

ON THE CRITICAL GALTON-WATSON PROCESS WITH IMMIGRATION

A. G. PAKES

Monash University

(Received 4th September 1969)

Communicated by P. D. Finch

1. Introduction

Consider a Galton-Watson process in which each individual reproduces independently of all others and has probability a_j ($j = 0, 1, \dots$) of giving rise to j progeny in the following generation and in which there is an independent immigration component where b_j ($j = 0, 1, \dots$) is the probability that j individuals enter the population at each generation. Then letting X_n ($n = 0, 1, \dots$) be the population size of the n -th generation, it is known (Heathcote [4], [5]) that $\{X_n\}$ defines a Markov chain on the non-negative integers. Unless otherwise stated, we shall consider only those offspring and immigration distributions that make the Markov chain $\{X_n\}$ irreducible and aperiodic.

Heathcote [5] has shown that in the case $\alpha = \sum_j j a_j < 1$, a necessary and sufficient condition for $\{X_n\}$ to be positive-recurrent is that $\sum_{j=1}^{\infty} b_j \log j < \infty$. Seneta [7] has shown that if $\alpha = 1$ and if $2\gamma = \sum_j j(j-1)a_j = \infty$ then it is possible for $\{X_n\}$ to be positive-recurrent.

In this paper we consider the case $\alpha = 1$ and $\beta, \gamma < \infty$ where $\beta = \sum_j j b_j$ is the mean of the immigration distribution. We shall show that the Markov chain $\{X_n\}$ may be either null-recurrent or transient. In the case of null-recurrence we obtain some information on the occupation times of the zero state. Finally, in the last section we show that X_n/n tends in distribution to a random variable having a gamma distribution.

2. Classification of the Markov chain (X_n)

Let $p_{ij}^{(n)}$ ($i, j, n = 0, 1, \dots$) be the n -step transition probability from state i to j and let $P_i^{(n)}(x) = \sum_{j=0}^{\infty} p_{ij}^{(n)} x^j$ ($|x| < 1$). Then letting $A(x) = \sum_{j=0}^{\infty} a_j x^j$ and $B(x) = \sum_{j=0}^{\infty} b_j x^j$ ($|x| < 1$) be the probability generating functions of the offspring and immigration distributions respectively, it is not difficult to show that

$$(1) \quad P_i^{(n)}(x) = [A_n(x)]^i \prod_{m=0}^{n-1} B[A_m(x)]$$

where $A_0(x) = x$ and $A_{n+1}(x) = A(A_n(x))$ ($n = 0, 1, \dots$), so that in particular we have

$$(2) \quad p_{00}^{(n)} = B(0) \prod_{m=1}^{n-1} B(A_m(0))$$

and it is clear that $p_{00}^{(n)}$ is a non increasing sequence.

Before stating theorem 1, we shall state a theorem which will play a key role in our work.

THEOREM A. (Kesten, Ney and Spitzer [6]). *If $\alpha = 1$ and $0 < \gamma < \infty$ and*

$$1/(1-x) + n\gamma - 1/[1 - A_n(x)] = h_n(x) \quad (0 \leq x < 1)$$

then $\lim_{n \rightarrow \infty} h_n(x)/n = 0$ uniformly in $0 \leq x < 1$. Furthermore, $h_n(x) = \sum_{m=0}^{n-1} \delta(A_m(x))$ where $\delta(x)$ satisfies the inequality

$$(3) \quad \frac{-\gamma^2(1-x)}{1-a_0} \leq \delta(x) \leq \varepsilon(x) \quad (0 \leq x < 1)$$

where $0 \leq \varepsilon(x) = \gamma - [A(x) - x]/(1-x)^2 \leq \gamma$ and $\varepsilon(x)$ is non-increasing in x and $\varepsilon(x) \downarrow 0$ ($x \uparrow 1$).

Observe that our assumption of irreducibility implies that $\gamma > 0$.

