On the Cramer-Rao Bound for Carrier Frequency Estimation in the Presence of Phase Noise

Alan Barbieri, Daniele Bolletta, and Giulio Colavolpe
Università di Parma, Dipartimento di Ingegneria dell'Informazione, Parma, Italy

***Abstract*— We consider the carrier frequency offset estimation in a digital burst-mode transmission affected by phase noise. The corresponding Cramer-Rao lower bound is analyzed for linear modulations under a Wiener phase noise model and in the hypothesis of knowledge of the transmitted data. Even if we resort to a Monte Carlo average, from a computational point of view the evaluation of the Cramer-Rao bound is very hard. We introduce a simple but very accurate approximation that allows to carry out this task in a very easy way. As it will be shown, the presence of the phase noise produce a remarkable performance degradation of the frequency estimation accuracy. In addition, we provide asymptotic closed-form expressions of the Cramer-Rao bound and we also gain some important hints on the estimators to be used in this scenario.**

I. Introduction

The Cramer-Rao bound (CRB) is a fundamental lower limit to the variance of any unbiased parameter estimator [1]. As such, it gives the ultimate accuracy that can be achieved in synchronization operations.

For the frequency offset estimation problem, this bound was computed under different assumptions. The CRBs for the frequency estimation of a single tone in the case of both a known and an unknown constant phase were computed in [2] based on a discrete-time observation model. These results can be also directly applied to the case of phase-shift keying (PSK) signals when transmitted data are perfectly known, i.e., when a data-aided (DA) frequency estimation is performed based on a known preamble. The CRBs in the case of non-data aided (NDA) operations for binary and quaternary PSK (BPSK and QPSK) were derived in [3] and extended to quadrature amplitude modulations (QAM) in [4]. In these papers, the phase offset was assumed known or the case of joint phase and frequency estimation was considered. Finally, for PSK signals, in [5] the CRBs for DA and NDA estimators considering both the case of unknown phase offset uniformly distributed in the interval $[0, 2\pi)$ and the case of joint phase and frequency estimation were computed. The comparison between the discrete-time model commonly used and the true continuous-time model was discussed in [5], showing that, although the correct observation model yields the smaller CRB, the difference between the CRBs resulting from the two models is apparent only at very low values of the signal-to-noise ratio (SNR).

All these papers, as well as the papers dealing with the algorithms for frequency estimation (see for example [2], [6]–[9], or [10] and references therein) refer to an idealized situation in which the phase offset is constant. However, in radio communications, and particularly in modern burst-mode satellite communications, it is common to incur in a strong time-varying phase noise due to the oscillator instabilities. In this case, it is interesting to quantify the resulting performance degradation. To this purpose, we consider the case of a burst-mode transmission using a linearly modulated signal. In this scenario, it is usual to have a first coarse carrier frequency acquisition to reduce the frequency error followed, after timing recovery, by a fine DA frequency estimator based on a known preamble [11]. Phase estimation and tracking is then performed after frequency compensation. Since we are interested in the operations of the fine DA frequency estimator, we consider this setting: known data, ideal timing, and a discrete observation model. In addition, the phase noise has to be considered as a nuisance parameter.

The computation of the resulting CRB is a formidable task. In fact, the likelihood function necessary for the CRB computation must be obtained by averaging over the phase noise. A closed-form expression does not exist and even if we resort to numerical methods, the computational effort is very hard. In this paper, we introduce a simple but very accurate closed form for the likelihood function and then we perform the expectation necessary to obtain the CRB by means of an arithmetical average over a number of computer-generated received samples. The result is in perfect agreement with the closed form asymptotic expressions of the CRB that we also compute in this paper. The derived approximated likelihood function can be also employed to derive new estimation algorithms and to gain new hints on the existing algorithms tailored for a constant phase offset.

