# On the coverings of Hantzsche-Wendt manifold 

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#### Abstract

There are only 10 Euclidean forms, that is flat closed three dimensional manifolds: six are orientable $\mathcal{G}_{1}, \ldots, \mathcal{G}_{6}$ and four are non-orientable $\mathcal{B}_{1}, \ldots, \mathcal{B}_{4}$. In the present paper we investigate the manifold $\mathcal{G}_{6}$, also known as Hantzsche-Wendt manifold; this is the unique Euclidean 3 -form with finite first homology group $H_{1}\left(\mathcal{G}_{6}\right)=\mathbb{Z}_{4}^{2}$.

The aim of this paper is to describe all types of $n$-fold coverings over $\mathcal{G}_{6}$ and calculate the numbers of non-equivalent coverings of each type. We classify subgroups in the fundamental group $\pi_{1}\left(\mathcal{G}_{6}\right)$ up to isomorphism. Given index $n$, we calculate the numbers of subgroups and the numbers of conjugacy classes of subgroups for each isomorphism type and provide the Dirichlet generating series for the above sequences.


## Introduction

Let $\mathcal{M}$ be a connected manifold with fundamental group $G=\pi_{1}(\mathcal{M})$. Two coverings

$$
p_{1}: \mathcal{M}_{1} \rightarrow \mathcal{M} \text { and } p_{2}: \mathcal{M}_{2} \rightarrow \mathcal{M}
$$

are said to be equivalent if there exists a homeomorphism $h: \mathcal{M}_{1} \rightarrow \mathcal{M}_{2}$ such that $p_{1}=p_{2} \circ h$. According to the general theory of covering spaces, any $n$-fold covering is uniquely determined by a subgroup of index $n$ in the group $G$. The equivalence classes of $n$-fold coverings of $\mathcal{M}$ are in one-to-one correspondence with the conjugacy classes of subgroups of index $n$ in the fundamental group $\pi_{1}(\mathcal{M})$. See, for example, (9], p. 67). In such a way the following natural problems arise: to describe the isomorphism classes of subgroups of finite index in the fundamental group of a given manifold and to enumerate the finite index subgroups and their conjugacy classes with respect to isomorphism type.

[^0]We use the following notations: let $s_{G}(n)$ denote the number of subgroups of index $n$ in the group $G$, and let $c_{G}(n)$ be the number of conjugacy classes of such subgroups. Similarly, by $s_{H, G}(n)$ denote the number of subgroups of index $n$ in the group $G$, which are isomorphic to $H$, and by $c_{H, G}(n)$ the number of conjugacy classes of such subgroups. So, $c_{G}(n)$ coincides with the number of nonequivalent $n$-fold coverings over a manifold $\mathcal{M}$ with fundamental group $\pi_{1}(\mathcal{M}) \cong G$, and $c_{H, G}(n)$ coincides with the number of nonequivalent $n$-fold coverings $p: \mathcal{N} \rightarrow \mathcal{M}$, where $\pi_{1}(\mathcal{N}) \cong H$ and $\pi_{1}(\mathcal{M}) \cong G$. The numbers $s_{G}(n)$ and $c_{G}(n)$, where $G$ is the fundamental group of closed orientable or nonorientable surface, were found in ([15], [16], [17]). In the paper [18], a general method for calculating the number $c_{G}(n)$ of conjugacy classes of subgroups in an arbitrary finitely generated group $G$ was given. Asymptotic formulas for $s_{G}(n)$ in many important cases were obtained in [14].

The values of $s_{G}(n)$ for the wide class of 3-dimensional Seifert manifolds were calculated in [12] and [13]. The present paper is a part of the series of our papers devoted to enumeration of finite-sheeted coverings over closed Euclidean 3-manifolds. These manifolds are also known as flat 3 -dimensional manifolds or Euclidean 3-forms.

The class of such manifolds is closely related to the notion of Bieberbach group. Recall that a subgroup of isometries of $\mathbb{R}^{3}$ is called Bieberbach group if it is discrete, cocompact and torsion free. Each 3 -form can be represented as a quotient $\mathbb{R}^{3} / G$, where $G$ is a Bieberbach group. In this case, $G$ is isomorphic to the fundamental group of the manifold, that is $G \cong \pi_{1}\left(\mathbb{R}^{3} / G\right)$. Classification of three dimensional Euclidean forms up to homeomorphism was obtained by W. Nowacki [19] and W. Hantzsche and H. Wendt [8]. There are only 10 Euclidean forms: six are orientable $\mathcal{G}_{1}, \ldots, \mathcal{G}_{6}$ and four are non-orientable $\mathcal{B}_{1}, \ldots, \mathcal{B}_{4}$ See monograph [25] for more details.

In our previous paper [3] we describe isomorphism types of finite index subgroups $H$ in the fundamental group $G$ of manifolds $\mathcal{B}_{1}$ and $\mathcal{B}_{2}$. Further, we calculate the respective numbers $s_{H, G}(n)$ and $c_{H, G}(n)$ for each isomorphism type $H$. In subsequent articles [4, [5] and [6] similar questions were solved for manifolds $\mathcal{G}_{2}, \mathcal{G}_{3}, \mathcal{G}_{4}, \mathcal{G}_{5}, \mathcal{B}_{3}$ and $\mathcal{B}_{4}$.

The aim of the present paper is to solve the same questions for the Hantzsche-Wendt manifold $\mathcal{G}_{6}$, undoubtedly the most weird among Euclidean 3-manifolds. This is the unique Euclidean 3 -form with finite first homology group $H_{1}\left(\mathcal{G}_{6}\right)=\mathbb{Z}_{4}^{2}$. Its fundamental set consists of two cubes described below. See also ([7], Table 4) for description of a cubical fundamental domain for $\mathcal{G}_{6}$. The Hantzsche-Wendt manifold is also known as Fibonacci manifold $M_{3}$. The Fibonacci manifold $M_{n}, n \geq 2$ is a closed orientable three-dimensional manifold whose fundamental group is the Fibonacci group $F(2,2 n)=$ $\left\langle x_{1}, \ldots, x_{2 n}: x_{i} x_{i+1}=x_{i+2}, i \bmod 2 n\right\rangle$. These manifolds were discovered by H. Helling, A.C. Kim and J. Mennicke [10]. It was shown by H.M. Hilden, M.T. Lozano and J.M. Montesinos [11] that $M_{n}$ is the $n$-fold cyclic covering of the three-dimensional sphere $\mathbb{S}^{3}$ branched over the figure-eight knot.

Also, by A.Yu. Vesnin and A.D. Mednykh [24], $M_{3}$ is the two fold covering of $\mathbb{S}^{3}$ branched over the Borromean rings. The outer automorphism group of the HantzscheWendt manifold was calculated by B. Zimmermann [26]. Its high dimensional analogues were investigated by A. Szczepański [23].

The description of $\mathcal{G}_{6}$ through Bieberbach group is the following: the group is gen-
erated by isometries

$$
\begin{aligned}
& S_{1}:(x, y, z) \mapsto(x+1,-y,-z+1), \\
& S_{2}:(x, y, z) \mapsto(-x+1, y+1,-z), \\
& S_{3}:(x, y, z) \mapsto(-x,-y+1, z+1) .
\end{aligned}
$$

In more geometric terms $\mathcal{G}_{6}$ can be described in the following way. We take the union of two cubes $[-1,0]^{3} \cup[0,1]^{3}$ as the fundamental domain of $\mathcal{G}_{6}$, we call this cubes positive and negative respectively. Now we have to provide six isometries to align each face of the negative cube with the respective face of the positive cube, we glue faces by this alignment.

- we align faces $z=0$ and $z=-1$ with faces $z=1$ and $z=0$ through $(x, y, z) \mapsto$ $(x+1,-y,-z+1)$ and $(x, y, z) \mapsto(x+1,-y,-z-1)$ respectively (these are isometries $S_{1}$ and $S_{3}^{-2} S_{1}$ respectively);
- align faces $x=0$ and $x=-1$ with faces $x=1$ and $x=0$ through $(x, y, z) \mapsto$ $(-x+1, y+1,-z)$ and $(x, y, z) \mapsto(-x-1, y+1,-z)$ respectively (these are isometries $S_{2}$ and $S_{1}^{-2} S_{2}$ respectively);
- finally, align faces $y=0$ and $y=-1$ with faces $z=1$ and $z=0$ through $(x, y, z) \mapsto(-x,-y+1, z+1)$ and $(x, y, z) \mapsto(-x,-y-1, z+1)$ respectively (these are isometries $S_{3}$ and $S_{2}^{-2} S_{3}$ respectively).

