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On the Creation of Scalar Particles in Some Anisotropic Universe

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Because of an importance of the particle creation (especially, its possible fulfilment of the black-body law with a definite temperature) in an early universe to various other cosmological problems, we study how the creation of scalar particles occurs in the Bianchi-type I anisotropic universe adopted in our previous works on the quantized scalar field. It is shown that, as in a special isotropic case dealt with in recent papers, the creation may occur at the sacrifice of the requirement that the quantization procedure should reproduce the usual theory for a free field in the limit when the anisotropic universe changes into the Minkowski space-time. It is further shown that the creation occurs in accordance with the black-body law only in a 2-dimensional hyper-surface relating to the anisotropic cosmic expansion, provided that we fix two arbitrary constants appearing in a general expression for the Feynman propagator in terms of a procedure similar to that in the isotropic case. A speculation on the isotropization of our model-universe is also made from the standpoint of seeking the attainment of the thermal equilibrium in the whole universe.

§ 1. Introduction

Since the pioneering works of Zel'dovich[1] and Parker,[2] the creation of particles in an early universe becomes one of the most fascinating problems in cosmology, because it may serve to elucidate the origin of a large specific entropy (the number of photons per baryon) in the present-day universe as well as its reaction effect to make the early universe with a large anisotropy (if it existed) isotropic. According to Parker's[2] canonical approach to the quantization of a scalar field, the creation of those particles in an isotropic universe with flat 3-space occurs in accordance with the black-body law, provided that the cosmic expansion began from an initial adiabatic state (whose existence is not necessarily permissible). To remedy the last defect in the above interesting result, on the basis of the path-integral quantization procedure, Chitre and Hartle[3] derived the Feynman propagator $K(x,x')$ for a massive ($m>0$) scalar field in a special universe with the metric

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=-dt^2+(t/t_0)^2\delta_{ij}dx^i dx^j \tag{1·1}$$

(so that the scalar curvature is of the form $R\equiv g^{\mu\nu}R_{\mu\nu}=6/t^2$, and $t_0=\mathrm{const}$) to show that the black-body temperature is given by $k_B T(t)=(\pi t)^{-1}$ (cf. $\hbar=c=1$ and k_B is the Boltzmann constant).

On the other hand, for examining the relation between DeWitt's[4] configuration-space approach and Parker's[2] momentum-space one to a cosmological gravitational field interacting with quantized matter fields, the author[5] studied the propagation behavior of a quantized scalar field in a special Bianchi-type I anisotropic universe adopted for mathematical simplicity. It is an easy matter to transfer the formalism into the one in a general isotropic universe with flat 3-space. In the light of this formalism, we[6] examined Chitre and Hartle's result to show that their quantization procedure cannot reproduce the usual theory for a free field in the limit when the universe changes into the Minkowski space-time. To resolve this complexity, we have to study in more detail how the vacuum state (without a unique physical meaning[7]) and the associated particle state at an initial epoch in the isotropic universe should be fixed, but there is at present no prospect to do so in a successful manner. Therefore, we prefer to accumulate other examples having a similar complexity to look for a clue for solving this difficult but important problem.

The purpose of this paper is to present such an example which is derivable from our quantization procedure in Ref. 5) by its replacement with another one similar to Chitre and Hartle's, i.e., an anisotropic version of the example dealt with in Ref. 6). In § 2 we summarize several geometric properties of the Bianchi-type I universe (which incidentally resemble those of the isotropic universe with the metric (1·1)), in addition to the discussion (for later usefulness) about the world line of a test particle in the universe. In § 3, by making use of the 4-dimensional commutation function $G(x,x')$ and the elementary solution $G^{(1)}(x,x')$ derived in Ref. 5), we construct the Feynman propagator $G_F(x,x')$ which depends in general upon two complex constants. A minor mistake in the fixation of those constants in Ref. 5) is first corrected, so that the pair-creation does not occur. Next we tend to another fixation in terms of which the pair-creation may occur in a 2-dimensional hyper-surface relating to the anisotropic cosmic expansion. Section 4 is devoted to the verification that the particle creation obeys a black-body law resembling closely the one in the isotropic case. In § 5 we speculate on the isotropization of the Bianchi-type I universe due to the creation of particles thus found. We show in the Appendix that, by virtue of the correction in § 3, our elementary solution is changed into the pseudo-elementary solution defined by Eq. (A·1) in Ref. 5).