II. System Model and the CRB

We consider the transmission of a sequence of complex modulation symbols $\{a_k\}_{k=0}^{K-1}$, belonging to an M-ary constellation of unit average energy, over an additive white Gaussian noise (AWGN) channel affected by carrier phase noise and a constant frequency offset ν. Symbols a_k are linearly modulated. Assuming Nyquist transmitted pulses, matched filtering, a small frequency offset and phase variations slow enough so as no intersymbol interference arises, the discrete-time baseband received signal is given by

$$r_k = a_k e^{j(2\pi\nu kT+\theta_k)} + w_k\,, \quad k = 0, 1, \ldots, K-1 \qquad (1)$$

where T is the symbol interval and the noise samples $\{w_k\}_{k=0}^{K-1}$ are independent and identically distributed (i.i.d.), complex, circularly symmetric Gaussian random variables (rvs), each with mean zero and variance equal to $2\sigma^2 = N_0/E_S$, N_0 being the one-sided noise power spectral density and E_S the received signal energy per information symbol. For the time-varying channel phase θ_k, we assume a random-walk (Wiener) model:

$$\theta_{k+1} = \theta_k + \Delta_k \qquad (2)$$

where $\{\Delta_k\}$ are real i.i.d. Gaussian rvs with mean zero and standard deviation σ_Δ,[1] and the rv θ_0 is uniformly distributed. The rvs $\{\theta_k\}$ are supposed unknown to the receiver, and statistically independent of symbols and noise. When $\sigma_\Delta = 0$ we obtain the classical case of a constant and uniformly distributed phase offset.

Some of the information symbols in the transmitted burst are known to the receiver (pilot symbols) and the frequency estimation is based on these symbols. For generality, we assume that the inserted N pilot symbols are $\{a_{k(n)}\}$ where $\{k(n)|0 \leq n \leq N-1\}$ is an index set for the sample times. These symbols and the corresponding received and phase samples are collected into three vectors $\mathbf{a} \triangleq \{a_{k(n)}\}_{n=0}^{N-1}$, $\mathbf{r} \triangleq \{r_{k(n)}\}_{n=0}^{N-1}$, and $\boldsymbol{\theta} \triangleq \{\theta_{k(n)}\}_{n=0}^{N-1}$.

The CRB for this estimation problem is defined as [1]

$$CRB_\nu^{-1} = E_{\mathbf{r}}\left[-\frac{\partial^2}{\partial\nu^2}\ln p(\mathbf{r}|\nu)\right] \tag{3}$$

where $p(\mathbf{r}|\nu)$ is the probability density function (pdf) of $\mathbf{r}$ given ν, the derivative is evaluated at the true value of ν, and $E_{\mathbf{r}}$ denotes statistical expectation with respect to the vector $\mathbf{r}$. This pdf can be obtained as

$$p(\mathbf{r}|\nu) = E_{\boldsymbol{\theta}}\{p(\mathbf{r}|\boldsymbol{\theta},\nu)\} = \int p(\mathbf{r}|\boldsymbol{\theta},\nu)p(\boldsymbol{\theta})\,d\boldsymbol{\theta}\,. \tag{4}$$

As already mention, the likelihood function $p(\mathbf{r}|\nu)$ cannot be expressed in a closed form. On the other hand, if the expectation in (3) can be easily performed by means of a Monte Carlo average, the computational effort required by the numerical evaluation of the expectation in (4) is much more intensive. In the next section, we describe an approximate but very accurate closed-form expression for this pdf.

In the technical literature, there is an alternative lower bound on the estimator error variance, the so-called modified CRB [12], easy to compute but in general quite looser. For the problem at hand, this bound is not useful, since it can be easily shown that it gives the same result is obtained when the phase noise is not present.

III. The Likelihood Function

In this section, we introduce an approximated closed-form expression for the pdf $p(\mathbf{r}|\nu)$ that will be used in the computation of the CRB, and also a couple of exact asymptotic closed-form expressions, in the absence of phase noise ($\sigma_\Delta = 0$) and in the absence of thermal noise ($\sigma = 0$), respectively.