In the present paper, we classify finite index subgroups in the fundamental group $\pi_{1}\left(\mathcal{G}_{6}\right)$ up to isomorphism. Given index $n$, we calculate the numbers of subgroups and the numbers of conjugacy classes of subgroups for each isomorphism type. Also, we provide the Dirichlet generating functions for all the above sequences.

Numerical methods to solve these and similar problems for the three-dimensional crystallographic groups were developed by the Bilbao group [2]. The convenience of language of Dirichlet generating series for this kind of problems was demonstrated in 22. The first homologies of all the three-dimensional crystallographic groups are determined in [20].

## Notations

Let $G$ be a group, $u, v$ are elements and $H, F$ are subgroups in $G$. We use $u^{v}$ instead of $v u v^{-1}$ and $[u, v]$ instead of $u v u^{-1} v^{-1}$ for the sake of brevity. By $H^{v}$ denote the subgroup $\left\{u^{v} \mid u \in H\right\}$. By $H^{F}$ denote the family of subgroups $H^{v}, v \in F$. By $A d_{v}: G \rightarrow G$ denote the automorphism given by $u \rightarrow u^{v}$.

By $s_{H, G}(n)$ we denote the number of subgroups of index $n$ in the group $G$ isomorphic to the group $H$; by $c_{H, G}(n)$ the number of conjugacy classes of subgroups of index $n$ in the group $G$ isomorphic to the group $H$. Through this paper usually $G$ and $H$ are fundamental groups of manifolds $\mathcal{G}_{i}$, in this case we omit $\pi_{1}$ in indices.

Also we will need the following number-theoretic functions. Given a fixed $n$ we widely use summation over all representations of $n$ as a product of two or three positive integer
factors $\sum_{a b=n}$ and $\sum_{a b c=n}$. The order of factors is important. We assume this sum vanishes if $n$ is not integer.

To start with, this is the natural language to express the function $\sigma_{0}(n)$ - the number of representations of number $n$ as a product of two factors $\sigma_{0}(n)=\sum_{a b=n} 1$. We will also need the following generalizations of $\sigma_{0}$ :

$$
\begin{aligned}
\sigma_{1}(n) & =\sum_{a b=n} a, & \sigma_{2}(n) & =\sum_{a b=n} \sigma_{1}(a)=\sum_{a b c=n} a \\
d_{3}(n) & =\sum_{a b=n} \sigma_{0}(a)=\sum_{a b c=n} 1, & \omega(n) & =\sum_{a b=n} a \sigma_{1}(a)=\sum_{a b c=n} a^{2} b
\end{aligned}
$$

## 1 Formulation of main results

The main goal of this paper is to prove the following two theorems.
Theorem 1. Every subgroup $\Delta$ of finite index $n$ in $\pi_{1}\left(\mathcal{G}_{6}\right)$ is isomorphic to either $\pi_{1}\left(\mathcal{G}_{6}\right)$, or $\pi_{1}\left(\mathcal{G}_{2}\right)$, or $\mathbb{Z}^{3}$. The respective numbers of subgroups are

$$
\begin{equation*}
s_{\mathcal{G}_{1}, \mathcal{G}_{6}}(n)=\omega\left(\frac{n}{4}\right) \tag{i}
\end{equation*}
$$

$$
\begin{equation*}
s_{\mathcal{G}_{6}, \mathcal{G}_{6}}(n)=n\left(d_{3}(n)-3 d_{3}\left(\frac{n}{2}\right)+3 d_{3}\left(\frac{n}{4}\right)-d_{3}\left(\frac{n}{8}\right)\right) \tag{iii}
\end{equation*}
$$

Theorem 2. Let $\mathcal{N} \rightarrow \mathcal{G}_{6}$ be an $n$-fold covering over $\mathcal{G}_{6}$. Then $\mathcal{N}$ is homeomorphic to one of $\mathcal{G}_{6}, \mathcal{G}_{2}$ or $\mathcal{G}_{1}$. The corresponding numbers of nonequivalent coverings are given by the following formulas:

$$
\begin{equation*}
c_{\mathcal{G}_{1}, \mathcal{G}_{6}}(n)=\frac{1}{4} \omega\left(\frac{n}{4}\right)+\frac{3}{4} \sigma_{2}\left(\frac{n}{4}\right)+\frac{9}{4} \sigma_{2}\left(\frac{n}{8}\right), \tag{i}
\end{equation*}
$$

(ii)
$c_{\mathcal{G}_{2}, \mathcal{G}_{6}}(n)=\frac{3}{2}\left(\sigma_{2}\left(\frac{n}{2}\right)+2 \sigma_{2}\left(\frac{n}{4}\right)-3 \sigma_{2}\left(\frac{n}{8}\right)+d_{3}\left(\frac{n}{2}\right)-d_{3}\left(\frac{n}{4}\right)-3 d_{3}\left(\frac{n}{8}\right)+5 d_{3}\left(\frac{n}{16}\right)-2 d_{3}\left(\frac{n}{32}\right)\right)$,

$$
\begin{equation*}
c_{\mathcal{G}_{6}, \mathcal{G}_{6}}(n)=d_{3}(n)-3 d_{3}\left(\frac{n}{2}\right)+3 d_{3}\left(\frac{n}{4}\right)-d_{3}\left(\frac{n}{8}\right) . \tag{iii}
\end{equation*}
$$

Remark. If $n$ is odd then $\mathcal{N} \cong \mathcal{G}_{6}$. If $n \equiv 2 \bmod 4$ then $\mathcal{N} \cong \mathcal{G}_{2}$. Finally, if $4 \mid n$ then $\mathcal{N} \cong \mathcal{G}_{2}$ or $\mathcal{N} \cong \mathcal{G}_{1}$.

Dirichlet generating series for the sequences provided by Theorems 1 and 2 are given in Table 2 in Appendix.

## 2 Preliminaries

In this section we have collected some known statements that will be used later.
Proposition 1. (i) The sublattices of index $n$ in the 2-dimensional lattice $\mathbb{Z}^{2}$ are in one-to-one correspondence with the matrices $\left(\begin{array}{ll}b & c \\ 0 & a\end{array}\right)$, where $a, b>0, a b=n$, $0 \leq c<b$. Consequently, the number of such sublattices is $\sigma_{1}(n)$.
(ii) The sublattices of index $n$ in the 3-dimensional lattice $\mathbb{Z}^{3}$ are in one-to-one correspondence with the integer matrices $\left(\begin{array}{ccc}c & e & f \\ 0 & b & d \\ 0 & d & a\end{array}\right)$, where $a, b, c>0, a b c=n, 0 \leq d<b$ and $0 \leq f, e<c$. Consequently, the number of such sublattices is $\omega(n)$.

For the proof see, for example, (4], Proposition 1).
Corollary 1. Let $\ell: \mathbb{Z}^{2} \mapsto \mathbb{Z}^{2}$ be an automorphism of $\mathbb{Z}^{2}$, given by $\ell(u, v)=(u,-v)$. The sublattices $\Delta$ of index $n$ in the 2-dimensional lattice $\mathbb{Z}^{2}$ such that $\ell(\Delta)=\Delta$ are in one-to-one correspondence with the matrices in the union of the two families of integer matrices, $\left(\begin{array}{ll}b & 0 \\ 0 & a\end{array}\right)$, where $a, b>0, a b=n$, and $\left(\begin{array}{cc}b & a / 2 \\ 0 & a\end{array}\right)$, where $a, b>0, a b=n$ and $a$ is even. Consequently, the number of such sublattices is $\sigma_{0}(n)+\sigma_{0}\left(\frac{n}{2}\right)$.

Corollary 2. Let $\ell: \mathbb{Z}^{3} \mapsto \mathbb{Z}^{3}$ be an automorphism of $\mathbb{Z}^{3}$, given by $\ell(u, v, w)=$ $(u, v,-w)$. Then the number of subgroups $\Delta$ of index $n$ in $\mathbb{Z}^{3}$ such that $\ell(\Delta)=\Delta$ is equal to $\sigma_{2}(n)+3 \sigma_{2}\left(\frac{n}{2}\right)$.

