x & \left(\frac{1}{2}-i\right)-y\\ \left(\frac{1}{2}+i\right)-y & 2-z\end{bmatrix}=a(w^3)^\top,\\ a(w^2)&=\begin{bmatrix}x & -\frac{1}{2}+y\\ -\frac{1}{2}+y & z\end{bmatrix}. \end{aligned} \quad (12)$$

The respective eigenvalues are

$$\begin{aligned} \operatorname{eig}\left\{a(w^0)\right\}&=4+\frac{x}{2}+\frac{z}{2}\pm\sqrt{\frac{9+(x-z)^2+4y(3+y)}{4}}\\ \operatorname{eig}\left\{a(w^1)\right\}&=2-\frac{x}{2}-\frac{z}{2}\pm\sqrt{\frac{5+(x-z)^2-4y(1-y)}{4}}\\ \operatorname{eig}\left\{a(w^2)\right\}&=\frac{x}{2}+\frac{z}{2}\pm\sqrt{\frac{1+(x-z)^2-4y(1-y)}{4}}. \end{aligned}$$

and the set where they are all positive is shown in Figure 1.

Next, we demonstrate the claimed zero-pattern for the inverse of the maximum entropy completion when blocks are partially specified. Set $z=1$ and leave the entries x and y in a_2 unspecified. The completion with maximal determinant corresponds to $x=0.3548$ and $y=0.4813$ while its inverse is $\lambda(S)$ with

$$\lambda_0=\begin{bmatrix}1.5507 & -0.0291\\ -0.0291 & 1.5869\end{bmatrix},\ \lambda_1=\begin{bmatrix}-0.6353 & -0.8163\\ 0.7344 & -0.1893\end{bmatrix},$$
$$\lambda_2=\begin{bmatrix}0 & 0\\ 0 & -0.9644\end{bmatrix},\ \lambda_3=\lambda_1^\top.$$

Once again, the zero-pattern is in agreement with Theorem 5.

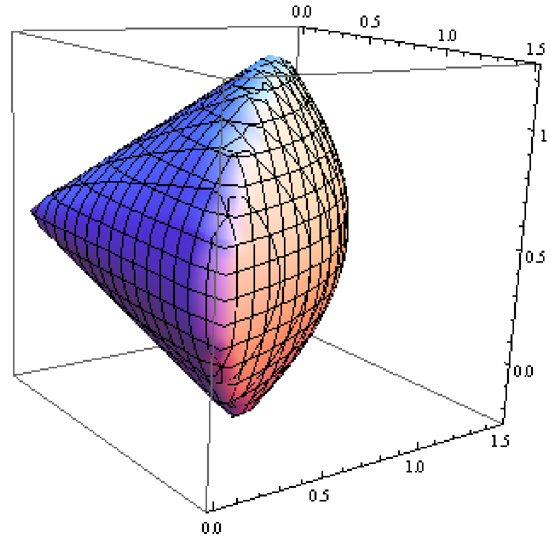

Fig. 1. Feasible set $\{(x,y,z)\mid a(S)\geq 0\}$.

Example 2: For our next example we take $m=1$ to highlight how polynomials in several variables, which happen to coincide with the determinant of a partially specified circulant matrix, can be readily factored via a Fourier transformation—an otherwise difficult task due to the irrationality of the factors in the absence of a suitable field extension. Thus, we consider a partially specified matrix $a(S)=\sum_{k=0}^{6}a_kS^k\in\mathcal{C}_7$ with real coefficients $a_0=2$, $a_1=a_6=1$, and $a_2=a_5=x$, $a_3=a_4=y$ left unspecified. The eigenvalues of $a(S)$ (see Corollary 3) are