THEOREM 1. *Let $\alpha = 1$ and $\beta, \gamma < \infty$, then the Markov chain $\{X_n\}$ is not positive-recurrent. Further, let $\sigma = \beta/\gamma$, then $\{X_n\}$ is null-recurrent if $\sigma < 1$ and transient if $\sigma > 1$. Define $\varepsilon(x)$ as in theorem A and let $\varepsilon(x) = O[(1-x)^\delta]$ ($x \uparrow 1$) for some $\delta > 0$. If $B''(1-) < \infty$ then $p_{00}^{(n)} \sim Cn^{-\sigma}$ as $n \rightarrow \infty$ where C is a finite, positive constant, and in particular, if $\sigma = 1$ then $\{X_n\}$ is null-recurrent.*

PROOF. By irreducibility and aperiodicity, the Markov chain $\{X_n\}$ is not positive-recurrent if $\lim_{n \rightarrow \infty} p_{00}^{(n)} = 0$, that is if the infinite product $\prod_{m=1}^\infty B(A_m(0))$ diverges to zero. (Observe that irreducibility implies that $B(0) > 0$.) It is known (Seneta [7]) that if $\beta < \infty$ then a necessary and sufficient condition for this is the divergence of the integral $\int_0^1 [(1-x)/(A(x)-x)]dx$. By Taylor's theorem, $A(x) = 1 - (1-x) + (1-x)^2 A''(\theta)/2$ ($x < \theta < 1$) and so if $0 < A''(1-) < \infty$, we see that the integrand is bounded below by $[(1-x)\gamma]^{-1}$ and so the integral diverges.

Thus the Markov chain will be transient or null-recurrent according as the series $\sum_{n=0}^\infty p_{00}^{(n)}$ converges or diverges, and by Raabe's test (Ferrar [2]) the first alternative will occur if $\lim_{n \rightarrow \infty} n(1 - p_{00}^{(n+1)}/p_{00}^{(n)}) = \lim_{n \rightarrow \infty} n[1 - B(A_n(0))] > 1$ and the second alternative occurs if this limit < 1 ; the equality in the last expression follows from equation (2). The hypotheses and Taylor's theorem show that for $0 \leq x < 1$, $B(x) = 1 - \beta(1-x) + o(1-x)$ ($x \uparrow 1$) and since $A_n(0) \uparrow 1$ ($n \rightarrow \infty$) (Harris [3]) we have

$$(4) \quad n[1 - B(A_n(0))] = \beta n(1 - A_n(0)) + no(1 - A_n(0)) \quad (n \rightarrow \infty)$$

Theorem A, with $x = 0$, shows that as $n \rightarrow \infty$ the right hand side of expression (4) tends to σ .

We shall now show that under all the conditions stated in the theorem, $0 < \lim_{n \rightarrow \infty} n^\sigma p_{00}^{(n)} < \infty$. From equation (2) we have

$$n^\sigma p_{00}^{(n)} = B(0) \prod_{m=1}^{n-1} \left(\frac{m+1}{m}\right)^\sigma B(A_m(0)) = B(0) \prod_{m=1}^{n-1} D_m$$

where $D_m = (1 + 1/m)^\sigma B(A_m(0))$. A necessary and sufficient condition for the required limit to exist and be finite and positive is that $-\infty < \sum_{m=1}^\infty (D_m - 1) < \infty$. Theorem A shows that $1 - A_m(0) = 1/(1 - h_m + m\gamma)$ where $h_m = o(m)$ ($m \rightarrow \infty$). Using this fact, a three term Taylor expansion of $B(x)$ to the left of $x = 1$, and the fact that $(1 + 1/m)^\sigma = 1 + \sigma/m + O(1/m^2)$ enables us to write

$$\begin{aligned} D_m - 1 &= \frac{\sigma}{m} - \frac{\beta}{m\gamma + 1 - h_m} + O(1/m^2) \\ &= \frac{\sigma(1 - h_m)/\gamma}{m^2 + m(1 - h_m)/\gamma} + O(1/m^2) \end{aligned}$$

It is clear that $\sum_{m=1}^\infty (D_m - 1)$ will converge to a finite limit if $\sum_{m=1}^\infty h_m/m^2$ does so. In fact Harris [3] shows that if $A'''(1-) < \infty$, that is $\delta \geq 1$, $[h_m] = O(\log m)$ so in this case the series converges. We now consider the case $0 < \delta < 1$.