§ 2. Geometric properties of the Bianchi-type I universe

According to Ref. 5), the Bianchi-type I universe is specified by the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + (t/t_0)^2(dx_1^2 + dx_2^2) + dx_3^2\,, \tag{2·1}$$

so that the metric determinant, the Ricci tensor and the scalar curvature are of the form

$$\begin{cases} (-g)^{1/2} = (t/t_0)^2, \quad R_0^{\ 0} = R_3^{\ 3} = 0, \quad R_1^{\ 1} = R_2^{\ 2} = 1/t^2, \\ R \equiv g^{\mu\nu} R_{\mu\nu} = 2/t^2, \end{cases} \tag{2·2}$$

where t_0 is a constant. The geodesic bi-scalar $\sigma(x, x')$ in the above universe is given by

$$\sigma(x, x') = \tfrac{1}{2}\{-t^2 - t'^2 + 2tt'\mathrm{ch}\xi_\perp + z^2\}, \tag{2·3}$$

which, together with $\bar{g}(x, x') \equiv \{g(x) g(x')\}^{1/2} = (tt'/t_0^2)^2$, leads to another bi-scalar

$$\Delta(x, x') \equiv -\{\bar{g}(x, x')\}^{-1}\det(\partial^2\sigma/\partial x^\mu \partial x'^\mu) = \mathrm{sh}\xi_\perp/\xi_\perp, \tag{2·4}$$

where $\xi_\perp \equiv \{(x_1 - x_1')^2 + (x_2 - x_2')^2\}^{1/2}/t_0$ and $z \equiv x_3 - x_3'$. As already shown,[5),8)] these bi-scalars play a significant role in the discussion of various bi-scalar propagators such as the 4-dimensional commutation function $G(x, x')$ and the elementary solution $G^{(1)}(x, x')$ for a quantized scalar field in the universe. The fact that both bi-scalars can be represented in the above explicit forms is the reason why we adopted such a special and unrealistic universe, like Chitre and Hartle's[3)] adoption of the metric (1·1).

If we designate the proper time of a test particle in the universe by $d\tau \equiv (-ds^2)^{1/2}$, its geodesic equations of motion are given by

$$\begin{cases} dx_a/d\tau = c_a (t_0/t)^2 \quad (a = 1, 2), \quad dx_3/d\tau = c_3, \\ dt/d\tau = \{1 + (c_1^2 + c_2^2)(t_0/t)^2 + c_3^2\}^{1/2}, \end{cases} \tag{2·5}$$

where $c_i\,(i = 1, 2, 3)$ are integration constants. However, Eq. (2·1) shows that the test particle has the physical 3-velocity defined by $v_a \equiv (t/t_0)(dx_a/dt)$ and $v_3 \equiv dx_3/dt$. Accordingly the corresponding Lorentz-like velocity is of the form

$$\begin{cases} V_a \equiv v_a/(1 - v_\perp^2 - v_3^2)^{1/2} = c_a (t_0/t), \quad (a = 1, 2) \\ V_3 \equiv v_3/(1 - v_\perp^2 - v_3^2)^{1/2} = c_3, \end{cases} \tag{2·6}$$

where $v_\perp^2 \equiv v_1^2 + v_2^2$.

Moreover, it would not be useless to point out that, if we stand on general relativity, both Chitre-Harlte's isotropic universe and our anisotropic one consist of such a fictitious fluid as

$$\begin{cases} \rho = -3p = 3/8\pi G t^2, \quad \text{(Chitre-Hartle)} \\ \rho = -3p = w/t = 1/8\pi G t^2, \quad \text{(Ours)} \end{cases} \tag{2·7}$$

where G is Newton's gravitation constant, and ρ, p and w stand for the density, pressure and coefficient of shear viscosity, respectively. The relation $\rho = -3p$ in Eq. (2·7) shows an unrealistic nature of both universes.