Proof in ([6], Corollary 3).
In the next two propositions we enumerate the subgroups $\Delta$ of index $n$ in $\pi_{1}\left(\mathcal{G}_{2}\right)$ with $\Delta \cong \pi_{1}\left(\mathcal{G}_{2}\right)$ and conjugacy classes of such subgroups. This statements correspond to (4, Proposition 3).
Proposition 2. The subgroups $\Delta$ of index $n$ in $\pi_{1}\left(\mathcal{G}_{2}\right)$ isomorphic to $\pi_{1}\left(\mathcal{G}_{2}\right)$ are in one-to-one correspondence with the triples $(k, H, h)$, where

- $k$ is an odd positive divisor of $n$,
- $H$ is a subgroup of index $\frac{n}{k}$ in $\mathbb{Z}^{2}$,
- $h$ is a coset in $\mathbb{Z}^{2} / H$.

Consequently, the number of the above described subgroups is $s_{\mathcal{G}_{2}, \mathcal{G}_{2}}(n)=\omega(n)-\omega\left(\frac{n}{2}\right)$.
Proposition 3. The conjugacy classes of subgroups $\Delta$ of index $n$ in $\pi_{1}\left(\mathcal{G}_{2}\right)$ isomorphic to $\pi_{1}\left(\mathcal{G}_{2}\right)$ are in one-to-one correspondence with the triples $(k, H, \bar{h})$, where

- $k$ is an odd positive divisor of $n$,
- $H$ is a subgroup of index $\frac{n}{k}$ in $\mathbb{Z}^{2}$,
- $\bar{h}$ is a coset in $\mathbb{Z}^{2} /\langle H,(2,0),(0,2)\rangle$.

Consequently, the number of conjugacy classes of the above described subgroups is $c_{\mathcal{G}_{2}, \mathcal{G}_{2}}(n)=$ $\sigma_{2}(n)+2 \sigma_{2}\left(\frac{n}{2}\right)-3 \sigma_{2}\left(\frac{n}{4}\right)$.

## 3 The structure of the groups $\pi_{1}\left(\mathcal{G}_{2}\right)$ and $\pi_{1}\left(\mathcal{G}_{6}\right)$

The groups $\pi_{1}\left(\mathcal{G}_{1}\right), \pi_{1}\left(\mathcal{G}_{2}\right)$ and $\pi_{1}\left(\mathcal{G}_{6}\right)$ is given by generators and relations in the following way

$$
\begin{align*}
& \pi_{1}\left(\mathcal{G}_{1}\right)=\mathbb{Z}^{3}=\left\langle x, y, z: x y x^{-1} y^{-1}=x z x^{-1} z^{-1}=y z y^{-1} z^{-1}=1\right\rangle, \\
& \pi_{1}\left(\mathcal{G}_{2}\right)=\left\langle x, y, z: x y x^{-1} y^{-1}=1, x^{z}=x^{-1}, y^{z}=y^{-1}\right\rangle,  \tag{3.1}\\
& \pi_{1}\left(\mathcal{G}_{6}\right)=\left\langle x, y, z: x y^{2} x^{-1} y^{2}=y x^{2} y^{-1} x^{2}=x y z=1\right\rangle .
\end{align*}
$$

See 25] or 21.
Remark. The above representation of the group $\pi_{1}\left(\mathcal{G}_{6}\right)$ is indeed symmetric with respect to permutations of $x, y$ and $z$. The relations $x z^{2} x^{-1} z^{2}=y z^{2} y^{-1} z^{2}=z x^{2} z^{-1} x^{2}=$ $z y^{2} z^{-1} y^{2}=1$ follow from given above.

Next proposition provides the canonical form of an element in $\pi_{1}\left(\mathcal{G}_{6}\right)$.
Proposition 4. (i) Each element of $\pi_{1}\left(\mathcal{G}_{6}\right)$ can be represented in the canonical form $g_{i} x^{2 a} y^{2 b} z^{2 c}$, where $g_{i} \in\{1, x, y, z\}$ and $a, b, c$ are some integers.
(ii) The subgroup $\left\langle x^{2}, y^{2}, z^{2}\right\rangle$ is normal in $\pi_{1}\left(\mathcal{G}_{6}\right)$ and isomorphic to $\mathbb{Z}^{3}$.
(iii) The following relations holds:

$$
\begin{align*}
x^{2 a} y^{2 b} z^{2 c} \cdot x & =x \cdot x^{2 a} y^{-2 b} z^{-2 c} \\
x^{2 a} y^{2 b} z^{2 c} \cdot y & =y \cdot x^{-2 a} y^{2 b} z^{-2 c}  \tag{3.2}\\
x^{2 a} y^{2 b} z^{2 c} \cdot z & =z \cdot x^{-2 a} y^{-2 b} z^{2 c}
\end{align*}
$$

(iv) The product $g_{i} g_{j}: g_{i}, g_{j} \in\{1, x, y, z\}$ is given by Table 1.

| $g_{i} g_{j}$ | $g_{j}=1$ | $g_{j}=x$ | $g_{j}=y$ | $g_{j}=z$ |
| :--- | :--- | :--- | :--- | :--- |
| $g_{i}=1$ | 1 | $x$ | $y$ | $z$ |
| $g_{i}=x$ | $x$ | $1 \cdot x^{2}$ | $z \cdot z^{-2}$ | $y \cdot x^{-2} z^{2}$ |
| $g_{i}=y$ | $y$ | $z \cdot x^{2} y^{-2}$ | $1 \cdot y^{2}$ | $x \cdot x^{-2}$ |
| $g_{i}=z$ | $z$ | $y \cdot y^{-2}$ | $x \cdot y^{2} z^{-2}$ | $1 \cdot z^{2}$ |
| Table 1 |  |  |  |  |

(v) The representation in the canonical form $w=g_{i} x^{a} y^{b} z^{c}$ for each element $w \in \pi_{1}\left(\mathcal{G}_{6}\right)$ is unique.

Proof. Items (i-iv) follow routinely from the representation (3.1) of the group $\pi_{1}\left(\mathcal{G}_{6}\right)$. To prove (v) consider the set $G$ of all the expressions $g_{i} x^{2 a} y^{2 b} z^{2 c}$, where $g_{i} \in\{1, x, y, z\}$ and $a, b, c$ are some integers. Define the multiplication by concatenation and further reduction to the described above form through the relations (iii) and (iv). Direct verification shows that $G$ is a group with respect to this operation. Since the relations (iii) and (iv) are derived from the relations of the group $\pi_{1}\left(\mathcal{G}_{6}\right)$, this group is a factor group
of the group $G$. By the other hand, one can verify that the relations of $\pi_{1}\left(\mathcal{G}_{6}\right)$ holds in $G$, thus $G \cong \pi_{1}\left(\mathcal{G}_{6}\right)$. In particular, different canonical representations represent different elements of $\pi_{1}\left(\mathcal{G}_{6}\right)$.

Notations. Denote the subgroup $\left\langle x^{2}, y^{2}, z^{2}\right\rangle \triangleleft \pi_{1}\left(\mathcal{G}_{6}\right)$ by $\Lambda$. Also denote the natural by homomorphism, of factorization $\pi_{1}\left(\mathcal{G}_{6}\right) \rightarrow \pi_{1}\left(\mathcal{G}_{6}\right) / \Lambda$ by $\phi$.

Definition 1. Let $g$ be an element of $\pi_{1}\left(\mathcal{G}_{6}\right)$. In case $g=x^{2 a} y^{2 b} z^{2 c}$ we say that $g$ has exponents $2 a, 2 b, 2 c$ at $x, y, z$ respectively. In case $g=z x^{2 a} y^{2 b} z^{2 c}$ we say the respective exponents are $2 a, 2 b, 2 c+1$. Similarly in cases $g=x \cdot x^{2 a} y^{2 b} z^{2 c}$ and $g=y x^{2 a} y^{2 b} z^{2 c}$. We denote the exponents of $g$ at $x, y, z$ by $\exp _{x}(g), \exp _{y}(g), \exp _{z}(g)$ respectively.

We widely use the following statement, too trivial to be a lemma. Let $g, h$ be some elements and $\exp _{y}(g), \exp _{z}(g), \exp _{y}(h), \exp _{z}(h)$ are even. Than $\exp _{x}(g h)=\exp _{x}(g)+$ $\exp _{x}(h)$.

Note that $\pi_{1}\left(\mathcal{G}_{6}\right) / \Lambda \cong \mathbb{Z}_{2}^{2}$, therefore there are only three possible isomorphism types of a subgroup in $\mathbb{Z}_{2}^{2}$, it is either trivial, or $\mathbb{Z}_{2}$, or $\mathbb{Z}_{2}^{2}$.