Using expression (3), the non-increasing nature of $\varepsilon(x)$ and the fact that $A_n(0) \uparrow 1$ ($n \rightarrow \infty$), we obtain

$$(5) \quad \frac{-\gamma^2}{1 - a_0} \sum_{m=1}^\infty \frac{1}{m^2} \sum_{k=0}^{m-1} (1 - A_k(0)) \leq \sum_{m=1}^\infty \frac{h_m}{m^2} \leq \sum_{m=1}^\infty \frac{1}{m^2} \sum_{k=0}^{m-1} \varepsilon(A_k(0))$$

For sufficiently large n there exist positive constants a, b such that $a/n < 1 - A_n(0) < b/n$ and so we see that the terms of the series on the left of equation (5) are $O[(\log m)/m^2]$ for large m . Using the condition on $\varepsilon(x)$ given in the statement of the theorem and also that $0 \leq \varepsilon(x) \leq \gamma$ ($0 \leq x < 1$), it is not difficult to show that the terms on the right hand of equation (5) are $O(m^{-1-\delta})$ for large m . The proof is now complete.

REMARKS

1. By an argument very similar to that used in the first part of the proof of lemma 8 of Kesten et al. (1966), it follows that the condition on $\varepsilon(x)$ given in theorem 1 above may be replaced by $\sum_{j=1}^\infty a_j j^2 \log j < \infty$.

2. In obtaining the asymptotic form of $p_{00}^{(n)}$ we have not made use of irreducibility; we only require $A(0), \gamma > 0$ and that $0 < B(0) < 1$, that is, there is positive probability of no immigrants in any generation.

3. By way of example, let $A(x) = 1/(2-x)$ and $B(x) = [1/(2-x)]^\nu$ ($\nu > 0$). Using the fractional linear generating function in Harris [3] p. 9, we see that $p_{00}^{(n)} \sim n^{-\nu}$ ($n \rightarrow \infty$). If instead we have a Poisson immigration component, that is $B(x) = e^{-\beta(1-x)}$, then $p_{00}^{(n)} \sim e^{-\eta\beta} n^{-\beta}$, where η is Euler's constant.

Since the Markov chain is transient when $\sigma > 1$, the zero state is entered only finitely often with probability one, that is, with probability one, after a finite number of generations have passed the population size will always be positive. The situation when $\sigma \leq 1$ is of course different, and the following theorem gives some information on this matter.

THEOREM 2. *Let $\alpha = 1$, $0 < B(0)$, $\beta, \gamma < \infty$, $B(0) < 1$ and either let $\varepsilon(x) = O[(1-x)^\delta]$ ($\delta > 0$, $0 \leq x < 1$) or $\sum_{j=1}^\infty a_j j^2 \log j < \infty$, so that $p_{00}^{(n)} \sim Cn^{-\sigma}$ ($0 < C < \infty$) and let $\sigma \leq 1$. Define the sequence $\{U_n\}$ ($n = 1, 2, \dots$) by $U_n = C\Gamma(1-\sigma)n^{1-\sigma}$ if $\sigma < 1$ and $U_n = C \log n$ if $\sigma = 1$. Finally let $V(\cdot)$ be the indicator function of the zero state, that is $V(j) = 1$ if $j = 0$, and $V(j) = 0$ otherwise. Then if $0 < \sigma \leq 1$, we have*

$$\lim_{n \rightarrow \infty} \Pr \left\{ \frac{1}{U_n} \sum_{m=0}^n V(X_m) \leq x \right\} = G_{1-\sigma}(x)$$

where $G_\xi(x)$ is the Mittag-Leffler distribution function given by

$$G_\xi(x) = \frac{1}{\pi\xi} \int_0^x \sum_{j=0}^\infty \frac{(-)^{j-1}}{j!} \sin(\pi\xi j) \Gamma(\xi j + 1) y^{j-1} dy$$

if $1 > \xi \geq 0$.

PROOF. Use of the inequality

$$\int_v^{n+1} x^{-\sigma} dx < \sum_{m=v}^n m^{-\sigma} < \int_{v-1}^n x^{-\sigma} dx$$

where v, n are positive integers and $n > v$ shows that

$$\sum_{m=0}^n p_{00}^{(m)} \sim \begin{cases} Cn^{1-\sigma}/(1-\sigma) & \text{if } \sigma < 1 \\ C \log n & \text{if } \sigma = 1 \end{cases} \quad (n \rightarrow \infty)$$

An Abelian theorem then shows that

$$\sum_{n=0}^\infty p_{00}^{(n)} x^n \sim \begin{cases} C\Gamma(1-\sigma)(1-x)^{-1+\sigma} & \text{if } \sigma < 1 \\ -C \log(1-x) & \text{if } \sigma = 1 \end{cases} \quad (x \uparrow 1)$$

The conditions of the occupation time theorem of Darling and Kac [1] are seen to be fulfilled and the theorem follows.