A. Approximated Closed-Form Expression

Let us denote by $g(\eta, \delta^2; x)$ a Gaussian distribution in x, with mean value η and variance δ^2, and by $t(\zeta; x)$ a Tikhonov distribution in x characterized by the complex parameter ζ, i.e.,

$$g(\eta, \delta^2; x) = \frac{1}{\sqrt{2\pi\delta^2}}e^{-\frac{(x-\eta)^2}{2\delta^2}} \tag{5}$$

$$t(\zeta; x) = \frac{1}{2\pi \mathrm{I}_0(|\zeta|)}e^{\mathrm{Re}[\zeta e^{-jx}]} \tag{6}$$

[1]Note that, since the channel phase is defined modulo 2π, the pdf of Δ_k can be approximated as Gaussian only if $\sigma_\Delta \ll 2\pi$.

where $\mathrm{I}_0(x)$ is the zero-th order modified Bessel function of the first kind. By using these definitions and taking into account the system model (1), we may express, discarding irrelevant proportionality factors independent of θ_k and ν

$$\begin{aligned} p(\mathbf{r}|\boldsymbol{\theta},\nu) &= \prod_{n=0}^{N-1} p(r_{k(n)}|\theta_{k(n)},\nu) \\ &\propto \prod_{n=0}^{N-1} \exp\left\{\frac{1}{\sigma^2}\mathrm{Re}[r_{k(n)}a^*_{k(n)}e^{-j(2\pi\nu k(n)T+\theta_{k(n)})}]\right\} \\ &= \prod_{n=0}^{N-1} t\left(z_{k(n)};\theta_{k(n)}\right) \end{aligned} \tag{7}$$

$$p(\boldsymbol{\theta}) = p(\theta_{k(0)})\prod_{n=1}^{N-1} p(\theta_{k(n)})|\theta_{k(n-1)}) \tag{8}$$

having defined $z_k \triangleq \frac{r_k a_k^*}{\sigma^2}e^{-j2\pi\nu kT}$. In (8), the pdf $p(\theta_{k(0)})$ is $p(\theta_{k(0)}) = 1/2\pi$, since the rv $\theta_{k(0)}$ is uniformly distributed, whereas the pdfs $p(\theta_{k(n)})|\theta_{k(n-1)})$ are Gaussian with zero mean and standard deviation $\delta(n) = \sigma_\Delta\sqrt{k(n)-k(n-1)}$ (we implicitly assume, for the adopted pilot distribution, $\delta(n) \ll 2\pi$). By substituting (7) and (8) into (4), observing that

$$t\left(z;\theta\right)t\left(u;\theta\right) = \frac{\mathrm{I}_0(|z+u|)}{2\pi\mathrm{I}_0(|z|)\mathrm{I}_0(|u|)}t\left(z+u;\theta\right) \tag{9}$$

and using the following approximation [13], [14][2]

$$\int t(\zeta,x)g(x,\delta^2;y)\,dx \simeq t\left(\frac{\zeta}{1+\delta^2|\zeta|};y\right) \tag{10}$$

discarding irrelevant multiplicative terms, after some manipulations we obtain the following expression of the likelihood function [15]:

$$p(\mathbf{r}|\nu) \mathrel{\tilde{\propto}} \prod_{n=0}^{N-2}\frac{\mathrm{I}_0\left(|z_{k(n)}+u_n|\right)}{\mathrm{I}_0\left(|u_n|\right)} \tag{11}$$

where coefficients u_n can be recursively computed as

$$\begin{aligned} u_n \;=\; & \frac{u_{n+1}+z_{k(n+1)}}{1+[k(n+1)-k(n)]\sigma_\Delta^2|u_{n+1}+z_{k(n+1)}|} \\ & n = N-2,\ldots,0\,. \end{aligned} \tag{12}$$

with initial condition $u_{N-1} = 0$.

B. Absence of Phase Noise

When $\sigma_\Delta = 0$, i.e., when a constant unknown phase offset is considered, we obtain an *exact* expression for the likelihood function which is equivalent to that derived in [9]. In fact, in this case (10) holds with equality and coefficients u_n can be expressed as

$$u_n = \sum_{\ell=n+1}^{N-2} z_{k(\ell)}\,. \tag{13}$$

[2]Note that, when $\delta = 0$, (10) holds with equality.