Definition 2. Let $\Delta$ be a subgroup of finite index in $\pi_{1}\left(\mathcal{G}_{6}\right)$. In case $\phi(\Delta)=1$ by $X_{\Delta}, Y_{\Delta}, Z_{\Delta}$ we refer to an arbitrary triple of generators of $\Delta$. If $\phi(\Delta)=\{1, x\}$ by $Z_{\Delta}$ denote an arbitrary element of $\Delta$ with the minimal positive odd exponent at $x$, and by $X_{\Delta}, Y_{\Delta}$ denote an arbitrary pair of generators of $\Delta \cap\left\langle x^{2}, y^{2}\right\rangle$. Similarly, in case $\phi(\Delta)=\{1, y\}$ and $\phi(\Delta)=\{1, z\} \quad\left(Z_{\Delta}\right.$ denote an element with the minimal positive odd exponent at $y$ and $z$ respectively). Finely, in case $\phi(\Delta)=\{1, x, y, z\}$ by $X_{\Delta}, Y_{\Delta}, Z_{\Delta}$ denote an arbitrary element with minimal positive odd exponent at $x, y$, $z$ respectively.

Proposition 5. Let $\Delta$ be a subgroup of finite index in $\pi_{1}\left(\mathcal{G}_{6}\right)$. Then $\Delta$ has one of the following three isomorphism types, defined by $|\phi(\Delta)|$. Subgroup $\Delta$ is isomorphic to $\mathbb{Z}^{3}$, $\pi_{1}\left(\mathcal{G}_{2}\right)$ and $\pi_{1}\left(\mathcal{G}_{6}\right)$ in case $|\phi(\Delta)|=1,|\phi(\Delta)|=2$ and $|\phi(\Delta)|=4$ respectively. In all cases $\Delta$ is generated by elements $X_{\Delta}, Y_{\Delta}, Z_{\Delta}$.

Proof. In case $|\phi(\Delta)|=1$ the triple $X_{\Delta}, Y_{\Delta}, Z_{\Delta}$ generates $\Delta$ by definition. Also, $\Delta$ is a subgroup of $\Lambda \cong \mathbb{Z}^{3}$, thus $\Delta$ is a free abelian group. Since $\Delta$ has finite index in $\Lambda$, we have $\Delta \cong \mathbb{Z}^{3}$.

In case $|\phi(\Delta)|=2$ without loss of generality assume that $\phi(\Delta)=\{\phi(1), \phi(x)\}$. Denote the exponent of $Z_{\Delta}$ at $x$ by $m$. Then for each $g \in \Delta$ its exponent at $x$ is a multiple of $m$. Otherwise multiplying either $g$ or $g^{-1}$ by the convenient power of $Z_{\Delta}$ we get an element of $\Delta$ with an odd exponent at $x$ strictly between 0 and $m$, which is the contradiction with the definition of $Z_{\Delta}$.

To prove that $X_{\Delta}, Y_{\Delta}, Z_{\Delta}$ generate $\Delta$ consider an arbitrary element $g \in \Delta$. Since its exponent at $x$ is divisible by $m, g Z_{\Delta}^{k} \in \Delta \cap\left\langle y^{2}, z^{2}\right\rangle$ for some $k$. Further, $\Delta \cap\left\langle y^{2}, z^{2}\right\rangle$ is generated by $X_{\Delta}, Y_{\Delta}$ by virtue of their definition. We claim that different expressions of the form $X_{\Delta}^{a} Y_{\Delta}^{b} Z_{\Delta}^{c}$ represent different elements $g \in \Delta$. Indeed, the exponent of $g$ at $x$ uniquely determines $c$, and $\Delta \cap\left\langle y^{2}, z^{2}\right\rangle \cong \mathbb{Z}^{2}$, so different pairs ( $a, b$ ) provide different elements $X_{\Delta}^{a} Y_{\Delta}^{b}$.

Note that the elements $X_{\Delta}, Y_{\Delta}, Z_{\Delta}$ yield the relations of the group $\pi_{1}\left(\mathcal{G}_{2}\right)$ hold for $x, y, z$, so we build the isomorphism $\pi_{1}\left(\mathcal{G}_{2}\right) \rightarrow \Delta$, given by $x \mapsto X_{\Delta}, y \mapsto Y_{\Delta}, z \mapsto Z_{\Delta}$.

In case $|\phi(\Delta)|=4$ we set: $X_{\Delta}=x^{m} y^{2 r} z^{2 s}, Y_{\Delta}=y^{k} x^{2 t} z^{2 u}$ and $Z_{\Delta}=z^{\ell} x^{2 v} y^{2 w}$. Here $m, k, \ell, r, s, t, u, v, w$ are integers, moreover $m, k, \ell$ are odd positives.

To prove that $\Delta$ is generated by $X_{\Delta}, Y_{\Delta}, Z_{\Delta}$ do the following.
Lemma 1. If for some element $g \in \Delta$ the numbers $\exp _{y}(g)$ and $\exp _{z}(g)$ are even, then $m \mid \exp _{x}(g)$.

The proof is similar to the case $|\phi(\Delta)|=2$. Analogous statements holds for any permutation of $x, y, z$.

Resume to the prove of Proposition 5. Note that $X_{\Delta}^{2}=x^{2 m}, Y_{\Delta}^{2}=y^{2 k}$ and $Z_{\Delta}^{2}=x^{2 \ell}$. By Lemma 1 if for some $g \in \Delta$ all three numbers $\exp _{x}(g), \exp _{y}(g), \exp _{z}(g)$ are even, then they are divisible by $2 m, 2 k$ and $2 \ell$ respectively; thus $g$ can be expressed through $X_{\Delta}^{2}, Y_{\Delta}^{2}, Z_{\Delta}^{2}$. If $g \in \Delta$ has one exponent odd, without loss of generality $\exp _{x}(g)$ is odd, then $g X_{\Delta}$ has all exponents even.

If $g \in \Delta$ has one exponent odd, without loss of generality we assume that $\exp _{x}(g)$ is odd, then $g X_{\Delta}$ has all exponents even.

To prove the isomorphism part note that the element $X_{\Delta} Y_{\Delta} Z_{\Delta}=x^{m-1-2 t+2 v} y^{1-k+2 w-2 r} z^{\ell-1+2 u-2 s}$ has all three exponents even, thus Lemma 1 implies $2 m|m-1-2 y+2 v, 2 k| 1-k+2 w-2 r$ and $2 \ell \mid \ell-1+2 u-2 s$. So, by replacing $X_{\Delta} \mapsto X_{\Delta} Y_{\Delta}^{2 i}$ for some integer $i$ and doing similar replacements for permuted generators, one can achieve that $X_{\Delta} Y_{\Delta} Z_{\Delta}=1$ and the property of $X_{\Delta}, Y_{\Delta}, Z_{\Delta}$ given in Definition 2 holds.

Now note that the elements $X_{\Delta}, Y_{\Delta}, Z_{\Delta}$ yield defining relations of the group $\pi_{1}\left(\mathcal{G}_{6}\right)$. Then the mapping $x \mapsto X_{\Delta}, y \mapsto Y_{\Delta}, z \mapsto Z_{\Delta}$ spawns the epimorphism $\psi: \pi_{1}\left(\mathcal{G}_{6}\right) \rightarrow \Delta$. We are going to prove that this epimorphism is indeed an isomorphism.

Each element $g$ of $\Delta$ can be represented in the form $g=g_{i} X_{\Delta}^{2 a} Y_{\Delta}^{2 b} Z_{\Delta}^{2 c}$, where $g_{i} \in\left\{1, X_{\Delta}, Y_{\Delta}, Z_{\Delta}\right\}$; whence such representation is possible in $\pi_{1}\left(\mathcal{G}_{6}\right)$. So it is sufficient to prove that the above representation is unique for each $g \in \Delta$. Assume the contrary, for some element $g$ there are two different representations $g=g_{i} X_{\Delta}^{2 a} Y_{\Delta}^{2 b} Z_{\Delta}^{2 c}=$ $g_{i}^{\prime} X_{\Delta}^{2 a^{\prime}} Y_{\Delta}^{2 b^{\prime}} Z_{\Delta}^{2 c^{\prime}}$.