§ 3. Propagators for a quantized scalar field in the universe

Let us represent the 4-dimensional commutation function $G(x, x')$ and the elementary solution $G^{(1)}(x,x')$ for a quantized scalar field (with mass m) in the universe under consideration as follows:[5)]

$$G(x,x') = -i\frac{\{\bar{g}(x,x')\}^{-1/2}}{(2\pi)^3}\int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\{f_k(t)f_k^*(t') - f_k(t')f_k^*(t)\} \quad (3\cdot1)$$

and

$$G^{(1)}(x,x') = \frac{\{\bar{g}(x,x')\}^{-1/2}}{(2\pi)^3}\int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\{f_k(t)f_k^*(t') + f_k(t')f_k^*(t)\}, \quad (3\cdot2)$$

where $\{\bar{g}(x,x')\}^{1/2} = (tt')/t_0^2$ as before, and $\boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{x}'$. Here, the time-dependent functions $f_k(t)$ and $f_k^*(t)$ are given by

$$\{d^2/dt^2 + U_k^2(t)\}\begin{Bmatrix} f_k(t) \\ f_k^*(t)\end{Bmatrix} = 0, \quad f_k(t)\dot{f}_k^*(t) - f_k^*(t)\dot{f}_k(t) = i \quad (3\cdot3)$$

with

$$U_k^2(t) \equiv (m^2 + k_3^2) + \{(k_\perp t_0)^2 + \lambda/3\}t^{-2}, \quad (3\cdot4)$$

where $k_\perp^2 \equiv k_1^2 + k_2^2$, and $\lambda = 1$ or 0 according as the scalar field (when $m = 0$) is conformally invariant or not. Both propagators satisfy one and the same homogeneous wave equation, e.g.,

$$(\Box - U)G(x,x')$$
$$\equiv \{-t^{-2}\partial_t(t^2\partial_t) + (t/t_0)^{-2}(\partial_{x_1}^2 + \partial_{x_2}^2) + \partial_{x_3}^2 - U\}G(x,x') = 0, \quad (3\cdot5)$$

where $U \equiv m^2 + \lambda R/6$. Then the Feynman propagator defined by

$$G_F(x,x') = -\tfrac{1}{2}\{\theta(t-t') - \theta(t'-t)\}G(x,x') + (i/2)G^{(1)}(x,x') \quad (3\cdot6)$$

satisfies the correct inhomogeneous wave equation

$$(\Box - U)G_F(x,x') = -\{(\bar{g}(x,x')\}^{-1/2}\delta^4(x-x'), \quad (3\cdot7)$$

where $\theta(t-t')$ is the step function.

For simplicity, we shall assume that $\lambda = 1$. Then it follows from Eqs. (3·3) and (3·4) that

$$\begin{cases} f_k(t) = \dfrac{\sqrt{\pi t}}{2}\{B_k\, e^{(2\nu-i)\pi/4}H_{i\nu}^{(2)}(\mu t) + C_k\, e^{-(2\nu-i)\pi/4}H_{i\nu}^{(1)}(\mu t)\}, \\ f_k^*(t) = \dfrac{\sqrt{\pi t}}{2}\{B_k^*\, e^{-(2\nu-i)\pi/4}H_{i\nu}^{(1)}(\mu t) + C_k^*\, e^{(2\nu-i)\pi/4}H_{i\nu}^{(2)}(\mu t)\} \end{cases} \quad (3\cdot8)$$

with

$$|B_k|^2 - |C_k|^2 = 1, \quad (B_k \text{ and } C_k \text{ are complex constants}) \quad (3\cdot9)$$

where μ and ν are an abbreviation of μ_3 and $\nu_\perp$ defined by

$$\mu_3 \equiv (m^2 + k_3^2)^{1/2}, \quad \nu_\perp \equiv \{(k_\perp t_0)^2 + 1/12\}^{1/2}, \quad (3\cdot10)$$

and $H_{i\nu}^{(a)}(\mu t)$ $(a=1, 2)$ stand for the Hankel functions. It is noticeable that Eqs. (3·8) and (3·9) closely resemble Eqs. (6) and (7) in Ref. 6). Inserting Eqs. (3·1) and (3·2) into Eq. (3·6) and making use of Eq. (3·8) and the expression for $\{\bar{g}(x, x')\}^{1/2}$, we obtain

$$G_F(x, x') = \frac{t_0^2}{(2\pi)^3}\int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\left(\frac{\pi i/4}{\sqrt{tt'}}\right)\{\theta(t-t')A_k(\mu t, \mu t') + \theta(t'-t)A_k(\mu t', \mu t)\} \tag{3·11}$$

with

$$\begin{aligned} A_k(\mu t, \mu t') \equiv\, & |B_k|^2 H_{i\nu}^{(2)}(\mu t) H_{i\nu}^{(1)}(\mu t') + |C_k|^2 H_{i\nu}^{(1)}(\mu t) H_{i\nu}^{(2)}(\mu t') \\ & + B_k^* C_k e^{i(i\nu+1/2)\pi} H_{i\nu}^{(1)}(\mu t) H_{i\nu}^{(1)}(\mu t') \\ & + B_k C_k^* e^{-i(i\nu+1/2)\pi} H_{i\nu}^{(2)}(\mu t) H_{i\nu}^{(2)}(\mu t'), \end{aligned} \tag{3·12}$$

where μ and ν are the abbreviation of μ_3 and $\nu_\perp$ given by Eq. (3·10) as before.