3. A limit theorem

It is easy to show from (1) that when $\alpha = 1$,

$$E(X_n | X_0 = i) = n\beta + i$$

and so it seems appropriate to investigate the limit in some sense of X_n/n as $n \rightarrow \infty$. If we consider convergence in distribution, the following theorem shows that we get a non-trivial result.

THEOREM 3. *Let $\alpha = 1$ and $0 < \gamma, \beta, \sigma < \infty$. Then the sequence $\{X_n/n\}$ ($n = 1, 2, \dots$) tends in distribution to the gamma variate having the density function*

$$f(t) = \frac{1}{\beta\Gamma(\sigma)} \left(\frac{t}{\beta}\right)^{\sigma-1} e^{-t/\beta} \quad (t > 0)$$

PROOF. It follows from equation (1) that ($\theta > 0$),

$$\psi_i^{(n)}(\theta) = E(e^{-\theta X_n/n} | X_0 = i) = [A_n(e^{-\theta/n})]^i \prod_{m=0}^{n-1} B[A_m(e^{-\theta/n})]$$

Since $A_n(x) \uparrow 1$ ($n \rightarrow \infty$), we see that the first term on the right tends to unity. Writing $b_{mn}(\theta) = B[A_m(e^{-\theta/n})]$, and using $\log(1-x) \geq -x-x^2/(1-x)$ ($0 \leq x < 1$), we have

$$\begin{aligned} (6) \quad \phi^{(n)}(\theta) &= \log \psi_0^{(n)}(\theta) = \sum_{m=0}^{n-1} \log [1 - (1 - b_{mn}(\theta))] \\ &= - \sum_{m=0}^{n-1} (1 - b_{mn}(\theta)) + R_1^{(n)}(\theta) \end{aligned}$$

where

$$\begin{aligned} 0 \geq R_1^{(n)}(\theta) &\geq - \sum_{m=0}^{n-1} [1 - b_{mn}(\theta)]^2 / b_{mn}(\theta) \\ &\geq - [(1 - b_{0n}(\theta)) / b_{0n}(\theta)] \sum_{m=0}^{n-1} (1 - b_{mn}(\theta)) \end{aligned}$$

since $b_{mn}(\theta)$ is non-decreasing in m for fixed n and θ . Since $1 - b_{0n}(\theta) \rightarrow 0$ ($n \rightarrow \infty$) it is clear that $R_1^{(n)}(\theta) \rightarrow 0$ if the first expression on the right hand side of equation (6) has a finite limit as $n \rightarrow \infty$.

For $0 \leq x < 1$, we have $1 - B(x) = \beta(1-x) - (1-x)f(x)$ where $0 \leq f(x) = (1-x)B''(\xi)/2$ and $x < \xi < 1$ and $f(x) = o(1)$ as $x \uparrow 1$. Then letting $a_{mn}(\theta) = A_m(e^{-\theta/n})$, equation (6) becomes

$$\phi^{(n)}(\theta) = -\beta \sum_{m=0}^{n-1} (1 - a_{mn}(\theta)) + R_1^{(n)}(\theta) + R_2^{(n)}(\theta)$$

where

$$\begin{aligned} 0 \leq R_2^{(n)}(\theta) &= \sum_{m=0}^{n-1} (1 - a_{mn}(\theta)) f[a_{mn}(\theta)] \\ &\leq (1 - e^{-\theta/n}) \sum_{m=0}^{n-1} f(a_{m1}(\theta)) \end{aligned}$$

since $f(x)$ is non-increasing. The last expression approaches zero ($n \rightarrow \infty$) since $1 - e^{-\theta/n} \sim \theta/n$ and $f[a_{m1}(\theta)] = o(1)$ ($n \rightarrow \infty$).