Note that for arbitrary $g \in \Delta$ hold $\left(x^{2}\right)^{g}=x^{ \pm 2},\left(y^{2}\right)^{g}=y^{ \pm 2}$ and $\left(z^{2}\right)^{g}=z^{ \pm 2}$, and the triple of signs in the exponents is solely determined by $g_{i}: 1 \mapsto(+,+,+)$, $x \mapsto(+,-,-), y \mapsto(-,+,-)$ and $z \mapsto(-,-,+)$. Thus $g_{i} X_{\Delta}^{2 a} Y_{\Delta}^{2 b} Z_{\Delta}^{2 c}=g_{i}^{\prime} X_{\Delta}^{2 a^{\prime}} Y_{\Delta}^{2 b^{\prime}} Z_{\Delta}^{2 c^{\prime}}$ implies $g_{i}=g_{i}^{\prime}$. Then $X_{\Delta}^{2 a} Y_{\Delta}^{2 b} Z_{\Delta}^{2 c}=X_{\Delta}^{2 a^{\prime}} Y_{\Delta}^{2 b^{\prime}} Z_{\Delta}^{2 c^{\prime}}$, or $x^{2 m\left(a-a^{\prime}\right)} y^{2 k\left(b-b^{\prime}\right)} z^{2 \ell\left(c-c^{\prime}\right)}=1$, which is a contradiction with Proposition 4 (iv).

## 4 Proof of Theorem 1 and Theorem 2

The isomorphism types of finite index subgroups are already provided by Proposition 5 , So we will consider isomorphism types separately in order to prove respective items of both theorems.

### 4.1 Case $\Delta \cong \mathbb{Z}^{3}$

Recall that $\Lambda=\left\langle x^{2}, y^{2}, z^{2}\right\rangle$. By Proposition 5 each subgroup $\Delta$ of index $n$ in $\pi_{1}\left(\mathcal{G}_{6}\right)$ with $\Delta \cong \mathbb{Z}^{3}$ has $\phi(\Delta)=\{1\}$, that is $\Delta \leqslant \Lambda$. Since $\left|\pi_{1}\left(\mathcal{G}_{6}\right): \Lambda\right|=4$, get $|\Lambda: \Delta|=\frac{n}{4}$. Applying Proposition 1 one gets

$$
s_{\mathcal{G}_{1}, \mathcal{G}_{6}}(n)=\omega\left(\frac{n}{4}\right) .
$$

Now we proceed to enumeration of the conjugacy classes of subgroups. Since $\Lambda$ is abelian, the group $\pi_{1}\left(\mathcal{G}_{6}\right)$ acts by conjugation on subgroups of $\Lambda$ as $\pi_{1}\left(\mathcal{G}_{6}\right) / \Lambda \cong \mathbb{Z}_{2}^{2}$. Thus each conjugacy class consists of one, two or four subgroups.

Definition 3. By $\mathcal{M}_{1}$ denote the family of all normal subgroups $\Delta$, by $\mathcal{M}_{2}$ and $\mathcal{M}_{4}$ denote the families of subgroups $\Delta$, which belong to conjugacy classes, containing two and four subgroups respectively. Also, by $\mathcal{M}_{x}$ denote the family of subgroups $\Delta$ such that $\Delta^{x}=\Delta . \mathcal{M}_{y}$ and $\mathcal{M}_{z}$ are defined similar way.

Each $\Delta \in \mathcal{M}_{1}$ belongs to all three of $\mathcal{M}_{x}, \mathcal{M}_{y}, \mathcal{M}_{z}$; while each $\Delta \in \mathcal{M}_{2}$ belongs to exactly one of $\mathcal{M}_{x}, \mathcal{M}_{y}, \mathcal{M}_{z}$. Needless to say that $\Delta \in \mathcal{M}_{4}$ does not belongs to any of $\mathcal{M}_{x}, \mathcal{M}_{y}, \mathcal{M}_{z}$. So $3\left|\mathcal{M}_{1}\right|+\left|\mathcal{M}_{2}\right|=\left|\mathcal{M}_{x}\right|+\left|\mathcal{M}_{y}\right|+\left|\mathcal{M}_{z}\right|$. Thus

$$
\begin{align*}
& c_{\mathcal{G}_{1}, \mathcal{G}_{6}}(n)=\left|\mathcal{M}_{1}\right|+\frac{\left|\mathcal{M}_{2}\right|}{2}+\frac{\left|\mathcal{M}_{4}\right|}{4}=\frac{\left|\mathcal{M}_{1}\right|+\left|\mathcal{M}_{2}\right|+\left|\mathcal{M}_{4}\right|}{4}+\frac{3\left|\mathcal{M}_{1}\right|+\left|\mathcal{M}_{2}\right|}{4}  \tag{4.3}\\
& =\frac{s_{\mathcal{G}_{1}, \mathcal{G}_{6}}(n)}{4}+\frac{\left|\mathcal{M}_{x}\right|+\left|\mathcal{M}_{y}\right|+\left|\mathcal{M}_{z}\right|}{4} .
\end{align*}
$$

Since in a suitable basis each of $A d_{x}, A d_{y}, A d_{z}$ takes the form $(a, b, c) \mapsto(-a, b, c)$, Corollary 2 claims $\left|\mathcal{M}_{x}\right|=\left|\mathcal{M}_{y}\right|=\left|\mathcal{M}_{z}\right|=\sigma_{2}\left(\frac{n}{4}\right)+3 \sigma_{2}\left(\frac{n}{8}\right)$. Thus

$$
c_{G_{1}, G_{6}}(n)=\frac{1}{4} \omega\left(\frac{n}{4}\right)+\frac{3}{4} \sigma_{2}\left(\frac{n}{4}\right)+\frac{9}{4} \sigma_{2}\left(\frac{n}{8}\right) .
$$

By definition, $\left|\mathcal{M}_{1}\right|$ is the number of normal subgroups of index $n$ in $\pi_{1}\left(\mathcal{G}_{6}\right)$ isomorphic to $\mathbb{Z}^{3}$. So it is interesting in itself. It is explicitly calculated in section 5.

### 4.2 Case $\Delta \cong \pi_{1}\left(\mathcal{G}_{2}\right)$

By Proposition 5 each subgroup $\Delta$ of index $n$ in $\pi_{1}\left(\mathcal{G}_{6}\right)$ with $\Delta \cong \pi_{1}\left(\mathcal{G}_{2}\right)$ has $|\phi(\Delta)|=2$. In other words, holds one of the inclusions $\Delta \leqslant\left\langle x, y^{2}, z^{2}\right\rangle=\Gamma_{x}, \Delta \leqslant\left\langle y, x^{2}, z^{2}\right\rangle=\Gamma_{y}$, $\Delta \leqslant\left\langle z, x^{2}, y^{2}\right\rangle=\Gamma_{z}$. Since the above groups have index 2 in $\pi_{1}\left(\mathcal{G}_{6}\right)$, subgroup $\Delta$ has index $\frac{n}{2}$ in the respective subgroup. Further, $\Delta$ belongs to just one of $\Gamma_{x}, \Gamma_{y}, \Gamma_{z}$ hence the intersection of each two of them is the abelian group $\Lambda=\left\langle x^{2}, y^{2}, x^{2}\right\rangle$.

Also, subgroups $\Gamma_{x}, \Gamma_{y}, \Gamma_{z}$ are permutable by some outer automorphism of $\pi_{1}\left(\mathcal{G}_{6}\right)$, which permutes $x, y, z$. Thus it is sufficient to enumerate the subgroups of $\Gamma=\Gamma_{x}$. Further during this subsection $\Delta$ denotes a subgroup of index $\frac{n}{2}$ in $\Gamma$ isomorphic to $\pi_{1}\left(\mathcal{G}_{2}\right)$. Pay attention, in spite of $\Delta \cong \Gamma, \Delta$ is a non-trivial subgroup in $\Gamma$.

Since $\Gamma \cong \pi_{1}\left(\mathcal{G}_{2}\right)$, the number of the above subgroups $\Delta$ in $\Gamma$ is provided by Proposition 2, thus

$$
s_{\mathcal{G}_{2}, \mathcal{G}_{6}}(n)=3 s_{\mathcal{G}_{2}, \mathcal{G}_{2}}\left(\frac{n}{2}\right)=3 \omega\left(\frac{n}{2}\right)-3 \omega\left(\frac{n}{4}\right) .
$$

To enumerate conjugacy classes we need one more definition.
Definition 4. Consider a subgroup $\Delta$. The set of subgroups $\left\{\Delta^{\gamma} \mid \gamma \in \Gamma\right\}$ we call a partial conjugacy class $\Delta^{\Gamma}$.