It is easily seen that an arbitrariness of the propagator $G_F(x, x')$ given by Eq. (3·11) (because of its dependence upon the complex constants $B_k \sim C_k^*$) has solely originated from the expression for $G^{(1)}(x, x')$ derivable from Eqs. (3·2) and (3·8). In Ref. 5), the arbitrariness was fixed by a suitable procedure which makes the propagator $G^{(1)}(x, x')$ reduce to its Minkowski counterpart $\Delta^{(1)}(x-x')$ in the limit when the cosmic expansion disappears. However, as will be shown in the Appendix, the paper included a minor mistake and the correct result is found to be $C_k = C_k^* = 0$ or $|B_k|^2 = 1$. Accordingly we can simplify Eq. (3·12) as follows:

$$A_k(\mu t, \mu t') = H_{i\nu}^{(2)}(\mu t) H_{i\nu}^{(1)}(\mu t'), \tag{3·13}$$

which prohibits the occurrence of the pair-creation process in the sense of Chitre and Hartle.[3)]

In view of this, we shall tend to another fixation satisfying the requirement that the coefficient $B_k C_k^* \exp\{(\nu - i/2)\pi\}$ of the term $H_{i\nu}^{(2)}(\mu t) H_{i\nu}^{(2)}(\mu t')$ in Eq. (3·12) is unity, just like the one in Ref. 6). Then, by virtue of Eq. (3·9), we obtain

$$\begin{cases} |B_k|^2 - 1 = |C_k|^2 = \frac{1}{2}\{(1+4e^{-2\nu\pi})^{1/2} - 1\}, \\ B_k C_k^* = ie^{-\nu\pi},\ B_k^* C_k = -ie^{-\nu\pi}. \end{cases} \tag{3·14}$$

A substitution of Eq. (3·14) in Eq. (3·12) gives

$$\begin{aligned} A_k(\mu t, \mu t') = & 2H_{i\nu}^{(2)}(\mu t) J_{i\nu}(\mu t') + e^{-2\nu\pi} H_{i\nu}^{(1)}(\mu t) H_{i\nu}^{(1)}(\mu t') \\ & + \tfrac{1}{2}\{(1+4e^{-2\nu\pi})^{1/2} - 1\}\{H_{i\nu}^{(1)}(\mu t) H_{i\nu}^{(2)}(\mu t') + H_{i\nu}^{(2)}(\mu t) H_{i\nu}^{(1)}(\mu t')\}, \end{aligned} \tag{3·15}$$

where $J_{i\nu}(z) \equiv \{H_{i\nu}^{(1)}(z) + H_{i\nu}^{(2)}(z)\}/2$. Equation (3·15) is derivable from Eq. (11) in Ref. 6) by replacing ik and mt in $H_{ik}^{(a)}(mt)$ $(a=1, 2)$ and $e^{-k\pi}$ with $i\nu$, μt and $e^{-\nu\pi}$, respectively.

§4. Pair-creation of particles described by Eqs. (3·11) and (3·15)

Equation (3·8) shows that, if $C_k=0$, the time-dependent function $f_k(t)$ oscillates in proportion to $\exp(-i\mu_3 t)$ when $\mu_3 t\gg 1$ (where we shall use the original symbols μ_3 and $\nu_\perp$ in what follows). Accordingly we can define a positive-frequency solution of the homogeneous wave equation $(\Box-U)\phi(x)=0$ by

$$\phi(x)=\frac{a^{-1}(t)}{(2\pi)^3}\int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\{f_k(t)\}_{C_k=0}\,,\quad \{a(t)\equiv t/t_0\} \tag{4·1}$$

where we have dropped the operator part in the integrand. Then its Fourier component $\phi_{\boldsymbol{k}}(t)$ may be represented as follows:

$$\phi_{\boldsymbol{k}}(t)=a^{-1}(t)\,\psi(\boldsymbol{k}_\perp)\frac{\exp(-i\mu_3 t)}{\sqrt{2\mu_3}}\qquad (\text{when } \mu_3 t\gg 1) \tag{4·2}$$

with the proviso that $\psi(\boldsymbol{k}_\perp)$ should obey the normalization condition $\int d\boldsymbol{k}_\perp|\psi(\boldsymbol{k}_\perp)|^2=1$ with $d\boldsymbol{k}_\perp\equiv dk_1dk_2$.