Hence, we have

$$p(\mathbf{r}|\nu) \propto \prod_{n=0}^{N-2} \frac{\mathrm{I}_0\left(|\sum_{\ell=n}^{N-2} z_{k(\ell)}|\right)}{\mathrm{I}_0\left(|\sum_{\ell=n+1}^{N-2} z_{k(\ell)}|\right)} . \quad (14)$$

C. Absence of Thermal Noise

We now consider the case of absence of thermal noise (i.e., $\sigma = 0$). This is an approximation of the case when the SNR is large enough so as the effect of thermal noise is negligible with respect to phase noise. In this case, an exact closed form of the likelihood function can be computed. Through straightforward manipulations, we find that [15]

$$p(\mathbf{r}|\nu) \propto \exp\left\{-\frac{(2\pi)^2 D}{2\sigma_\Delta^2}[\nu T - \frac{1}{2\pi D}\sum_{n=1}^{N-1} \arg(r_{k(n)} a^*_{k(n)} r^*_{k(n-1)} a_{k(n-1)})]^2\right\} \quad (15)$$

having defined $D = k(N-1) - k(0)$. Hence, in this case the likelihood function is Gaussian and does not depend on the number and position of pilot symbols, but only on the distance D between the first and the last pilot symbol.

IV. The Cramer-Rao Bound

We now describe the computation of the CRB for the problem at hand.

As already mentioned, a first computationally intensive method is based on a numerical evaluation, through Monte Carlo average, of both the expectations in (3) and (4). The corresponding result, denoted as CRB_{MC}, is used to verify the accuracy of the CRB obtained through the use of the simplified approximated closed-form expression of the likelihood function (11) and denoted as CRB_{simp}. In this latter case, the Monte Carlo average is only used to compute the expectation in (3). In the case of absence of phase noise, by using the closed-form expression (14), the CRB_{simp} is *exact* and gives the same result obtained in [5, eqn. (29)] for known data, a constant and unknown phase offset, and the discrete-time observation model.

The low and high SNR limits of the CRB can be also computed in closed form. By observing that for low SNR values the arguments of the Bessel functions in (11) assume low values, we can use the limiting form for small arguments $\ln \mathrm{I}_0(x) \simeq x^2/4$, obtaining [15]

$$CRB_L = \frac{\sigma^4/}{2\pi^2 T^2 \sum_{n=0}^{N-2}\sum_{\ell=n+1}^{N-1} F(n,\ell)} \quad (16)$$

where

$$F(n,\ell) \triangleq [k(\ell)-k(n)]^2 |a_{k(l)}|^2 |a_{k(n)}|^2 e^{-\frac{1}{2}[k(\ell)-k(n)]\sigma_\Delta^2} . \quad (17)$$

For $\sigma_\Delta = 0$, PSK signals, and N consecutive pilots, i.e., $k(n) = n$, $n = 0, 1, \ldots, N$, this result coincides with the low SNR limit in [5]. For $\sigma_\Delta > 0$, the CRB increases and this means that there is a performance degradation due to the phase noise.

For high SNR values, by using (15) in (3) we obtain [15]

$$CRB_H = \frac{1}{DT^2}\left(\frac{\sigma_\Delta}{2\pi}\right)^2 . \quad (18)$$

Note that this result is exact since no approximation (excepting that of high SNR values) is involved in the derivation of (15). This high SNR limit allows to draw some important considerations. First of all, in the presence of a time-varying phase, the CRB has a floor, i.e., it is not possible to reach the desired estimation accuracy simply increasing the SNR value. In addition, the asymptotic CRB only depends on the positions of the first and last pilot symbols (as the asymptotic likelihood function (15)) and is completely independent of the actual pilot distribution. Let us now consider the particular pilot distribution characterized by $k(n) = nL$, where $L \geq 1$ is an integer constant which plays the role of the distance between two consecutive pilot symbols. It it worth noting that $L = 1$ depicts the situation of N consecutive pilot symbols. In this case, being $D = (N-1)L$, the high SNR limit assumes the form

$$CRB_H = \frac{1}{(N-1)LT^2}\left(\frac{\sigma_\Delta}{2\pi}\right)^2 . \quad (19)$$