An enumeration of partial conjugacy classes of subgroups $\Delta$ is given by Proposition 3. To enumerate conjugacy classes note that $\pi_{1}\left(\mathcal{G}_{6}\right)=\Gamma \cup y \Gamma$, so $\Gamma$ has index 2 in $\pi_{1}\left(\mathcal{G}_{6}\right)$. Consequently $\Gamma$ is normal in $\pi_{1}\left(\mathcal{G}_{6}\right)$. Thus for each $\Delta$ its conjugacy class consists of one or two partial conjugacy classes $\Delta^{\Gamma}$ and $\left(\Delta^{\Gamma}\right)^{y}$ depending upon whether partial conjugacy classes $\Delta^{\Gamma}$ and $\left(\Delta^{\Gamma}\right)^{y}$ coincide or not.

Notation. By $\mathcal{K}_{1}$ denote the set of partial conjugacy classes $\Delta^{\Gamma}$, such that the equality $\Delta^{\Gamma}=\left(\Delta^{\Gamma}\right)^{y}$ holds. By $\mathcal{K}_{2}$ denote the set of partial conjugacy classes $\Delta^{\Gamma}$ with $\Delta^{\Gamma} \neq\left(\Delta^{\Gamma}\right)^{y}$.

In the introduced notation

$$
\begin{equation*}
c_{\mathcal{G}_{2}, \mathcal{G}_{6}}(n)=3\left(\left|\mathcal{K}_{1}\right|+\frac{\left|\mathcal{K}_{2}\right|}{2}\right)=3\left(\frac{\left|\mathcal{K}_{1}\right|+\left|\mathcal{K}_{2}\right|}{2}+\frac{\left|\mathcal{K}_{1}\right|}{2}\right) . \tag{4.4}
\end{equation*}
$$

Proposition 3 implies $\left|\mathcal{K}_{1}\right|+\left|\mathcal{K}_{2}\right|=\sigma_{2}\left(\frac{n}{2}\right)+2 \sigma_{2}\left(\frac{n}{4}\right)-3 \sigma_{2}\left(\frac{n}{8}\right)$. All that's left is to calculate $\left|\mathcal{K}_{1}\right|$, this is done in Lemma 3. First we need the following auxiliary statement.

Lemma 2. The following identity holds

$$
d_{3}(n)-3 d_{3}\left(\frac{n}{2}\right)+3 d_{3}\left(\frac{n}{4}\right)-d_{3}\left(\frac{n}{8}\right)= \begin{cases}d_{3}(n) & \text { if } n \text { is odd } \\ 0 & \text { if } n \text { is even }\end{cases}
$$

Proof. Consider all factorizations $n=a b c$ and use inclusion-exclusion formula for all triples of parities of $a, b, c$.

Proof. Recall that by definition $d_{3}(n)$ is the number of ordered positive integer factorizations $a b c=n$. Then in case of an odd $n$ equality $d_{3}(n)-3 d_{3}\left(\frac{n}{2}\right)+3 d_{3}\left(\frac{n}{4}\right)-d_{3}\left(\frac{n}{8}\right)=d_{3}(n)$ holds because terms $d_{3}\left(\frac{n}{2}\right), d_{3}\left(\frac{n}{4}\right), d_{3}\left(\frac{n}{8}\right)$ vanish.

Assume $n$ is even. Note that positive integer factorizations $a b c=n$ with even $a$ are enumerated by $d_{3}\left(\frac{n}{2}\right)$. Indeed, they bijectively correspond to factorizations $\frac{a}{2} b c=\frac{n}{2}$. Same holds for factorizations $a b c=n$ with even $b$, and factorizations $a b c=n$ with even $c$. Similarly, factorizations $a b c=n$ with even $a$ and $b$ simultaneously are enumerated by $d_{3}\left(\frac{n}{4}\right)$. The same holds for permuted $a, b, c$. Finally, factorizations $a b c=n$ with even $a, b, c$ are enumerated by $d_{3}\left(\frac{n}{8}\right)$. Applying inclusion-exclusion formula we get that $d_{3}(n)-3 d_{3}\left(\frac{n}{2}\right)+3 d_{3}\left(\frac{n}{4}\right)-d_{3}\left(\frac{n}{8}\right)$ enumerates factorizations $a b c=n$, where all three $a, b, c$ are odd. Since $n$ is even such factorization is impossible, the above expression vanishes.

Next lemma finally calculates $\left|\mathcal{K}_{1}\right|$.

## Lemma 3.

$$
\left|\mathcal{K}_{1}\right|=d_{3}(n / 2)-d_{3}(n / 4)-3 d_{3}(n / 8)+5 d_{3}(n / 16)-2 d_{3}(n / 32) .
$$

Remark. The above formula for $\left|\mathcal{K}_{1}\right|$ looks horribly; actually it means the following. Let $n=2^{q} r$ where $2 \nmid s$. Then

- if $q=0$ then $\left|\mathcal{K}_{1}\right|=0$;
- if $q=1$ then $\left|\mathcal{K}_{1}\right|=d_{3}(r)$;
- if $q=2$ then $\left|\mathcal{K}_{1}\right|=2 d_{3}(r)$;
- if $q>2$ then $\left|\mathcal{K}_{1}\right|=0$.

Proof. Let $\Delta$ be a subgroup of even index $n$ in $\pi_{1}\left(\mathcal{G}_{6}\right)$, such that $\Delta \cong \pi_{1}\left(\mathcal{G}_{2}\right)$ and $\left(\Delta^{\Lambda}\right)^{y}=\Delta^{\Lambda}$.

Let $Z_{\Delta}=x^{k} y^{2 s} z^{2 t}$, where $k$ is an odd positive, and $k \left\lvert\, \frac{n}{2}\right.$. We set $H_{\Delta}=\Delta \cap\left\langle y^{2}, z^{2}\right\rangle$. Further we identify $\left\langle y^{2}, z^{2}\right\rangle$ with $\mathbb{Z}^{2}$, that is we address an element $y^{2 a} z^{2 b}$ as $(a, b)$.

Proposition 3 implies that the condition $\left(\Delta^{\Lambda}\right)^{y}=\Delta^{\Lambda}$ means that the groups $\Delta$ and $\Delta^{y}$ have the same triples $(k, H, \bar{h})$ of invariants. Consider two conditions: (i) the groups $\Delta$ and $\Delta^{y}$ share the same invariant $H$, (ii) the groups $\Delta$ and $\Delta^{y}$ share the same invariant $\bar{h}$.

Condition (i) means that $\Delta^{y} \cap\left\langle y^{2}, z^{2}\right\rangle=\Delta \cap\left\langle y^{2}, z^{2}\right\rangle$, i.e. $A d_{y}(H)=H$. Since the action of $A d_{y}$ on $\left\langle y^{2}, z^{2}\right\rangle$ is given by $(u, v) \mapsto(u,-v)$, Corollary 1 claims that either $H=$ $\langle(a, 0),(0, b)\rangle$ or $H=\langle(a, 0),(a / 2, b)\rangle$, where $a, b>0$ and $a b=\frac{n}{2 k}$; additionally $a$ is even in the second case. We say that a subgroups $H$ is of the first type if $H=\langle(a, 0),(0, b)\rangle$, likewise $H$ is of the second type if $H=\langle(a, 0),(a / 2, b)\rangle$.

Consider condition (ii). Note that $\left(Z_{\Delta}\right)^{y}=\left(x^{k} y^{2 s} z^{2 t}\right)^{y}=x^{-k} y^{2 s-2} z^{-2 t-2}$. Thus the element $\left(Z_{\Delta}^{y}\right)^{-1} \in \Delta^{y}$ satisfies the definition of the element $Z_{\Delta^{y}}$. So the corresponding value of $\bar{h}$ is $(s-1,-t-1)$. Then the condition (ii) is reformulated as $(s, t) \in(s-$ $1,-t-1)+\langle H,(2,0),(0,2)\rangle$, or equivalently $(1,1) \in\langle H,(2,0),(0,2)\rangle$.