$$\begin{aligned} a(w^0)&=2(2+x+y) && (13a)\\ a(w^1)&=2-2y\text{Cos}\left[\frac{\pi}{7}\right]-2x\text{Sin}\left[\frac{\pi}{14}\right]+2\text{Sin}\left[\frac{3\pi}{14}\right] && (13b)\\ a(w^2)&=-2\left(-1+x\text{Cos}\left[\frac{\pi}{7}\right]+\text{Sin}\left[\frac{\pi}{14}\right]-y\text{Sin}\left[\frac{3\pi}{14}\right]\right) && (13c)\\ a(w^3)&=-2\left(-1+\text{Cos}\left[\frac{\pi}{7}\right]+y\text{Sin}\left[\frac{\pi}{14}\right]-x\text{Sin}\left[\frac{3\pi}{14}\right]\right) && (13d)\\ a(w^4)&=a(w^4),\ a(w^5)=a(w^2),\ a(w^6)=a(w^1), && (13e) \end{aligned}$$

and the feasible set is the interior of a polyhedron. The determinant of $a(S)$ is a polynomial of degree 7,

$$\begin{aligned} \det(a(S))&=4+42x+56x^2-294x^3+140x^4+84x^5-28x^6\\ &+2x^7-14y-28xy+350x^2y-196x^3y-112x^4y-84x^5y\\ &+14x^6y-168xy^2+56x^2y^2+238x^3y^2+112x^4y^2+14x^5y^2\\ &+28y^3-238x^2y^3-28x^3y^3-42x^4y^3+98xy^4-14y^5\\ &+28x^2y^5-14xy^6+2y^7 \end{aligned} \quad (14)$$

in x and y. Over the ring of polynomials with rational coefficients it factors as (e.g., using Matlab or Mathematica)

$$\begin{aligned} \det(a(S))=2(2+x+y)&\left(1+5x-8x^2+x^3-2y+5xy\right.\\ &\left.+3x^2y-y^2-4xy^2+y^3\right)^2. \end{aligned}$$

However, using (13a-13e), we already know that

$$\begin{aligned}\det(a(S)) &= 2(2+x+y)\times\\ &\left[2-2y\cos\left(\frac{\pi}{7}\right)-2x\sin\left(\frac{\pi}{14}\right)+2\sin\left(\frac{3\pi}{14}\right)\right]^2\\ &\left[-2\left(-1+x\cos\left(\frac{\pi}{7}\right)+\sin\left(\frac{\pi}{14}\right)-y\sin\left(\frac{3\pi}{14}\right)\right)\right]^2\\ &\left[-2\left(-1+\cos\left(\frac{\pi}{7}\right)+y\sin\left(\frac{\pi}{14}\right)-x\sin\left(\frac{3\pi}{14}\right)\right)\right]^2. \quad (15)\end{aligned}$$

Provided we know a suitable field extension of $\mathbb{Q}$ which contains the coefficients of the factors, i.e., $\mathbb{Q}[\cos\left(\frac{\pi}{7}\right), \sin\left(\frac{\pi}{14}\right), \text{etc.}]$, the factorization can be carried out with standard methods [1]. Finding such an extension, in general, is a challenging problem. Of course, expressing a given rational polynomial as the determinant of a circulant matrix with rational coefficients may an equally challenging one, in general. Yet, we hope that the above observations may provide alternative ways to factor polynomials in certain suitable cases.

V. Concluding remarks

The main contribution in this work is the proof and insight that has been gained by Theorem 5 and Remark 3 into the problem of completion of partially specified circulant covariance matrices. While such matrices have been widely used in the signal processing literature [3], [8], the case for completion problems has only been brought forth in [2]. The present work builds on [2] and exposes the finer structure of the feasible set of such completions, namely, that maximizers of the determinant have the Dempster property in general. This fact is expected to prove advantageous in tailoring max-det algorithms to the case of circulant matrices. In particular, for algorithms that trace determinantal-maximizers through intermediate steps, as would be the case when adapting the homotopy-based techniques in [6], the convenient representation of a sparse inverse together with a possible equalization of numerical errors to retain such a sparse structure, is expected to prove beneficial from a numerical standpoint.

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