Now it is an easy matter to follow Chitre and Hartle's procedure[3] for describing the pair-creation in question. In fact, by the use of the above $\phi_{\boldsymbol{k}}(t)$ and the Feynman propagator $G_F(x,x')$ given by Eqs. (3·11) and (3·15), we can represent the amplitude that a pair of particles whose wave-number are $\boldsymbol{k}_\perp$ (with state i) and $-\boldsymbol{k}_\perp$ (with state j) are created in the form

$$A_{ij}=-A_0\int d\boldsymbol{k}_\perp \psi_i{}^*(\boldsymbol{k}_\perp)\,\psi_j{}^*(-\boldsymbol{k}_\perp)\,e^{-\pi\nu_\perp} \tag{4·3}$$

(just like Eq. (4·6) in Ref. 3)), where A_0 stands for the amplitude that no particle is created. On the other hand, Eqs. (4·2) and (4·3) in Ref. 3) hold in the present case, too: They represent the amplitude $A_{i_1\cdots i_{2n}}$ for producing n-pairs and the unitarity condition satisfied by those amplitudes. Accordingly we obtain the probability that n-pairs with the wave-numbers $\boldsymbol{k}_\perp$ and $-\boldsymbol{k}_\perp$ are created as follows:

$$P_n(\boldsymbol{k}_1)=e^{-2\pi n\nu_\perp}(1-e^{-2\pi\nu_\perp})\,, \tag{4·4}$$

so that the average number of pairs in the mode $\boldsymbol{k}_\perp$ becomes

$$N(\boldsymbol{k}_\perp)\equiv\sum_{n=0}^{\infty} nP_n(\boldsymbol{k}_\perp)=(e^{2\pi\nu_\perp}-1)^{-1}\,, \tag{4·5}$$

where $\nu_\perp\equiv\{(k_\perp t_0)^2+1/12\}^{1/2}$ as before. Then the number density $\langle N_\perp\rangle$ (i.e., the number of pairs per unit area in the 2-dimensional hyper-surface relating to the anisotropic cosmic expansion) is defined by

$$\langle N_\perp\rangle\equiv\int N(\boldsymbol{k}_\perp)\,d\boldsymbol{k}_\perp=2\pi(t/t_0)^2\int_0^\infty\frac{p_\perp dp_\perp}{[\exp\{2\pi\sqrt{(p_\perp t)^2+1/12}\}-1]}\,, \tag{4·6}$$

where $p_a\equiv k_a/a(t)$ $(a=1,2)$ stand for physical momenta of a particle in the k_1- and k_2-directions, and $p_\perp\equiv(p_1{}^2+p_2{}^2)^{1/2}$.

The appearance of the numerical constant 1/12 (when $\lambda=1$) in the expression for $\nu_\perp = \{(k_\perp t_0)^2 + 1/12\}^{1/2}$ makes Eq. (4·4)$\sim$(4·6) deviate from the black-body formulas, like in the case of gaseous matter with a non-vanishing chemical potential. However, the validity of Eq. (4·4) would become worse for small value of $k_\perp t_0$. Accordingly we shall discard the numerical constant 1/12 in the expression for $\nu_\perp$, i.e., we shall put $\nu_\perp = k_\perp t_0 = p_\perp t$, in what follows. If so, we can regard Eq. (4·5) as the black-body law specified by the temperature (in the c.g.s. unit)

$$T_\perp(t) = \hbar/k_B \pi t\,. \tag{4·7}$$

Similarly, we can reduce Eq. (4·6) to

$$\langle N_\perp \rangle = \frac{(ct_0)^{-2}}{2\pi} \int_0^\infty \frac{x\,dx}{(e^x - 1)} = (\pi/12)\,(ct_0)^{-2}\,. \tag{4·8}$$

In contrast with the black-body temperature $T(t) = \hbar/k_B \pi t$ given by Chitre and Hartle,[3] our formula (4·7) for $T_\perp(t)$ is formally the same as theirs, but the temperature in the k_3-direction does not exist or $T_3(t) = 0$, because the pair-creation does not occur in that direction. Moreover, the counterpart of Eq. (4·8) in the Chitre-Hartle case is given by

$$\langle N \rangle = \int N(\boldsymbol{k})\, d\boldsymbol{k} = (\pi/25.79)\,(ct_0)^{-3}\,,$$

where we have used the relation $\sum_{r=1}^{\infty} r^{-3} = \pi^3/25.79$.