In case $H=\langle(a, 0),(0, b)\rangle$ this implies $a$ and $b$ are odd, thus $\frac{n}{2}=k a b$ is odd ( $k$ is odd due to Proposition (2). Vice versa, in case $\frac{n}{2}$ is odd, an arbitrary positive factorization $\frac{n}{2}=k a b$ spawns the unique group $H=\langle(a, 0),(0, b)\rangle$, that defines the unique coset $\stackrel{\mathbb{Z}}{ }^{2} /\langle H,(2,0),(0,2)\rangle$, so we get the unique partial conjugacy class $\Delta^{\Lambda}$ with $\Delta^{\Lambda}=\left(\Delta^{\Lambda}\right)^{y}$ corresponding to each factorization $\frac{n}{2}=k a b$. Thus there are $d_{3}\left(\frac{n}{2}\right)$ conjugacy classes of the first type if $\frac{n}{2}$ is odd. Then by Lemma 2 the first type provides $d_{3}\left(\frac{n}{2}\right)-3 d_{3}\left(\frac{n}{4}\right)+$ $3 d_{3}\left(\frac{n}{8}\right)-d_{3}\left(\frac{n}{16}\right)$ partial conjugacy classes in $\mathcal{K}_{1}$.

In case $H=\langle(a, 0),(a / 2, b)\rangle$ condition $(1,1) \in\langle H,(2,0),(0,2)\rangle$ implies $b$ and $\frac{a}{2}$ are odd. Then $\left|\mathbb{Z}^{2} /\langle(a, 0),(a / 2, b),(2,0),(0,2)\rangle\right|=2$. Vice versa, if $\frac{n}{2}$ is even but not divisible by 4 , then each factorization $\frac{n}{2}=k a b$, where $k, b$ are odd provides the unique subgroup $H$ of the second type. In turn, each subgroup $H$ provides two cosets $\bar{h}$ because $\left|\mathbb{Z}^{2} /\langle H,(2,0),(0,2)\rangle\right|=2$. Thus if $\frac{n}{4}$ is odd there are $2 d_{3}\left(\frac{n}{4}\right)$ partial conjugacy classes of the second type in $\mathcal{K}_{1}$, again by Lemma 2 this amount is equal to $2 d_{3}\left(\frac{n}{4}\right)-6 d_{3}\left(\frac{n}{8}\right)+$ $6 d_{3}\left(\frac{n}{16}\right)-2 d_{3}\left(\frac{n}{32}\right)$.

Summing up one gets $\left|\mathcal{K}_{1}\right|=d_{3}(n / 2)-d_{3}(n / 4)-3 d_{3}(n / 8)+5 d_{3}(n / 16)-2 d_{3}(n / 32)$.
Substituting the result of Lemma 3 into equation (4.4) we get

$$
\begin{aligned}
c_{\mathcal{G}_{2}, \mathcal{G}_{6}}(n) & =3\left(\frac{\left|\mathcal{K}_{1}\right|+\left|\mathcal{K}_{2}\right|}{2}+\frac{\left|\mathcal{K}_{1}\right|}{2}\right) \\
& =\frac{3}{2}\left(\sigma_{2}\left(\frac{n}{2}\right)+2 \sigma_{2}\left(\frac{n}{4}\right)-3 \sigma_{2}\left(\frac{n}{8}\right)+d_{3}\left(\frac{n}{2}\right)-d_{3}\left(\frac{n}{4}\right)-3 d_{3}\left(\frac{n}{8}\right)+5 d_{3}\left(\frac{n}{16}\right)-2 d_{3}\left(\frac{n}{32}\right)\right) .
\end{aligned}
$$

### 4.3 Case $\Delta \cong \pi_{1}\left(\mathcal{G}_{6}\right)$

We claim that the following two propositions holds.
Notation. Given integers $m, n$ with $n>0$, by $[m]_{n}$ denote the integer number, defined by $0 \leq[m]_{n}<n$ and $m \equiv[m]_{n} \bmod n$.

Proposition 6. The subgroups $\Delta$ of index $n$ in $\pi_{1}\left(\mathcal{G}_{6}\right)$ isomorphic to $\pi_{1}\left(\mathcal{G}_{6}\right)$ are in one-to-one correspondence with the 6-plets $(k, \ell, m, u, v, w), 0 \leq v<m, 0 \leq u<\ell, 0 \leq w<$ $k$ where $k, \ell, m$ are odd positives and $k \ell m=n$. Moreover, a subgroup $\Delta$ is generated by elements $X_{\Delta}=x^{m} y^{[1-k+2 w]_{2 k}} z^{[\ell-1+2 u]_{2 \ell}}, Y_{\Delta}=y^{k} x^{[m-1+2 v]_{2 m}} z^{2 u}$ and $Z_{\Delta}=z^{\ell} x^{2 v} y^{2 w}$.

Proof. The correspondence from the set of subgroups $\Delta$ onto the set of 6 -plets of the above form is built in the proof of Proposition 5.

To build the back correspondence consider the subgroup generated by $X_{\Delta}, Y_{\Delta}, Z_{\Delta}$. Consider the following set

$$
\left\langle x^{2 m}, y^{2 k}, z^{2 \ell}\right\rangle \cup X_{\Delta}\left\langle x^{2 m}, y^{2 k}, z^{2 \ell}\right\rangle \cup Y_{\Delta}\left\langle x^{2 m}, y^{2 k}, z^{2 \ell}\right\rangle \cup Z_{\Delta}\left\langle x^{2 m}, y^{2 k}, z^{2 \ell}\right\rangle
$$

Direct verification shows that it is the subgroup. That is, the group $\left\langle X_{\Delta}, Y_{\Delta}, Z_{\Delta}\right\rangle$ is a subgroup of index $k \ell m=n$ in $\pi_{1}\left(\mathcal{G}_{6}\right)$, isomorphic to $\pi_{1}\left(\mathcal{G}_{6}\right)$ itself, by virtue of Proposition 5.

Proposition 7. The conjugacy classes of subgroups $\Delta$ of index $n$ in $\pi_{1}\left(\mathcal{G}_{6}\right)$ isomorphic to $\pi_{1}\left(\mathcal{G}_{6}\right)$ are in one-to-one correspondence with the triples $(k, \ell, m)$, where $k, \ell, m$ are odd positive integers and $k l m=n$.

Proof. By Proposition 6, a subgroup $\Delta$ of the above type is defined by the 6 -plet $(k, \ell, m, u, v, w)$. Note that the conjugation with the elements $x^{2}, y^{2}, z^{2}$ acts on the above 6 -plets in the following way:

$$
\begin{aligned}
& A d_{x^{2}}:(k, \ell, m, u, v, w) \mapsto\left(k, \ell, m, u,[v-2]_{m}, w\right), \\
& A d_{y^{2}}:(k, \ell, m, u, v, w) \mapsto\left(k, \ell, m, u, v,[w-2]_{k}\right) \\
& A d_{z^{2}}:(k, \ell, m, u, v, w) \mapsto\left(k, \ell, m,[u-2]_{\ell}, v, w\right) .
\end{aligned}
$$

Thus each two subgroups with the same triple $(k, \ell, m)$ are conjugated by a suitable element of $\left\langle x^{2}, y^{2}, z^{2}\right\rangle$. Obviously the conjugation with any element can not change the triple $(k, \ell, m)$.

## Corollary 3.

$$
\begin{gathered}
s_{\mathcal{G}_{6}, \mathcal{G}_{6}}(n)=n\left(d_{3}(n)-3 d_{3}\left(\frac{n}{2}\right)+3 d_{3}\left(\frac{n}{4}\right)-d_{3}\left(\frac{n}{8}\right)\right), \\
c_{\mathcal{G}_{6}, \mathcal{G}_{6}}(n)=d_{3}(n)-3 d_{3}\left(\frac{n}{2}\right)+3 d_{3}\left(\frac{n}{4}\right)-d_{3}\left(\frac{n}{8}\right) .
\end{gathered}
$$

Proof. To get the second formula use Lemma 2. To proceed to the first formula note that for any triple $(k, \ell, m)$ there are $m$ choices of $v, k$ choices of $w$ and $\ell$ choices of $u$. By the second formula there exist $d_{3}(n)-3 d_{3}\left(\frac{n}{2}\right)+3 d_{3}\left(\frac{n}{4}\right)-d_{3}\left(\frac{n}{8}\right)$ triples $(k, \ell, m)$, each corresponds to exactly $n$ different 6 -plets $(k, \ell, m, u, v, w)$.

## 5 Additional Notes

The purpose of this section is to enumerate the normal subgroups $\Delta$ of index $n$ in $\pi_{1}\left(\mathcal{G}_{6}\right)$, such that $\Delta \cong \mathbb{Z}^{3}$. In the notations of Section 4.1 the following holds.