Hence, for high SNR values, the CRB goes as N^{-1} in the presence of phase noise whereas it goes as N^{-3} for a constant phase offset [9]. As a consequence, an increase in the estimation window has still a beneficial effect on the estimation accuracy, mitigated by the fact that the presence of a time-varying phase leads to almost independent received samples if the window becomes larger.

Similarly, the CRB goes as L^{-1} for a time-varying phase whereas it can be shown that it is goes as L^{-2} for a constant phase offset. This behavior is due to the fact that increasing the distance between two consecutive pilot symbols has the same effect of increasing the phase noise variance.

V. Estimation Algorithms

By using the expressions for the likelihood function derived in Section III we can design a couple of maximum likelihood (ML) estimation algorithms for this scenario. We consider the above mentioned pilot distribution $k(n) = nL$. In this case, for the considered discrete-time signal model, values of frequency offset which differ of $1/LT$ are indistinguishable since they produce the same received samples $r_{k(n)} = r_{nL}$. Hence, the likelihood functions are $\frac{1}{L}$-periodic with respect to the normalized frequency offset νT. This means that the valid estimation range must be small enough so as no more than one global maximum appear in the likelihood function, that is, the possible values of the frequency offset must be inside the range $[-\frac{1}{2LT}, \frac{1}{2LT}]$.

By considering the likelihood function (11), we obtain the following estimator

$$\begin{aligned}\hat{\nu} &= \operatorname*{argmax}_{\nu} p(\mathbf{r}|\nu) = \operatorname*{argmax}_{\nu} \ln p(\mathbf{r}|\nu) \\ &= \operatorname*{argmax}_{\nu} \sum_{n=0}^{N-2} [\ln \mathrm{I}_0(|z_{nL} + u_n|) - \ln \mathrm{I}_0(|u_n|)] .\end{aligned} \quad (20)$$

The search for the maximum of the log-likelihood function can be accomplished, as for the Rife and Boorstyn algorithm [2], in two steps. In a first coarse search, the log-likelihood function is evaluated for some values of the frequency offset in the range $[-\frac{1}{2LT}, \frac{1}{2LT}]$ and the value ν_{cs} which corresponds to the maximum value is obtained. Then, with a fine search the value of ν closest to ν_{cs} which maximizes the log-likelihood function is located, for example by using the secant method. This estimator will be denoted as E_{PN}. Obviously, it is quite complex for a receiver implementation. In the numerical results it will be used as a term of comparison to evaluate the performance that can be obtain with a practical estimator.

Let us now take into account the asymptotic expression of the likelihood function. A ML estimator based upon (15) is characterized by this simple estimation rule ($D = (N-1)L$ for the above mentioned pilot distribution)

$$\hat{\nu} = \frac{\sum_{n=1}^{N-1} \arg\left[r_{nL} a_{nL}^* r_{(n-1)L}^* a_{(n-1)L}\right]}{2\pi(N-1)LT} \qquad (21)$$

that is very similar to the Kay estimator excepting for the weighting coefficients [6], [10]. We will denote this estimator as E_{asymp}. It is straightforward to show that, for high SNR values, this estimator is unbiased.

In the numerical results we will compare the performance of these two algorithms with that of some of the algorithms designed in the literature in the absence of phase noise.

VI. Numerical Results

Although the results in the previous sections can be applied to general linear modulations, in the numerical results we consider M-PSK signals since in this case the performance does not depend on the adopted pilot sequence. We show the accuracy of the CRB computed by using the simplified closed-form expression of the likelihood function. The performance of the derived estimators is also shown and compared with the CRB and with the performance of "classical" frequency estimators.