Here let us define the mass density of the created particles by

$$\rho_c(t) \equiv \begin{cases} m\langle N \rangle = (\pi/25.79)\, m\,(ct_0)^{-3}\,, & \text{(Chitre-Hartle)} \\ m\langle N_\perp \rangle^{3/2} = (\pi/12)^{3/2} m\,(ct_0)^{-3}\,. & \text{(Ours)} \end{cases} \tag{4·9}$$

Equations (2·7) and (4·9) show that both universes are dominated by the created particles ($\rho_c > \rho$), provided that

$$t/t_0 > A\,(ct_0/\lambda_m)^{1/2}\,(\lambda_m/\lambda_G) \tag{4·10}$$

with

$$A = \begin{cases} 0.990, & \text{(Chitre-Hartle)} \\ 0.545, & \text{(Ours)} \end{cases} \tag{4·11}$$

where $\lambda_G \equiv (G\hbar/c^3)^{1/2} \sim 10^{-33}$ cm is the Planck length and $\lambda_m \equiv \hbar/mc$ the Compton wave length of a created particle.

§ 5. Speculation on the isotropization of our model-universe

It was first insisted upon by Zel'dovich[1] that the particle creation in an anisotropic universe would effectively serve to its isotropization, while his analysis did not take the reaction effect of those created particles to the background universe into consideration. The subject of this section is to speculate on the isotropization of our model-universe with the metric (2·1) in the level similar to Zel'dovich's.

As already shown, the assemblage of created particles in the universe obeys the black-body law with the temperature $T_{\perp}(t)$ given by Eq. (4·7), but its temperature in the k_3-direction, $T_3(t)$, is vanishing. In other words, those particles are in a state deviating strongly from thermal equilibrium, so that the thermal energy in the $k_{\perp}$-plane begins (at $t=t_i$) to flow into the k_3-direction till some epoch $t=t_f(>t_i)$ at which the equality $T_3(t_f)=T_{\perp}(t_f)$ is established. Then we can regard the epoch t_f as the one at which the isotropization in question is attained.

In order that the above picture may hold in spite of the unrealistic nature (cf. Eq. (2·7)) of our model-universe, we have only to impose the inequality condition (4·10) on $t=t_i$. However, the resulting inequality for t_i/t_0 is shown to be very severe as soon as we estimate it numerically, because the factor (λ_m/λ_G) on the right-hand side of Eq. (4·10) becomes as large as 10^{20} for the creation of particles with nucleon mass $m\sim10^{-24}$g. (It is a matter of course that such a situation arises also in the isotropic case dealt with by Chitre and Hartle, as Eqs. (4·10) and (4·11) show.) In view of this, we shall henceforth discard the inequality condition (4·10) on the assumption that the fictitious fluid specified by Eq. (2·7) is a mere ether without any effect to the created particles.

Now let us describe the interaction among those created particles by means of such a collision time that

$$\tau(t)\equiv1/n\sigma v(t), \tag{5·1}$$

where σ is the collision cross-section defined by $\lambda_m{}^2\equiv(\hbar/mc)^2$, $n\equiv\rho_c/m=\langle N_{\perp}\rangle^{3/2}$ (cf. Eq. (4·9)) is the number density and $v(t)$ stands for the r.m.s. of the 3-velocities given by Eq. (2·6) for test particles in the universe, i.e.,

$$v(t)\equiv\{\langle V_{\perp}{}^2\rangle+\langle V_3^2\rangle\}^{1/2}=\{\langle c_{\perp}{}^2\rangle(t_0/t)^2+\langle c_3^2\rangle\}^{1/2}. \tag{5·2}$$

Then we obtain

$$\tau(t)=\alpha t/\{1+(\beta t/t_0)^2\}^{1/2}, \tag{5·3}$$

where

$$\alpha\equiv7.465(ct_0/\lambda_m)^2\{\langle c_{\perp}{}^2\rangle/c^2\}^{-1/2},\quad \beta\equiv\{\langle c_3^2\rangle/\langle c_{\perp}{}^2\rangle\}^{1/2}. \tag{5·4}$$