Proposition 8. The number of normal subgroups of index $n$ in $\pi_{1}\left(\mathcal{G}_{6}\right)$, isomorphic to $\mathbb{Z}^{3}$ is given by the formula

$$
\left|\mathcal{M}_{1}\right|=d_{3}(n / 4)+4 d_{3}(n / 8)+d_{3}(n / 16)+2 d_{3}(n / 32) .
$$

Proof. As it was shown above, a subgroup $\Delta$ of described type is a subgroup of index $\frac{n}{4}$ in $\Lambda$, so we use Proposition 11. The matrix $\left(\begin{array}{lll}a & d & f \\ 0 & b & e \\ 0 & 0 & c\end{array}\right)$ determines a normal subgroup if and only if $(2 d, 0,0) \in\langle(a, 0,0)\rangle$ and $(2 f, 2 e, 0) \stackrel{0}{\in}\langle(a, 0,0),(d, b, 0)\rangle$. Thus we have to find the number of integer matrixes among the following eight: $\left(\begin{array}{lll}a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c\end{array}\right),\left(\begin{array}{ccc}a & 0 & a / 2 \\ 0 & b & 0 \\ 0 & 0 & c\end{array}\right),\left(\begin{array}{ccc}a & a / 2 & 0 \\ 0 & b & 0 \\ 0 & 0 & c\end{array}\right)$, $\left(\begin{array}{ccc}a & a / 2 & a / 2 \\ 0 & b & 0 \\ 0 & 0 & c\end{array}\right),\left(\begin{array}{ccc}a & 0 & 0 \\ 0 & b & b / 2 \\ 0 & 0 & c\end{array}\right),\left(\begin{array}{ccc}a & 0 & a / 2 \\ 0 & b & b / 2 \\ 0 & 0 & c\end{array}\right),\left(\begin{array}{cccc}a & a / 2 & a / 4 \\ 0 & b & b / 2 \\ 0 & 0 & c\end{array}\right),\left(\begin{array}{ccc}a & a / 2 & 3 a / 4 \\ 0 & b & b / 2 \\ 0 & 0 & c\end{array}\right)$. The first matrix is always integer, that is appears $d_{3}\left(\frac{n}{4}\right)$ times, once in each factorization of the type $a b c=\frac{n}{4}$. The next three matrices are integer if $a$ is even, that is they appear in $d_{3}\left(\frac{n}{8}\right)$ factorizations $2 a b c=$ $\frac{n}{4}$. Analogously the fifth matrix is integer if $b$ is even, so this matrix is counted $d_{3}\left(\frac{n}{8}\right)$ times. The sixth matrix is integer if $a$ and $b$ are both even, it is counted $d_{3}\left(\frac{n}{16}\right)$ times. The seventh and eighth matrices are integer if $4 \mid a$ and $b$ is even, they are counted $d_{3}\left(\frac{n}{32}\right)$ times. So

$$
\left|\mathcal{M}_{1}\right|=d_{3}(n / 4)+4 d_{3}(n / 8)+d_{3}(n / 16)+2 d_{3}(n / 32) .
$$

Remark. The respective Dirichlet generating function is $2^{-2 s}\left(1+4 \cdot 2^{-s}+2^{-2 s}+\right.$ $\left.2 \cdot 2^{-3 s}\right) \zeta^{3}(s)$, see Appendix for details.

Proposition 9. The number of normal subgroups $\Delta$ of index $n$ in $\pi_{1}\left(\mathcal{G}_{6}\right)$ isomorphic to $\pi_{1}\left(\mathcal{G}_{2}\right)$ equals to 3 in case $n$ is of the form $4 m+2 ; 6$ in case $n$ is of the form $8 m+4 ; 0$ in all other cases.

Proof. In this proof we follow notations and overall ideas of Section 4.2 (see first two paragraphs). So it is sufficient to enumerate the subgroups $\Delta$ in $\Gamma_{x}$ of the type considered in Proposition 9. Proposition 2 claims that a subgroup $\Delta$ of the above type is uniquely defined by a triple $(k, H, h)$, while Proposition 3 describes the transformations of such triple under conjugation of the group $\Delta$ with an element $g \in \Gamma_{x}$. Similarly, the proof of Lemma 3 describes the transformation of triple $(k, H, h)$ induced by the conjugation of the group $\Delta$ by an element $y$. Since $\pi_{1}\left(\mathcal{G}_{6}\right)=\Gamma_{x} \cup y \Gamma_{x}$, each conjugation can be achieved as a composition of described above.

Summarizing, we get that the invariant $h$ is preserved by any conjugation if and only if $(2,0),(0,2),(1,1) \in H$. This holds for just two subgroups $H$ having index 1 and 2 in $\left\langle y^{2}, z^{2}\right\rangle \cong \mathbb{Z}^{2}$ respectively. Both subgroups are normal in $\mathcal{G}_{6}$, thus in both cases $H$ is automatically preserved by any conjugation. So, case $k=\frac{n}{2}$, where $k$ is odd, provides one subgroup of $\Gamma_{x}$ which is normal in $\pi_{1}\left(\mathcal{G}_{6}\right)$. Case $k=\frac{n}{4}$, where $k$ is odd, provides two subgroups of $\Gamma_{x}$ which are normal in $\pi_{1}\left(\mathcal{G}_{6}\right)$. No other values of $k$ provides normal subgroups.

Proposition 10. Any normal subgroup $\Delta \unlhd \pi_{1}\left(\mathcal{G}_{6}\right)$ isomorphic to $\pi_{1}\left(\mathcal{G}_{6}\right)$ coincide with whole $\pi_{1}\left(\mathcal{G}_{6}\right)$.

Actually this is shown in the proof of Proposition 7.

## Appendix

Given a sequence $\{f(n)\}_{n=1}^{\infty}$, the formal power series

$$
\widehat{f}(s)=\sum_{n=1}^{\infty} \frac{f(n)}{n^{s}}
$$

is called a Dirichlet generating function for $\{f(n)\}_{n=1}^{\infty}$. To reconstruct the sequence $f(n)$ from $\widehat{f}(s)$ one can use Perron's formula ([1], Th. 11.17). Given sequences $f(n)$ and $g(n)$ we call their convolution $(f * g)(n)=\sum_{k \mid n} f(k) g\left(\frac{n}{k}\right)$. In terms of Dirichlet generating series the convolution of sequences corresponds to the multiplication of generating series $\widehat{f * g}(s)=\widehat{f}(s) \widehat{g}(s)$. For the above facts see, for example, ([1], Ch. 11-12).

Here we present the Dirichlet generating functions for the sequences $s_{H, G}(n)$ and $c_{H, G}(n)$. Since Theorems 1 and 2 provide the explicit formulas, the remainder is done by direct calculations.

Consider the Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty} \frac{1}{n^{s}}$. Following [1] note that

$$
\widehat{\sigma}_{0}(s)=\zeta^{2}(s), \quad \widehat{\sigma}_{2}(s)=\zeta^{2}(s) \zeta(s-1), \quad \widehat{d}_{3}(s)=\zeta^{3}(s), \quad \widehat{\omega}(s)=\zeta(s) \zeta(s-1) \zeta(s-2)
$$

Table 2. Dirichlet generating functions for the sequences $s_{H, \mathcal{G}_{6}}(n)$ and $c_{H, \mathcal{G}_{6}}(n)$.

| $H$ | $s_{H, \mathcal{G}_{6}}$ | $c_{H, \mathcal{G}_{6}}$ |
| :---: | :---: | :--- |
| $\pi_{1}\left(\mathcal{G}_{1}\right)$ | $4^{-s} \zeta(s) \zeta(s-1) \zeta(s-2)$ | $4^{-s-1} \zeta(s) \zeta(s-1)\left(\zeta(s-2)+3\left(1+3 \cdot 2^{-s}\right) \zeta(s)\right)$ |
| $\pi_{1}\left(\mathcal{G}_{2}\right)$ | $2^{-s}\left(1-2^{-s}\right) \zeta(s) \zeta(s-1) \zeta(s-2)$ | $3 \cdot 2^{-s-1}\left(1-2^{-s}\right) \zeta^{2}(s)\left(\left(1+3 \cdot 2^{-s}\right) \zeta(s-1)+\right.$ |
|  |  | $\left.\left(1-2^{-s}\right)^{2}\left(1+2^{-s+1}\right) \zeta(s)\right)$ |
| $\pi_{1}\left(\mathcal{G}_{1}\right)$ | $\left(1-2^{-s+1}\right)^{3} \zeta^{3}(s-1)$ | $\left(1-2^{-s}\right)^{3} \zeta^{3}(s)$ |

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