Theorem A shows that we can write

$$1 - A_m(x) = \frac{1-x}{1+\gamma m(1-x)} [1 + g_m(x)]$$

where $g_m(x) \rightarrow 0$ uniformly in $0 \leq x < 1$ ($m \rightarrow \infty$) and $g_m(1) = 0$. Thus we have

$$\phi^{(n)}(\theta) = -\beta \sum_{m=0}^{n-1} \frac{1 - e^{-\theta/n}}{1 + \gamma m(1 - e^{-\theta/n})} + R_1^{(n)}(\theta) + R_3^{(n)}(\theta) + o(1) \quad (n \rightarrow \infty)$$

where

$$R_3^{(n)}(\theta) = -\beta \sum_{m=0}^n \frac{1 - e^{-\theta/n}}{1 + \gamma m(1 - e^{-\theta/n})} g_m(e^{-\theta/n}) \quad (7)$$

It follows from the uniform convergence of the $g_m(\cdot)$ that there exists $M(\varepsilon)$ such that $|g_m(e^{-\theta/n})| < \varepsilon$ ($n = 1, 2, \dots$) if $m > M(\varepsilon)$. Breaking the summation in expression (7) into the form $\sum_{m=0}^{M(\varepsilon)} + \sum_{m=M(\varepsilon)+1}^n$ and using the fact that

$$[1 - e^{-\theta/n}]/[1 + \gamma m(1 - e^{-\theta/n})] \leq 1 - e^{-\theta/n} \sim \theta/n \quad (n \rightarrow \infty)$$

shows that $R_3^{(n)}(\theta) = o(1)$ ($n \rightarrow \infty$).

It is easily seen that

$$0 \leq \frac{\theta/n}{1 + \gamma m\theta/n} - \frac{1 - e^{-\theta/n}}{1 + \gamma m(1 - e^{-\theta/n})} \leq \frac{\theta^2}{2n^2}$$

so that finally we have

$$\phi^{(n)}(\theta) = -\beta \sum_{m=0}^{n-1} \frac{\theta/n}{1 + \gamma m\theta/n} + R_1^{(n)}(\theta) + o(1) \quad (n \rightarrow \infty)$$

The sum in this expression can be recognized as an upper Darboux sum of the Riemann integral

$$\theta \int_0^1 (1 + \gamma\theta x)^{-1} dx,$$

so that we obtain

$$\phi^{(n)}(\theta) = -\sigma \log(1 + \gamma\theta) + o(1) \quad (n \rightarrow \infty)$$

and thus $\lim_{n \rightarrow \infty} \psi_i^{(n)}(\theta) = (1 + \gamma\theta)^{-\sigma}$ which is the Laplace transform of the density function given in the statement of the theorem. The convergence in distribution assertion follows from the continuity theorem for Laplace-Stieltjes transforms.

It is clear that the theorem is true under the conditions given in remark 2 following theorem 1.

Added in proof. Since submitting this paper, the author has learned that theorem 3 was obtained independently by E. Seneta in *J. Roy. Stat. Soc.* 32 B (1970), 149–52.

Acknowledgement

This work was carried out during the tenure of a Department of Supply Post-graduate Studentship.

References

- [1] D. A. Darling, and M. Kac, On occupation times for Markov processes. *Trans. Amer. Math. Soc.* 84 (1957), 444–458.
- [2] W. L. Ferrar, *A Textbook of Convergence* (O.U.P. Oxford, 1937).
- [3] T. E. Harris, *The Theory of Branching Processes*. (Springer-Verlag, Berlin, 1963).
- [4] C. R. Heathcote, A branching process allowing immigration. *J. Roy. Stat. Soc. Ser. B.* 27 (1965), 138–43.
- [5] C. R. Heathcote, Corrections and comments on the paper ‘A branching process allowing immigration’. *J. Roy. Stat. Soc. Ser. B.* 28 (1966), 213–17.
- [6] H. Kesten, P. Ney, and F. Spitzer, The Galton-Watson process with mean one and finite variance. *Teor. Veroyatnost. i Primenen.*, 11 (1966) 579–611.
- [7] E. Seneta, The stationary distribution of a branching process allowing immigration: a remark on the critical case. *J. Roy. Stat. Soc., Ser B.* 30 (1968) 176–9.

Department of Mathematics
Monash University
Melbourne, Australia