In Fig. 1, for $\sigma_\Delta = 0, 2, 6$ degrees, we show the CRB_{simp}, together with the derived low and high SNR asymptotic expressions, as a function of the SNR in the case of $N = 64$ consecutive pilot symbols (hence $L = 1$). The CRB_{MC} is also shown. We may observe that the derived simplified method has a very good accuracy since the CRB_{simp} coincides with the CRB_{MC}. The high SNR value CRB_H is reached for values of E_S/N_0 around 10 dB, whereas the low asymptotic value CRB_L, which slightly depends on the value of σ_Δ, is reached only at very low SNR values. The performance degradation for high SNR values due to the presence of phase noise is significant already for $\sigma_\Delta = 2$ degrees, as shown by the presence of the floor predicted by our high SNR asymptote.

We now consider the performance of the estimators described in the previous section and compare it with the performance of the best algorithm for frequency estimation in the presence of constant phase offset, i.e., the Rife & Boorstyn (R&B) algorithm, denoted in the figures as E_{RB}. Since we verified that in the considered operating conditions all the estimators are unbiased, we show the estimator variance, normalized to $1/T^2$, which coincide with the mean square estimation error. All the following simulation results have been obtained by generating a random frequency offset in the range $[-2 \cdot 10^{-2}, 2 \cdot 10^{-2}]$, independently frame by frame.

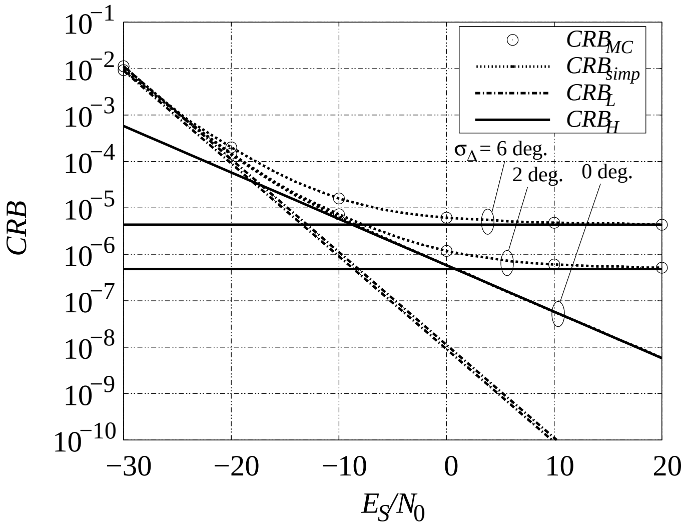

Fig. 1. CRB in the case of $N = 64$ consecutive pilot symbols and different values of σ_Δ.

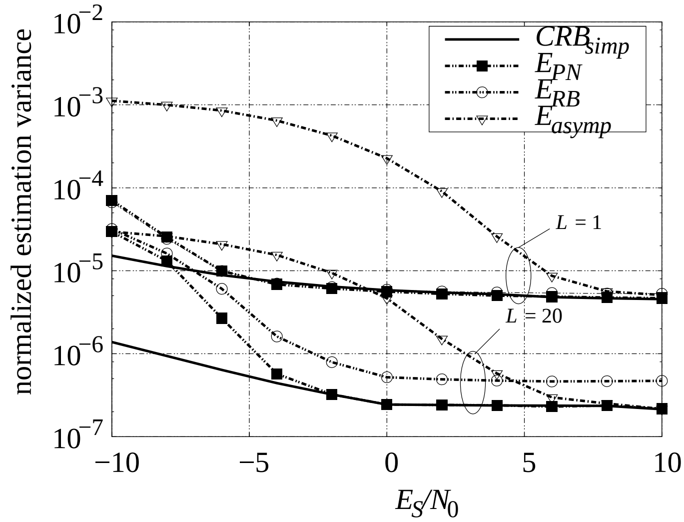

Fig. 2. Normalized estimator variance for $N = 64$ and $\sigma_\Delta = 6$ degrees.