The next task is to set up a suitable system of the dynamical equations for $T_{\perp}(t)$ and $T_3(t)$ in which the collision time given by Eq. (5·3) would play a significant role. For simplicity, we shall adopt the following system:

$$dT_{\perp}/dt=-T_{\perp}/\tau(t),\ dT_3/dt=(T_3+T_*)/\tau(t) \tag{5·5}$$

on the prescription that $T_3(t_i)=0$ and $T_*\equiv T_{\perp}(t_i)=\hbar/k_B\pi t_i$ (cf. Eq. (4·7)). An integration of the above differential equations gives

$$\begin{cases} T_{\perp}(t)=T_*\exp\left\{-\int_{t_i}^{t}dt/\tau(t)\right\}, \\ T_3(t)=T_*\left[\exp\left\{\int_{t_i}^{t}dt/\tau(t)\right\}-1\right]. \end{cases} \tag{5·6}$$

In order that both temperatures may be equal to each other at $t=t_f$, i.e., $T_3(t_f)=T_\perp(t_f)\equiv T(t_f)$, we must have

$$T(t_f)/T_*=(\sqrt{5}-1)/2=0.618 \tag{5·7}$$

and

$$e^{X_f}\left(\frac{X_f-1}{X_f+1}\right)^{1/2}=\left(\frac{\sqrt{5}+1}{2}\right)^{\alpha}e^{X_i}\left(\frac{X_i-1}{X_i+1}\right)^{1/2}, \tag{5·8}$$

where

$$X_n\equiv\{1+(\beta t_n/t_0)^2\}^{1/2}. \quad (n=i,f) \tag{5·9}$$

The parameters α and β defined by Eq. (5·4) should satisfy the inequality

$$(\alpha^2-1)^{1/2}\geq\beta t_i/t_0 \quad (>0), \tag{5·10}$$

because $|-T_\perp(t)/\dot{T}_\perp(t)|=\tau(t)$ when $t\geq t_i$ must not be smaller than its counterpart when $t_i\geq t$, i.e., $|-T_\perp(t)/\dot{T}_\perp(t)|=t$ (cf. Eq. (4·7)).

Remark. It is clear that Eq. (5·5) ceases to be valid at $t=t_f$, because it cannot assure the condition $T_3(t)=T_\perp(t)\equiv T(t)$ for $t\geq t_f$ (but not for a particular epoch $t=t_f$). This is, of course, due to our disregard of any non-linear term in setting up the dynamical equations for $T_\perp(t)$ and $T_3(t)$. In spite of such a defect, Eqs. (5·7) and (5·8) would provide us with some useful information at the instant when the thermal equilibrium (and, therefore, the isotropization of our model-universe) is attained.

To estimate the numerical values of $T_*=\hbar/k_B\pi t_i$ and t_f/t_i on the above premise, let us tentatively assume that

$$t_i=t_0=\hbar/mc^2, \quad \langle c_\perp{}^2\rangle^{1/2}=c,$$

so that Eq. (5·4) gives $\alpha=7.465$. If $m=1.673\times10^{-24}$ g (the nucleon mass) in particular, we obtain

$$t_i=7.0\times10^{-25}\,\text{sec}, \quad T_*=3.5\times10^{12}\text{K}. \tag{5·11}$$

Then it follows from Eqs. (5·8)$\sim$(5·10) that

$$t_f/t_i=\begin{Bmatrix}4.1 & \text{for} & \beta=1\\ 12.5 & \text{for} & \beta=1/5\end{Bmatrix}. \tag{5·12}$$

The numerical values given by Eqs. (5·7), (5·11) and (5·12) permit us to conclude that, so far as the creation of scalar particles with $m\sim10^{-24}$g and $\langle c_\perp{}^2\rangle^{1/2}\sim c$ is concerned, their assemblage reaches quickly to the thermal equilibrium.

§ 6. Concluding remarks

The universe specified by the metric (2·1) is an anisotropic version of Chitre-Hartle's model-universe used in their discussion of the pair-creation of scalar particles. These universes permit us to obtain not only an explicit form of the geodesic bi-scalar $\sigma(x,x')$, but also the one for a quantized massive scalar field $\phi(x)$ on which the 4-dimensional commutation function $G(x,x')$ and the elementary solution $G^{(1)}(x,x')$ stand. Accordingly the example presented in this paper (together with the one in Ref. 6)) will play an important role in our future work on the relation between Chitre and Hartle's path-integral quantization procedure and our canonical one.