In Fig. 2 we show the normalized error variance as a function of the SNR, for $\sigma_\Delta = 6$ degrees and $N = 64$ pilot symbols. The cases of $L = 1$ (consecutive pilots) and $L = 20$ have been considered. The R&B estimator, which is optimum for a constant phase, does not seem to be able to reach the CRB for high SNR. On the contrary, estimators E_{PN} and E_{asymp}, designed taking into account the phase noise statistics, are asymptotically optimal. At very low SNR, all the estimator exhibits a larger variance with respect to the bound due to the occurrence of outliers [2], [10]—the corresponding threshold depending on the estimator and on the value of L. In particular, the estimator E_{asymp} has a very high threshold. However, we would like to point out that for E_S/N_0 larger than few dBs,

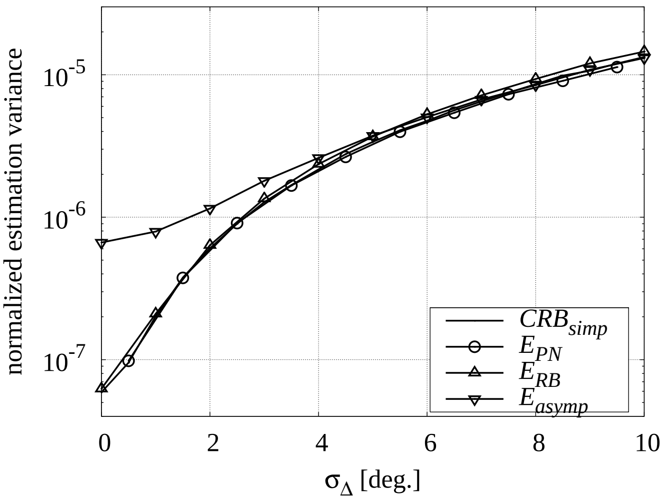

Fig. 3. Normalized estimator variance for $E_S/N_0 = 10$ dB and $N = 64$ consecutive pilot symbols.

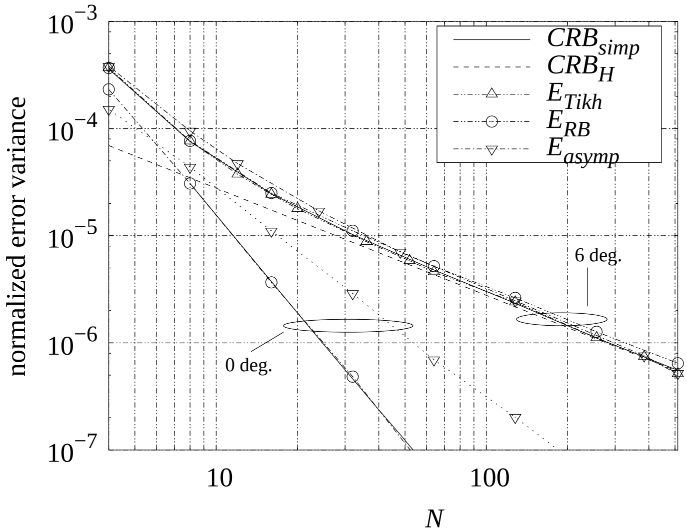

Fig. 4. Normalized estimator variance for $E_S/N_0 = 10$ dB and $\sigma_\Delta = 6$ degrees.

it is convenient to use the estimator E_{asymp}, which is able to reach the CRB and presents a noticeable smaller complexity with respect to the other estimators.

The performance degradation due to phase noise is highlighted in Fig. 3, where the normalized estimation variance for the considered estimators is reported, together with the CRB, as a function of the phase noise standard deviation for $E_S/N_0 = 10$ dB and $N = 64$ consecutive pilot symbols.

Finally, in Fig. 4, for $E_S/N_0 = 10$ dB, $L = 1$, and $\sigma_\Delta = 0$ and 6 degrees, we show the performance of the estimators as a function of the number of pilots N. We may observe that in the presence of phase noise, the normalized error variance decreases as N^{-1} as predicted by the high SNR asymptote of the CRB.