Moreover, the example is also interesting from the standpoint of Zel'dovich's idea that the particle creation in an anisotropic universe will serve to its isotropization, because we can now deal with the problem in terms of the thermodynamical language. As regards the contents of § 5, however, there are several weak points, especially the linearity of Eq. (5·5) for $T_\perp(t)$ and $T_3(t)$. A more reliable set of the dynamical equations for them will, therefore, be sought for in a separate paper.

Appendix

——*Examination of the Procedure in Ref. 5) for Fixing the Arbitrary Constants* $B_k \sim C_k^*$ *in Eq.* (3·12)——

As in Ref. 5), let us put

$$\begin{cases} a_k(t) \equiv \{f_k(t) - f_k^*(t)\}/i\sqrt{2}\,, \quad b_k(t) \equiv \{f_k(t) + f_k^*(t)\}/\sqrt{2}\,, \\ \qquad a_k(t)\dot{b}_k(t) - b_k(t)\dot{a}_k(t) = 1\,, \end{cases} \tag{A·1}$$

where $f_k(t)$ and $f_k^*(t)$ are the time-dependent functions given by Eq. (3·8). Then we have

$$\begin{aligned} -i\{f_k(t)f_k^*(t') - f_k(t')f_k^*(t)\} &= a_k(t)b_k(t') - a_k(t')b_k(t) \\ &= (1/\mu)\{j_{i\nu}(\mu t)y_{i\nu}(\mu t') - j_{i\nu}(\mu t')y_{i\nu}(\mu t)\} \end{aligned} \tag{A·2}$$

and

$$\begin{aligned} \{f_k(t)f_k^*(t') + f_k(t')f_k^*(t)\} &= a_k(t)a_k(t') + b_k(t)b_k(t') \\ &= (1/\mu)[(|B_k|^2 + |C_k|^2)\{j_{i\nu}(\mu t)j_{i\nu}(\mu t') + y_{i\nu}(\mu t)y_{i\nu}(\mu t')\} \\ &\quad - (B_k^*C_k e^{-\nu\pi} + B_k C_k^* e^{\nu\pi})\{j_{i\nu}(\mu t)y_{i\nu}(\mu t') + j_{i\nu}(\mu t')y_{i\nu}(\mu t)\} \\ &\quad + i(B_k^*C_k e^{-\nu\pi} - B_k C_k^* e^{\nu\pi})\{j_{i\nu}(\mu t)j_{i\nu}(\mu t') - y_{i\nu}(\mu t)y_{i\nu}(\mu t')\}], \end{aligned} \tag{A·3}$$

where

$$\begin{Bmatrix} j_{i\nu}(z) \\ iy_{i\nu}(z) \end{Bmatrix} \equiv \frac{1}{2}(\pi z/2)^{1/2}\{H_{i\nu}^{(1)}(z) + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} H_{i\nu}^{(2)}(z)\}. \tag{A·4}$$

Equation (A·2) shows that the commutation function $G(x,x')$ given by Eq. (3·1) is identical with the previous one, i.e., Eq. (4·9) in Ref. 5). On the other hand, the elementary solution $G^{(1)}(x,x')$ given by Eq. (5·4) in Ref. 5) should be identical with the one derivable from Eqs. (3·2) and (A·3) on the requirement that it must be symmetric with respect to $j_{i\nu}(\mu t)j_{i\nu}(\mu t')$ and $y_{i\nu}(\mu t)y_{i\nu}(\mu t')$. In reality, however, we dropped (by a careless mistake) the factors $\exp(\mp\nu\pi)$ in Eq. (A·3), so that the erroneous condition $B_k{}^*C_k=B_kC_k{}^*$ (on which Eq. (5·4) in Ref. 5) stands) was obtained. As is easily seen, the correct condition derivable from the above symmetry requirement is $C_k=C_k{}^*=0$ or $|B_k|^2=1$ (cf. Eq. (3·9)) which corresponds to the case $P_k(\mu t_0)=1$ and $Q_k(\mu t_0)=0$ in Ref. 5). In other words, the 2-point function $\widehat{G}^{(1)}(x,x')$ given by Eq. (A·1) in Ref. 5) is the correct elementary solution to be derived from the symmetry requirement mentioned above.

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