VII. Conclusions

In this paper, the Cramer-Rao lower bound for frequency estimation in the presence of phase noise has been computed. Although it is not possible to derive a closed-form expression, we have shown an approximation that leads to a simple, fast but very accurate evaluation of the bound by using a Monte Carlo average. The asymptotic closed-form expressions of the bound for low and high values of signal-to-noise ratio have been also provided. These expressions are very useful to better understand the effects of the phase noise on the frequency offset estimation accuracy. In particular, we demonstrated that in the presence of the phase noise it is not possible to reach the desired estimation accuracy simply decreasing the signal-to-noise ratio. These asymptotic expressions of the bound allow also to quantify the effect of the pilot distribution parameters and phase noise variance. Finally, a couple of ML-based algorithms specifically tailored for this scenario have been designed and compared with the algorithms designed in the literature for the case of absence of phase noise.

References

[1] H. L. Van Trees, *Detection, Estimation, and Modulation Theory - Part I*. John Wiley & Sons, 1968.

[2] D. C. Rife and R. R. Boorstyn, "Single tone parameter estimation from discrete-time observations," *IEEE Trans. Inform. Theory*, vol. 20, pp. 591–598, September 1974.

[3] W. G. Cowley, "Phase and frequency estimation for PSK packets: bounds and algorithms," *IEEE Trans. on Commun.*, vol. 44, pp. 26–28, Jan. 1996.

[4] F. Rice, B. Cowley, B. Moran, and M. Rice, "Phase and frequency estimation for PSK packets: bounds and algorithms," *IEEE Trans. on Commun.*, vol. 49, pp. 1582–1591, Sept. 2001.

[5] N. Noels, H. Steendam, and M. Moeneclaey, "The true Cramer-Rao bound for carrier frequency estimation from a PSK signal," *IEEE Trans. on Commun.*, vol. 52, pp. 834–844, May 2004.

[6] S. M. Kay, "A fast and accurate single frequency estimator," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 1987–1990, Dec. 1989.

[7] M. Luise and R. Reggianini, "Carrier frequency recovery in all-digital modems for burst-mode transmissions," *IEEE Trans. on Commun.*, vol. 43, pp. 1169–1178, Mar. 1995.

[8] U. Mengali and M. Morelli, "Data-aided frequency estimation for burst digital transmission," *IEEE Trans. on Commun.*, vol. 45, pp. 23–25, Jan. 1997.

[9] N. Noels, H. Steendam, and M. Moeneclaey, "Pilot-symbol assisted iterative carrier synchronization for burst transmission," in *Proc. IEEE Intern. Conf. Commun.*, Paris, France, June 2004, pp. 509–513.

[10] M. Morelli and U. Mengali, "Feedforward frequency estimation for PSK: a tutorial review," *European Trans. Telecommun.*, vol. 9, pp. 103–116, March/April 1998.

[11] S. Benedetto, C. Berrou, C. Douillard, R. Garello, D. Giancristofaro, A. Ginesi, L. Giugno, M. Luise, and G. Montorsi, "MHOMS: High speed ACM modem for satellite applications," *IEEE Wireless Communications*, vol. 12, Feb. 2005.

[12] A. N. D'Andrea, U. Mengali, and R. Reggiannini, "The modified Cramer-Rao bound and its application to synchronization problems," *IEEE Trans. on Commun.*, vol. 42, pp. 1391–1399, Feb. 1994.

[13] G. Colavolpe, A. Barbieri, and G. Caire, "Algorithms for iterative decoding in the presence of strong phase noise," IEEE J. Selected Areas in Communications, to appear, 2005. Available at *http://www.tlc.unipr.it/people/colavolpe/public.html*.

[14] A. Barbieri, G. Colavolpe, and G. Caire, "Joint iterative detection and decoding in the presence of phase noise and frequency offset," in *Proc. IEEE Intern. Conf. Commun.*, May 2005.

[15] A. Barbieri, D. Bolletta, and G. Colavolpe, "On the Cramer-Rao bound for carrier frequency estimation in the presence of phase noise," *submitted to Trans. on Comm.*, 2005.